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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Via E.Orabona, 4
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+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
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J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
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A novel approach for solving fully fuzzy linear programming problem with LR flat fuzzy numbers[†]

Zeng-Tai Gong[a,*], Wen-Cui Zhao[a,b]
[a] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*
[b] *Group of Mathematics, YiFu Experiment Middle School of Tianshui, Tianshui 741200, China*

Abstract: The fuzzy linear programming problem with triangular fuzzy numbers in its objective functions or constraints has been discussed by many scholars based on using Zadeh's decomposition theorem of fuzzy numbers and transforming it into some crisp linear programming problems. However, existed methods and results are limited if the fuzzy linear programming problem with generalized fuzzy numbers in its objective functions and constraints. In this paper, we discuss fully fuzzy linear programming (FFLP) problems of which all parameters are LR flat fuzzy numbers and a simple but practical method is developed to solve it. In this method, the approximate representation of fully fuzzy constraints is investigated by means of the arithmetic operations on LR flat fuzzy numbers space firstly. Meanwhile, we constructed a auxiliary multi-objective programming to solve the FFLP problems. After that, three approaches are proposed to solve the constructed auxiliary multi-objective programming, including optimistic approach, pessimistic approach and linear sum approach based on membership function. Finally, the obtained results are compared with the existing works and numerical example is given to illustrate the effectiveness of the proposed method.
Keywords: LR flat fuzzy number; multi-objective linear programming; fully fuzzy linear programming

1. Introduction

Linear programming (LP) is an essential mathematical tool in science and technology. Although, it has been investigated and expanded for more than six decades by many researchers from various point of views, it is still useful to develop new approaches in order to better fit the real world problems within the framework of linear programming. In conventional approach, parameters of linear programming models must be well defined and precise. However, in real world environment, this is not a realistic assumption. Usually, most of information is not deterministic and in this situation human has a capability to make a rational decision based on this uncertainty. This is hard challenge for decision makers to design an intelligent system which make a decision the same as the human. In fact, some of parameters of the system may be represented by fuzzy quantities rather than crisp ones in practice. Hence, it is necessary to develop mathematical theory and numerical schemes to handle fuzzy linear programming (FLP) problems. Bellman and Zadeh [1] proposed the concept of decision making in fuzzy environments. Since then, a number of researchers have exhibited their interest to various types of the FLP problems and proposed several approaches for solving these problems [2-11]. FLP model with triangular fuzzy numbers (TFNs) [6]. Lai and Hwang [6] developed a new approach to some possibilistic linear programming problems with TFNs, they transformed the fuzzy linear programming into a multi-objective linear programming model, involving three objective functions: minimizing the low loss, maximizing the most possible value and maximizing the upper the profit. FLP model with trapezoidal fuzzy numbers (TrFNs) [7 – 11]. For example, Ganesan [7] and Ebrahimnejad [8] studied the fuzzy linear programs with TrFNs, but the constructed fuzzy linear programming models are only suitable for the symmetrical TrFNs. Mahadavi-Amiri [9] and Campos [10] utilized the ranking function to solve the fuzzy linear programming models with TrFNs. Wan [11] developed a method which taken advantage of multi-objective linear programming model to solve FLP problems with TrFNs. However, in all of the above mentioned works, those cases of

[†]This work is supported by the National Natural Science Foundations of China (11461062, 61262022).
[*]Corresponding Author: Zeng-Tai Gong. Tel.:+86 09317971430.
E-mail addresses: gongzt@nwnu.edu.cn, zt-gong@163.com(Zeng-Tai Gong).

FLP problems have been studied in which not all parts of the problem were assumed to be fuzzy, e.g., only the right hand side or the objective function coefficients were fuzzy but the variables were not fuzzy. The FLP problems in which all the parameters as well as the variables are represented by fuzzy numbers are known as fully fuzzy linear programming (FFLP) problems. Many authors [12-14] have proposed different methods for solving FFLP problems.

Lotfi [12] proposed a novel method to obtain the approximate solution of FFLP problems by using the concept of the symmetric triangular fuzzy numbers and introduce an approach to defuzzify a general fuzzy quantity. In Kumar's article [13], an exact optimal solution is achieved using a linear ranking function. In this method, the linear ranking function has been used to convert the fuzzy objective function to a crisp objective function. The shortcoming exists of it is that the fuzziness of objective function has been neglected by the linear ranking function. Ezzati [14] proposed a new algorithm to solve FFLP problems with TFNs. To the best of our knowledge, till now there is no method in the literature to obtain the exact solution of FFLP problems in which all the parameters as well as the variables are represented by LR flat fuzzy numbers. The LR flat fuzzy number and its operations were first introduced by Dubois [15]. We know that triangular fuzzy numbers are just specious cases of LR flat fuzzy numbers. In 2006, Dehgham [16] discussed the computational methods for fully fuzzy linear systems whose coefficient matrix and the right-hand side vector are denoted by LR fuzzy numbers. In this paper, the approximate representation of fully fuzzy constraints (max(min) $\widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}$, s.t. $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}},\ \widetilde{\boldsymbol{x}} \geqslant \boldsymbol{0}$, where $\widetilde{\boldsymbol{c}}^T$, $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{c}}$ are fuzzy matrixs which consist of LR flat fuzzy numbers) is investigated by means of the arithmetic operations on LR flat fuzzy numbers space firstly. Meanwhile, we constructed a auxiliary multi-objective programming to solve the FFLP problems. After that, three approaches are proposed to solve the constructed auxiliary multi-objective programming, including optimistic approach, pessimistic approach and linear sum approach based on membership function. Finally, the results obtained are compared with the existing works and numerical example is given to illustrate the effectiveness of the proposed method.

The structure of this paper is organized as follows. In Section 2, we review some basic concepts and introduce the interval objective programming. In Section 3, the approximate representation of fully fuzzy constraints is given. The method for solving the FFLP problem is discussed in Section 4. In Section 5, the numerical example is solved and the obtained results are compared with the existing works. Conclusion is drawn in Section 6.

2. Preliminaries

2.1 Basic definitions and arithmetic operations

In this section, some basic definitions and arithmetic operations of LR flat fuzzy numbers are presented.

Definition 2.1.[17] A fuzzy number $\widetilde{A}$, defined on universal set of real numbers R, denoted as $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$u_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & x < m,\ \alpha > 0, \\ R(\frac{x-n}{\beta}) & x > n,\ \beta > 0, \\ 1 & m \leqslant x \leqslant n, \end{cases}$$

where the closed interval $[m, n]$ is the mode of $\widetilde{A}$, α and β are the left and right spreads, respectively. The function $L(\cdot)$, which is called the left shape function satisfying:

(1) $L(x) = L(-x)$;

(2) $L(0) = 1$ and $L(1) = 0$;

(3) $L(x)$ is non-increasing on $[0, +\infty)$.

The definition of the right shape function $R(\cdot)$ is usually similar to that of $L(\cdot)$.

Usually, we could define the predetermined left spreads shape functions $L(\cdot)$ and right spreads shape functions $R(\cdot)$ as follows according to the need of mathematical modeling:

(1) $L(x) = \max\{0, 1 - |x|^p\}(p > 0)$;

(2) $L(x) = \max\{0, \frac{1}{e-1}(e^{1-|x|^p} - 1)\}(p > 0)$;

(3) $L(x) = \max\{0, \frac{2}{1+|x|^p} - 1\}(p > 0)$,

and similarly to $R(\cdot)$.

Definition 2.2.[18] An LR flat fuzzy number $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be non-negative LR flat fuzzy number if $m - \alpha \geqslant 0$ and is said to be non-positive LR flat fuzzy number if $n + \beta \leqslant 0$.

Definition 2.3.[17] LR flat fuzzy numbers $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are said to be equal, i.e., $\widetilde{A_1} = \widetilde{A_2}$ if and only if $m_1 = m_2,\ n_1 = n_2,\ \alpha_1 = \alpha_2,\ \beta_1 = \beta_2$.

Remark 2.1. If $m = n$, then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be an LR fuzzy number and is denoted as $(m, \alpha, \beta)_{LR}$ or $(n, \alpha, \beta)_{LR}$.

Remark 2.2. If $m \neq n$ and $L(x) = R(x) = \max\{0, 1 - |x|\}$, then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a trapezoidal fuzzy number and is denoted as (m, n, α, β).

Remark 2.3. If $m = n$ and $L(x) = R(x) = \max\{0, 1 - |x|\}$, then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number and is denoted as (m, α, β).

The arithmetic operations between two LR flat fuzzy numbers are defined by the extension principle as follows[17]:

Let $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$, $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be any LR flat fuzzy numbers and $\widetilde{A_3} = (m_3, n_3, \alpha_3, \beta_3)_{RL}$ be any RL flat fuzzy number. Then,

(i) $\widetilde{A_1} \oplus \widetilde{A_2} = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$.

(ii) $\widetilde{A_1} \ominus \widetilde{A_3} = (m_1 - n_3, n_1 - m_3, \alpha_1 + \beta_3, \alpha_3 + \beta_1)_{LR}$.

(iii) If $\widetilde{A_1}$ and $\widetilde{A_2}$ are non-negative, then
$\widetilde{A_1} \otimes \widetilde{A_2} \cong (m_1 m_2, n_1 n_2, m_1\alpha_2 + m_2\alpha_1, n_1\beta_2 + n_2\beta_1)_{LR}$.

(iv) If $\widetilde{A_1}$ is non-positive and $\widetilde{A_2}$ is non-negative, then
$\widetilde{A_1} \otimes \widetilde{A_2} \cong (m_1 n_2, n_1 m_2, \alpha_1 n_2 - m_1\beta_2, m_2\beta_1 - n_1\alpha_2)_{LR}$.

(v) $\lambda \otimes \widetilde{A_1} \cong \begin{cases} (\lambda m_1, \lambda n_1, \lambda\alpha_1, \lambda\beta_1)_{LR}, & \lambda \geqslant 0, \\ (\lambda n_1, \lambda m_1, -\lambda\beta_1, -\lambda\alpha_1)_{RL}, & \lambda < 0. \end{cases}$

Definition 2.4. A matrix $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$ $(i = 1, 2, \cdots m, j = 1, 2, \cdots n)$ is said to be a fuzzy matrix if each element of $\widetilde{\boldsymbol{A}}$ is a fuzzy number. If for every element $\widetilde{a}_{ij} \geqslant 0$ (or $\widetilde{a}_{ij} \leqslant 0$), then $\widetilde{\boldsymbol{A}}$ is said to be a non-negative (or non-positive) fuzzy matrix, denoted by $\widetilde{\boldsymbol{A}} \geqslant \boldsymbol{0}$ (or $\widetilde{\boldsymbol{A}} \leqslant \boldsymbol{0}$).

Let all element of $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$ $(i = 1, 2, \cdots m, j = 1, 2, \cdots n)$ are LR flat fuzzy numbers, i.e., $\widetilde{a}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$. Then we can represent $m \times n$ fuzzy matrix $\widetilde{\boldsymbol{A}}$ with new notation $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{N})$, where $\boldsymbol{A} = (m_{ij}), \boldsymbol{B} = (n_{ij}), \boldsymbol{M} = (\alpha_{ij})$ and $\boldsymbol{N} = (\beta_{ij})$ are four $m \times n$ matrices.

For brevity, a crisp matrix consisting of real numbers is written as a matrix directly throughout this paper, similar to linear systems, linear equations, matrix equations, and so on.

Definition 2.5.[16,19] Let $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$ and $\widetilde{\boldsymbol{B}} = (\widetilde{b}_{ij})_{n\times p}$. Then

$$\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{D}} = (\widetilde{d}_{ij})_{m\times p},$$

where

$$\widetilde{d}_{ij} = \sum_{k=1,\cdots,n}^{\oplus} \widetilde{a}_{ik} \otimes \widetilde{b}_{kj}.$$

2.2 Interval objective programming

Ishibuchi and Tanaka [20] gave the definitions of the maximization and minimization problems with the interval objective functions, which are introduced in Definitions 2.6 and 2.7 as follows.

Definition 2.6.[20] Let $\widetilde{a} = [a_l, a_u]$ be an interval. The maximization problem with the interval objective function is described as follow:

$$\begin{aligned} \max \quad & \{\widetilde{a}\}, \\ \text{s.t.} \quad & \widetilde{a} \in \Omega, \end{aligned}$$

which is equivalent to the following bi-objective mathematical programming problem:

$$\begin{aligned} \max \quad & \{a_l, \frac{1}{2}(a_l + a_u)\}, \\ \text{s.t.} \quad & \widetilde{a} \in \Omega, \end{aligned}$$

where Ω is a set of constraints in which the variable $\widetilde{a}$ should satisfy according to requirements in real situations.

Definition 2.7.[20] Let $\widetilde{a} = [a_l, a_u]$ be an interval. The minimization problem with the interval objective function is described as follow:

$$\begin{aligned} \min \quad & \{\widetilde{a}\}, \\ \text{s.t.} \quad & \widetilde{a} \in \Omega, \end{aligned}$$

which is equivalent to the following bi-objective mathematical programming problem:

$$\begin{aligned} \min \quad & \{a_u, \frac{1}{2}(a_l + a_u)\}, \\ \text{s.t.} \quad & \widetilde{a} \in \Omega. \end{aligned}$$

3. Fully fuzzy linear programming problem

Linear programming is one of the most frequently applied operations research technique. In the conventional approach, the value of the parameters and variables of linear programming models must be well defined and precise. However, this is not a realistic assumption in the real world environment. There may exists uncertainty about the parameters and variables in the real life problems. So, the fuzzy numbers and fuzzy variables should be used in the LP problem. Therefore, we encounter with the FFLP problems.

Consider the standard form of FFLP problem which with m constraints and n variables as follows:

$$\begin{aligned} \max(\min)\widetilde{Z} &= \widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}, \\ \text{s.t.} \quad & \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}, \\ & \widetilde{\boldsymbol{x}} \ \text{ is non-negative fuzzy number vector}, \end{aligned} \tag{0.1}$$

where $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$, $\widetilde{\boldsymbol{c}}^T = (\widetilde{c}_j)_{1\times n}$, $\widetilde{\boldsymbol{x}} = (\widetilde{x}_j)_{n\times 1}$, $\widetilde{\boldsymbol{b}} = (\widetilde{b}_i)_{m\times 1}$, and $\widetilde{a}_{ij} \geqslant 0$ (or $\widetilde{a}_{ij} < 0$), $\widetilde{c}_j \geqslant 0$ (or $\widetilde{c}_j < 0$), $\widetilde{b}_i$, $\widetilde{x}_j$ are LR flat fuzzy numbers.

It should be noted that $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \leqslant \widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \geqslant \widetilde{\boldsymbol{b}}$ can be transformed to the standard form by introducing a vector variable $\widetilde{\boldsymbol{T}} = (\widetilde{t_1}, \widetilde{t_2}, \cdots, \widetilde{t_m})$, where $\widetilde{t_j}$ $(j = 1, 2, \cdots, m)$ are LR flat fuzzy numbers, as $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{T}} = \widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \ominus \widetilde{\boldsymbol{T}} = \widetilde{\boldsymbol{b}}$, respectively.

Definition 3.1. The fuzzy exact optimal solution of FFLP problem (1) will be a fuzzy number vector $\widetilde{\boldsymbol{x}}^*$ if it satisfies the following characteristics:

(1) $\widetilde{\boldsymbol{x}}^* = (\widetilde{x_j}^*)_{n\times 1} \geqslant \boldsymbol{0}$, where $\widetilde{x_j}^* (j = 1, 2, \cdots, n)$ are LR flat fuzzy numbers;

(2) $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}}^* = \widetilde{\boldsymbol{b}}$;

(3) For any $\widetilde{\boldsymbol{x}} \in \widetilde{\boldsymbol{S}} = \{ \widetilde{\boldsymbol{x}} | \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}, \ \widetilde{\boldsymbol{x}} = (\widetilde{x_j})_{n\times 1} \geqslant \boldsymbol{0}$, where $\widetilde{x_j}$ are LR flat fuzzy numbers$\}$, we have $\widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}^* \geqslant \widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}$ (in case of maximization problem), $\widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}^* \leqslant \widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}$ (in case of minimization problem).

Remark 3.1. Let $\widetilde{\boldsymbol{x}}^*$ be an exact optimal solution of FFLP problem (1). If there exists an $\widetilde{\boldsymbol{x}}' \in \widetilde{\boldsymbol{S}}$ such that $\widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}^* = \widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}}'$, then $\widetilde{\boldsymbol{x}}'$ is also an exact optimal solution of FFLP problem (1) and is called an alternative exact optimal solution.

Note that the elements ($\widetilde{a}_{ij}$) in coefficient matrix $\widetilde{\boldsymbol{A}}$ of the FFLP problem (1) have two forms, i.e., (1) $\widetilde{a}_{ij} \geqslant 0$, (2) $\widetilde{a}_{ij} < 0$. So we define the coefficient matrix $\widetilde{\boldsymbol{A}}$ as follows:

$$(\widetilde{\boldsymbol{A}}_1)_{ij} = \begin{cases} \widetilde{a}_{ij}, & \widetilde{a}_{ij} \geqslant 0, \\ \widetilde{0}, & \widetilde{a}_{ij} < 0; \end{cases}$$

$$(\widetilde{\boldsymbol{A}}_2)_{ij} = \begin{cases} \widetilde{a}_{ij}, & \widetilde{a}_{ij} \leqslant 0, \\ \widetilde{0}, & \widetilde{a}_{ij} > 0, \end{cases}$$

where $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$.

Obviously,

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2,$$

$$\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{A}}_1 \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{A}}_2 \otimes \widetilde{\boldsymbol{x}}.$$

Similarly, we define the coefficient matrix $(\widetilde{\boldsymbol{c}}^T)$ of objective function of FFLP problem (1) as follows:

$$(\widetilde{\boldsymbol{c}}_1^T)_j = \begin{cases} \widetilde{c}_j, & \widetilde{c}_j \geqslant 0, \\ \widetilde{0}, & \widetilde{c}_j < 0; \end{cases}$$

$$(\widetilde{\boldsymbol{c}}_2^T)_j = \begin{cases} \widetilde{c}_j, & \widetilde{c}_j \leqslant 0, \\ \widetilde{0}, & \widetilde{c}_j > 0, \end{cases}$$

where $1 \leqslant j \leqslant n$.

Obviously,

$$\widetilde{\boldsymbol{c}}^T = \widetilde{\boldsymbol{c}}_1^T \oplus \widetilde{\boldsymbol{c}}_2^T,$$
$$\widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{c}}_1^T \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{c}}_2^T \otimes \widetilde{\boldsymbol{x}}.$$

In the FFLP problems (1), let $\widetilde{\boldsymbol{c}} = (\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{q})_{LR} = \widetilde{\boldsymbol{c}}_1 \oplus \widetilde{\boldsymbol{c}}_2$ (where $\widetilde{\boldsymbol{c}}_1 = (\boldsymbol{c}_1, \boldsymbol{d}_1, \boldsymbol{p}_1, \boldsymbol{q}_1)_{LR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{c}}_2 = (\boldsymbol{c}_2, \boldsymbol{d}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)_{LR} \leqslant \boldsymbol{0}$), $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR}$. Then, according to the operations of the LR flat fuzzy numbers, we have

$$\begin{aligned} \widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{c}}_1^T \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{c}}_2^T \otimes \widetilde{\boldsymbol{x}} &= (\boldsymbol{c}_1, \boldsymbol{d}_1, \boldsymbol{p}_1, \boldsymbol{q}_1)_{LR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR} \oplus (\boldsymbol{c}_2, \boldsymbol{d}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)_{LR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR} \\ &\cong (\boldsymbol{c}_1\boldsymbol{x}, \boldsymbol{d}_1\boldsymbol{y}, \boldsymbol{c}_1\boldsymbol{s} + \boldsymbol{p}_1\boldsymbol{x}, \boldsymbol{d}_1\boldsymbol{t} + \boldsymbol{q}_1\boldsymbol{y})_{LR} \oplus (\boldsymbol{c}_2\boldsymbol{y}, \boldsymbol{d}_2\boldsymbol{x}, \boldsymbol{p}_2\boldsymbol{y} - \boldsymbol{c}_2\boldsymbol{t}, \boldsymbol{q}_2\boldsymbol{x} - \boldsymbol{d}_2\boldsymbol{s})_{LR} \\ &= (\boldsymbol{c}_1\boldsymbol{x} + \boldsymbol{c}_2\boldsymbol{y}, \boldsymbol{d}_1\boldsymbol{y} + \boldsymbol{d}_2\boldsymbol{x}, \boldsymbol{c}_1\boldsymbol{s} - \boldsymbol{c}_2\boldsymbol{t} + \boldsymbol{p}_1\boldsymbol{x} + \boldsymbol{p}_2\boldsymbol{y}, \boldsymbol{q}_2\boldsymbol{x} + \boldsymbol{q}_1\boldsymbol{y} + \boldsymbol{d}_1\boldsymbol{t} - \boldsymbol{d}_2\boldsymbol{s})_{LR}. \end{aligned}$$

For brevity, we substitute $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$ for $(\boldsymbol{c}_1\boldsymbol{x} + \boldsymbol{c}_2\boldsymbol{y}, \boldsymbol{d}_1\boldsymbol{y} + \boldsymbol{d}_2\boldsymbol{x}, \boldsymbol{c}_1\boldsymbol{s} - \boldsymbol{c}_2\boldsymbol{t} + \boldsymbol{p}_1\boldsymbol{x} + \boldsymbol{p}_2\boldsymbol{y}, \boldsymbol{q}_2\boldsymbol{x} + \boldsymbol{q}_1\boldsymbol{y} + \boldsymbol{d}_1\boldsymbol{t} - \boldsymbol{d}_2\boldsymbol{s})_{LR}$ throughout this paper.

Theorem 3.1. (Approximate representation of fully fuzzy constraints) Let $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{N})_{LR} = \widetilde{\boldsymbol{A}_1} \oplus \widetilde{\boldsymbol{A}_2}$, $\widetilde{\boldsymbol{b}} = (\boldsymbol{b}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{f})_{LR}$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR} \geqslant \boldsymbol{0}$, where $\widetilde{\boldsymbol{A}_1} = (\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{LR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{A}_2} = (\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{LR} \leqslant \boldsymbol{0}$. Then $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$ can be represented approximately as follows:

$$\begin{cases} \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{y} = \boldsymbol{b}, \\ \boldsymbol{B}_1\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{x} = \boldsymbol{g}, \\ \boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{y} + \boldsymbol{A}_1\boldsymbol{s} - \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{h}, \\ \boldsymbol{N}_2\boldsymbol{x} + \boldsymbol{N}_1\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{t} - \boldsymbol{B}_2\boldsymbol{s} = \boldsymbol{f}. \end{cases} \tag{0.2}$$

Proof. Since $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{N})_{LR} = \widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2$, $\widetilde{\boldsymbol{A}}_1 = (\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{LR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{A}}_2 = (\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{LR} \leqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR} \geqslant \boldsymbol{0}$, we have

$$\begin{aligned} \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = (\widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2) \otimes \widetilde{\boldsymbol{x}} &= \widetilde{\boldsymbol{A}}_1 \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{A}}_2 \otimes \widetilde{\boldsymbol{x}} \\ &= (\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{LR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR} \oplus (\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{LR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR} \\ &\cong (\boldsymbol{A}_1\boldsymbol{x}, \boldsymbol{B}_1\boldsymbol{y}, \boldsymbol{A}_1\boldsymbol{s} + \boldsymbol{M}_1\boldsymbol{x}, \boldsymbol{B}_1\boldsymbol{t} + \boldsymbol{N}_1\boldsymbol{y})_{LR} \oplus (\boldsymbol{A}_2\boldsymbol{y}, \boldsymbol{B}_2\boldsymbol{x}, \boldsymbol{M}_2\boldsymbol{y} - \boldsymbol{A}_2\boldsymbol{t}, \boldsymbol{N}_2\boldsymbol{x} - \boldsymbol{B}_2\boldsymbol{s})_{LR} \\ &= (\boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{y}, \boldsymbol{B}_1\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{s} - \boldsymbol{A}_2\boldsymbol{t} + \boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{y}, \boldsymbol{N}_2\boldsymbol{x} + \boldsymbol{N}_1\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{t} - \boldsymbol{B}_2\boldsymbol{s})_{LR} \\ &= \widetilde{\boldsymbol{b}} = (\boldsymbol{b}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{f})_{LR}. \end{aligned}$$

According to Definition 2.3, we have

$$\boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{y} = \boldsymbol{b}, \qquad \boldsymbol{B}_1\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{x} = \boldsymbol{g},$$

$$\boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{y} + \boldsymbol{A}_1\boldsymbol{s} - \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{h}, \qquad \boldsymbol{N}_2\boldsymbol{x} + \boldsymbol{N}_1\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{t} - \boldsymbol{B}_2\boldsymbol{s} = \boldsymbol{f}.$$

Hence, $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$ can be represented approximately as (2).

4. Proposed method to find the fuzzy exact optimal solution of FFLP problem

In this section, in order to find an effective fuzzy solution of the type of Eq. (1) problem, we are going to introduce a method based on the definition of the interval objective programming and the arithmetic operations of LR flat fuzzy numbers.

We consider the case of maximizing fuzzy objective function at first.

In Eq. (1), the fuzzy objective $\widetilde{Z} = \widetilde{\boldsymbol{c}}^T \otimes \widetilde{\boldsymbol{x}} = (\boldsymbol{c}_1\boldsymbol{x} + \boldsymbol{c}_2\boldsymbol{y}, \boldsymbol{d}_1\boldsymbol{y} + \boldsymbol{d}_2\boldsymbol{x}, \boldsymbol{c}_1\boldsymbol{s} - \boldsymbol{c}_2\boldsymbol{t} + \boldsymbol{p}_1\boldsymbol{x} + \boldsymbol{p}_2\boldsymbol{y}, \boldsymbol{q}_2\boldsymbol{x} + \boldsymbol{q}_1\boldsymbol{y} + \boldsymbol{d}_1\boldsymbol{t} - \boldsymbol{d}_2\boldsymbol{s})_{LR} = (Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$ is an LR flat fuzzy number. This fuzzy objective is fully defined by four corner points $(Z_m, 1)$, $(Z_n, 1)$, $(Z_m - Z_\alpha, 0)$ and $(Z_n + Z_\beta, 0)$, geometrically. Thus, maximizing the fuzzy objective can be obtained by pushing these four critical points in the direction of the right-hand side. Fortunately, the vertical coordinates of the critical points are fixed at either 1 or 0. The only considerations then are the four horizontal coordinates. Therefore, our problem is to solve

$$\begin{aligned} \max \quad & \{Z_m - Z_\alpha, Z_m, Z_n, Z_n + Z_\beta\}, \\ \text{s.t.} \quad & \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}, \\ & \widetilde{\boldsymbol{x}} \geqslant \boldsymbol{0}. \end{aligned} \tag{0.3}$$

However, the above four objectives $(Z_m - Z_\alpha)$, Z_m, Z_n and $(Z_n + Z_\beta)$ should always preserve the form of the LR flat fuzzy number $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$ during the optimization process. Thus, in order to keep the LR flat fuzzy number shape (normal and convex) of the possibility distribution, it is necessary to make a little change.

For the mode of LR flat fuzzy number $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$, since the objective function of Eq. (1) is to maximize $\widetilde{Z}$, it is natural to maximize the interval $[Z_m, Z_n]$ for this objective function. According to Definition 2.6, in order to maximize the interval $[Z_m, Z_n]$, we need maximize the left endpoint Z_m and maximize the middle point $\frac{1}{2}(Z_m + Z_n)$ of this interval simultaneously. For the lower and upper limits of LR flat fuzzy number $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$, we minimize Z_α and maximize Z_β instead of maximizing the lower $Z_m - Z_\alpha$ and the upper $Z_n + Z_\beta$, respectively.

Therefore, combining Theorem 3.1, Eq. (3) can be transformed into the following multi-objective programming model:

$$\begin{aligned} \min \quad & Z_1 = Z_\alpha, \\ \max \quad & Z_2 = Z_m, \\ \max \quad & Z_3 = \frac{1}{2}(Z_m + Z_n), \\ \max \quad & Z_4 = Z_\beta, \\ \text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{y} = \boldsymbol{b}, \\ & \boldsymbol{B}_1\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{x} = \boldsymbol{g}, \\ & \boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{y} + \boldsymbol{A}_1\boldsymbol{s} - \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{h}, \\ & \boldsymbol{N}_2\boldsymbol{x} + \boldsymbol{N}_1\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{t} - \boldsymbol{B}_2\boldsymbol{s} = \boldsymbol{f}, \\ & \boldsymbol{y} - \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x} - \boldsymbol{s} \geqslant \boldsymbol{0},\ \boldsymbol{s} \geqslant \boldsymbol{0},\ \boldsymbol{t} \geqslant \boldsymbol{0}. \end{aligned} \tag{0.4}$$

Although Eq. (4) is also a multi-objective linear programming model, it can effectively keep the LR flat fuzzy number shape of objective function $\widetilde{Z}$. To solve Eq. (4), we may use any MOLP technique [21] such as utility theory, goal programming fuzzy programming or interactive approaches. In this paper, we propose three kinds of approaches to solve this multi-objective linear programming model. Since the objective function Z_i is the function of the decision variable vector $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})_{LR}$, simply denote by $Z_i = Z_i(\widetilde{\boldsymbol{x}})$ $(i = 1, 2, 3, 4)$. Let $Z_1^{\min}$, $Z_i^{\max}$ $(i = 2, 3, 4)$ and $\widetilde{\boldsymbol{x}}_i'$ $(i = 1, 2, 3, 4)$ respectively be the minimum objective, maximum objective value and the optimal solution for the following single objective linear programming model:

$$\begin{aligned} \min \quad & Z_1 = Z_1(\widetilde{\boldsymbol{x}}), \\ \text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{y} = \boldsymbol{b}, \\ & \boldsymbol{B}_1\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{x} = \boldsymbol{g}, \\ & \boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{y} + \boldsymbol{A}_1\boldsymbol{s} - \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{h}, \\ & \boldsymbol{N}_2\boldsymbol{x} + \boldsymbol{N}_1\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{t} - \boldsymbol{B}_2\boldsymbol{s} = \boldsymbol{f}, \\ & \boldsymbol{y} - \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x} - \boldsymbol{s} \geqslant \boldsymbol{0},\ \boldsymbol{s} \geqslant \boldsymbol{0},\ \boldsymbol{t} \geqslant \boldsymbol{0}, \end{aligned} \tag{0.5}$$

and

$$\begin{aligned} \max \quad & Z_i = Z_i(\widetilde{\boldsymbol{x}}),\ (i=2,3,4) \\ \text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x}+\boldsymbol{A}_2\boldsymbol{y}=\boldsymbol{b}, \\ & \boldsymbol{B}_1\boldsymbol{y}+\boldsymbol{B}_2\boldsymbol{x}=\boldsymbol{g}, \\ & \boldsymbol{M}_1\boldsymbol{x}+\boldsymbol{M}_2\boldsymbol{y}+\boldsymbol{A}_1\boldsymbol{s}-\boldsymbol{A}_2\boldsymbol{t}=\boldsymbol{h}, \\ & \boldsymbol{N}_2\boldsymbol{x}+\boldsymbol{N}_1\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{t}-\boldsymbol{B}_2\boldsymbol{s}=\boldsymbol{f}, \\ & \boldsymbol{y}-\boldsymbol{x}\geqslant\boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{t}\geqslant\boldsymbol{0}. \end{aligned} \tag{0.6}$$

Then, set $Z_1^{\max}=\max\{Z_1(\widetilde{\boldsymbol{x}}_1'),Z_1(\widetilde{\boldsymbol{x}}_2'),Z_1(\widetilde{\boldsymbol{x}}_3'),Z_1(\widetilde{\boldsymbol{x}}_4')\}$, $Z_i^{\min}=\min\{Z_i(\widetilde{\boldsymbol{x}}_1'),Z_i(\widetilde{\boldsymbol{x}}_2'),Z_i(\widetilde{\boldsymbol{x}}_3'),Z_i(\widetilde{\boldsymbol{x}}_4')\}$ $(i=2,3,4)$. The linear membership function of the objective function Z_1 can be calculated as follows:

$$\mu_{z_1}(\widetilde{\boldsymbol{x}})=\begin{cases} 1 & \text{if } Z_1<Z_1^{\min}, \\ \frac{Z_1^{\max}-Z_1}{Z_1^{\max}-Z_1^{\min}} & \text{if } Z_1^{\min}\leqslant Z_1\leqslant Z_1^{\max}, \\ 0 & \text{if } Z_1>Z_1^{\max}. \end{cases}$$

The linear membership function of the objective function $Z_i(i=2,3,4)$ can be calculated as follows:

$$\mu_{z_i}(\widetilde{\boldsymbol{x}})=\begin{cases} 0 & \text{if } Z_i<Z_i^{\min}, \\ \frac{Z_i-Z_i^{\min}}{Z_i^{\max}-Z_i^{\min}} & \text{if } Z_i^{\min}\leqslant Z_i\leqslant Z_i^{\max}, \\ 1 & \text{if } Z_i>Z_i^{\max}. \end{cases}$$

Thus, Eq. (4) can be solved by the following linear programming model:

$$\begin{aligned} \max \quad & \mu, \\ \text{s.t.} \quad & 4\mu_{z_i}(\widetilde{\boldsymbol{x}})+\sum_{i=1}^{4}\mu_{z_i}(\widetilde{\boldsymbol{x}})\geqslant 8\mu\ (i=1,2,3,4), \\ & \boldsymbol{A}_1\boldsymbol{x}+\boldsymbol{A}_2\boldsymbol{y}=\boldsymbol{b}, \\ & \boldsymbol{B}_1\boldsymbol{y}+\boldsymbol{B}_2\boldsymbol{x}=\boldsymbol{g}, \\ & \boldsymbol{M}_1\boldsymbol{x}+\boldsymbol{M}_2\boldsymbol{y}+\boldsymbol{A}_1\boldsymbol{s}-\boldsymbol{A}_2\boldsymbol{t}=\boldsymbol{h}, \\ & \boldsymbol{N}_2\boldsymbol{x}+\boldsymbol{N}_1\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{t}-\boldsymbol{B}_2\boldsymbol{s}=\boldsymbol{f}, \\ & \boldsymbol{y}-\boldsymbol{x}\geqslant\boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{t}\geqslant\boldsymbol{0}, \end{aligned} \tag{0.7}$$

or

$$\begin{aligned} \max \quad & \mu, \\ \text{s.t.} \quad & 4\mu_{z_i}(\widetilde{\boldsymbol{x}})+\sum_{i=1}^{4}\mu_{z_i}(\widetilde{\boldsymbol{x}})\leqslant 8\mu\ (i=1,2,3,4), \\ & \boldsymbol{A}_1\boldsymbol{x}+\boldsymbol{A}_2\boldsymbol{y}=\boldsymbol{b}, \\ & \boldsymbol{B}_1\boldsymbol{y}+\boldsymbol{B}_2\boldsymbol{x}=\boldsymbol{g}, \\ & \boldsymbol{M}_1\boldsymbol{x}+\boldsymbol{M}_2\boldsymbol{y}+\boldsymbol{A}_1\boldsymbol{s}-\boldsymbol{A}_2\boldsymbol{t}=\boldsymbol{h}, \\ & \boldsymbol{N}_2\boldsymbol{x}+\boldsymbol{N}_1\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{t}-\boldsymbol{B}_2\boldsymbol{s}=\boldsymbol{f}, \\ & \boldsymbol{y}-\boldsymbol{x}\geqslant\boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{t}\geqslant\boldsymbol{0}, \end{aligned} \tag{0.8}$$

or

$$\begin{aligned} \max \quad & w_1\mu_{z_1}(\widetilde{\boldsymbol{x}})+w_2\mu_{z_2}(\widetilde{\boldsymbol{x}})+w_3\mu_{z_3}(\widetilde{\boldsymbol{x}})+w_4\mu_{z_4}(\widetilde{\boldsymbol{x}}), \\ \text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x}+\boldsymbol{A}_2\boldsymbol{y}=\boldsymbol{b}, \\ & \boldsymbol{B}_1\boldsymbol{y}+\boldsymbol{B}_2\boldsymbol{x}=\boldsymbol{g}, \\ & \boldsymbol{M}_1\boldsymbol{x}+\boldsymbol{M}_2\boldsymbol{y}+\boldsymbol{A}_1\boldsymbol{s}-\boldsymbol{A}_2\boldsymbol{t}=\boldsymbol{h}, \\ & \boldsymbol{N}_2\boldsymbol{x}+\boldsymbol{N}_1\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{t}-\boldsymbol{B}_2\boldsymbol{s}=\boldsymbol{f}, \\ & \boldsymbol{y}-\boldsymbol{x}\geqslant\boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{s}\geqslant\boldsymbol{0},\ \boldsymbol{t}\geqslant\boldsymbol{0}, \end{aligned} \tag{0.9}$$

where $\boldsymbol{w} = (w_1, w_2, w_3, w_4)^T$ is the weight vector of objective Z_i $(i = 1, 2, 3, 4)$, satisfies that $w_i \geqslant 0$ $(i = 1, 2, 3, 4)$ and $\sum_{i=1}^{4} w_i = 1$. Eq. (7) is a kind of pessimistic approach which shows that the decision maker (DM) is very conservative, whereas Eq. (8) is a kind of optimistic approach which shows that the (DM) is very aggressive. Eq. (9) is the linear sum approach based on membership function.

Analogously, for the case of minimization fuzzy objective function programming, using Definition 2.7, it can be transformed into the following multi-objective programming model:

$$\begin{aligned}
\max \quad & Z_1 = Z_\alpha, \\
\min \quad & Z_2 = Z_n, \\
\min \quad & Z_3 = \frac{1}{2}(Z_m + Z_n), \\
\min \quad & Z_4 = Z_\beta, \\
\text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{y} = \boldsymbol{b}, \\
& \boldsymbol{B}_1\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{x} = \boldsymbol{g}, \\
& \boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{y} + \boldsymbol{A}_1\boldsymbol{s} - \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{h}, \\
& \boldsymbol{N}_2\boldsymbol{x} + \boldsymbol{N}_1\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{t} - \boldsymbol{B}_2\boldsymbol{s} = \boldsymbol{f}, \\
& \boldsymbol{y} - \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x} - \boldsymbol{s} \geqslant \boldsymbol{0},\ \boldsymbol{s} \geqslant \boldsymbol{0},\ \boldsymbol{t} \geqslant \boldsymbol{0},
\end{aligned} \tag{0.10}$$

which can be solved by using a similar approach (as previously described).

5. Examples

In this section, we will demonstrate efficiency and superiority of the proposed method using numerical examples. At the same time, the shortcomings of the existing methods [11-14] for solving FFLP problems with equality constraints are pointed out.

Example 5.1. Let $L(x) = \max\{0, 1 - |x|\}, R(x) = \max\{0, \frac{1}{e-1}(e^{1-x^2} - 1)\}$. Consider the following FFLP:

$$\begin{aligned}
\max \ & Z = (2,3,1,3) \otimes \widetilde{x}_1 \oplus (-3,-2,2,1) \otimes \widetilde{x}_2 \oplus (0,1,0,0) \otimes \widetilde{x}_3, \\
\text{s.t.} \ & (2,2,1,0) \otimes \widetilde{x}_1 \oplus (-2,-1,2,1) \otimes \widetilde{x}_2 \oplus (-2,-1,1,0) \otimes \widetilde{x}_3 = (2,9,16,4), \\
& (-3,-1,1,1) \otimes \widetilde{x}_1 \oplus (2,2,1,1) \otimes \widetilde{x}_2 \oplus (-2,-1,0,1) \otimes \widetilde{x}_3 = (-13,2,11,12), \\
& \widetilde{x}_1 \geqslant 0,\ \widetilde{x}_2 \geqslant 0,\ \widetilde{x}_3 \geqslant 0.
\end{aligned} \tag{0.11}$$

Let $\widetilde{x}_1 = (x_1, y_1, s_1, t_1)_{LR},\ \widetilde{x}_2 = (x_2, y_2, s_2, t_2)_{LR},\ \widetilde{x}_3 = (x_3, y_3, s_3, t_3)_{LR}$. Then

$$\widetilde{\boldsymbol{c}}^T = ((2,3,1,3),(-3,-2,2,1),(0,1,0,0)), \qquad \widetilde{\boldsymbol{b}}^T = ((2,9,16,4),(-13,2,11,12)),$$

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} (2,2,1,0) & (-2,-1,2,1) & (-2,-1,1,0) \\ (-3,-1,1,1) & (2,2,1,1) & (-2,-1,0,1) \end{pmatrix},$$

$$\boldsymbol{A} = \begin{pmatrix} 2,-2,-2 \\ -3,2,-2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 2,-1,-1 \\ -1,2,-1 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 1,2,1 \\ 1,1,0 \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} 0,1,0 \\ 1,1,1 \end{pmatrix},$$

$$\boldsymbol{A}_1 = \begin{pmatrix} 2,0,0 \\ 0,2,0 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} 0,-2,-2 \\ -3,0,-2 \end{pmatrix}, \quad \boldsymbol{B}_1 = \begin{pmatrix} 2,0,0 \\ 0,2,0 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} 0,-1,-1 \\ -1,0,-1 \end{pmatrix},$$

$$\boldsymbol{M}_1 = \begin{pmatrix} 1,0,0 \\ 0,1,0 \end{pmatrix}, \quad \boldsymbol{M}_2 = \begin{pmatrix} 0,2,1 \\ 1,0,0 \end{pmatrix}, \quad \boldsymbol{N}_1 = \begin{pmatrix} 0,0,0 \\ 0,1,0 \end{pmatrix}, \quad \boldsymbol{N}_2 = \begin{pmatrix} 0,1,0 \\ 1,0,1 \end{pmatrix}.$$

$$\begin{aligned}
&\boldsymbol{c}^T = (2,-3,0), \quad \boldsymbol{d}^T = (3,-2,1), \quad \boldsymbol{p}^T = (1,2,0), \quad \boldsymbol{q}^T = (3,1,0), \\
&\boldsymbol{c}_1^T = (2,0,0), \quad \boldsymbol{c}_2^T = (0,-3,0), \quad \boldsymbol{d}_1^T = (3,0,1), \quad \boldsymbol{d}_2^T = (0,-2,0), \\
&\boldsymbol{p}_1^T = (1,0,0), \quad \boldsymbol{p}_2^T = (0,2,0), \quad \boldsymbol{q}_1^T = (3,0,0), \quad \boldsymbol{q}_2^T = (0,1,0), \\
&\boldsymbol{b}^T = (2,-13), \quad \boldsymbol{g}^T = (9,2), \quad \boldsymbol{h}^T = (16,11), \quad \boldsymbol{f}^T = (4,12).
\end{aligned}$$

By Eq.(4), Eq. (11) can be solved by the auxiliary MOLP model as follows:

$$\begin{aligned}
\min\ & Z_1 = x_1 + 2y_2 + 2s_1 + 3t_2,\\
\max\ & Z_2 = 2x_1 - 3y_2,\\
\max\ & Z_3 = x_1 - x_2 + 1.5y_1 - 1.5y_2 + 0.5y_3,\\
\max\ & Z_4 = x_2 + 3y_1 + 3t_1 + t_3 + 2s_2,\\
\text{s.t. } & 2x_1 - 2y_2 - 2y_3 = 2,\\
& 2x_2 - 3y_1 - 2y_3 = -13,\\
& 2y_1 - x_2 - x_3 = 9,\\
& 2y_2 - x_1 - x_3 = 2,\\
& x_1 + 2y_2 + y_3 + 2s_1 + 2t_2 + 2t_3 = 16,\\
& x_2 + y_1 + 2s_2 + 3t_1 + 2t_3 = 11,\\
& x_2 + 2t_1 + s_2 + s_3 = 4,\\
& x_1 + x_3 + y_2 + s_1 + s_3 + 2t_2 = 12,\\
& y_i - x_i \geqslant 0,\ x_i - s_i \geqslant 0,\ s_i \geqslant 0,\ t_i \geqslant 0\ (i = 1, 2, 3, 4).
\end{aligned} \tag{0.12}$$

Using Eq. (5,6), it follows that

$$Z_1^{\max} = 17.20, \quad Z_2^{\max} = -0.40, \quad Z_3^{\max} = 6.10, \quad Z_4^{\max} = 23.86,$$
$$Z_1^{\min} = 16.35, \quad Z_2^{\min} = -0.85, \quad Z_3^{\min} = 5.93, \quad Z_4^{\min} = 22.60.$$

Then, we can get

$$\mu_{z_1}(\widetilde{\boldsymbol{x}}) = \begin{cases} 1 & \text{if } Z_1 < 16.35,\\ \frac{17.20 - Z_1}{0.85} & \text{if } 16.35 \leqslant Z_1 \leqslant 17.20,\\ 0 & \text{if } Z_1 > 17.20; \end{cases}$$

$$\mu_{z_2}(\widetilde{\boldsymbol{x}}) = \begin{cases} 1 & \text{if } Z_2 > -0.40,\\ \frac{Z_2 + 0.85}{0.45} & \text{if } -0.85 \leqslant Z_2 \leqslant -0.40,\\ 0 & \text{if } Z_2 < -0.85; \end{cases}$$

$$\mu_{z_3}(\widetilde{\boldsymbol{x}}) = \begin{cases} 1 & \text{if } Z_3 > 6.10,\\ \frac{Z_3 - 5.93}{0.17} & \text{if } 5.93 \leqslant Z_3 \leqslant 6.10,\\ 0 & \text{if } Z_3 < 5.93; \end{cases}$$

$$\mu_{z_4}(\widetilde{\boldsymbol{x}}) = \begin{cases} 1 & \text{if } Z_4 > 23.86,\\ \frac{Z_4 - 22.60}{1.26} & \text{if } 22.60 \leqslant Z_4 \leqslant 23.86,\\ 0 & \text{if } Z_4 < 22.60. \end{cases}$$

Next, we use three approach (i.e., Eqs. (7)-(9)) to solving MOLP (12), respectively.

First, according to Eq. (7) (i.e., pessimistic approach), we can obtain the optimal solution of the MOLP (12) as follows:

$$x_1^* = 5.28,\ x_2^* = 2.96,\ x_3^* = 0.15,\ y_1^* = 6.06,\ y_2^* = 3.72,\ y_3^* = 0.38,$$
$$s_1^* = 0.00,\ s_2^* = 0.88,\ s_3^* = 0.15,\ t_1^* = 0.00,\ t_2^* = 1.35,\ t_3^* = 0.11.$$

Therefore, the optimal solution of the FFLP (11) is:

$$\widetilde{\boldsymbol{x}}^* = \begin{pmatrix} (5.28, 6.06, 0.00, 0.00)_{LR} \\ (2.96, 3.72, 0.88, 1.35)_{LR} \\ (0.15, 0.38, 0.15, 0.11)_{LR} \end{pmatrix}.$$

Substituted the above optimal solution into the objective function of FFLP (11), the optimal objective value is obtained as $(-0.60, 12.64, 16.77, 23.01)_{LR}$. Namely, the most likely value of the objective is between -0.60 and 12.64, the upper and lower limits of the objective value are -17.37 and 35.65, respectively. Its membership function is given by

$$u_{\widetilde{Z}}(t) = \begin{cases} L(\frac{-0.60 - t}{16.77}) & t < -0.60,\\ 1 & -0.60 \leqslant t \leqslant 12.64,\\ R(\frac{t - 12.64}{23.01}) & t > 12.64. \end{cases}$$

That is

$$u_{\widetilde{Z}}(t)=\begin{cases}1+\frac{0.60+t}{16.77} & -17.37<t<-0.60,\\ 1 & -0.60\leqslant t\leqslant 12.64,\\ \frac{1}{e-1}(e^{1-\frac{(t-12.64)^2}{23.01^2}}-1) & 12.64<t<35.65,\\ 0, & \text{others.}\end{cases}$$

Second, solving the MOLP (12) by using Eq. (8) (i.e., optimistic approach), we obtain the optimal solution as follows:

$$x_1^*=4.09,\ x_2^*=1.14,\ x_3^*=0.01,\ y_1^*=5.08,\ y_2^*=3.05,\ y_3^*=0.02,$$

$$s_1^*=0.72,\ s_2^*=0.54,\ s_3^*=0.00,\ t_1^*=1.16,\ t_2^*=2.07,\ t_3^*=0.11.$$

Therefore, the optimal solution of the FFLP (11) is:

$$\widetilde{\boldsymbol{x}}^*=\begin{pmatrix}(4.09,5.08,0.72,1.16)_{LR}\\ (1.14,3.05,0.54,2.07)_{LR}\\ (0.01,0.02,0.00,0.11)_{LR}\end{pmatrix}.$$

Substituted the above optimal solution into the objective function of FFLP (11), the optimal objective value is obtained as $(-0.97,12.98,17.84,21.05)_{LR}$. Namely, the most likely value of the objective is between -0.97 and 12.98, the upper and lower limits of the objective value are -18.81 and 34.03, respectively. Its membership function is given by

$$u_{\widetilde{Z}}(t)=\begin{cases}L(\frac{-0.97-t}{17.84}) & t<-0.97,\\ 1 & -0.97\leqslant t\leqslant 12.98,\\ R(\frac{t-12.98}{21.05}) & t>12.98,\end{cases}$$

That is

$$u_{\widetilde{Z}}(t)=\begin{cases}1+\frac{0.97+t}{17.84} & -18.81<t<-0.97,\\ 1 & -0.97\leqslant t\leqslant 12.98,\\ \frac{1}{e-1}(e^{1-\frac{(t-12.98)^2}{21.05^2}}-1) & 12.98<t<34.03,\\ 0, & \text{others.}\end{cases}$$

Finally, solving the MOLP (12) by using Eq. (9) (i.e., linear sum approach based on membership function), we obtain the optimal solution for $\boldsymbol{w}=(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})^T$ as follows:

$$x_1^*=5.32,\ x_2^*=2.84,\ x_3^*=0.05,\ y_1^*=5.95,\ y_2^*=3.68,\ y_3^*=0.42,$$

$$s_1^*=0.00,\ s_2^*=1.11,\ s_3^*=0.05,\ t_1^*=0.00,\ t_2^*=1.45,\ t_3^*=0.00.$$

Therefore, the optimal solution of the FFLP (11) is:

$$\widetilde{\boldsymbol{x}}^*=\begin{pmatrix}(5.32,5.95,0.00,0.00)_{LR}\\ (2.84,3.68,1.11,1.45)_{LR}\\ (0.05,0.42,0.05,0.00)_{LR}\end{pmatrix}.$$

Substituted the above optimal solution into the objective function of FFLP (11), the optimal objective value is obtained as $(-0.40,12.59,17.03,22.91)_{LR}$. Namely, the most likely value of the objective is between -0.40 and 12.59, the upper and lower limits of the objective value are -17.43 and 35.50, respectively. Its membership function is given by

$$u_{\widetilde{Z}}(t)=\begin{cases}L(\frac{-0.40-t}{17.03}) & t<-0.40,\\ 1 & -0.40\leqslant t\leqslant 12.59,\\ R(\frac{t-12.59}{22.91}) & t>12.59,\end{cases}$$

That is

$$u_{\widetilde{Z}}(t)=\begin{cases}1+\frac{0.40+t}{17.03} & -17.43<t<-0.40,\\ 1 & -0.40\leqslant t\leqslant 12.59,\\ \frac{1}{e-1}(e^{1-\frac{(t-12.59)^2}{22.91^2}}-1) & 12.59<t<35.50,\\ 0, & \text{others.}\end{cases}$$

To illustrate the influence of the weight vector $\boldsymbol{w}$ on the optimal value in this example, we use a different weight vector $\boldsymbol{w}$ to solve the MOLP (12) according to Eq. (9). Generally speaking, we need to consider some special cases. One is an average weight, i.e, $\boldsymbol{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$. Secondly, the objective functions Z_2 and Z_3 are relative to the mode of the flat fuzzy number Z, which are the most possible values of Z. Thus, more weights should be assigned to Z_2 and Z_3, i.e., $\boldsymbol{w} = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^T$. Thirdly, similar to the Olympic games, which discarded a maximum point and a minimum point, we can set $\boldsymbol{w} = (0, \frac{1}{2}, \frac{1}{2}, 0)^T$. Finally, $\boldsymbol{w} = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^T$ emphasizes both ends and reduces the middle. All the computation results are shown in Table 1.

Table 1: The optimal solution and optimal objective value for FFLP (11) with different approaches.

variable	x_1	x_2	x_3	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	optimal valueZ
by Eq.(6)	5.28	2.96	0.15	6.06	3.72	0.38	0.00	0.88	0.15	0.00	1.35	0.11	$(-0.60, 12.64, 16.77, 23.01)_{LR}$
by Eq.(7)	4.09	1.14	0.01	5.08	3.05	0.02	0.72	0.54	0.00	1.16	2.07	0.11	$(-0.97, 12.98, 17.84, 21.05)_{LR}$
by Eq.(8)													
$\boldsymbol{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$	5.32	2.84	0.05	5.95	3.68	0.42	0.00	1.11	0.05	0.00	1.45	0.00	$(-0.40, 12.59, 17.03, 22.91)_{LR}$
$\boldsymbol{w} = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^T$	5.32	2.84	0.05	5.95	3.68	0.42	0.00	1.11	0.05	0.00	1.45	0.00	$(-0.40, 12.59, 17.03, 22.91)_{LR}$
$\boldsymbol{w} = (0, \frac{1}{2}, \frac{1}{2}, 0)^T$	5.20	2.60	0.00	5.80	3.60	0.40	0.00	1.00	0.00	0.20	1.60	0.00	$(-0.40, 12.60, 17.20, 22.60)_{LR}$
$\boldsymbol{w} = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^T$	5.45	3.55	0.36	6.45	3.91	0.36	0.00	0.45	0.00	0.00	1.14	0.05	$(-0.83, 12.61, 16.69, 23.85)_{LR}$

It can be seen from Table 1 that, the optimal solutions and optimal objective value of Z for $\boldsymbol{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$, $\boldsymbol{w} = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^T$ and $\boldsymbol{w} = (0, \frac{1}{2}, \frac{1}{2}, 0)^T$ are completely identical. The optimal solution and optimal objective value of Z for $\boldsymbol{w} = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^T$ are not the same as that for the other weight vectors. This signifies that the weight vector $\boldsymbol{w}$ does affect the optimal solutions and optimal objective value.

In addition, Table 1 shows that applying different approaches to solving the multi-objective programming may result in different optimal solutions and optimal objective values. The DM can choose the proper approach to solving the multi-objective programming according to his/her risk preference and actual requirements.

Note that for a fully fuzzy linear programming problem, the fuzzy optimal solution $\widetilde{\boldsymbol{x}}^*$ is more general than the result which are calculated in Lotfi [12], Kumar [13] and Ezzati [14] when the left spreads shape functions $L(\cdot)$ and right spreads shape functions $R(\cdot)$ are linear functions. Furthermore, even the uncertain elements in a fuzzy linear programming problem were extended fuzzy numbers, we can construct a corresponding auxiliary multi-objective programming problem and solve it by use of three approaches (i.e., optimistic approach, pessimistic approach and linear sum approach based on membership function).

In Wan [11], the FLP problems with trapezoidal fuzzy numbers have been studied in which not all parts of the problem were assumed to be fuzzy (the variables were not fuzzy). Lotfi [12] proposed a novel method to find the optimal solution of FFLP problems, this method was obtained by using the concept of the symmetric triangular fuzzy numbers and introduce an approach to defuzzify a general fuzzy quantity. However, it can be applied only if the elements of the coefficient matrix are symmetric fuzzy numbers. In Kumar's article [13], the fuzzy optimal solution of FFLP problem can be obtained by using the arithmetic operations of triangular fuzzy numbers and linear ranking function which is used to convert the fuzzy objective function to the real objective function. Although the ranking function is convenient for the specific numerical computation, the fuzziness of the objective function is neglected by it. Ezzati [14] proposed a new algorithm to solve FFLP problems with TFNs. It can be applied only if all the parameters as well as the variables are represented by TFNs which is the specious case of LR flat fuzzy number, for the generalized fuzzy numbers, it is unsuitable.

6. Conclusion

In this paper, we propose a simple and practical method to solve a full fuzzy linear programming problem. The corresponding auxiliary MOLP problem is constructed and we solve it by use of three approaches. By numerical example, the obtained results of proposed algorithm with Wan [11], Lotfi [12], Kumar [13] and R. Ezzati [14] have been compared and shown the reliability and applicability of our

algorithm.

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Generalized Bateman's $G-$function and its bounds

Mansour Mahmoud[1] and Hanan Almuashi[2]

[1]Mansoura University, Faculty of Science, Mathematics Department, Mansoura 35516, Egypt.

[2]King Abdulaziz University, Faculty of Science, Mathematics Department, Jeddah 21589, Saudi Arabia.

[1] mansour@mans.edu.eg

[2] beautyrose_12_2006@hotmail.com

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Abstract

In this paper, we presented some functional equations of the generalized Bateman's $G-$function $G_h(x)$ and its relation with the hypergeometric series ${}_3F_2$. We deduced an asymptotic expansion of the function $G_h(x)$ and studied the completely monotonic property of some functions involving it. Also, we presented some new bounds of the function $G_h(x)$ and the double inequality

$$\ln\left(1+\frac{h}{x+\beta}\right) < G_h(x) - \frac{2h}{x(x+h)} < \ln\left(1+\frac{h}{x+\alpha}\right), \quad x>0;\ 0<h<2$$

where the constants $\alpha = 1$ and $\beta = \frac{h}{e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}-1}$ are the best possible.

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1 Introduction.

The ordinary gamma function $\Gamma(x)$ is defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad x>0$$

and the Psi or digamma function $\psi(x)$ is given by

$$\psi(x) = \frac{d}{dx}\log\Gamma(x).$$

The gamma function and its logarithmic derivatives $\psi^{(n)}(x)$ are of the most widely used special functions encountered in advanced mathematics . For more details about the properties of

these functions and their bounds, please refer to [2], [3], [8], [9], [15]-[18], [23]-[27] and plenty of references therein.

The Bateman's $G-$function is defined by [7]

$$G(x)=\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right), \qquad x\neq 0,-1,-2,\ldots . \tag{1}$$

The function $G(x)$ is very useful in estimating and summing certain numerical and algebraic series. For more details about the properties, bounds and applications of the $G(x)$, please refer to [7], [12]-[14], [15], [19], [28] and the references therein.

The function $G(x)$ satisfies the following relations [7]

$$G(x)=2\sum_{k=0}^{\infty}\frac{(-1)^k}{k+x}, \tag{2}$$

$$G(x+1)+G(x)=2x^{-1}, \tag{3}$$

$$G(nx)=2n^{-1}\sum_{k=0}^{n-1}(-1)^{k+1}\psi\left(x+\frac{k}{n}\right), \qquad n=2,4,6,\ldots \tag{4}$$

$$G(nx)=n^{-1}\sum_{k=0}^{n-1}(-1)^{k}G\left(x+\frac{k}{n}\right), \qquad n=1,3,5,\ldots \tag{5}$$

$$G(x)=2\int_0^{\infty}\frac{e^{-xt}}{1+e^{-t}}dt, \qquad x>0 \tag{6}$$

$$G(x)=2x^{-1}\ {}_2F_1(1,x;x+1;-1), \tag{7}$$

where

$${}_rF_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_r)_k}{(\beta_1)_k\ldots(\beta_s)_k}\frac{x^k}{k!}$$

is the generalized hypergeometric series [3] defined for $r,s\in\mathbb{N}$, $\alpha_j,\beta_j\in\mathbb{C}$, $\beta_j\neq 0,-1,-2,\ldots$ and the Pochhammer or shifted symbol $(\alpha)_n$ is given by

$$(\alpha)_0=1 \quad \text{and} \quad (\alpha)_m=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}, \qquad m\geq 1.$$

Qiu and Vuorinen [28] presented the double inequality

$$\frac{4(3/2-\ln 4)}{x^2}<G(x)-x^{-1}<\frac{1}{2x^2}, \qquad x>0.5\ . \tag{8}$$

Mahmoud and Agarwal [12] deduced the following asymptotic formula for Bateman's G-function

$$G(x)\sim x^{-1}+\sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k}, \qquad x\to\infty \tag{9}$$

and they presented the inequality

$$\frac{1}{2x^2+1.5} < G(x) - x^{-1} < \frac{1}{2x^2}, \qquad x > 0 \tag{10}$$

which improved the lower bound of the inequality (8) for $x > \sqrt{\frac{9-6\ln 4}{8\ln 4-11}}$. Also, Mahmoud and Almuashi [13] proved the following double inequality of the Bateman's $G-$function

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{n} x^{-2n} < G(x) - x^{-1} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{n} x^{-2n}, \qquad m \in \mathbb{N} \tag{11}$$

with the best possible bounds, where $B_m{}'s$ are the Bernoulli numbers [11].

Mortici [15] presented the double inequality

$$0 < \psi(x+\lambda) - \psi(x) \leq \psi(\lambda) + \gamma - \lambda + \lambda^{-1}, \qquad x \geq 1;\ 0 < \lambda < 1 \tag{12}$$

where γ is the Euler constant, which also improves the double inequality (8). Also, Alzer deduced the inequality [2]

$$x^{-1} - T_r(\lambda; x) - \omega_r(\lambda; x) < \psi(x+\lambda) - \psi(x) < x^{-1} - T_r(\lambda; x),$$

where $x > 0$, $r = 0, 1, 2, \ldots$, $0 < \lambda < 1$,

$$T_r(\lambda; x) = (1-\lambda)\left[\frac{1}{\lambda+r+1} + \sum_{i=0}^{r-1} \frac{1}{(x+i+1)(x+i+\lambda)}\right]$$

and

$$\omega_r(\lambda; x) = \frac{1}{x+r+\lambda} \log \frac{(x+r)^{(x+r)(1-\lambda)}(x+r+1)^{(x+r+1)\lambda}}{(x+r+\lambda)^{x+r+\lambda}}.$$

Mahmoud, Talat and Moustafa [14] presented the following family of approximations of the function $G(x)$

$$M(\mu, x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)}, \qquad x > 0;\ 1 \leq \mu \leq 2$$

which is of an order of convergence of $O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$ for $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ and is asymptotically equivalent to $G(x)$ as $x \to \infty$.

In this paper, we presented some functional equations of the generalized Bateman's $G-$function

$$G_h(x) = \psi\left(\frac{x+h}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad 0 < h < 2;\ x \neq -2m, -2m-h \text{ for } m = 0, 1, 2, \ldots \tag{13}$$

and its relation with the hypergeometric function ${}_3F_2$. We deduced an asymptotic expansion of the function $G_h(x)$ and studied the completely monotonic property of the function $G_h(x) - \frac{s}{x^r}$ for different values of the parameter s, r and h for $x > 0$. Also, some new bounds and best possible bounds of the generalized Bateman's $G-$function are given.

2 Some relations of the function $G_h(x)$.

Lemma 2.1. *The function $G_h(x)$ satisfies the functional equation*

$$G_h(x+1)+G_h(x)=2\left(\psi(x+h)-\psi(x)\right), \qquad x>0. \tag{14}$$

Proof. Using the integral representation [3]

$$\psi(z)=-\gamma+\int_0^\infty \frac{e^{-t}-e^{-tz}}{1-e^{-t}}dt, \qquad R(z)>0 \tag{15}$$

we get

$$G_h(x)=2\int_0^\infty \frac{1-e^{-ht}}{1-e^{-2t}}\,e^{-xt}dt, \qquad x>0. \tag{16}$$

Also,

$$\begin{aligned}\psi(x+h)-\psi(x) &= \int_0^\infty \frac{1-e^{-ht}}{1-e^{-t}}\,e^{-xt}dt\\ &= \int_0^\infty \frac{1-e^{-ht}}{1-e^{-2t}}\,e^{-(x+1)t}dt+\int_0^\infty \frac{1-e^{-ht}}{1-e^{-2t}}\,e^{-xt}dt\\ &= \frac{1}{2}\left[G_h(x+1)+G_h(x)\right].\end{aligned}$$

□

In case of $h=1$ and using the functional equation $\psi(x+1)=\frac{1}{x}+\psi(x)$, we get the relation (3).

Lemma 2.2. *The function $G_h(x)$ satisfies the functional equation*

$$G_h(mx)=\frac{1}{m}\sum_{r=0}^{m-1} G_{\frac{h}{m}}\left(x+\frac{2r}{m}\right), \qquad x>0;\ m\in\mathbb{N}. \tag{17}$$

Proof.

$$\begin{aligned}\sum_{r=0}^{m-1} G_{\frac{h}{m}}\left(x+\frac{2r}{m}\right) &= \int_0^\infty \left(\sum_{r=0}^{m-1} e^{\frac{-2rt}{m}}\right)\frac{1-e^{\frac{-ht}{m}}}{1-e^{-2t}}\,e^{-xt}dt\\ &= \int_0^\infty \left(\frac{1-e^{-2t}}{1-e^{\frac{-2t}{m}}}\right)\frac{1-e^{\frac{-ht}{m}}}{1-e^{-2t}}\,e^{-xt}dt\\ &= \int_0^\infty \frac{1-e^{\frac{-ht}{m}}}{1-e^{\frac{-2t}{m}}}\,e^{-xt}dt\\ &= m\,G_h(mx).\end{aligned}$$

□

Remark 1. The following new functional equation of the ordinary function $G(x)$ in terms of the generalized function $G_h(x)$ can be obtained in case of $h=1$.

Corollary 2.3. *The function $G(x)$ satisfies the functional equation*

$$G(mx) = \frac{1}{m}\sum_{r=0}^{m-1} G_{\frac{1}{m}}\left(x+\frac{2r}{m}\right), \qquad x>0;\ m\in\mathbb{N}. \tag{18}$$

The following result relates the function $G_h(x)$ and the hypergeometric function ${}_3F_2$.

Lemma 2.4. *The function $G_h(x)$ satisfies*

$$G_h(x) = \frac{h}{x+h}\,{}_3F_2\left(1,1,\frac{h+2}{2};2,\frac{x+h+2}{2};1\right), \qquad x>0. \tag{19}$$

Proof. Using the integral representation [3]

$$\psi(z) = -\gamma + \int_0^1 \frac{1-t^{z-1}}{1-t}dt, \qquad R(z)>0$$

we get

$$G_h(x) = \int_0^1 \frac{t^{\frac{x-2}{2}} - t^{\frac{x+h-2}{2}}}{1-t}dt = \int_0^1 \left(t^{\frac{x-2}{2}} - t^{\frac{x+h-2}{2}}\right)\left(\sum_{n=0}^{\infty} t^n\right)dt, \qquad x>0$$

and then

$$G_h(x) = \sum_{n=0}^{\infty}\frac{2h}{(x+2n)(x+h+2n)}, \qquad x>0. \tag{20}$$

Using the relation

$$x+n = \frac{x(x+1)_n}{(x)_n},$$

we obtain

$$\begin{aligned} G_h(x) &= \frac{2h}{x(x+h)}\sum_{n=0}^{\infty}\frac{\left(\frac{x+h}{2}\right)_n\left(\frac{x}{2}\right)_n}{\left(\frac{x+h+2}{2}\right)_n\left(\frac{x+2}{2}\right)_n} \\ &= \frac{2h}{x(x+h)}\,{}_3F_2\left(1,\frac{x}{2},\frac{x+h}{2};\frac{x+2}{2},\frac{x+h+2}{2};1\right), \qquad x>0. \end{aligned}$$

Now using the two-term Thomae transformation formula [30], [21]

$${}_3F_2\left(\alpha,\beta,\sigma;\delta,\eta;1\right) = \frac{\Gamma(\delta)\Gamma(\theta-\sigma)}{\Gamma(\theta)\Gamma(\delta-\sigma)}\,{}_3F_2\left(\eta-\alpha,\eta-\beta,\sigma;\theta,\eta;1\right), \quad \theta=\delta+\eta-\alpha-\beta$$

with

$$\alpha=\frac{x}{2},\ \ \beta=\frac{x+h}{2},\ \ \sigma=1,\ \ \eta=\frac{x+h+2}{2},\ \ \delta=\frac{x+2}{2}$$

we have

$${}_3F_2\left(1,\frac{x}{2},\frac{x+h}{2};\frac{x+2}{2},\frac{x+h+2}{2};1\right) = \frac{x}{2}\,{}_3F_2\left(1,1,\frac{h+2}{2};2,\frac{x+h+2}{2};1\right),$$

which complete the proof. □

Remark 2. From the formulas (7) and (19) for $h=1$, we can conclude that

$${}_3F_2\left(1,1,3/2;2,\frac{x+3}{2};1\right) = \frac{2(x+1)}{x}\,{}_2F_1\left(1,x;x+1;-1\right), \qquad x>0. \tag{21}$$

3 An asymptotic expansion of the function $G_h(x)$.

Ii is well known that the Psi function has the asymptotic expansion [6]

$$\psi(z) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} \frac{1}{z^k}$$

and its generalization is given by

$$\psi(z+l) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k(l)}{k} \frac{1}{z^k},$$

where $B_k(l)$ are the Bernoulli polynomials defined by the generating function [11]

$$\frac{ze^{zt}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(l)}{k!} z^k$$

and the Bernoulli constants $B_k = B_k(0)$. Using the operations of the asymptotic expansions [5]; [20], we obtain

$$\psi(z+l) - \psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[B_k(l) - B_k\right] \frac{1}{z^k}.$$

For more details about the general theory of the asymptotic expansion of the function $f(z+t)$ by the asymptotic expansion of the function $f(z)$ using Appell polynomials, we refer to [4]. Now, using the identity [11]

$$B_k(l) = \sum_{r=0}^{k} \binom{k}{r} B_r l^{k-r},$$

we get

$$\psi(z+l) - \psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\sum_{r=0}^{k-1} \binom{k}{r} B_r l^{k-r}\right] \frac{1}{z^k}.$$

Then we obtain the following result.

Lemma 3.1. *The following asymptotic series holds:*

$$G_h(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \left[B_r\left(\frac{h}{2}\right) - B_r\right] \frac{1}{x^n}, \qquad x \to \infty. \tag{22}$$

or

$$G_h(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\sum_{r=0}^{n-1} \binom{n}{r} 2^r B_r h^{n-r}\right] \frac{1}{x^n}, \qquad x \to \infty. \tag{23}$$

Remark 3. As a special case at $h = 1$, we obtain

$$G(x) \sim \frac{1}{x} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \left[B_r\left(\frac{1}{2}\right) - B_r\right] \frac{1}{x^n}, \qquad x \to \infty$$

and using the identities [1]

$$B_n\left(\frac{1}{2}\right) = \left(2^{1-n}-1\right)B_n, \qquad n = 0, 1, 2, ...$$

and

$$B_{2n+1} = 0, \qquad n = 1, 2, ...$$

then we get the asymptotic series (9).

Now we will study the completely monotonic property of the function $G_h(x) - \frac{s}{x^r}$ for different values of the parameters s, h and r for positive values of x.

Lemma 3.2. *The functions*

$$\chi_{s,1}(x,h) = G_h(x) - \frac{s}{x} \qquad s \le h;\ x > 0;\ 0 < h < 2, \tag{24}$$

and

$$\chi_{s,r}(x,h) = G_h(x) - \frac{s}{x^r} \qquad s < 0;\ x > 0;\ 0 < h < 2;\ r = 2, 3, 4, ... \tag{25}$$

are strictly completely monotonic.

Proof. Using the relation (16) and the known formula

$$(r-1)!\, x^{-m} = \int_0^\infty v^{m-1}e^{-xv}dv, \qquad m \in \mathbb{N} \tag{26}$$

we get

$$(-1)^n \chi_{s,r}^{(n)}(x,h) = \int_0^\infty \phi_{h,s}(r,t)\frac{t^n e^{-xt}}{e^{2t}-1}dt, \qquad n = 0, 1, 2, 3, ... \tag{27}$$

where

$$\phi_{h,s}(r,t) = 2\left(e^{2t} - e^{(2-h)t}\right) - \frac{s\ t^{r-1}}{(r-1)!}\left(e^{2t}-1\right).$$

Then

$$\phi_{h,s}(r,t) = \sum_{k=1}^{\infty}\frac{2^{k+1}t^k}{k!}P_{h,s}(r,t),$$

where

$$P_{h,s,k}(r,t) = 1 - \left(1-\frac{h}{2}\right)^k - \frac{s}{2(r-1)!}t^{r-1}.$$

Firstly, if $r = 1$, we obtain

$$P_{h,s,k}(1,t) = 1 - \left(1-\frac{h}{2}\right)^k - \frac{s}{2} > 0 \quad \text{iff} \quad \frac{s}{2} \le 1 - \left(1-\frac{h}{2}\right)^k \qquad k = 1, 2, 3, ...$$

But

$$\frac{h}{2} \le 1 - \left(1-\frac{h}{2}\right)^k \qquad 0 < h < 2;\ k = 1, 2, 3, ...$$

and thus, $\phi_{h,s}(1,t) > 0$ for all $t \ge 0$ iff $s \le h$. Secondly, when $r = 2, 3, 4, ...$, then $P_{h,s,k}(r,t)$ is increasing as a function of t if $s < 0$ with $P_{h,s,k}(r,0) = 1 - \left(1-\frac{h}{2}\right)^k > 0$ for $0 < h < 2$ and $k = 1, 2, 3, ...$. Thus $\phi_{h,s}(r,t) > 0$ for all $t \ge 0$, $r = 2, 3, ...$ iff $s < 0$. □

As a result of the strictly completely monotonicity of the function $\chi_{s,1}(x,h)$ and the relation (23), we obtain

$$\chi_{s,1}(x,h) > \lim_{x\to\infty}(\chi_{s,1}(x,h)) = 0, \qquad s \le h.$$

Hence, we have the following result:

Corollary 3.3. *The following inequality holds*

$$G_h(x) > \frac{h}{x}, \qquad x > 0;\ 0 < h < 2. \tag{28}$$

4 Some Bounds of the function $G_h(x)$.

Lemma 4.1.

$$G_h(x) < \frac{2}{x} + \frac{h(2-h)}{2x^2}, \qquad x > 0;\ 0 < h < 2. \tag{29}$$

Proof. By using the formulas (16) and (26), we get for $x > 0$ that

$$\begin{aligned} G_h(x) - \frac{2}{x} - \frac{h(2-h)}{2x^2} &= \int_0^\infty \left(2\left(e^{2t} - e^{(2-h)t}\right) - 2\left(e^{2t}-1\right) - \frac{h(2-h)}{2}\left(e^{2t}-1\right)t\right)\frac{e^{-xt}}{e^{2t}-1}dt \\ &= \int_0^\infty \left(2\left(1 - e^{(2-h)t}\right) - \frac{h(2-h)}{2}\left(e^{2t}-1\right)t\right)\frac{e^{-xt}}{e^{2t}-1}dt \\ &< 0 \quad \text{for } 0 < h < 2. \end{aligned}$$

□

Theorem 1.

$$G_h(x) < \frac{h}{x} + \frac{h(2-h)}{2x^2}, \qquad x > 0;\ 1 \le h < 2. \tag{30}$$

Proof. Using the two formulas (16) and (26), we have

$$G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2} = \int_0^\infty \rho_h(t)\frac{e^{-xt}}{e^{2t}-1}dt,$$

where

$$\rho_h(t) = 2\left(e^{2t} - e^{(2-h)t}\right) - h\left(e^{2t}-1\right) - \frac{h(2-h)}{2}\left(e^{2t}-1\right)t \qquad t > 0.$$

Then

$$\rho_h''(t) = 2(h-2)e^{(2-h)t}Q_h(t)$$

with

$$Q_h(t) = 2 - h + e^{ht}(h - 2 + ht).$$

The function $Q_h(t)$ is convex function with minimum value at $t_0 = \frac{1-h}{h}$, which is non positive for $1 \le h < 2$ and $Q_h(0) = 0$. Hence $Q_h(t) > 0$ for $1 \le h < 2$. Hence $\rho_h(t)$ is concave for $1 \le h < 2$ and its has maximum value at $t = 0$. Then

$$\rho_h(t) < 0, \qquad 1 \le h < 2;\ t > 0.$$

Then the function $G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2}$ is strictly increasing function for $1 \leq h < 2$ and $x > 0$ and using the asymptotic expansion (23), we get

$$\lim_{x\to\infty}\left(G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2}\right) = 0,$$

which complete the proof. □

Remark 4. In case of $h = 1$, the inequality (30) will prove the right-hand side of the inequality (8).

To obtain our next result, we will apply the following monotone form of L'Hôpital's rule [10] (see also [22] and [29]).

Theorem 2. *Let $-\infty < \alpha < \beta < \infty$ and $L, U : [\alpha, \beta] \to \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable on (α, β). Let $U'(x) \neq 0$ on (α, β). If $L'(x)/U'(x)$ is increasing (decreasing) on (α, β), then so are*

$$\frac{L(x) - L(\alpha)}{U(x) - U(\alpha)} \quad \text{and} \quad \frac{L(x) - L(\beta)}{U(x) - U(\beta)}. \tag{31}$$

If $L'(x)/U'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Theorem 3.

$$G_h(x) > \frac{h}{x} + \frac{h(2-h)}{2\left(x^2 + 3h^2\right)}, \qquad x > 0;\ 0 < h < 2. \tag{32}$$

Proof. Using the two formulas (16) and (26) and the Laplace transformation of the sine function, we get

$$G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2\left(x^2 + 3h^2\right)} = \int_0^\infty \xi_h(t)\frac{e^{-xt}}{6\left(e^{2t} - 1\right)}dt,$$

where

$$\xi_h(t) = 6\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right) + \sqrt{3}e^{ht}\left(-1 + e^{2t}\right)(-2+h)\sin\left(\sqrt{3}ht\right).$$

Now consider the function

$$\tau_h(t) = \frac{2\sqrt{3}e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right)}{\left(-1 + e^{2t}\right)(2-h)} \qquad t > 0;\ 0 < h < 2.$$

The function

$$\frac{2\sqrt{3}\frac{d}{dt}\left(e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right)\right)}{\frac{d}{dt}\left(\left(-1 + e^{2t}\right)(2-h)\right)} = 2\sqrt{3}e^{-ht}(-1 + e^{ht})$$

is increasing function for $t > 0$. Using the monotone form of L'Hôpital's rule, we get that the function $\tau_h(t)$ is increasing. Similarly, the function

$$H_h(t) = \frac{\tau_h(t)}{\sqrt{3}ht}, \qquad t > 0;\ 0 < h < 2.$$

is increasing function and

$$\lim_{t\to\infty} H_h(t) = 1.$$

Then

$$2\sqrt{3}e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right) > ht\left(-1+e^{2t}\right)(2-h), \qquad t>0;\ 0<h<2$$

and using Jordan's inequality

$$\frac{2z}{\pi} \le \sin z \le z, \qquad x \in [0, \pi/2]$$

we have

$$2\sqrt{3}e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right) > ht\left(-1+e^{2t}\right)(2-h)\sin\left(\sqrt{3}ht\right), \qquad t>0;\ 0<h<2.$$

Hence

$$\xi_h(t) > 0, \qquad t>0;\ 0<h<2.$$

Then the function $G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2(x^2+3h^2)}$ is strictly decreasing function for $0 \le h < 2$ and $x > 0$. Also, using the asymptotic expansion (23), we get

$$\lim_{x\to\infty}\left(G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2\left(x^2+3h^2\right)}\right) = 0,$$

which complete the proof. □

Remark 5. In case of $h = 1$, the inequality (32) will prove the left-hand side of the inequality (10).

Remark 6. Using the inequalities (29), (30) and (32) with the relation (20), we get the following estimations

$$\frac{1}{2x} + \frac{2-h}{4\left(x^2+3h^2\right)} < \sum_{n=0}^{\infty}\frac{1}{(x+2n)(x+h+2n)} < \frac{1}{2x} + \frac{2-h}{4x^2}, \qquad x>0;\ 1\le h<2$$

and

$$\frac{1}{2x} + \frac{2-h}{4\left(x^2+3h^2\right)} < \sum_{n=0}^{\infty}\frac{1}{(x+2n)(x+h+2n)} < \frac{1}{hx} + \frac{2-h}{4x^2}, \qquad x>0;\ 0<h<2.$$

5 Sharp double inequality of the function $G_h(x)$.

Theorem 4. *For $x > 0$, we have*

$$\ln\left(1+\frac{h}{x+2}\right) + \frac{2h}{x(x+h)} \le G_h(x) \le \ln\left(1+\frac{h}{x}\right) + \frac{2h}{x(x+h)}. \tag{33}$$

Proof. Consider the following function for $x > 0$ and $0 < h < 2$

$$A_{x,h}(t) = \frac{2h}{(x+h+2t)(x+2t)}, \qquad t \geq 0$$

which is strictly decreasing since

$$\frac{d}{dt}A_{x,h}(t) = -\frac{4h(2x+h+4t)}{(x+2t)^2(x+h+2t)^2}.$$

The function $A_{x,h}(t)$ is continuous for $t \geq 0$, then its lower Riemann sum $L(A_{x,h}(t), P)$ with respect to the partition $P = \{0, 1, ..., n\}$ of the interval $[0, n]$, $n \in N$, satisfies

$$L(A_{x,h}(t), P) < \int_0^n A_{x,h}(t)dt.$$

Hence

$$\sum_{i=1}^{n} A_{x,h}(i) < \ln\left(\frac{(x+h)(x+2n)}{x(x+h+2n)}\right)$$

or

$$\sum_{i=0}^{n} \frac{2h}{(x+h+2i)(x+2i)} < \ln\left(\frac{(x+h)(x+2n)}{x(x+h+2n)}\right) + \frac{2h}{x(x+h)}.$$

Then as $n \to \infty$, we get

$$G_h(x) \leq \ln\left(1+\frac{h}{x}\right) + \frac{2h}{x(x+h)}.$$

Also, the function $A_{x,h}(t+1)$ is continuous for $t \geq -1$, then its upper Riemann sum $U(A_{x,h}(t+1), P)$ with respect to the partition P of the interval $[0, n]$, $n \in N$, satisfies

$$\int_0^n A_{x,h}(t+1)dt < U(A_{x,h}(t+1), P).$$

Hence

$$\ln\left(\frac{(x+h+2)(x+2+2n)}{(x+2)(x+h+2+2n)}\right) < \sum_{i=0}^{n-1} A_{x,h}(i+1) = \sum_{i=1}^{n} A_{x,h}(i)$$

or

$$\ln\left(\frac{(x+h+2)(x+2+2n)}{(x+2)(x+h+2+2n)}\right) + \frac{2h}{x(x+hx)} < \sum_{i=0}^{n} \frac{2h}{(x+h+2i)(x+2i)}.$$

Then as $n \to \infty$, we get

$$\ln\left(1+\frac{h}{x+2}\right) + \frac{2h}{x(x+h)} \leq G_h(x).$$

□

Our next step, will be the investigation of the best bounds of the double inequality (33). To obtain this, we will deduce some auxiliary Lemmas.

Lemma 5.1. *The function*

$$f_h(x) = e^{G_h(x) - \frac{2h}{x(x+h)}} - 1, \qquad x > 0;\ 0 < h < 2$$

satisfies

$$\lim_{x\to\infty} \left(\frac{h}{f_h(x)} - x \right) = 1, \tag{34}$$

$$\lim_{x\to 0} \left(\frac{h}{f_h(x)} - x \right) = \frac{h}{e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)}}, \tag{35}$$

$$\lim_{x\to\infty} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) = 1, \tag{36}$$

and

$$\lim_{x\to 0} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) = \delta(h), \tag{37}$$

where

$$\delta(h) = \frac{e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} \left(24 + h^2\pi^2 - 6h^2\psi'\left(\frac{h}{2}\right) \right)}{12h \left(e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} - 1 \right)^2}.$$

Proof. Using the expansion

$$f_h(x) = \frac{h}{x} - \frac{h}{x^2} + O(x^{-3}), \tag{38}$$

we get

$$\begin{aligned} \lim_{x\to\infty} \left(\frac{h}{f_h(x)} - x \right) &= \lim_{x\to\infty} \left(\frac{h}{\frac{h}{x} - \frac{h}{x^2} + O(x^{-3})} - x \right) \\ &= \lim_{x\to\infty} \left(\frac{\frac{h}{x} - O(x^{-2})}{\frac{h}{x} - \frac{h}{x^2} + O(x^{-3})} \right) = 1. \end{aligned}$$

The expansion

$$G_h(x) - \frac{2h}{x(x+h)} = \gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right) + \left(-\frac{2}{h^2} - \frac{\pi^2}{12} + \frac{1}{2}\psi'\left(\frac{h}{2}\right) \right) x + O(x^2) \tag{39}$$

gives us that

$$\begin{aligned} \lim_{x\to 0} \left(\frac{h}{f_h(x)} - x \right) &= \lim_{x\to 0} \left(\frac{h}{e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right) + \left(-\frac{2}{h^2} - \frac{\pi^2}{12} + \frac{1}{2}\psi'\left(\frac{h}{2}\right) \right) x + O(x^2)} - 1} - x \right) \\ &= \frac{h}{e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} - 1}. \end{aligned}$$

Now using the expansion (38) and the expansion

$$G_h'(x) + \frac{2h(h+2x)}{x^2(x+h)^2} = -\frac{h}{x^2} + O(x^{-3}),$$

we have

$$\begin{aligned}\lim_{x\to\infty}\frac{d}{dx}\left(\frac{h}{f_h(x)}\right) &= \lim_{x\to\infty}\left(\frac{-h}{\left[\frac{h}{x}-\frac{h}{x^2}+O(x^{-3})\right]^2}\left[1+\frac{h}{x}-\frac{h}{x^2}+O(x^{-3})\right]\left[-\frac{h}{x^2}+O(x^{-3})\right]\right)\\ &= \lim_{x\to\infty}\left(\frac{h^2+O\left(\frac{1}{x}\right)}{h^2-O\left(\frac{1}{x}\right)}\right)\\ &= 1.\end{aligned}$$

Also, using the expansions

$$e^{G_h(x)-\frac{2h}{x(x+h)}}=e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}+O(x)$$

and

$$G_h'(x)+\frac{2h(h+2x)}{x^2(x+h)^2}=\frac{-2}{h^2}-\frac{\pi^2}{12}+\frac{1}{2}+\psi'\left(\frac{h}{2}\right)+O(x),$$

we obtain

$$\begin{aligned}\lim_{x\to 0}\frac{d}{dx}\left(\frac{h}{f_h(x)}\right) &= \lim_{x\to 0}\left(\frac{-h}{\left[e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}-1+O(x)\right]^2}\left[e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}+O(x)\right]\right.\\ &\qquad\left.\left[\frac{-2}{h^2}-\frac{\pi^2}{12}+\frac{1}{2}+\psi'\left(\frac{h}{2}\right)+O(x)\right]\right)\\ &= \delta(h).\end{aligned}$$

□

Lemma 5.2.

$$\delta(h)=\frac{e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}\left(24+h^2\pi^2-6h^2\psi'\left(\frac{h}{2}\right)\right)}{12h\left(e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}-1\right)^2}<1,\qquad 0<h<2. \tag{40}$$

Proof. Consider the two functions

$$P_1(h)=\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)$$

and

$$P_2(h)=24+h^2\pi^2-6h^2\psi'\left(\frac{h}{2}\right).$$

Using the integral representation (15), we have

$$P_1'(h)=\frac{-2}{h^2}+\frac{1}{2}\psi'\left(\frac{h}{2}\right)=\int_0^\infty\frac{2t}{e^{2t}-1}e^{-xt}dt\ >0$$

and then the function $P_1(h)$ is increasing and positive since

$$\lim_{h\to 0}P_1(h)=0,$$

where

$$\psi\left(\frac{h}{2}\right) = -\gamma - \frac{2}{h} + \frac{\pi^2}{12}h + O(h^2).$$

Also,

$$P_2'(h) = hP_3(h),$$

where

$$P_3(h) = 2\pi^2 - 12\psi'\left(\frac{h}{2}\right) - 3h\psi''\left(\frac{h}{2}\right).$$

Using the integral representation (15), we obtain

$$P_3(h) = 2\pi^2 - \int_0^\infty \frac{24e^{2t}(e^{2t} - 1 + t)}{(e^{2t} - 1)^2} e^{-xt} dt$$

and then

$$P_3'(h) = \int_0^\infty \frac{24e^{2t}(e^{2t} - 1 + t)}{(e^{2t} - 1)^2} te^{-xt} dt \ > 0.$$

Hence the function $P_3(h)$ is strictly increasing and using the expansions

$$\psi'\left(\frac{h}{2}\right) = \frac{4}{h^2} + \frac{\pi^2}{6} + O(h)$$

and

$$\psi''\left(\frac{h}{2}\right) = \frac{-16}{h^3} + \psi'(1) + O(h),$$

we get

$$\lim_{h\to 0} P_3(h) = 0.$$

Then $P_3(h) > 0$ and hence the function $P_2(h)$ is strictly increasing and positive since

$$\lim_{h\to 0} P_2(h) = \lim_{h\to 0}\left(24 + h^2\pi^2 - 6h^2\left(\frac{4}{h^2} + \frac{\pi^2}{6} + O(h)\right)\right) = 0.$$

Now consider the two functions

$$P_4(h) = 24(e-1)^2 e^{P_1(h)} P_2(h)$$

and

$$P_5(h) = 288eh(e^{P_1(h)} - 1)^2.$$

Using the properties of the two functions $P_1(h)$ and $P_2(h)$, we conclude that $P_4(h)$ and $P_5(h)$ are strictly increasing positive functions and

$$P_4(0) = P_5(0), \qquad P_4(2) = P_5(2) = 576e(e-1)^2,$$

$$P_4(1) = 12(e-1)^2(\pi^2 - 12)e^2 \ < \ P_5(1) = 18e(e^2-4)^2,$$

where $\psi\left(\frac{1}{2}\right) = -\gamma - \ln 4$ (see [3]). Then we get

$$P_4(h) < P_5(h), \qquad 0 < h < 2$$

which complete the proof. □

Now, we are in position to prove the following result.

Theorem 5. *For all $x \in (0,\infty)$ and a fixed $h \in (0,2)$, we have*

$$\ln\left(1+\frac{h}{x+\beta}\right) < G_h(x) - \frac{2h}{x(x+h)} < \ln\left(1+\frac{h}{x+\alpha}\right), \tag{41}$$

where the constants $\alpha = 1$ and $\beta = \frac{h}{e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}-1}$ are the best possible.

Proof. Using the inequality (33), we have

$$0 < \frac{h}{e^{G_h(x)-\frac{2h}{x(x+h)}}-1} - x < 2.$$

Let

$$M_h(x) = \frac{h}{f_h(x)} - x, \qquad x > 0;\ 0 < h < 2.$$

Using the relation

$$f_h'(x) = e^{G_h(x)-\frac{2h}{x(x+h)}}\left(G_h'(x) + \frac{2h(h+2x)}{x^2(x+h)^2}\right)$$

and the integral representation

$$G_h(x) = 2\int_0^\infty \frac{1-e^{-ht}}{1-e^{-2t}}e^{-xt}dt, \qquad x > 0$$

we have

$$G_h'(x) + \frac{2h(h+2x)}{x^2(x+h)^2} = -\int_0^\infty \frac{2e^{-ht}(e^{ht}-1)t}{e^{2t}-1}e^{-xt}dt < 0$$

then $f_h(x)$ is strictly decreasing positive function. Hence $\frac{h}{f_h(x)}$ is strictly increasing positive function. Also, $f_h(x)$ is strictly convex function since

$$f_h''(x) = e^{G_h(x)-\frac{2h}{x(x+h)}}\left[\left(G_h'(x) + \frac{2h(h+2x)}{x^2(x+h)^2}\right)^2 + \int_0^\infty \frac{2e^{-ht}(e^{ht}-1)t^2}{e^{2t}-1}e^{-xt}dt\right] > 0.$$

From the relations (36) and (37) with the inequality (40), we conclude that the function $\frac{h}{f_h(x)}$ is convex and $\frac{d}{dx}\left(\frac{h}{f_h(x)}\right)$ is increasing function. Thus we get

$$\frac{d}{dx}\left(\frac{h}{f_h(x)}\right) < \lim_{x\to\infty}\frac{d}{dx}\left(\frac{h}{f_h(x)}\right) = 1, \qquad x > 0.$$

Then $M_h(x)$ is strictly decreasing function for all $x > 0$, where

$$\frac{d}{dx}M_h(x) = \frac{d}{dx}\frac{h}{f_h(x)} - 1 < 0.$$

Hence $\lim_{x\to\infty} M_h(x) < M_h(x) < \lim_{x\to 0^+} M_h(x)$ and using the limits (34) and (35), we have

$$1 < M_h(x) < \frac{h}{e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}} \tag{42}$$

with best bounds. □

Remark 7. Using the double inequality (41) with the relation (19) or the relation (20), we obtain the following estimation

$$\left(\frac{x+h}{h}\right)\ln\left(1+\frac{h}{x+\beta}\right)+\frac{2}{x}<\ {}_3F_2\left(1,1,\frac{h+2}{2};2,\frac{x+h+2}{2};1\right)$$
$$<\left(\frac{x+h}{h}\right)\ln\left(1+\frac{h}{x+\alpha}\right)+\frac{2}{x}\qquad x>0;\ 0<h<2$$

or

$$\frac{1}{2h}\ln\left(1+\frac{h}{x+\beta}\right)+\frac{1}{x(x+h)}<\sum_{k=0}^{\infty}\frac{1}{(x+2k)(x+h+2k)}$$
$$<\frac{1}{2h}\ln\left(1+\frac{h}{x+\alpha}\right)+\frac{1}{x(x+h)},\qquad x>0;\ 0<h<2$$

with sharp bounds $\alpha=1$ and $\beta=\frac{h}{e^{\gamma+\frac{2}{h}+\psi\left(\frac{h}{2}\right)}-1}$.

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Interval-valued fuzzy quasi-metric spaces *

Hanchuan Lu[a],[b],⋆ Shenggang Li[c]

a School of Mathematical Sciences, Nanjing Normal University,210023 Nanjing , P. R. China

b Guizhou Provincial Key Laboratory of Public Big Data, Guizhou University,550025, Guiyang, P. R. China

c College of Mathematics and Information Science, Shaanxi Normal University, 710062, Xi'an, P. R. China

Abstract — Under the context of quasi-metric, promoted the concept of interval-valued fuzzy metric space, the main results are as follows :(1) topology induced by quasi-metric is consistent with which induced via a standard interval-valued fuzzy quasi-metric; (2) proved that every quasi-metrizable topological space admits a compatible interval-valued fuzzy quasi-metric. On the contrary, topology generated by interval-valued fuzzy quasi-metric is quasi-metrizable; (3) discussed some properties of interval-valued fuzzy quasi-metric space which is bicompletion. proved that if an interval-valued fuzzy quasi-metric space has bicompletion, then it is unique up to isometry. In addition, we define a fuzzy contraction mapping of interval-valued fuzzy metric space, promote the Banach and Edelstein fixed point theorem to interval fuzzy metric space.

Keywords —Interval-valued fuzzy quasi-metric spaces; Quasi-metric; Quasi-uniformity; Bicompletion; Isometry

AMS classification 54A40

1 Introduction

In 1975, Kramosil and Michalek introduced the concept of fuzzy metric spaces which is closely related with probabilistic metric spaces in [1], it is also known as generalized Menger space. Currently, many mathematical workers use the concept of fuzzy sets giving different fuzzy metric space ideas [1-5]. Among them, A.George and P.Veeramani improved the concept of fuzzy metric space defined by Kramosil and Michalek in [2, 6], proposed the concept of continuous t-norm, and they used fuzzy sets to represent the uncertainty of the distance between two points in a fuzzy metric space, thus obtained a stronger form of fuzzy metric which has a Hausdorff topology. It is clearly that

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⋆ Corresponding author. E-mail: luhanchuan2004@163.com (H.-C. Lu)

the concept of fuzzy metric space is different with the paper [1, 3, 4, 5] definates. In particular, every metric induces a fuzzy metric space which is under A.George and P.Veeramani significance. Instead, each fuzzy metric space in A.George and P.Veeramani significance generate a metrizable topology [7, 8]. As early as in 1975, Zadah has given the concept of interval-valued fuzzy sets [9]. It characterizes fuzzy set by interval-valued membership functions, which Is another generalize of fuzzy set. In 2012, based on the concept of A.George and P.Veeramani's fuzzy metric space, Shen [10] given a conception of continuous interval value t- norm and interval-valued fuzzy metric space, and also discussed some topological properties of such metric spaces. In addition, it is known that quasi-metric space constitutes a very effective tool in discuss and resolve topology algebraic, approximation theory, theory computational science and other aspects [11,18]. In the context of quasi-metric, this article generalized interval-valued fuzzy metric that the paper [10] defined, proposed the concept of interval-valued fuzzy quasi-metric, as a basis for the study. Structure of the article is divided into five parts. The first part is to introduce the background knowledge; the second part is prior knowledge; the third section discuss the quasi-metrizable of interval-valued fuzzy metric space; part IV discuss the bicompletion of interval-valued fuzzy metric space; section V discuss the fixed point theorem of interval-valued fuzzy metric space.

2 Preliminaries

Interval analysis (see http://www.cs.utep.edu/interval-comp/main.html) leaded by interval numbers is an area of active research in mathematics, numerical analysis and computer science began in the late 1950s. An interval number is a point $\langle a^-, a^+\rangle$ in the 2-dimensional Euclidean space R^2 which satisfies $a^- \leq a^+$. The set of all interval numbers is denoted by $\mathbb{I}(R)$ [1]. For any $\langle a^-, a^+\rangle, \langle b^-, b^+\rangle \in \mathbb{I}(R)$ and each nonnegative real number r, define $\langle a^-, a^+\rangle \oplus \langle b^-, b^+\rangle = \langle a^- + b^-, a^+ + b^+\rangle$, $\langle a^-, a^+\rangle \ominus \langle b^-, b^+\rangle = \langle \min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}\rangle$, and $r\langle a^-, a^+\rangle = \langle ra^-, ra^+\rangle$. We write $\langle a^-, a^+\rangle = \boldsymbol{a}$ when $a^- = a^+ = a$, and $\mathbb{I}(I) = \{\langle a^-, a^+\rangle \in \mathbb{I}(R) \mid a^-, a^+ \in I\}$ (where $I = [0, 1]$ is the ordinary closed unit interval). In this paper we only involve the notion of interval-valued fuzzy set used generally to dispose uncertainness (for an overview of

[1] Unless confusion arise, we identify a real number $a \in R$ with an interval number $\boldsymbol{a}$, and identify a closed interval $[a, b]$ of R and a point $\langle a, b\rangle$ in R^2 since there exists a nature one-to-one correspondence between the set of all closed intervals of R and $\mathbb{I}(R)$; we will also use $\mathbb{a}, \mathbb{b}, \mathbb{c}, \cdots$ to denote interval numbers.

interval analysis, its relationship to fuzzy set theory, and possible areas of further fruitful research, please see [10,12,13,14,15,17,19]).

Definition 2.1 An interval-valued fuzzy set (resp., fuzzy set) on a set X is exactly a mapping $A : X \longrightarrow \mathbb{I}(I)$ (resp., $A : X \longrightarrow I$). The set of all interval-valued fuzzy sets on X is denoted by $\mathrm{IVF}(X)$.

Each $A \in \mathrm{IVF}(X)$ induces two fuzzy sets $A^- : X \longrightarrow I$ and $A^+ : X \longrightarrow I$ whose values are determined by $A(x) = \langle A^-(x), A^+(x)\rangle \ \ (\forall x \in X)$. For $A, B \in \mathrm{IVF}(X)$, the infimum $A \wedge B \in \mathrm{IVF}(X)$ of A and B in $\mathrm{IVF}(X)$ is given by

$$(A \wedge B)(x) = \langle \min\{A^-(x), B^-(x)\}, \min\{A^+(x), B^+(x)\}\rangle \ \ (\forall x \in X),$$

the suprmum $A \vee B \in \mathrm{IVF}(X)$ of A and B in $\mathrm{IVF}(X)$ is given by

$$(A \vee B)(x) = \langle \max\{A^-(x), B^-(x)\}, \max\{A^+(x), B^+(x)\}\rangle \ \ (\forall x \in X),$$

and the complement $A' \in \mathrm{IVF}(X)$ of A in $\mathrm{IVF}(X)$ is given by

$$A'(x) = \langle 1 - A^+(x), 1 - A^-(x)\rangle \ \ (\forall x \in X).$$

Definition 2.2[10] An interval-valued t-norm is a binary operation $* : \mathbb{I}(I) \times \mathbb{I}(I) \longrightarrow \mathbb{I}(I)$ which satisfies the following conditions:

(i) $\mathbb{a} * \mathbb{b} = \mathbb{b} * \mathbb{a} \ \ (\forall \mathbb{a}, \mathbb{b} \in \mathbb{I}(I))$.

(ii) $\mathbb{a} * (\mathbb{b} * \mathbb{c}) = (\mathbb{a} * \mathbb{b}) * \mathbb{c} \ \ (\forall \mathbb{a}, \mathbb{b}, \mathbb{c} \in \mathbb{I}(I))$.

(iii) $\mathbb{a} * \mathbb{b} \leq \mathbb{a} * \mathbb{c}$ whenever $\mathbb{b} \leq \mathbb{c}$.

(iv) $\mathbb{a} * \mathbf{1} = \mathbb{a} \ \ (\forall \mathbb{a} \in \mathbb{I}(I))$; $\langle a^-, a^+\rangle * \langle 0, 1\rangle = \langle 0, a^+\rangle \ \ (\forall \langle a^-, a^+\rangle \in \mathbb{I}(I))$.

Proposition 2.3[10] An interval-valued t-norm $*$ has the following properties:

(i) $\mathbf{0} * \mathbb{a} = \mathbb{a} * \mathbf{0} = \mathbf{0}$; $\langle 0, 1\rangle * \langle a^-, a^+\rangle = \langle 0, a^+\rangle$; $\mathbf{1} * \mathbb{a} = \mathbb{a}$.

(ii) $\mathbb{a} * \mathbb{b} \leq \mathbb{c} * \mathbb{d}$ whenever $\mathbb{a} \leq \mathbb{c}$ and $\mathbb{b} \leq \mathbb{d}$.

(iii) $\mathbf{0} = \mathbb{a} * \mathbf{0} \leq \mathbb{a} * \mathbb{b} \leq \mathbb{a} * \mathbf{1} = \mathbb{a}, \mathbf{0} = \mathbb{a} * \mathbf{0} \leq \mathbf{0} * \mathbb{b} \leq \mathbf{1} * \mathbb{b} = \mathbb{b}, \mathbf{0} = \mathbb{a} * \mathbf{0} \leq \mathbb{a} * \mathbb{b} \leq \mathbb{a} \wedge \mathbb{b}$.

Definition 2.4[10] (1) Let $\mathbb{a} = \langle a^-, a^+\rangle$ and $\mathbb{a}_n = \langle a_n^-, a_n^+\rangle$ are in $\mathbb{I}(I)$ for each $n \in N$ (the natural number set). $\{\mathbb{a}_n\}_{n\in N}$ (briefly, $\{\mathbb{a}_n\}$) is said to be convergent to $\mathbb{a}$ (write as $\lim\limits_{n\to\infty} \mathbb{a}_n = \mathbb{a}$) if $\lim\limits_{n\to\infty} a_n^- = a^-$ and $\lim\limits_{n\to\infty} a_n^+ = a^+$.

(2) An interval-valued t-norm $*$ is said to be continuous in its first component if $\lim\limits_{n\to\infty} (\mathbb{a}_n * \mathbb{b}) = (\lim\limits_{n\to\infty} \mathbb{a}_n) * \mathbb{b} = \mathbb{a} * \mathbb{b}$ for all $\mathbb{b} \in \mathbb{I}(I)$ whenever $\lim\limits_{n\to\infty} \mathbb{a}_n = \mathbb{a} \ \ (\{\mathbb{a}_n\}_{n\in N} \subseteq \mathbb{I}(I), \mathbb{a} \in \mathbb{I}(I))$.

Proposition 2.5[10] The followings hold for a continuous interval-valued t-norm $*$:

(i) For any $\mathtt{a}_1, \mathtt{a}_2 \in \mathbb{I}(I)$ with $\mathtt{a}_1 > \mathtt{a}_2$, there exists a $\mathtt{a}_3 \in \mathbb{I}(I)$ such that $\mathtt{a}_1 * \mathtt{a}_3 \geq \mathtt{a}_2$.

(ii) For any $\mathtt{a}_4 \in \mathbb{I}(I)$, there exists a $\mathtt{a}_5 \in \mathbb{I}(I)$ such that $\mathtt{a}_5 * \mathtt{a}_5 \geq \mathtt{a}_4$.

Definition 2.6[10] An interval-valued fuzzy metric space is a triple $(X, M, *)$, where $*$ is a continuous t-norm and M (called an interval-valued fuzzy metric on X) is an interval-valued fuzzy set on $X^2 \times (0, \infty)$ having the following properties $(x, y, z \in X)$:

(IV1) $M(x, y, t) > \mathbf{0}$.

(IV2) $M(x, y, t) = \mathbf{1}$ if and only if $x = y$.

(IV3) $M(x, y, t) = M(y, x, t)$.

(IV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$.

(IV5) $M(x, y, \cdot) : (0, \infty) \longrightarrow \mathbb{I}(I) - \{\mathbf{0}\}$ is continuous, i.e. both $M^-(x, y, \cdot) = p_1 \circ M(x, y, \cdot) : (0, \infty) \longrightarrow (0, 1]$ and $M^+(x, y, \cdot) = p_2 \circ M(x, y, \cdot) : (0, \infty) \longrightarrow (0, 1]$ are continuous, where $p_i : I^2 \longrightarrow I$ is the projective mapping $(i = 1, 2)$, and $M^-(x, y, t)$ and $M^+(x, y, t)$ denote the lower nearness and upper nearness degree between x and y with respect to t, respectively.

(IV6) $\lim\limits_{t \to \infty} M(x, y, t) = \mathbf{1}$.

Remark 2.7[10] (1) An interval-valued fuzzy metric space $(X, M, *)$ will degenerate into an ordinary fuzzy metric space if $M^-(x, y, t) = M^+(x, y, t)$ for all $t > 0$.

(2) Let $(X, M, *)$ be an interval-valued fuzzy metric space.

(i) The set $B(x, \mathtt{r}, t) = \{y \in X \mid M(x, y, t) > \mathbf{1} \ominus \mathtt{r}\}$ is called a open ball with center $x \in X$, where $t > 0$ and $\mathtt{r} \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}$.

(ii) $\mathcal{J}_M = \{A \subseteq X \mid \forall x \in A, \exists t_x > 0, \exists \mathtt{r}_x \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}, B(x, \mathtt{r}_x, t_x) \subseteq A\}$ is a topology on X, and $\{B(x, \mathtt{r}, t) \mid x \in X, \mathtt{r} \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}, t > 0\}$ is a base of $\mathcal{J}_M$.

(iii) For each $(x, y, t, \mathtt{r}) \in X^2 \times (0, +\infty) \times (\mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\})$, there exists a $t_0 \in (0, t)$ such that $M(x, y, t_0) > \mathbf{1} \ominus \mathtt{r}$ whenever $M(x, y, t) > \mathbf{1} \ominus \mathtt{r}$.

(iv) For a sequence $\{x_n\} \subseteq X$, $\lim\limits_{n \to \infty} x_n = x$ in $(X, \mathcal{J}_M)$ if and only if $\lim\limits_{n \to \infty} M(x, x_n, t) = \mathbf{1}$.

(v) A sequence $\{x_n\}$ in $(X, M, *)$ is called a Cauchy sequence if for all $\boldsymbol{\varepsilon} > \mathbf{0}$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > \mathbf{1} \ominus \varepsilon$ for all $n, m \geq n_0$.

(vi) $(X, M, *)$ is said to be complete if every Cauchy sequence in it is convergent.

Remark 2.8[16] (1) A quasi-pseudo-metric space is a pair (X, d), where X is a set and d (called a quasi-pseudo-metric on X) is a mapping $d : X \longrightarrow [0, +\infty)$ satisfying the following conditions:

(i) $d(x, x) = 0 \ \ (\forall x \in X)$;

(ii) $d(x,z) \leq d(x,y) + d(y,z)$ $(\forall x,y,z \in X)$.

(2) A quasi-metric space is a pair (X,d), where X is a set and d (called a quasi-metric on X) is a quasi-pseudo-metric on X satisfying the following condition:

(iii) $x = y \Longleftrightarrow d(x,y) = d(y,x) = 0$.

Each quasi-metric d on X induces a T_0 topology $\mathcal{J}_d$ on X which has as a base the family of open balls $\{B_d(x,r) \mid x \in X, r > 0\}$, where $B_d(x,r) = \{y \in X \mid d(x,y) < r\}$ $(x \in X, r > 0)$.

(3) A topological space $(X,\mathcal{J})$ is said to be quasi-metrizable if there is a quasi-metric d on X such that $\mathcal{J}_d = \mathcal{J}$ (in this case, we say that d is compatible with $\mathcal{J}$, and that $\mathcal{J}$ is a quasi-metrizable topology on X), where $\mathcal{J}_d = \langle\langle\{B_d(x,r) \mid x \in X, r > 0\}\rangle\rangle$ (i.e. $\mathcal{J}_d$ is the smallest topology on X containing $\{B_d(x,r) \mid x \in X, r > 0\}$).

(4) If d is a quasi-pseudo-metric on X, then d^{-1} (called the conjugate of d which is defined by $d^{-1}(x,y) = d(y,x)$ $(\forall x,y \in X)$) and d^s (defined by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ $(\forall x,y \in X)$) are also quasi-pseudo-metrics on X. A quasi-metric space (X,d) is said to be bicomplete if (X,d^s) is a complete metric space. In this case we say that d is a bicomplete quasi-metric on X.

3 Quasi-metrizable of interval-valued fuzzy quasi-metric space

Definition 3.1 Let X be a set, $*$ a continuous t-norm, and M an interval-valued fuzzy set on $X^2 \times (0,\infty)$ having the following properties:

(IV1) $M(x,y,t) > \mathbf{0}$ $(x,y \in X, t \in (0,\infty))$.

(IV2) $M(x,x,t) = \mathbf{1}$ $(x \in X, t \in (0,\infty))$.

(IV3) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ $(x,y,z \in X, s,t \in (0,\infty))$.

(IV4) $M(x,y,\cdot) : (0,\infty) \longrightarrow \mathbb{I}(I) - \{\mathbf{0}\}$ is continuous $(x,y \in X)$.

(IV5) $\lim\limits_{t\to\infty} M(x,y,t) = \mathbf{1}$ $(x,y \in X)$.

(1) Such a triple $(X,M,*)$ is called an interval-valued fuzzy quasi-pseudo-metric space, and $(M,*)$ is called an interval-valued fuzzy quasi-pseudo-metric on X.

(2) This triple $(X,M,*)$ is called an interval-valued fuzzy quasi-metric space (and $(M,*)$ is called an interval-valued fuzzy quasi-metric on X) if M satisfies the following condition:

(IV1′) $x = y$ if and only if $M(x,y,t) = M(y,x,t) = \mathbf{1}$ for all $t > 0$.

(3) This triple $(X, M, *)$ is called a T_1 interval-valued fuzzy quasi-metric space (and $(M, *)$ is called a T_1 interval-valued fuzzy quasi-metric on X) if M satisfies the following condition:

(IV2′) $x = y$ if and only if $M(x, y, t) = \mathbf{1}$ for all $t > 0$,

(4) This triple $(X, M, *)$ is called an interval-valued fuzzy pseudo-metric space (and $(M, *)$ is called an interval-valued fuzzy pseudo-metric on X) if M satisfies the following condition:

(IV6) $M(x, y, t) = M(y, x, t)$for all $t > 0$.

Definition 3.1' An interval-valued fuzzy (pseudo-) metric on X is a interval-valued fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for each $x, y \in X$:

(IV6) $M(x, y, t) = M(y, x, t)$for all $t > 0$.

Remark 3.2 (1) There are many interval-valued fuzzy quasi-metric spaces which are not interval-valued fuzzy metric space.

(2) There are many interval-valued fuzzy pseudo-metric spaces which are not interval-valued fuzzy metric space.

Interval-valued fuzzy quasi-pseudo metric space is a weakened form of the interval value fuzzy metric Spaces, which is a generalization of the interval-valued fuzzy metric Spaces, will have a greater scope in the application.

Remark 3.4 It is clear that every interval-valued fuzzy metric is a T_1 interval-valued fuzzy quasi-metric; every T_1 interval-valued fuzzy quasi-metric is an interval-valued fuzzy quasi-metric, and every interval-valued fuzzy quasi-metric is an interval-valued fuzzy quasi-pseudo-metric.

Definition 3.5 An interval-valued fuzzy quasi-(pseudo-)metric space is a triple $(X, M, *)$ such that X is a (nonempty) set and $(M, *)$ is an interval-valued fuzzy quasi-(pseudo-)metric on X.

The notions of a T_1 interval-valued fuzzy quasi-metric space and of a interval-valued fuzzy metric space are defined in the obvious manner.

Remark 3.6 If $(M, *)$ is an interval-valued fuzzy quasi-(pseudo-)metric on a set X, it is immediate to show that $(M^{-1}, *)$ is also an interval-valued fuzzy quasi-(pseudo-)metric on X, where M^{-1} is the interval-valued fuzzy set in $X \times X \times (0, +\infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote M^i the interval-valued fuzzy set in $X \times X \times (0, +\infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is, clearly, an interval-valued fuzzy (pseudo-)metric on X.

Thus, conditions (IV1') of the definition 3.2 above is equivalent to the following:

$M(x,x,t)=\mathbf{1}$ for all $x\in X$ and $t>0$,and $M^i(x,y,t)<\mathbf{1}$ for all $x\neq y$ and $t>0$.

Definition 3.7 Let $(X,M,*)$ be an interval-valued fuzzy quasi-pseudo-metric space. We define open ball $B_M(x,\mathbb{r},t)=\{y\in X|M(x,y,t)>\mathbf{1}-\mathbb{r}\}$ for $t>0$ with center $x\in X$ and the interval number $\mathbb{r}$, $\mathbf{0}<\mathbb{r}<\mathbf{1}$, $t>0$.

Similar to the [10], we get the following conclusion:

Proposition 3.8 Let $(X,M,*)$ be an interval-valued fuzzy quasi-pseudo-metric space. Then every open ball $B_M(x,\mathbb{r},t)$ is an open set.

Proof. Let $B_M(x,\mathbb{r},t)$ is an open ball. For any $y\in B_M(x,\mathbb{r},t)$, we know that $M(x,y,t)>\mathbf{1}-\mathbb{r}$. Therefore, there exists a $t_0\in(0,t)$ such that $M(x,y,t_0)>\mathbf{1}-\mathbb{r}$. Set $\mathbb{r}_0=M(x,y,t_0)$. Since $\mathbb{r}_0>\mathbf{1}-\mathbb{r}$, there exists a $\mathbb{s}\in\mathbb{I}(I)$ such that $\mathbb{r}_0>\mathbf{1}-\mathbb{s}>\mathbf{1}-\mathbb{r}$. Now for given $\mathbb{r}_0$ and $\mathbb{s}$ with $\mathbb{r}_0>\mathbf{1}-\mathbb{s}$, there exists a $\mathbb{r}_1\in\mathbb{I}(I)$such that $\mathbb{r}_0*_I\mathbb{r}_1\geq\mathbf{1}-\mathbb{s}$. Consider the open ball $B_M(y,\mathbf{1}-\mathbb{r}_1,t-t_0)$, we will obtain that $B_M(y,\mathbf{1}-\mathbb{r}_1,t-t_0)\subset B_M(x,\mathbb{r},t)$. In fact, for every $z\in B_M(y,\mathbf{1}-\mathbb{r}_1,t-t_0)$, we have $M(y,z,t-t_0)>\mathbb{r}_1$. Therefore,

$$M(x,z,t)\geq M(x,y,t_0)*_I M(y,z,t-t_0)\geq\mathbb{r}_0*_I\mathbb{r}_1\geq\mathbf{1}-\mathbb{s}>\mathbf{1}-\mathbb{r}$$

Thus $x\in B_M(x,\mathbb{r},t)$, and hence $B_M(y,\mathbf{1}-\mathbb{r}_1,t-t_0)\subset B_M(x,\mathbb{r},t)$.

Similar to the [10], we get the following conclusion:

Theorem 3.9 Let $(X,M,*)$ be an interval-valued fuzzy quasi-pseudo-metric space. Define

$$\tau_M=\{A\subset X\mid\forall x\in A,\exists\mathbb{r}\in\mathbb{I}(I)-\{\mathbf{0},\mathbf{1}\},\text{ and }t>0\text{ such that }B_M(x,\mathbb{r},t)\subset A\}$$

Then τ_M is a topology on X.

Proposition 3.10 A sequence $\{x_n\}_n$ in an interval-valued fuzzy quasi-pseudo-metric space $(X,M,*)$ is said to be a Cauchy sequence if and only if $\lim_n M(x_{n+p},x_n,t)=\mathbf{1}$, for all $p>0,t>0$.

A sequence $\{x_n\}_n$ in an interval-valued fuzzy quasi-pseudo-metric space $(X,M,*)$ is converging to x in X,denoted by $x_n\longrightarrow x$, if and only if $\lim_n M(x,x_n,t)=\mathbf{1}$ for all $t>0$.

A interval-valued fuzzy quasi-pseudo-metric space $(X,M,*)$ is said to be complete if and only if every Cauchy sequence is convergent.

Proposition 3.11 Let $(X,M,*)$ be an interval-valued fuzzy quasi-pseudo-metric space. Then, for each $x,y\in X$ the function $M(x,y,-)$ is nondecreasing.

Proof. Let $x, y \in X$ and $0 \leq t < s$. Then

$$M(x, y, s) \geq M(x, x, s - t) * M(x, y, t) = \mathbf{1} * M(x, y, t) = M(x, y, t).$$

Proposition 3.12 Let $(X, M, *)$ be an interval-valued fuzzy quasi-metric space. Then, for each $t > 0$ the function

$$M(-, -, t) : (X \times X, \tau_{M^i} \times \tau_{M^i}) \longrightarrow \mathbb{I}(I) - \{\mathbf{0}\}$$

is continuous.

Proof. Fit $t > 0$. Let $x, y \in X$ and let $(x'_n, y'_n)_n$ be a sequence in $X \times X$ that converges to (x, y) with respect to $\tau_{M^i} \times \tau_{M^i}$. Then, it will be sufficient to show that $M(x, y, t) = \lim_n M(x_n, y_n, t)$ for some subsequence $(x_n, y_n)_n$ of $(x'_n, y'_n)_n$.

Indeed, since $(M(x'_n, y'_n, t))_n$ is sequence in $\mathbb{I}(I) - \{\mathbf{0}\}$, there is a subsequence $(x_n, y_n)_n$ of $(x'_n, y'_n)_n$ such that the sequence $(M(x_n, y_n, t))_n$ converges to some ε of $\mathbb{I}(I)$. Fix $\delta > 0$ such that $2\delta < t$. Then

$$M(x_n, y_n, t) \geq M(x_n, x, \delta) * M(x, y, t - 2\delta) * M(y, y_n, \delta),$$

and

$$M(x, y, t + 2\delta) \geq M(x, x_n, \delta) * M(x_n, y_n, t) * M(y_n, y, \delta).$$

Since $\lim_n M^i(x, x_n, \delta) = \lim_n M^i(y, y_n, \delta) = \mathbf{1}$, we deduce that

$$M(x, y, t + 2\delta) \geq \lim_n M(x_n, y_n, t) \geq M(x, y, t - 2\delta).$$

Finally, it follows from the continuity of $M(x, y, -)$ that $M(x, y, t) = \lim_n M(x_n, y_n, t)$. This completes the proof.

Definition 3.13 Let (X, d) be a quasi-metric space. Define the interval-valued t-norm

$$\mathfrak{a} * \mathfrak{b} = \langle a^- \wedge b^-, a^+ \wedge b^+ \rangle$$

and interval-valued fuzzy quasi-metric

$$M_d(x, y, t) = \langle M^-(x, y, t), M^+(x, y, t) \rangle = \langle \frac{t}{t + ld(x, y)}, \frac{t}{t + md(x, y)} \rangle$$

$$(\forall x, y \in X, \forall t, l, m \in \mathbb{R}^+).$$

Then $(X, M_d, *)$ is an interval-valued fuzzy quasi-metric space called the standard interval-valued fuzzy quasi-metric space and $(M_d, *)$ is the interval-valued fuzzy quasi-metric induced by d.

Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^i = M_{d^s}$, where $d^{-1}(x,y) = d(y,x), d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$.

In the paper [10], it is proved that every metric can induce an interval-valued fuzzy metric. Moreover, if (X,d) is a metric space ,then the topology generated by d coincides with the topology τ_{M_d} generated by the interval-valued fuzzy metric. Finally, from proposition 3.8, 3.10, 3.11 and definition 3.13, it follows that the topology τ_d, generated by d, coincides with the topology τ_{M_d} generated by the induced standard interval-valued fuzzy quasi-metric $(M_d, *)$.

Definition 3.14 We say that a topological space (X,τ) admits a compatible interval-valued fuzzy quasi-metric if there is an interval-valued fuzzy quasi-metric $(M,*)$ on X such that $\tau = \tau_M$.

It follows from definition 3.11 that every quasi-metrizable topological space admits a compatible interval-valued fuzzy quasi-metric. Then, conversely, the topology generated by an interval-valued fuzzy quasi-metric space is quasi-metrizable.

Lemma 3.15 Let $(X, M_d, *)$ be an interval-valued fuzzy quasi-metric space. Then $\{U_n : n = 2,3,\cdots\}$ is a base for a quasi-uniformity $\mathcal{U}_{\mathcal{M}}$ on X compatible with τ_M, where $U_n = \{(x,y) \in X \times X : M(x,y,1/n) > \langle 1 - 1/n, 1\rangle\}$, for $n = 2,3,\cdots$.

Moreover the conjugate quasi-uniformity $(u_M)^{-1}$ coincides with $u_{M^{-1}}$ and it is compatible with $\tau_{M^{-1}}$. Form definition 3.11, Lemma 3.13 and the well-known result that the topology generated by a quasi-uniformity with a countable base is quasi-pseudo-metrizable, we immediately deduce the following.

Theorem 3.16 For a topological space (X,τ) the following are equivalent .

(i) (X,τ) is quasi-metrizable.

(ii) (X,τ) admits a compatible interval-valued fuzzy quasi-metric.

4 Bicomplete interval-valued fuzzy quasi-metric space

Definition 4.1 An interval-valued fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if $(X, M^i, *)$ is a complete interval-valued fuzzy metric space. In this case, we say that $(M,*)$ is a bicomplete interval-valued fuzzy quasi-metric on X.

Proposition 4.2

(1) Let $(X, M, *)$ be a bicomplete interval-valued fuzzy quasi-metric space.Then (X, τ_M) admits a compatible bicomplete quasi-metric space.

(2)) Let (X, d) be a bicomplete quasi-metric space. Then $(X, M_d, *)$ is a bicomplete interval-valued fuzzy quasi-metric space.

Proof. (1) Let d be a quasi-metric on X inducing the quasi-uniformity $\mathcal{U}_{\mathcal{M}}$. Then d is compatible with τ_M. Now let $(x_n)_n$ be a Cauchy sequence in (X, d^s). Clearly $(x_n)_n$ is a Cauchy sequence in the interval-valued fuzzy metric space $(X, M^i, *)$. So it converges to a point $y \in X$ with respect to τ_{M^i}. Hence $(x_n)_n$ converges to y with respect to τ_{d^s}. Consequently d is bicomplete.

(2) This part is almost obvious because $(M_d)^i = M_{d^s}$ (see definition 3.11) and thus each Cauchy sequence in $(X, (M_d)^i, *)$ is clearly a Cauchy sequence in (X, d^s).

Definition 4.3 Let $(X, M, *)$ and $(Y, N, \star_I)$ be two interval-valued fuzzy quasi-metric space.Then

(i) A mapping f from X to Y is called an isometry if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$.

(ii) $(X, M, *)$ and $(Y, N, \star_I)$ are called isometric if there is an isometry from X onto Y.

Definition 4.4 Let $(X, M, *)$ be an interval-valued fuzzy quasi-metric space. An interval-valued fuzzy quasi-metric bicompletion of $(X, M, *)$ is a bicomplete interval-valued fuzzy quasi-metric space $(Y, N, \star_I)$ such that $(X, M, *)$ is isometric to a $\tau_{N^i}-$dense subspace of Y.

Lemma 4.5 Let $(X, M, *)$ be an interval-valued fuzzy quasi-metric space and $(Y, N, \star_I)$ a bicomplete interval-valued fuzzy quasi-metric space.If there is a $\tau_{M^i}-$dense subset A of X and an isometry $f : (A, M, *) \longrightarrow (Y, N, \star_I)$, then there exists a unique isometry $F : (X, M, *) \longrightarrow (Y, N, \star_I)$ such that $F|_A = f$.

Proof. It is clear that f is a quasi-uniformly continuous mapping from the quasi-uniform space $(A, \mathcal{U}_{\mathcal{M}}|\mathcal{A} \times \mathcal{A})$ to the quasi-uniform space $(Y, \mathcal{U}_{\mathcal{N}})$. By Theorem 3.29 of [16], f has a unique quasi-uniformly continuous extension $F : (X, \mathcal{U}_{\mathcal{M}}) \longrightarrow (\mathcal{Y}, \mathcal{U}_{\mathcal{N}})$. We shall show that actually F is an isometry from $(X, M, *)$ to $(Y, N, \star_I)$. Indeed, let $x, y \in X$ and $t > 0$. Then, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in A such that $(x_n)_n \longrightarrow x$ and $(y_n)_n \longrightarrow y$ with respect to τ_{M^i}. Thus $F(x_n) \longrightarrow F(x)$ and $F(y_n) \longrightarrow F(y)$ with respect to τ_{N^i}.Choose $\varepsilon \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}$ with $0 < \varepsilon < t$. Therefore, there is n_ε such that for $n > n_\varepsilon$,

$$M(x, x_n, \varepsilon/2) > \mathbf{1} - \varepsilon, M(y_n, y, \varepsilon/2) > \mathbf{1} - \varepsilon,$$

$$N(F(x_n), F(x), \varepsilon/2) > \mathbf{1} - \varepsilon, N(F(y), F(y_n), \varepsilon/2) > \mathbf{1} - \varepsilon.$$

Thus

$$\begin{aligned} M(x,y,t) &\geq M(x,x_n,\varepsilon/2) * M(x_n,y_n,t-\varepsilon) * M(y_n,y,\varepsilon/2) \\ &\geq (\mathbf{1}-\varepsilon) * N(F(x_n),F(y_n),t-\varepsilon) * (\mathbf{1}-\varepsilon) \\ &\geq (\mathbf{1}-\varepsilon) * \langle (\mathbf{1}-\varepsilon) \star_I N(F(x),F(y),t-2\varepsilon) \star_I (\mathbf{1}-\varepsilon)\rangle * (\mathbf{1}-\varepsilon) \end{aligned}$$

By continuity of $*$and $\star_I$ and by continuity of $N(F(x),F(y),-)$, it follows that $M(x,y,t) \geq N(F(x),F(y),t)$. Similarly we can show that $N(F(x),F(y),t) \geq M(x,y,t)$. Consequently F is an isometry from $(X,M,*)$ to $(Y,N,\star_I)$.

Theorem 4.6 Suppose that $(Y_1,N_1,\star_{I_1})$ and $(Y_2,N_2,\star_{I_2})$ are two interval-valued fuzzy quasi-metric bicompletions of $(X,M,*)$. Then $(Y_1,N_1,\star_{I_1})$ and $(Y_2,N_2,\star_{I_2})$ are isometric. Thus, if an interval-valued fuzzy quasi-metric space has an interval-valued fuzzy quasi-metric bicompletion, it is unique in the mean of isometry.

Proof. Since $(Y_2,N_2,\star_{I_2})$ is an interval-valued fuzzy quasi-metric bicompletion of $(X,M,*)$, there is an isometry f from $(X,M,*)$ onto a dense subspace of $(Y_2,N_2,\star_{I_2})$. By Lemma 4.5, f admits a (unique) extension F to $(Y_1,N_1,\star_{I_1})$ which is also an isometry. So, it remains to see that F is onto. But this fact follows from standard arguments. Indeed, given $y_2 \in Y_2$, there is a sequence $(x_n)_n$ in X such that $F(x_n) \longrightarrow y_2$. Since F is an isometry, $(x_n)_n$ is a Cauchy sequence, so it converges to some point $y_1 \in Y_1$. Consequently $F(y_1) = y_2$. The proof is complete.

5 Fixed point theorems in interval-valued fuzzy metric spaces

We will discuss the interval-valued fuzzy metric space fixed point problem in this section. Firstly, the definition of interval-valued fuzzy contraction mapping in interval-valued fuzzy metric space, then a generalization of Banach and Edelstein fixed point theorems is given.

In an interval-valued fuzzy metric space $(X,M,*)$, whenever $M(x,y,t) > \mathbf{1}-\mathbb{r}$ for all $x,y \in X, t > 0, \mathbb{r} \in \mathbb{I}(I) - \{\mathbf{0},\mathbf{1}\}$, there exists a t_0 with $0 < t_0 < t$ such that $M(x,y,t_0) > \mathbf{1}-\mathbb{r}$.

Definition 5.1 Let $(X,M,*)$ be an interval-valued fuzzy metric space. We will say the mapping $f : X \to X$ is t-uniformly continuous if for each $\varepsilon \in \mathbb{I}(I)$, there exists a $\mathbb{r} \in (I)$ such that $M(x,y,t) > \mathbf{1}-\mathbb{r}$ implies $M(f(x),f(y),t) \geq \mathbf{1}-\varepsilon$, for each $x,y \in X$ and $t > 0$.

Clearly if f is t-uniformly continuous it is uniformly continuous for the uniformity generated by M, and then continuous for the topology deduced from M.

Definition 5.2 Let $(X, M, *)$ be an interval-valued fuzzy metric space. We will say the mapping $f : X \to X$ is fuzzy contractive if there exists a $k \in (0, 1)$ such that for each $x, y \in X$, and $t > 0$,

$$M(x, y, t) * (\mathbf{1} - M(f(x), f(y), t) \leq k(\mathbf{1} - M(x, y, t)) * M(f(x), f(y), t),$$

where k is called the contractive constant of f.

The above definition is justified by the next Proposition 5.4.

Proposition 5.3 Let $(X, M, *)$ be an interval-valued fuzzy metric space. If $f : X \to X$ is fuzzy contractive then f is t-uniformly continuous.

Proposition 5.4 Let (X, d) be a metric space. The mapping $f : X \to X$ is contractive (a contraction) on the metric space (X, d) with contractive constant k if and only if f is fuzzy contractive, with contractive constant k, on the standard interval-valued fuzzy metric space $(X, M_d, *)$, induced by d.

Recall that a sequence (x_n) in a metric space (X, d) is said to be contractive if there exists a $k \in (0, 1)$ such that $d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Now, we give the following definition (compare with Definition 5.2).

Definition 5.5 Let $(X, M, *)$ be an interval-valued fuzzy metric space. We will say that the sequence (x_n) in X is fuzzy contractive if there exists $k \in (0, 1)$ such that for all $t > 0, n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, t) * (\mathbf{1} - M(x_{n+1}, x_{n+2}, t)) \leq k(\mathbf{1} - M(x_n, x_{n+1}, t)) * M(x_{n+1}, x_{n+2}, t).$$

Proposition 5.6 Let $(X, M_d, *)$ be the standard interval-valued fuzzy metric space induced by the metric d on X. The sequence (x_n) in X is contractive in (X, d) if and only if (x_n) is fuzzy contractive in $(X, M_d, *)$.

Research of the fixed point theorem in fuzzy metric space has attracted the attention of many scholars [20,23-29], below we will discuss the fuzzy metric space fixed point theorems in interval-valued fuzzy metric space.

Definition 5.7 A sequence $\{x_n\}$ in an interval-valued fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for each $\varepsilon > \mathbf{0}$ and each $t > 0$ there exists a $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > \mathbf{1} - \varepsilon$ for all $n, m \geq n_0$.

Definition 5.8 An interval-valued fuzzy metric space $(X, M, *)$ in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Theorem 5.9 A sequence $\{x_n\}$ in an interval-valued fuzzy metric space $(X, M, *)$ converges to x if and only if $M(x, x_n, t) \to \mathbf{1}$ as $n \to \infty$.

Result 5.10 The metric space (X, d) is complete if and only if the standard interval-valued fuzzy metric space $(X, M_d, *)$ is complete.

Proof may refer to A. George and P. Veeramani literature [5].

Next, we extend the Banach fixed point theorem to fuzzy contractive mappings of interval-valued fuzzy metric spaces.

Theorem 5.11 (fuzzy Banach contraction theorem) Let $(X, M, *)$ be an interval-valued fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \to X$ be a fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.

Proof. Fix $x \in X$. Let $x_n = T^n(x), n \in \mathbb{N}$. We have for $t > 0$,

$$M(x, x_1, t) * (\mathbf{1} - M(T(x), T^2(x), t)) \leq k(\mathbf{1} - M(x, x_1, t)) * M(T(x), T^2(x), t),$$

and by induction, for any $n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, t) * (\mathbf{1} - M(x_{n+1}, x_{n+2}, t)) \leq k(\mathbf{1} - M(x_n, x_{n+1}, t)) * M(x_{n+1}, x_{n+2}, t),$$

Then (x_n) is a fuzzy contractive sequence, so it is a Cauchy sequence and, hence, (x_n) converges to y, for some $y \in X$. We will see y is a fixed point for T. By Theorem 5.9, we have

$$M(y, x_n, t) * (\mathbf{1} - M(T(y), T(x_n), t)) \leq k(\mathbf{1} - M(y, x_n, t)) * M(T(y), T(x_n), t)) \longrightarrow \mathbf{0}$$

as $n \longrightarrow \infty$, then $\lim_n M(T(y), T(x_n), t)) = \mathbf{1}$ for each $t > 0$, and, therefore, $\lim_n T(x_n) = T(y)$, i.e., $\lim_n x_{n+1} = T(y)$ and then $T(y) = y$.

To show uniqueness, assume $T(z) = z$ for some $z \in X$. Then for $t > 0$ we have

$$\begin{array}{rl} & M(y, z, t) * (\mathbf{1} - M(y, z, t)) \\ = & M(y, z, t) * (\mathbf{1} - M(T(y), T(z), t)) \\ \leq & k[M(T(y), T(z), t)) * (\mathbf{1} - M(y, z, t))] \\ = & k[M(y, z, t) * (\mathbf{1} - M(T(y), T(z), t))] \\ \leq & k^2[M(T(y), T(z), t)) * (\mathbf{1} - M(y, z, t))] \\ \leq & \cdots \\ \leq & k^n[M(T(y), T(z), t)) * (\mathbf{1} - M(y, z, t))] \end{array}$$

then $M(y, z, t) * (\mathbf{1} - M(y, z, t)) \to \mathbf{0}$ as $n \to \infty$.

Hence, $M(y, z, t) = \mathbf{1}$ and then$y = z$.

Now suppose $(X, M_d, *)$ is a complete standard interval-valued fuzzy metric space induced by the metric d on X . From Result 5.10 (X, d) is complete, then if (x_n) is a fuzzy contractive sequence, by Proposition 5.6 it is contractive in (X, d), hence convergent.

So, from Theorem 5.11 we have the following corollary, which can be considered as the fuzzy version of the classic Banach contraction theorem on complete interval-valued metric spaces.

Corollary 5.12 Let $(X, M_d, *)$ be a complete standard interval-valued fuzzy metric space and let $T : X \to X$ a fuzzy contractive mapping. Then T has a unique fixed point.

Definition 5.13[10] An interval-valued fuzzy metric space $(X, M, *)$ is compact space if X is a compact set.

Lemma 5.14 Let $(X, M_d, *)$ be an interval-valued fuzzy metric space. If $\lim_{n\in\mathbb{N}} x_n = x$ and $\lim_{n\in\mathbb{N}} y_n = y$, then for all $t > 0$ and $0 < \varepsilon < \frac{t}{2}$,

$$M(x, y, t-\varepsilon) \leq \lim_{n\in\mathbb{N}} M(x_n, y_n, t) \leq M(x, y, t+\varepsilon).$$

Proof. By Definition 3.1(IV3),

$$M(x_n, y_n, t) \geq M(x_n, x, \frac{\varepsilon}{2}) * M(x, y, t-\varepsilon) * M(y, y_n, \frac{\varepsilon}{2}).$$

Thus,

$$\lim_{n\in\mathbb{N}} M(x_n, y_n, t) \geq \mathbf{1} * M(x, y, t-\varepsilon) * \mathbf{1} = M(x, y, t-\varepsilon).$$

On the other hand,

$$M(x, y, t+\varepsilon) \geq M(x, y, t-\varepsilon) * M(x_n, y_n, t) * M(y_n, y, \frac{\varepsilon}{2}),$$

hence

$$M(x, y, t+\varepsilon) \geq \lim_{n\in\mathbb{N}} M(x_n, y_n, t).$$

Corollary 5.15 Let $(X, M, *)$ be an interval-valued fuzzy metric space. If $\lim_{n\in\mathbb{N}} x_n = x$ and $\lim_{n\in\mathbb{N}} y_n = y$, then $\lim_{n\in\mathbb{N}} M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.

Theorem 5.16 (fuzzy Edelstein contraction theorem). Let $(X, M_d, *)$ be a compact interval-valued fuzzy metric space. Let $T : X \to X$ be a mapping satisfying for all $x \neq y$ and $t > 0$,

$$M(Tx, Ty, t) > M(x, y, t),$$

then T has a unique fixed point.

Proof. Let $x \in X$ and $x_n = T^n x (n \in \mathbb{N})$. Assume $x_n \neq x_{n+1}$ for each n (if not, $T(x_n) = x_n$). Now, assume $x_n \neq x_m (n \neq m)$. Otherwise we get

$$M(x_n, x_{n+1}, t) = M(x_m, x_{m+1}, t) > M(x_{m-1}, x_{m+1}, t) > \cdots > M(x_n, x_{n+1}, t),$$

where $m > n$, a contradiction. Since X is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$. Let $y = \lim_{i\in\mathbb{N}} x_{n_i}$. We also assume that $y, Ty \notin \{x_n : n \in \mathbb{N}\}$ (if not, choose a subsequence with such a property). According to the above assumptions we may now write

$$M(Tx_{n_i}, Ty, t) > M(x_{n_i}, y, t)$$

for all $i \in \mathbb{N}$ and $t > 0$. Since $M(x, y, \cdot)$ is continuous for all $x, y \in X$,by Corollary 5.15 we obtain

$$\lim_{i\in\mathbb{N}} M(Tx_{n_i}, Ty, t) \geq \lim_{i\in\mathbb{N}} M(x_{n_i}, y, t) = \mathbf{1}$$

for each $t > 0$, hence

$$\lim_{i\in\mathbb{N}} Tx_{n_i} = Ty, \cdots\cdots\cdots\cdots (1).$$

Similarly, we obtain

$$\lim_{i\in\mathbb{N}} T^2x_{n_i} = T^2y, \cdots\cdots\cdots\cdots\cdots (2),$$

since $Ty \neq Tx_{n_i}$ for all i. Now, observe that

$M(x_{n_1}, Tx_{n_1}, t) < M(Tx_{n_1}, Tx_{n_1}^2, t) < \cdots < M(x_{n_i}, Tx_{n_i}, t) < M(Tx_{n_i}, T^2x_{n_i}, t) < \cdots < M(x_{n_{i+1}}, Tx_{n_{i+1}}, t) < M(Tx_{n_{i+1}}, T^2x_{n_{i+1}}, t) < \cdots < \mathbf{1}$ for all $t > 0$.

Thus $\{M(x_{n_i}, Tx_{n_i}, t)\}$ and $\{M(Tx_{n_i}, T^2x_{n_i}, t)\}$ $(t > 0)$ are convergent to a common limit [$cf.$[29]). So, by (1),(2) and Corollary 5.15 we get

$$\begin{aligned} M(y, Ty, t) &= M(\lim_i x_{n_i}, T(\lim_i x_{n_i}), t) \\ &= \lim_i M(x_{n_i}, Tx_{n_i}, t) \\ &= \lim_i M(Tx_{n_i}, T^2x_{n_i}, t) \\ &= M(\lim_i Tx_{n_i}, \lim_i T^2x_{n_i}, t) \\ &= M(Ty, T^2y, t) \end{aligned}$$

for all $t > 0$. Suppose $y \neq Ty$. Then, by $M(Tx, Ty, t) > M(x, y, t)$, $M(y, Ty, t) > M(Ty, T^2y, t)$,$(t > 0)$, a contradiction. Hence $y = Ty$, a fixed point. Uniqueness follows at once from $M(Tx, Ty, t) > M(x, y, t)$.

6 Concluding remarks

In this paper, the concept of interval-valued fuzzy metric space is defined, and it is proved that the topology induced by quasi-metric is consistent with which induced via a standard interval-valued fuzzy quasi-metric, every quasi-metrizable topological space admits a compatible interval-valued fuzzy quasi-metric. On the contrary, topology generated by interval-valued fuzzy quasi-metric is quasi-metrizable. Furthermore,

some properties of interval-valued fuzzy quasi-metric space which is bicompletion are discussed, and it is proved that if an interval-valued fuzzy quasi-metric space has bicompletion, then it is unique in the mean of isometry. Finally, a fuzzy contraction mapping of interval-valued fuzzy metric space is defined, and the Banach and Edelstein fixed point theorem to interval fuzzy metric space are promoted.

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BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER BASED ON QUASI-SUBORDINATE CONDITIONS INVOLVING WRIGHT HYPERGEOMETRIC FUNCTIONS

N.E.CHO[1,*] G. MURUGUSUNDARAMOORTHY[2] AND K.VIJAYA[3]

[1]Department of Applied Mathematics
Pukyong National University
Busan 608-737, Korea
E-mail: necho@pknu.ac.kr.

[2,3]School of Advanced Sciences, VIT University,
Vellore 632014, Tamilnadu, India
E-mail: gmsmoorthy@yahoo.com; kvijaya@vit.ac.in

[*]Corresponding Author

Abstract. In the present paper, we introduce and investigate a new subclass of bi-univalent functions of complex order defined in the open unit disk, which are associated with Wright hypergeometric functions and satisfying quasi-subordinate conditions. Furthermore, we find estimates on the second and the third coefficients of the Taylor-Maclaurin series for functions in the new subclass. Several special consequences of the results are also pointed out.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$.

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.2}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Now we briefly recall the definitions of the special functions and operators used in this paper.

For complex parameters $\alpha_1, \ldots, \alpha_l$ ($\frac{\alpha_j}{A_j} \neq 0, -1, \ldots; j = 1, 2, \ldots l$) and $\beta_1, \ldots, \beta_m$ ($\frac{\beta_j}{B_j} \neq 0, -1, \ldots; j = 1, 2, \ldots m$), Fox's H-function (for details, see [23]) we mean the Wright's generalized hypergeometric functions ${}_l\Psi_m$ with $A_j, B_j > 0$, give (rather general and typical examples of $H-$functions, not reducible to $G-$functions):

$$\begin{aligned} {}_l\Psi_m\left(\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_m, A_l) \\ (\beta_1, B_1), \ldots, (\beta_m, B_m) \end{array} : z \right) &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \ldots \Gamma(\alpha_l + nA_l)}{\Gamma(\beta_1 + nB_1) \ldots \Gamma(\beta_m + nB_m)} \frac{z^n}{n!} \\ &= H_{l,m+1}^{1,l}\left[-z \middle| \begin{array}{c} (1-\alpha_1, A_1), \ldots, (1-\alpha_l, A_l) \\ (0,1), (1-\beta_1, B_1), \ldots, (1-\beta_m, B_m) \end{array} \right], \end{aligned} \tag{1.3}$$

where $1 + \sum_{n=1}^{m} B_n - \sum_{n=1}^{l} A_n \geq 0$, $(l, m \in \mathbb{N} = \{1, 2, ...\})$ and for suitably bounded values of $|z|$.

We note that when $A_1 = \cdots = A_l = B_1 = \cdots = B_m = 1$, they turn into the generalized hypergeometric functions

$$\begin{aligned} {}_l\Psi_m\left(\begin{array}{c} (\alpha_1, 1), \ldots, (\alpha_l, 1) \\ (\beta_1, 1), \ldots, (\beta_m, 1) \end{array} : z \right) &= \left[\frac{\prod_{j=1}^{l} \Gamma(\alpha_j)}{\prod_{i=1}^{m} \Gamma(\beta_j)} \right] {}_lF_m(\alpha_1, \ldots, \alpha_l : \beta_1, \ldots, \beta_m : z) \\ (l \leq m+1;\ l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}). \end{aligned} \tag{1.4}$$

Now we state the linear operator due to Srivastava [23] (see [6]) and Wright [24] in terms of the Hadamard product (or convolution) involving the generalized hypergeometric function. Let $l, m \in \mathbb{N}$ and suppose that the parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ are also positive real numbers. Then, corresponding to a function

$${}_l\Phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z]$$

defined by

$${}_l\Phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z] = \Omega z_l\Psi_m[(\alpha_j, A_j)_{1,l}(\beta_j, B_j)_{1,m}; z] \tag{1.5}$$

where $\Omega = \left(\prod_{j=1}^{l} \Gamma(\alpha_j)\right)^{-1} \left(\prod_{j=1}^{m} \Gamma(\beta_j)\right)$, we consider a linear operator

$$\mathcal{W}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}] : \mathcal{A} \to \mathcal{A}$$

defined by the following Hadamard product (or convolution)

$$\mathcal{W}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}]f(z) \;:=\; z\ {}_l\Phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z] * f(z).$$

We observe that , for $f(z)$ of the form(1.1),we have

$$\mathcal{W}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}]f(z) = z + \sum_{n=2}^{\infty} \varphi_n\ a_n z^n \tag{1.6}$$

where

$$\varphi_n = \frac{\Omega\ \Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_l + A_l(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_m + B_m(n-1))} \tag{1.7}$$

If, for convenience, we write

$$\mathcal{W}_m^l f(z) = \mathcal{W}[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z). \tag{1.8}$$

We state the following remark due to Srivastava [23] (see [6]) and Wright [24].

Remark 1.1. Other interesting and useful special cases of the Fox-Wright generalized hypergeometric function ${}_l\Psi_m$ defined by (1.3) include (for example) the generalized Bessel function

$${}_0\Psi_1(-; (\nu+1, \mu); -z) \equiv \mathcal{J}_\mu^\nu = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\ \Gamma(n\mu + \nu + 1)}.$$

For $\mu = 1$, corresponds essentially to the classical Bessel function $\mathcal{J}^\nu(z)$, and the generalized Mittag-Leffler function

$${}_1\Psi_1((1,1); (\mu, \lambda); z) \equiv \mathbf{E}_\mu^\lambda = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(n\lambda + \mu)}.$$

Remark 1.2. By setting $A_j = 1 (j = 1, ..., l)$ and $B_j = 1 (j = 1, ..., m)$ in (1.5), we are led immediately to the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\Omega_l F_m(z) \equiv \Omega_l F_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n}\ \frac{z^n}{n!} \tag{1.9}$$

$$(l \leq m+1;\ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\alpha)_n$ is the Pochhammer symbol.

In view of the relationship (1.9), the linear operator(1.6) includes the Dziok-Srivastava operator (see [5]), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi [3], Carlson and Shaffer [4], Libera [11], Livingston [12], Ruscheweyh [19] and Srivastava and Owa [22].

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.10}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

For some intriguing examples of functions and characterization of the class Σ, one could refer Srivastava et al., [21] and the references stated therein (see also, [9]). Recently there has been triggering interest to study the bi-univalent function class Σ (see [8, 9, 13, 15, 17, 21]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$ is presumably still an open problem.

In 1970 Robertson[18] introduced the concept of quasi-subordination. An analytic function $f(z)$ is quasi-subordinate to an analytic function $\phi(z)$, in the open unit disk if there exist analytic functions $h(z)$ and w, with $w(0) = 0$ such that $|h(z)| \leq 1,\ |w(z)| < 1$ and $f(z) = h(z)\phi[w(z)]$. Then we write $f(z) \prec_{\widetilde{q}} \phi(z)$. If $h(z) = 1$, then the quasi-subordination reduces to the subordination. Also,if $w(z) = z$ then $f(z) = h(z)\phi(z)$ and in this case we say that $f(z)$ is majorized by $\phi(z)$ and it is written as $f(z) << \phi(z)$ in $\mathbb{U}$. Hence it is obvious that quasi-subordination is the generalization of subordination as well as majorization. It is unfortunate that the concept quasi-subordination is so far an underlying concept in the area of complex function theory although it deserves much attention as it unifies the concept of both subordination and majorization.

Through out this paper it is assumed that ϕ is analytic in $\mathbb{U}$ with $\phi(0) = 1$ and let

$$\phi(z) = 1 + C_1z + C_2z^2 + C_3z^3 + \cdots \quad (C_1 > 0). \tag{1.11}$$

also let

$$\psi(z) = D_0 + D_1z + D_2z^2 + D_3z^3 + \cdots \quad (|\psi(z) \leq 1|;\ z \in \mathbb{U}). \tag{1.12}$$

Motivated by the earlier work of Deniz [7] (see [1, 17, 20]) in the present paper, we introduce new subclass of the function class Σ of complex order $\gamma \in \mathbb{C}\backslash\{0\}$, involving Wright hypergeometric functions $\mathcal{W}_m^l$, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered , and connection to earlier known results are made.

Definition 1.3. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{G}_\Sigma^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1) \tag{1.13}$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda z(\mathcal{W}_m^l g(w))'} - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \tag{1.14}$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq \lambda < 1,\ z \in \mathbb{U}$ and the function g is given by (1.10).

Example 1. For $\lambda = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{l,m}(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{\mathcal{W}_m^l f(z)} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1) \tag{1.15}$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{\mathcal{W}_m^l g(w)} - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \tag{1.16}$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.10).

On specializing the parameters l, m one can state the various new subclasses of Σ (or $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$), as illustrations, we present some examples for the case with $A_j = 1$ $(j = 1, 2, ..., l)$; $B_j = 1$ $(j = 1, 2, ..., m)$.

Example 2. If $l \leq m + 1,\ l, m \in N_0 := N \cup \{0\}$, and $\gamma \in \mathbb{C}\backslash\{0\}$, then a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{H}_m^l f(z))'}{(1-\lambda)(\mathcal{H}_m^l f(z)) + \lambda(\mathcal{H}_m^l f(z))'} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{H}_m^l g(w))'}{(1-\lambda)(\mathcal{H}_m^l g(w)) + \lambda(\mathcal{H}_m^l g(w))'} - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \quad (0 \leq \lambda < 1),$$

where $\mathcal{H}_m^l f(z) := \left(z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \frac{(z)^n}{n!}\right) * f(z) \equiv \mathcal{W}[(\alpha_t, 1)_{1,l}; (\beta_t, 1)_{1,m}] f(z)$ is a well-known Dziok-Srivastava operator [5], the function g is given by (1.10) and $z, w \in \mathbb{U}$.

Example 3. If $l = 2$ and $m = 1$ with $\alpha_1 = a\,(a > 0)$, $\alpha_2 = b, (b > 0)$ $\beta_1 = c\,(c > 0)$, and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{a,c,b}(\gamma, \lambda, \phi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{I}_c^{a,b} f(z))'}{(1-\lambda)(\mathcal{I}_c^{a,b} f(z)) + \lambda(\mathcal{I}_c^{a,b} f(z))'} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{I}_c^{a,b} g(w))'}{(1-\lambda)(\mathcal{I}_c^{a,b} g(w)) + \lambda(\mathcal{I}_c^{a,b} g(w))'} - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \quad (0 \leq \lambda < 1; z, w \in \mathbb{U}),$$

where $\mathcal{I}_c^{a,b} f(z) := \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} z^n\right) * f(z) \equiv \mathcal{H}_1^2(a, b; c) f(z)$, is a well-known Hohlov operator [10] and the function g is given by (1.10).

Example 4. If $l = 2$ and $m = 1$ with $\alpha_1 = a\,(a > 0)$, $\alpha_2 = 1$, $\beta_1 = c\,(c > 0)$, and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{a,c}(\gamma, \lambda, \phi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{L}(a, c) f(z))'}{(1-\lambda)(\mathcal{L}(a, c) f(z)) + \lambda(\mathcal{L}(a, c) f(z))'} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{L}(a,c)g(w))'}{(1-\lambda)(\mathcal{L}(a,c)g(w))+\lambda(\mathcal{L}(a,c)g(w))'}-1\right)\prec_{\widetilde{q}}(\phi(w)-1),\quad(0\leq\lambda<1;z,w\in\mathbb{U}),$$

where $\mathcal{L}(a,c)f(z):=\left(z+\sum_{n=2}^{\infty}\frac{(a)_n}{(c)_n}z^n\right)*f(z)\equiv\mathcal{H}_1^2(a,1;c)f(z)$, is a well-known Carlson-Shaffer operator [4] and the function g is given by (1.10).

Example 5. If $l=2$ and $m=1$ with $\alpha_1=\delta+1\,(\delta\geq-1)$, $\alpha_2=1$, $\beta_1=1$, and $\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{\delta}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{D}^{\delta}f(z))'}{(1-\lambda)(\mathcal{D}^{\delta}f(z))+\lambda(\mathcal{D}^{\delta}f(z))'}-1\right)\prec_{\widetilde{q}}(\phi(z)-1)$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{D}^{\delta}g(w))'}{(1-\lambda)(\mathcal{D}^{\delta}g(w))+\lambda(\mathcal{D}^{\delta}g(w))'}-1\right)\prec_{\widetilde{q}}(\phi(w)-1),\quad(0\leq\lambda<1;z,w\in\mathbb{U}),$$

where $\mathcal{D}^{\delta}$ is called Ruscheweyh derivative[19] of order δ $(\delta\geq-1)$ and $\mathcal{D}^{\delta}f(z):=\frac{z}{(1-z)^{\delta+1}}*f(z)\equiv\mathcal{H}_1^2(\delta+1,1;1)f(z)$ and the function g is given by (1.10).

Example 6. If $l=2$ and $m=1$ with $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, and $\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{*}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda f'(z)}-1\right)\prec_{\widetilde{q}}(\phi(z)-1)$$

and

$$\frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda g'(w)}-1\right)\prec_{\widetilde{q}}(\phi(w)-1),$$

where $0\leq\lambda<1$, $z,w\in\mathbb{U}$ and the function g is given by (1.10).

Example 7. If $l=2$ and $m=1$ with $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, and $\lambda=0;\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{*}(\gamma,\phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec_{\widetilde{q}}(\phi(z)-1)$$

and

$$\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}-1\right)\prec_{\widetilde{q}}(\phi(w)-1),$$

where $0\leq\lambda<1$, $z,w\in\mathbb{U}$ and the function g is given by (1.10).

Remark 1.4. For $\lambda=0$ and $\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1.1), as in Example 1, one can state various analogous subclasses defined in Examples 2 to 4.

Definition 1.5. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1) \tag{1.17}$$

and

$$\frac{1}{\gamma}\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \tag{1.18}$$

where $\gamma \in \mathbb{C} \setminus \{0\}, \lambda \geq 0, z, w \in \mathbb{U}$ and the function g is given by(1.10).

On specializing the parameters λ one can define the various new subclasses of Σ associated with Wright hypergeometric functions $\mathcal{W}_m^l$, as illustrated in the following examples.

Example 8. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1.1), $\mathcal{B}_{\Sigma}^{l,m}(\gamma, 0, \phi) \equiv \mathcal{S}_{\Sigma}^{l,m}(\gamma, \phi)$.

Example 9. For $\lambda = 1$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{l,m}(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\mathcal{W}_m^l f(z))' - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1) \tag{1.19}$$

and

$$\frac{1}{\gamma}\left((\mathcal{W}_m^l g(w))' - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \tag{1.20}$$

where $\gamma \in \mathbb{C} \setminus \{0\}; z, w \in \mathbb{U}$ and the function g is given by(1.10).

It is of interest to note that for $\gamma = 1$ the class $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ reduces to the following new subclass $\mathcal{B}_{\Sigma}^{l,m}(\lambda, \phi)$.

Example 10. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{l,m}(\lambda, \phi)$ if the following conditions are satisfied:

$$\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\right) \prec_{\widetilde{q}} (\phi(z) - 1) \tag{1.21}$$

and

$$\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\right) \prec_{\widetilde{q}} (\phi(w) - 1), \tag{1.22}$$

where $\lambda \geq 0,\ z, w \in \mathbb{U}$ and the function g is given by (1.10).

Remark 1.6. On specializing the parameters l, m one can state the various new subclasses of $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$, as illustrated in Examples 1 to Examples 7, with $A_j = 1$ $(j = 1, 2, ..., l)$; $B_j = 1$ $(j = 1, 2, ..., m)$.

In the following section, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ of the function class Σ.

2. Coefficient Bounds for the Function Class $\mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$

In order to derive our main results, we shall need the following lemma.

Lemma 2.1. (see [16]) *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{U}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$.

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{G}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$. Then*

$$|a_2| \leq \frac{|\gamma|\ |D_0| C_1 \sqrt{C_1}}{\sqrt{|[\gamma\ D_0\ (\lambda^2-1)C_1^2 + (1-\lambda)^2(C_1-C_2)]\varphi_2^2 + 2\gamma(1-\lambda)D_0 C_1^2 \varphi_3|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2\ C_1^2}{(1-\lambda)^2\varphi_2^2} + \frac{|\gamma D_0|\ C_1}{2(1-\lambda)\varphi_3} + \frac{|\gamma D_1| C_1}{2(1-\lambda)\varphi_3}. \tag{2.2}$$

Proof. It follows from (1.17) and (1.18) that

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\right) = \psi(z)[\phi(u(z)) - 1] \tag{2.3}$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda z(\mathcal{W}_m^l g(w))'} - 1\right) = \psi(w)[\phi(v(w)) - 1], \tag{2.4}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$\psi(z)[\phi(u(z))-1]=\frac{1}{2}D_0C_1p_1z+\left[\frac{1}{2}D_1\ C_1\ p_1+\frac{1}{2}D_0C_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{D_0C_2}{4}p_1^2\right]z^2+\cdots \tag{2.5}$$

and

$$\psi(w)[\phi(v(w))-1]=\frac{1}{2}D_0C_1q_1w+\left[\frac{1}{2}D_1\ C_1\ q_1+\frac{1}{2}D_0C_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{D_0C_2}{4}q_1^2\right]w^2+\cdots \tag{2.6}$$

respectively. Now, equating the coefficients in (2.3) and (2.4), we get

$$\frac{(1-\lambda)}{\gamma}\varphi_2a_2=\frac{1}{2}D_0C_1p_1, \tag{2.7}$$

$$\frac{(\lambda^2-1)}{\gamma}\varphi_2^2a_2^2+\frac{2(1-\lambda)}{\gamma}\varphi_3a_3=\frac{1}{2}D_1\ C_1p_1+\frac{1}{2}D_0C_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{D_0C_2}{4}p_1^2, \tag{2.8}$$

$$-\frac{(1-\lambda)}{\gamma}\varphi_2a_2=\frac{1}{2}D_0C_1q_1 \tag{2.9}$$

and

$$\frac{(\lambda^2-1)}{\gamma}\varphi_2^2a_2^2+\frac{2(1-\lambda)}{\gamma}\varphi_3(2a_2^2-a_3)=\frac{1}{2}D_1\ C_1q_1+\frac{1}{2}D_0C_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{D_0C_2}{4}q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we find that

$$a_2=\frac{\gamma D_0C_1p_1}{2(1-\lambda)\varphi_2}=\frac{-\gamma D_0C_1q_1}{2(1-\lambda)\varphi_2}, \tag{2.11}$$

which implies

$$p_1=-q_1. \tag{2.12}$$

and

$$8(1-\lambda)^2\varphi_2^2a_2^2=\gamma^2D_0^2C_1^2(p_1^2+q_1^2). \tag{2.13}$$

Adding (2.8) and (2.10), by using(2.12)and(2.13) ,we obtain

$$4\left([\gamma D_0(\lambda^2-1)C_1^2+(1-\lambda)^2(C_1-C_2)]\varphi_2^2+2\gamma D_0(1-\lambda)C_1^2\varphi_3\right)a_2^2=\gamma^2D_0^2C_1^3(p_2+q_2). \tag{2.14}$$

Thus,

$$a_2^2=\frac{\gamma^2D_0^2C_1^3(p_2+q_2)}{4\left([\gamma D_0(\lambda^2-1)C_1^2+(1-\lambda)^2(C_1-C_2)]\varphi_2^2+2\gamma D_0(1-\lambda)C_1^2\varphi_3\right)}. \tag{2.15}$$

Applying Lemma 2.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2\leq\frac{|\gamma|^2|D_0|^2C_1^3}{|[\gamma D_0(\lambda^2-1)C_1^2+(1-\lambda)^2(C_1-C_2)]\varphi_2^2+2\gamma D_0(1-\lambda)C_1^2\varphi_3|}. \tag{2.16}$$

Since $C_1>0$,the last inequality gives the desired estimate on $|a_2|$ given in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get

$$\frac{4(1-\lambda)}{\gamma}\varphi_3 a_3 - \frac{4(1-\lambda)}{\gamma}\varphi_3 a_2^2 = \frac{D_1\ C_1(p_1-q_1)}{2} + \frac{D_0 C_1(p_2-q_2)}{2} + \frac{D_0(C_2-C_1)(p_1^2-q_1^2)}{4}. \quad (2.17)$$

It follows from (2.11), (2.12) and (2.17) that

$$a_3 = \frac{\gamma^2 D_0^2 C_1^2(p_1^2+q_1^2)}{8(1-\lambda)^2\varphi_2^2} + \frac{\gamma D_1\ C_1(p_1-q_1)}{8(1-\lambda)\varphi_3} + \frac{\gamma C_1 D_0(p_2-q_2)}{8(1-\lambda)\varphi_3}.$$

Applying Lemma 2.1 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leq \frac{|\gamma D_0|^2\ C_1^2}{(1-\lambda)^2\varphi_2^2} + \frac{|\gamma D_0|\ C_1}{2(1-\lambda)\varphi_3} + \frac{|\gamma D_1| C_1}{2(1-\lambda)\varphi_3}.$$

This completes the proof of Theorem 2.2. □

Putting $\lambda = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathfrak{S}_\Sigma^{l,m}(\gamma,\phi)$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1\sqrt{C_1}}{\sqrt{|[(C_1-C_2)-\gamma D_0 C_1^2]\varphi_2^2 + 2\gamma D_0 C_1^2\varphi_3|}} \quad (2.18)$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{\varphi_2^2} + \frac{|\gamma D_0| C_1}{2\varphi_3} + + \frac{|\gamma D_1| C_1}{2\varphi_3}. \quad (2.19)$$

Now we state the following corollaries for the function classes $\mathfrak{S}_\Sigma^*(\gamma,\lambda,\phi)$ and $\mathfrak{S}_\Sigma^*(\gamma,\lambda)$ defined in Example 5 and Example 6, respectively.

Corollary 2.4. *Let the function $f(z)$ given by (1.1) be in the class $\mathfrak{S}_\Sigma^*(\gamma,\lambda,\phi)$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1\sqrt{C_1}}{(1-\lambda)\sqrt{|(C_1-C_2)+\gamma D_0 C_1^2|}} \quad (2.20)$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-\lambda)^2} + \frac{|\gamma D_0| C_1}{2(1-\lambda)} + \frac{|\gamma D_1| C_1}{2(1-\lambda)}. \quad (2.21)$$

Corollary 2.5. *Let the function $f(z)$ given by (1.1) be in the class $\mathfrak{S}_\Sigma^*(\gamma,\phi)$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1\sqrt{C_1}}{\sqrt{|(C_1-C_2)+\gamma D_0 C_1^2|}} \quad (2.22)$$

and

$$|a_3| \leq |\gamma D_0|^2 C_1^2 + \frac{|\gamma D_0| C_1}{2} + \frac{|\gamma D_1| C_1}{2}. \quad (2.23)$$

3. Coefficient Bounds for the Function Class $\mathcal{B}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{\mu,b}(\gamma,\lambda,\phi)$. Then*

$$|a_2| \leq \frac{|\gamma|\ |D_0|C_1\sqrt{2C_1}}{\sqrt{|\gamma D_0 C_1^2[(\lambda-1)(\lambda+2)\varphi_2^2+2(\lambda+2)\varphi_3]-2(C_2-C_1)(1+\lambda)^2\varphi_2^2|}} \tag{3.1}$$

and

$$|a_3| \leq \left(\frac{|\gamma D_0|C_1}{(1+\lambda)\varphi_2}\right)^2 + \frac{|\gamma D_0|C_1}{(\lambda+2)\varphi_3} + \frac{|\gamma D_1|C_1}{2(1+\lambda)\varphi_3}. \tag{3.2}$$

Proof. Let $f \in \mathcal{B}_{\Sigma}^{l,m}(\gamma,\lambda,\phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \triangle \longrightarrow \triangle$ with $u(0) = 0 = v(0)$, satisfying

$$\frac{1}{\gamma}\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\right) = \psi(z)[\phi(u(z)) - 1] \tag{3.3}$$

and

$$\frac{1}{\gamma}\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\right) = \psi(w)[\phi(u(w)) - 1]. \tag{3.4}$$

In light of (1.1) - (1.11), from (2.5) and (2.6), it is evident that

$$\begin{aligned}&\frac{(\lambda+1)}{\gamma}\varphi_2 a_2 z + \frac{1}{\gamma}\left[(\lambda+2)\varphi_3 a_3 + \frac{(\lambda-1)(\lambda+2)}{2}\varphi_2^2 a_2^2\right] z^2 + \cdots \\ &\quad = \frac{1}{2}D_0 C_1 p_1 z + \left[\frac{1}{2}D_1\ C_1 p_1 + \frac{1}{2}D_0 C_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{D_0 C_2}{4}p_1^2\right] z^2 + \cdots\end{aligned}$$

and

$$\begin{aligned}&-\frac{(\lambda+1)}{\gamma}\varphi_2 a_2 w + \frac{1}{\gamma}\left[-(\lambda+2)\varphi_3 a_3 + \left(\frac{(\lambda-1)(\lambda+2)}{2}\varphi_2^2 + 2(\lambda+2)\varphi_3\right) a_2^2\right] w^2 + \cdots \\ &\quad = \frac{1}{2}D_0 C_1 q_1 w + \left[\frac{1}{2}D_1\ C_1 q_1 + \frac{1}{2}D_0 C_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{D_0 C_2}{4}q_1^2\right] w^2 + \cdots\end{aligned}$$

Now proceeding on lines similar to Theorem 2.2 we get the desired results. □

Choosing $\lambda = 0$ and $\lambda = 1$ we state the initial Taylor coefficients for the function classes $\mathcal{S}_{\Sigma}^{l,m}(\gamma,\phi)$ and $\mathcal{H}_{\Sigma}^{l,m}(\gamma,\phi)$.

4. Concluding remarks

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1), \tag{4.1}$$

which gives

$$C_1 = 2\alpha \quad \text{and} \quad C_2 = 2\alpha^2.$$

On the other hand, for $-1 \leq B \leq A < 1$ if we take

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + \cdots, \tag{4.2}$$

then we have

$$C_1 = (A-B),\ C_2 = -B(A-B).$$

By taking, $A = (1-2\beta)$ where $0 \leq \beta < 1$ and $B = -1$ in (4.2), we get

$$\begin{aligned} \phi(z) &= \frac{1+(1-2\beta)z}{1-z} \\ &= 1 + 2(1-\beta)z + 2(1-\beta)z^2 + 2(1-\beta)z^3 + \cdots. \end{aligned} \tag{4.3}$$

Hence, we have

$$C_1 = C_2 = 2(1-\beta).$$

Further, by taking $\beta = 0$, in (4.3), we get

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \cdots \quad , \tag{4.4}$$

Hence,

$$C_1 = C_2 = 2.$$

Various Choices of ϕ as mentioned above and suitably choosing the values of C_1 and C_2, we state some interesting results analogous to Theorem 2.2 , Theorem 3.1 and the Corollaries 2.3 to 2.5 for various new subclasses of Σ.

Remark 4.1. Setting $A_j = 1$ $(j = 1, ..., l)$ and $B_j = 1$ $(j = 1, ..., m)$ and if $l = 2$ and $m = 1$ with $\alpha_1 = \mu + 1 (\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu + 2$, where $\mathcal{J}_\mu$ is a Bernardi operator [3] defined by

$$\mathcal{J}_\mu f(z) := \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t)dt \equiv H_1^2(\mu+1, 1; \mu+2)f(z).$$

Note that the operator J_1 was studied earlier by Libera [11] and Livingston [12] and various other interesting corollaries and consequences of our main results (which are asserted by Theorem 2.2 and Theorem 3.1 above) can be derived similarly. Further, by setting $A_j = 1$ $(j = 1, ..., l)$ and $B_j = 1$ $(j = 1, ..., m)$ and suitably choosing l, m, λ, we can state the various results for the new classes defined in Examples 1 to 10, but the details involved may be left as an exercise for the interested reader.

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On uni-soft mighty filters of BE-algebras

Jeong Soon Han[1] and Sun Shin Ahn[2,*]

[1]**Department of Applied Mathematics, Hanyang University, Ahnsan, 15588, Korea**

[2]**Department of Mathematics Education, Dongguk University, Seoul 04620, Korea**

Abstract. The notion of uni-soft mighty filters of a BE-algebra is introduced, and related properties are investigated. The problem of classifying uni-soft mighty by their γ-exclusive filter is solved. We construct a new quotient structure of a transitive BE-algebra using a unit soft filter and consider some properties of it.

1. Introduction

Kim and Kim [7] introduced the notion of a BE-algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [12]. In response to this situation Zadeh [13] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [14]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [10]. Maji et al. [9] and Molodtsov [10] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [10] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [9] described the application of soft set

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* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409

[0]**E-mail**: han@hanyang.ac.kr (J. S. Han); sunshine@dongguk.edu (S. S. Ahn)

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theory to a decision making problem. Maji et al. [8] also studied several operations on the theory of soft sets.

H. R. Lee and S. S. Ahn [6] introduced the notion of a mighty filter in BE-algebras, and investigated some properties of it. Jun et al. [5] introduced the notion of a uni-soft filter of a BE-algebra, and studied their properties.

In this paper, we introduce the notion of a uni-soft mighty filter of a BE-algebra, and investigate their properties. We solve the problem of classifying uni-soft mighty by their γ-exclusive filter. Also we construct a new quotient structure of a transitive BE-algebra using a unit soft filter and study some properties of it.

2. Preliminaries

We recall some definitions and results discussed in [7].

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1 \in F$;
(F2) $x * y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in X$.

F is a *mighty filter* [6] of X if it satisfies (F1) and

(F3) $z * (y * x) \in F$ and $z \in F$ imply $((x * y) * y) * x \in F$ for all $x, y, z \in X$.

Proposition 2.2. *Let $(X; *, 1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.3. *Let $(X; *, 1)$ be a self distributive BE-algebra. Then following hold: for any $x, y, z \in X$,*

(i) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.*
(ii) $y * z \leq (z * x) * (y * z)$.
(iii) $y * z \leq (x * y) * (x * z)$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.3(iii).

Theorem 2.4.([6]) *A filter F of a BE-algebra X is mighty if and only if*

(2.1) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

A soft set theory is introduced by Molodtsov [10]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and $A, B, C, \cdots \subseteq X$.

Definition 2.5. A *soft set* (f, A) of X over U is defined to be the set of ordered pairs

$$(f, A) := \{(x, f(x)) : x \in X,\ f(x) \in \mathscr{P}(U)\},$$

where $f : X \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For a soft set (f, A) of X and a subset γ of U, the γ-*exclusive set* of (f, A), denoted by $e_A(f; \gamma)$, is defined to be the set

$$e_A(f; \gamma) := \{x \in A \mid f(x) \subseteq \gamma\}.$$

For any soft sets (f, X) and (g, X) of X, we call (f, X) a *soft subset* of (g, X), denoted by $(f, X) \tilde{\subseteq} (g, X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of (f, X) and (g, X), denoted by $(f, X) \tilde{\cup} (g, X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of X over U in which $f \tilde{\cup} g$ is defined by

$$(f \tilde{\cup} g)(x) = f(x) \cup g(x) \text{ for all } x \in X.$$

The *soft intersection* of (f, X) and (g, X), denoted by $(f, X) \tilde{\cap} (g, X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of X over U in which $f \tilde{\cap} g$ is defined by

$$(f \tilde{\cap} g)(x) = f(x) \cap g(x) \text{ for all } x \in M.$$

3. Uni-soft mighty filters

In what follows, we take a BE-algebra X, as a set of parameters unless specified.

Definition 3.1.([5]) A soft set (f, X) of X over U is called a *union-soft filter* (briefly, *uni-soft filter*) of X if it satisfies:

(US1) $(\forall x \in X)(f(1) \subseteq f(x))$,

(US2) $(\forall x, y \in X)(f(y) \subseteq f(x * y) \cup f(x))$.

Proposition 3.2.([5]) Every uni-soft filter (f, X) of X over U satisfies the following properties:

(i) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \subseteq f(x))$.

(ii) $(\forall x, y, z \in X)(z \leq x * y \Rightarrow f(y) \subseteq f(x) \cup f(z))$.

Definition 3.3. A soft set (f, X) of X over U is called a *union-soft mighty filter* (briefly, *uni-soft mighty filter*) of X if it satisfies (US1) and

(US3) $(\forall x, y, z \in X)(f(((x * y) * y) * x) \subseteq f(z * (y * x)) \cup f(z)))$.

Example 3.4. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra [7] with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over $U := \mathbb{Z}$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{N} & \text{if } x \in \{1, a, b\} \\ \mathbb{Z} & \text{if } x \in \{c, d, 0\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft mighty filter of X.

Proposition 3.5. *Every uni-soft mighty filter of a BE-algebra X is a uni-soft filter of X.*

Proof. Let (f, X) be a uni-soft mighty filter of X. Setting $y := 1$ in (US3), we have $f(x) = (((x * 1) * 1) * x) \subseteq f(z * (1 * x)) \cup f(z) = f(z * x) \cup (z)$. Hence (US2) holds. Therefore (f, X) is a uni-soft filter of X. □

The converse of Proposition 3.5 is not true in general as seen in the following example.

Example 3.6. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d\}$ is a BE-algebra [7] with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let (f, X) be a soft set of X over $U := \mathbb{Z}$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 3\mathbb{N} & \text{if } x \in \{1\} \\ 4\mathbb{N} & \text{if } x \in \{a\} \\ 3\mathbb{Z} & \text{if } x \in \{b, c, d\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft filter of X. But it is not a uni-soft mighty filter of X, since $f(((a * c) * c) * a) = f(a) = 4\mathbb{N} \not\subseteq f(1 * (c * a)) \cup f(1) = f(1) = 3\mathbb{N}$.

We provide conditions for a uni-soft filter to be a uni-soft mighty filter.

Theorem 3.7. *Any uni-soft filter of a BE-algebra X is mighty if and only if it satisfies*

(3.1) $(\forall x, y \in X)(f(((x * y) * y) * x)) \subseteq f(y * x)$.

Proof. Assume that a uni-soft filter (f, X) of X is mighty. Putting $z := 1$ in (US3), we have $f(((x * y) * y) * x) \subseteq f(1 * (y * x)) \cup f(1) = f(y * x)$. Hence (3.1) holds.

Conversely, suppose that the uni-soft filter (f, X) of X satisfies the condition (3.1). Using (3.1) and (US2), we have $f(((x * y) * y) * x) \subseteq f(y * x) \subseteq f(z * (y * x)) \cup f(z)$ for any $x, y \in X$. Hence (f, X) is an uni-soft mighty filter of X. □

Proposition 3.8. *Let (f, X) be a uni-soft mighty filter of a BE-algebra X. Then $X_f := \{x \in X | f(x) = f(1)\}$ is a mighty filter of X.*

Proof. Clearly, $1 \in X_f$. Let $z * (y * x), z \in X_f$. Then $f(z * (y * x)) = f(1)$ and $f(z) = f(1)$. It follows from (US3) that $f(((x * y) * y) * x) \subseteq f(z * (y * x)) \cup f(z) = f(1)$. By (US1), we get $f(((x * y) * y) * x) = f(1)$. Hence $((x * y) * y) * x \in X_f$. Therefore X_f is a mighty filter of X. □

Theorem 3.9. *Let (f, X) and (g, X) be uni-soft filters of a transitive BE-algebra such that $(f, X)\tilde{\subseteq}(g, X)$ and $f(1) = g(1)$. If (g, X) is mighty, then so is (f, X).*

Proof. Let $x, y \in X$. Note that $y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since (g, X) is a uni-soft mighty filter of a BE-algebra X, by (3.1) and $(f, X)\tilde{\subseteq}(g, X)$ we have $g(1) = g(y * ((y * x) * x)) \supseteq g(((((y * x) * x) * y) * y) * ((y * x) * x)) \supseteq f(((((y * x) * x) * y) * y) * ((y * x) * x))$. Since $f(1) = g(1)$, we get $f((y * x) * ((((y * x) * x) * y) * y) * x)) = f(((((y * x) * x) * y) * y) * ((y * x) * x)) = f(1)$. It follows from (US1) and (US2) that

$$\begin{aligned} f(y * x) =& f(1) \cup f(y * x) \\ =& f((y * x) * (((((y * x) * x) * y) * y) * x)) \cup f(y * x) \\ \supseteq& f(((((y * x) * x) * y) * y) * x). \end{aligned} \tag{3.2}$$

Since X is transitive, we get

$$\begin{aligned} [((((y * x) * x) * y) * y) * x] * [((x * y) * y) * x] & \\ \geq ((x * y) * y) * ((((y * x) * x) * y) * y) & \\ \geq (((y * x) * x) * y) * (x * y) & \\ \geq x * ((y * x) * x) & \\ = (y * x) * (x * x) & \\ = (y * x) * 1 = 1. & \end{aligned}$$

It follows from Proposition 3.2 that $f(((((y*x)*x)*y)*y)*x) \cup f(1) = f(((((y*x)*x)*y)*y)*x) \supseteq f(((x*y)*y)*x)$. Using (3.2), we have $f(y*x) \supseteq f(((((y*x)*x)*y)*y)*x) \supseteq f(((x*y)*y)*x)$. Therefore $f(y * x) \supseteq f(((x * y) * y) * x)$. By Theorem 3.7, (f, X) is a uni-soft mighty filter of X. □

Let (f, X) be a uni-soft filter of a transitive BE-algebra X. Define a binary relation " $\sim^f$ " on X by putting $x \sim^f y$ if and only if $f(x*y) = f(y*x) = f(1)$ for any $x, y \in X$.

Lemma 3.10. *The relation " $\sim^f$ " is an equivalence relation on a transitive BE-algebra X.*

Proof. For any $x \in X$, $x*x = 1$ by (BE1). So $f(x*x) = f(1)$, hence $x \sim^f x$, which $\sim^f$ is reflexive. Suppose that $x \sim^f y$ for any $x, y \in X$. Then $f(x*y) = f(y*x) = f(1)$. Hence $\sim^f$ is symmetric. Assume that $x \sim^f y$ and $y \sim^f z$ for any $x, y, z \in X$. Then $f(x*y) = f(y*x) = f(1)$ and $f(y*z) = f(z*y) = f(1)$. By transitivity, $(x*y)*[(y*z)*(x*z)] = 1$ and $(z*y)*[(y*x)*(z*x)] = 1$. By Proposition 3.2, we have $f(x*z) \subseteq f(y*z) \cup f(x*y) = f(1)$ and $f(z*x) \subseteq f(y*x) \cup f(z*y) = f(1)$. Hence $f(z*x) = f(z*x) = f(1)$, i.e., $x \sim^f z$. Thus $\sim^f$ is an equivalence relation on X. □

Lemma 3.11. *The relation " $\sim^f$ " is a congruence relation on a transitive BE-algebra X.*

Proof. If $x \sim^f y$ and $u \sim^f v$ for any $x, y, u, v \in X$, then $f(x*y) = f(y*x) = f(1)$ and $f(u*v) = f(v*u) = f(1)$. By transitivity, $(u*v)*[(x*u)*(x*v)] = 1$ and $(v*u)*[(x*v)*(x*u)] = 1$, it follows from Proposition 3.2 that $f(1) = f(u*v) \supseteq f((x*u)*((x*v))$ and $f(1) = f(v*u) \supseteq f((x*v)*(x*u))$. Hence $f((x*u)*(x*v)) = f(1)$ and $f((x*v)*(x*u)) = f(1)$. Therefore $x*u \sim^f x*v$. By a similar way, we can prove that $x*v \sim^f y*v$. Hence $x*u \sim^f y*v$. Therefore $\sim^f$ is a congruence relation on X. □

X is decomposed by the congruence relation $\sim^f$. The class containing x is denoted by f_x. Denote $X/f := \{f_x | x \in X\}$. We define a binary relation $\bullet$ on X/f by $f_x \bullet f_y := f_{x*y}$. This definition is well defined since $\sim^f$ is a congruence relation on X

Lemma 3.12. $f_1 = X_f$.

Proof. $f_1 = \{x \in X | 1 \sim^f x\} = \{x \in X | f(1*x) = f(x*1) = f(1)\} = \{x \in X | f(x) = f(1)\} = X_f$. □

Theorem 3.13. *Let (X, f) be a uni-soft filter of a transitive BE-algebra X. Then $(X/f; \bullet, f_1)$ is a transitive BE-algebra.*

Proof. It is straightforward. □

Theorem 3.14. *Let X be a transitive BE-algebra. If every filter of the quotient algebra X/f is mighty, then a uni-soft filter of X is mighty.*

Proof. Suppose that every filter of the quotient algebra X/f is mighty and let $x, y \in X$ be such that $y*x \in f_1$. Then $f(y*x) = f(1)$ and so $f_y \bullet f_x \in f_1$. Since $\{f_1\}$ is a mighty filter of X/f, it follows from Theorem 2.4 that $f_{((x*y)*y)*x} = ((f_x \bullet f_y) \bullet f_y) \bullet f_x \in f_1$. Hence $f((((x*y)*y)*)*x) = f(1)$. Therefore $f(y*x) = f(((x*y)*y))*x)$. Thus (f, X) is a uni-soft mighty filter by Theorem 3.7. □

Theorem 3.15. *A uni-soft set (X, f) of a BE-algebra X is a uni-soft mighty filter of X if and only if the set $e_X(f;\gamma) := \{x \in X | f(x) \subseteq \gamma\}$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ whenever it is nonempty.*

Proof. Suppose that (f, X) is a uni-soft mighty filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $z * (y * x) \in e_X(f;\gamma)$ and $z \in e_X(f;\gamma)$. Then $f(z * (y * x)) \subseteq \gamma$ and $f(z) \subseteq \gamma$. It follows from (US1) and (US3) that $f(1) \subseteq f_X(((x * y) * y) * x) \subseteq f(z * (y * x)) \cup f(z) \subseteq \gamma$. Hence $1 \in e_X(f;\gamma)$ and $((x * y) * y) * x \in e_X(f;\gamma)$, and therefore $e_X(f;\gamma)$ is a mighty filter of X.

Conversely, assume that $e_X(f;\gamma)$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $e_X(f;\gamma) \neq \emptyset$. For any $x \in X$, let $f(x) = \gamma$. Then $x \in e_X(f;\gamma)$. Since $e_X(f;\gamma)$ is a mighty filter of X, we have $1 \in e_X(f;\gamma)$ and so $f(x) = \gamma \supseteq f(1)$. For any $x, y, z \in X$, let $f(z * (y * x)) = \gamma_{z*(y*x)}$ and $f(z) = \gamma_z$. Let $\gamma := \gamma_{z*(y*x)} \cup \gamma_z$. Then $z * (y * x) \in e_X(f;\gamma)$ and $z \in e_X(f;\gamma)$ which imply that $((x * y) * y) * x \in e_X(f;\gamma)$. Hence $f(((x * y) * y) * x) \subseteq \gamma = \gamma_{z*(y*x)} \cup \gamma_z = f(z * (y * x)) \cup f(z)$. Thus (f, X) is a uni-soft mighty filter of X. □

We call $e_X(f;\gamma)$ a *γ-exclusive filter* of a BE-algebra X.

Theorem 3.16. *Every filter of a BE-algebra can be represented as a γ-exclusive set of a uni-soft mighty filter, i.e., given a filter F a BE-algebra X, there exists a uni-soft mighty filter (f, X) of X over U such that F is the γ-exclusive set of (f, X) for a non-empty subset γ of U.*

Proof. Let F be a filter of a BE-algebra X. For a subset γ of U, define a soft set (f, X) over U by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ U & \text{if } x \notin F. \end{cases}$$

Obviously, $F = e_X(f;\gamma)$. We now prove that (f, X) is a uni-soft mighty filter of X. Since $1 \in F = e_X(f;\gamma)$, we have $f(1) = \gamma \subseteq f(x)$ for all $x \in X$. Let $x, y, z \in X$. If $z * (y * x), z \in F$, then $((x*y)*y)*x \in F$ because F is a mighty filter of X. Hence $f(z*(y*x)) = f(z) = f(((x*y)*y)*x) = \gamma$, and so $f(z*(y*x)) \cup f(z) \supseteq f(((x*y)*y)*x)$. If $z*(y*x) \in F$ and $z \notin F$, then $f(z*(y*x)) = \gamma$ and $f(z) = U$ which imply that $f(z*(y*x))) \cup f(z) = \gamma \cup U = U \supseteq f(((x*y)*y)*x)$. Similarly, if $z * (y * x) \notin F$ and $z \in F$, then $f(z * (y * x))) \cup f(z) \supseteq f(((x * y) * y) * x)$. Obviously, if $z * (y * x) \notin F$ and $z \notin F$, then $f(z * (y * x)) \cup f(z) \supseteq f(((x * y) * y) * x)$. Therefore (f, X) is a uni-soft mighty filter of X. □

Any filter of a BE-algebra X cannot be represented as a γ-exclusive set of a uni-soft mighty filter (f, X) of X over U (see Example 3.17).

Example 3.17. Let $X = \{1, a, b, c, d, 0\}$ be the BE-algebra as in Example 3.4. Given $U = X$, consider a soft set (f, X) over U which is defined by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{c\} & \text{if } x \in \{1, a, b\}, \\ \{1, c\} & \text{if } x \in \{c, d, 0\}. \end{cases}$$

Then (f, X) is a uni-soft mighty filter of X. The γ-exclusive sets of (f, X) are described as follows:

$$e_X(f;\gamma) = \begin{cases} X & \text{if } \gamma \supseteq \{1, c\}, \\ \{1, a, b\} & \text{if } \{c\} \subseteq \gamma \subsetneq \{1, c\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The filter $\{1\}$ cannot be a γ-exclusive set $e_X(f;\gamma)$, since there is no $\gamma \subseteq U$ such that $e_X(f;\gamma) = \{1\}$.

Proposition 3.18. *Let (f, X) be a uni-soft filter of a transitive BE-algebra X. The mapping $\gamma : X \to X/f$, given by $\gamma(x) := f_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = f_1\} = X_f$.*

Proof. Let $f_x \in X/f$. Then there exists an element $x \in X$ such that $\gamma(x) = f_x$. Hence γ is surjective. For any $x, y \in X$, we have

$$\gamma(x * y) = f_{x*y} = f_x \bullet f_y = \gamma(x) \bullet \gamma(y).$$

Thus γ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = f_1\} = \{x \in X | x \sim^f 1\} = \{x \in X | f(x) = f(1)\} = X_f$. □

Example 3.19. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra [6] with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in \{1, b, c\} \\ \gamma_2 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then (f, X) is a uni-soft mighty filter over U. By Proposition 3.5, it is a uni-soft filter over U. Then $X_f = \{x \in X | f(x) = f(1)\} = \{1, b, c\}$. Define $x \sim^f y$ if and only if $f(x * y) = f(y * x) = f(1)$. Then $f_1 = \{x \in X | x \sim^f 1\} = \{x \in X | f(x * 1) = f(1 * x) = f(1)\} = \{1, b, c\}$, $f_a = \{x \in X | x \sim^f a\} = \{x \in X | f(x * a) = f(a * x) = f(1)\} = \{a\}$, $f_d = \{x \in X | x \sim^f d\} = \{x \in X | f(x * d) = f(d * x) = f(1)\} = \{d\}$, and $f_0 = \{x \in X | x \sim^f 0\} = \{x \in X | f(x * 0) = f(0 * x) = f(1)\} = \{0\}$. Hence $X/f = \{f_1, f_a, f_d, f_0\}$. Let $\varphi : X \to X/f$ be a map defined by $\varphi(1) = \varphi(b) = \varphi(c) = f_1$, $\varphi(a) = f_a$, $\varphi(d) = f_d$, $\varphi(0) = f_0$. It is easy to check

that φ is a homomorphism and $Ker\varphi = \{x \in X|\varphi(x) = f_1\} = \{x \in X|x \sim^f 1\} = \{x \in X|f(x) = f(1)\} = X_f$.

Proposition 3.20. *Let $\mu : X \to Y$ be an epimorphism of BE-algebras. If (f, Y) is a uni-soft filter of Y, then $(f \circ \mu, X)$ is a uni-soft filter of X.*

Proof. For any $x \in X$, we have $(f \circ \mu)(x) = f(\mu(x)) \supseteq f(1_Y) = f(\mu(1_X)) = (f \circ \mu)(1_X)$ and $(f \circ \mu)(y) = f(\mu(y)) \subseteq f(a *_Y \mu(y)) \cup f(a)$ for any $a \in Y$. Let x be any preimage of a under μ. Then

$$\begin{aligned}(f \circ \mu)(y) \subseteq & f(a *_Y \mu(y)) \cup f(a)\\ = & f(\mu(x) *_Y \mu(y)) \cup f(\mu(x))\\ = & f(\mu(x *_X y)) \cup f(\mu(x))\\ = & (f \circ \mu)(x *_X y) \cup (f \circ \mu)(x).\end{aligned}$$

Therefore $(f \circ \mu, X)$ is a uni-soft filter of X. □

Proposition 3.21. *Let (f, X) be a uni-soft filter of a transitive BE-algebra X. If J is a filter of X, then J/f is a filter of X/f.*

Proof. Let (f, X) be a uni-soft filter of X and let J be a filter of X. For any $x * y, x \in J$, we obtain $y \in J$. Hence for any $f_x \bullet f_y = f_{x*y}, f_x \in J/f$, we have $f_y \in J/f$. Thus J/f is a filter of X/f. □

Theorem 3.22. *Let (f, X) be a uni-soft filter of a transitive BE-algebra X. If J^* is a filter of a transitive BE-algebra X/f, then there exists a filter $J = \{x \in X|f_x \in J^*\}$ in X such that $J/f = J^*$.*

Proof. Since J^* is a filter of X/f, so $f_x \bullet f_y = f_{x*y}, f_x \in J^*$ imply $f_y \in J^*$ for any $f_x, f_y \in J^*$. Therefore $x * y, x \in J$ imply $y \in J$ for any $x, y \in J$. Therefore J is a filter of X. By Proposition 3.18, we have

$$\begin{aligned}J/f = & \{f_j|j \in J\}\\ = & \{f_j|\exists f_x \in J^* \text{ such that } j \sim^f x\}\\ = & \{f_j|\exists f_x \in J^* \text{ such that } f_x = f_j\}\\ = & \{f_j|f_j \in J^*\} = J^*.\end{aligned}$$

Theorem 3.23. *Let (f, X) be a uni-soft filter of a transitive BE-algebra X. If J is a filter of X, then $\dfrac{X/f}{J/f} \cong X/J$.*

Proof. Note that $\dfrac{X/f}{J/f} = \{[f_x]_{J/f}|f_x \in X/f\}$. If we define $\varphi : \dfrac{X/f}{J/f} \to X/J$ by $\varphi([f_x]_{J/f}) = [x]_J = \{y \in X|x \sim^J y\}$, then it is well defined. In fact, suppose that $[f_x]_{J/f} = [f_y]_{J/f}$. Then

$f_x \sim^{J/f} f_y$ and so $f_{x*y} = f_x \bullet f_y \in J/f$. Hence $x * y \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[f_x]_{J/f}, [f_y]_{J/f} \in \dfrac{X/f}{J/f}$, we have

$$\begin{aligned}\varphi([f_x]_{J/f} \bullet [f_y]_{J/f}) =& \varphi([f_x \bullet f_y]_{J/f}) \\ =& [x * y]_J = [x]_J * [y]_J \\ =& \varphi([f_x]_{J/f}) * \varphi([f_y]_{J/f}).\end{aligned}$$

Hence φ is a homomorphism.

Obviously, φ is onto. Finally, we show that φ is one-to-one. If $\varphi([f_x]_{J/f}) = \varphi([f_y]_{J/f})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $f_a \in [f_x]_{J/f}$, then $f_a \sim^{J/f} f_x$ and hence $f_{a*x} \in J/f$. It follows that $a * x \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y \in J$ and so $f_{a*y} \in J/f$. Therefore $f_a \sim^{J/f} f_y$. Hence $f_a \in [f_y]_{J/f}$. Thus $[f_x]_{J/f} \subseteq [f_y]_{J/f}$. Similarly, we obtain $[f_y]_{J/f} \subseteq [f_x]_{J/f}$. Therefore $[f_x]_{J/f} = [f_y]_{J/f}$. It is completes the proof. □

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INCLUSION RELATIONSHIPS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH GENERALIZED BESSEL FUNCTIONS

K.A. SELVAKUMARAN, H. A. AL-KHARSANI, D. BALEANU, S.D. PUROHIT, AND K.S. NISAR

Abstract. This paper introduces new subclasses of analytic functions and investigate the inclusion properties of these subclasses using the generalized Bessel functions of the first kind. We also derive a variety of special cases and corollaries of the main results.

1. Introduction

Let $\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ be an open disk and let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \tag{1.1}$$

which are analytic in $\mathcal{U}$ and satisfy the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Let $\mathcal{S}$ subclasses of $\mathcal{A}$ containing all functions which are univalent, $\mathcal{C}$ is the close-to-convex, $\mathcal{S}^*(\alpha)$ starlike of order α and $\mathcal{K}(\alpha)$ is the convex of order α in $\mathcal{U}$. For functions $f_j \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{n=1}^{\infty} a_{n+1,j}\ z^{n+1}, \qquad (j = 1, 2)$$

we state the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) := z + \sum_{n=1}^{\infty} a_{n+1,1}\ a_{n+1,2}\ z^{n+1}, \quad (z \in \mathcal{U}).$$

$f(z)$ is named subordinate to $g(z)$ if $\exists$ $w(z)$ analytic in $\mathcal{U}$ in such a way that

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U}) \quad \text{and} \quad f(z) = g(w(z)) \qquad (z \in \mathcal{U}).$$

and we write $f(z) \prec g(z)$. Let $\mathcal{M}$ be the class of analytic functions $\varphi(z)$ in $\mathcal{U}$ with $\varphi(0) = 1$. We consider that $\mathcal{S}$ denote the subclasses of $\mathcal{A}$ containing all functions which are univalent, $\mathcal{C}$ is the close-to-convex, $\mathcal{S}^*(\alpha)$ starlike of order α and $\mathcal{K}(\alpha)$ denote the convex of order α in $\mathcal{U}$. Using the subordination between analytic functions, now we

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define the subclasses of $\mathcal{A}$ for $0 \leq \alpha,\ \beta < 1$ and $\varphi,\ \psi \in \mathcal{N}$ (cf., [4, 5, 9, 11]):

$$\mathcal{S}^*(\alpha;\ \varphi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \prec \varphi(z),\ z \in \mathcal{U} \right\},$$

$$\mathcal{K}(\alpha;\ \varphi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\alpha}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) \prec \varphi(z),\ z \in \mathcal{U} \right\}$$

and

$$\mathcal{C}(\alpha, \beta;\ \varphi, \psi) := \left\{ f \in \mathcal{A} : \text{for some}\, g \in \mathcal{S}^*(\alpha;\ \varphi)\ \text{s.t.}\ \frac{1}{1-\beta}\left(\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z),\ z \in \mathcal{U} \right\}.$$

clearly

$$f(z) \in \mathcal{K}(\alpha;\ \varphi) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha;\ \varphi).$$

The particular choices of φ and ψ yields the familier subclasses of $\mathcal{A}$ as:

$$\mathcal{S}^*\left(\alpha;\ \frac{1+z}{1-z}\right) = \mathcal{S}^*(\alpha), \quad \mathcal{K}\left(\alpha;\ \frac{1+z}{1-z}\right) = \mathcal{K}(\alpha) \quad \text{and} \quad \mathcal{C}\left(0, 0;\ \frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = \mathcal{C}.$$

Recently, Deniz et al. [7] gave the transformation $\phi_{l,b,c}(z)$ of generalized Bessel function of first kind of order l (cf. [1]):

$$\mathcal{W}_{l,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{k!\Gamma(l + k + \frac{b+1}{2})}\left(\frac{z}{2}\right)^{2k+l} \qquad (z \in \mathbb{C},\ l, b, c \in \mathbb{R}). \tag{1.2}$$

by

$$\begin{aligned}\phi_{l,b,c}(z) &= 2^l\,\Gamma\left(l + \frac{b+1}{2}\right) z^{1-l/2}\,\mathcal{W}_{l,b,c}(z^{\frac{1}{2}}) \\ &= z + \sum_{k=1}^{\infty} \frac{(-c)^k}{4^k \cdot (\nu)_k}\frac{z^{k+1}}{k!} \qquad \left(\nu = l + \frac{b+1}{2} \notin \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}\right),\end{aligned}$$

where $(\lambda)_k$ represents the Pochhammer symbol given by

$$(\lambda)_k = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1) \quad (k \in \mathbb{N} := \{1, 2, 3, \cdots\}) \qquad \text{and} \qquad (\lambda)_0 = 1$$

Subsequently, by using $\phi_{l,b,c}(z)$, Deniz [6] developed the operator B_ν^c as follows:

$$B_\nu^c f(z) = \phi_{l,b,c}(z) * f(z) = z + \sum_{k=1}^{\infty} \frac{(-c)^k a_{k+1}}{4^k \cdot (\nu)_k}\frac{z^{k+1}}{k!} \qquad (z \in \mathbb{C}). \tag{1.3}$$

clearly from (1.3),

$$z\left(B_{\nu+1}^c f(z)\right)' = \nu B_\nu^c f(z) - (\nu - 1)B_{\nu+1}^c f(z), \tag{1.4}$$

where

$$\nu = l + \frac{b+1}{2} \notin \mathbb{Z}_0^-.$$

Indeed, the operator B_ν^c given by (1.3) provides an elementary transform of the generalized hypergeometric function,

$$B_\nu^c f(z) = z\,{}_0F_1\left(\text{—}\,;\nu;-\frac{c}{4}\,z\right) * f(z)$$

and

$$\phi_{\nu,c}\left(-\frac{c}{4}\,z\right) = z\,{}_0F_1(\text{—}\,;\nu;z).$$

In this article, we investigate several inclusion relationships for each of the following subclasses of $\mathcal{A}$, associated with $\mathcal{W}_{l,b,c}(z)$ (see also [15], [16] and [17] for inclusion relationships of various other function classes). Indeed, for $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $0 \leq \alpha, \beta < 1$ and $\varphi, \psi \in \mathcal{N}$ we define

$$\mathcal{S}_\nu^c(\alpha;\, \varphi) := \{f \in \mathcal{A} : B_\nu^c f(z) \in \mathcal{S}^*(\alpha;\, \varphi),\ z \in \mathcal{U}\},$$

$$\mathcal{K}_\nu^c(\alpha;\, \varphi) := \{f \in \mathcal{A} : B_\nu^c f(z) \in \mathcal{K}(\alpha;\, \varphi),\ z \in \mathcal{U}\}$$

$$\mathcal{C}_\nu^c(\alpha, \beta;\, \varphi, \psi) := \{f \in \mathcal{A} : B_\nu^c f(z) \in \mathcal{C}(\alpha, \beta;\, \varphi, \psi),\ z \in \mathcal{U}\}.$$

Also,

$$f(z) \in \mathcal{K}_\nu^c(\alpha;\, \varphi) \Longleftrightarrow zf'(z) \in \mathcal{S}_\nu^c(\alpha;\, \varphi). \tag{1.5}$$

Particularly, we set

$$\mathcal{S}_\nu^c\left(\alpha;\, \left(\frac{1+Az}{1+Bz}\right)^\delta\right) = \mathcal{S}_\nu^c(\alpha;\, A, B;\, \delta), \quad (0 < \delta \leq 1;\ -1 \leq B < A \leq 1)$$

and

$$\mathcal{K}_\nu^c\left(\alpha;\, \left(\frac{1+Az}{1+Bz}\right)^\delta\right) = \mathcal{K}_\nu^c(\alpha;\, A, B;\, \delta), \quad (0 < \delta \leq 1;\ -1 \leq B < A \leq 1).$$

We need the following results for the investigation of our inclusion properties.

Lemma 1. [8] *Let $\phi(z)$ be analytic and convex univalent in $\mathcal{U}$ with $\phi(0) = 1$ and $Re\{\eta\phi(z) + \sigma\} > 0 \quad (\eta, \sigma \in \mathbb{C})$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then the subordination*

$$p(z) + \frac{zp'(z)}{\eta p(z) + \sigma} \prec \phi(z) \qquad (z \in \mathcal{U})$$

implies that

$$p(z) \prec \phi(z) \qquad (z \in \mathcal{U}).$$

Lemma 2. [12] *Let $h(z)$ be convex in $\mathcal{U}$ with $h(0) = 1$. Let $Q(z)$ be analytic in $\mathcal{U}$ with $Re\{Q(z)\} \geq 0 \quad (z \in \mathcal{U})$. If $q(z)$ is analytic in $\mathcal{U}$ such that $q(0) = h(0)$, then we have*

$$q(z) + Q(z)zq'(z) \prec h(z) \qquad (z \in \mathcal{U})$$

which implies that

$$q(z) \prec h(z) \qquad (z \in \mathcal{U}).$$

2. The Main Inclusion Relationships

Theorem 3. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1 \quad (0 \leq \alpha < 1)$. Then*

$$f \in \mathcal{S}_\nu^c(\alpha;\, \varphi) \Longrightarrow f \in \mathcal{S}_{\nu+1}^c(\alpha;\, \varphi)$$

or equivalently,

$$\mathcal{S}_\nu^c(\alpha;\, \varphi) \subset \mathcal{S}_{\nu+1}^c(\alpha;\, \varphi) \qquad (\varphi \in \mathcal{N}).$$

Proof. Let $f \in \mathcal{S}_\nu^c(\alpha;\, \varphi)$ and set

$$q(z) = \frac{1}{1-\alpha}\left(\frac{z\left(B_{\nu+1}^c f(z)\right)'}{B_{\nu+1}^c f(z)} - \alpha\right), \tag{2.1}$$

where $q(z)$ is analytic in $\mathcal{U}$ with $q(0) = 1$. From (1.4) we get,

$$\nu \frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)} = \frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c f(z)} + (\nu - 1). \tag{2.2}$$

By using (2.1) and (2.2), we have

$$\frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)} = \frac{1}{\nu}\big[(1-\alpha)q(z) + \alpha + \nu - 1\big]. \tag{2.3}$$

Now, by applying logarithmic differentiation on (2.3), we get

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c f(z)} = \frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c f(z)} + \frac{(1-\alpha)zq'(z)}{(1-\alpha)q(z) + \alpha + \nu - 1}$$

in view of (2.1), yields

$$\frac{1}{1-\alpha}\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c f(z)} - \alpha\right) = q(z) + \frac{zq'(z)}{(1-\alpha)q(z) + \alpha + \nu - 1} \qquad (z \in \mathcal{U}). \tag{2.4}$$

Finally, by applying Lemma 1 and (2.4), we have $q(z) \prec \varphi(z)$, hence $f \in \mathcal{S}_{\nu+1}^c(\alpha;\, \varphi)$. □

Theorem 4. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1 \quad (0 \le \alpha < 1)$. Then*

$$f \in \mathcal{K}_\nu^c(\alpha;\, \varphi) \Longrightarrow f \in \mathcal{K}_{\nu+1}^c(\alpha;\, \varphi)$$

or equivalently,

$$\mathcal{K}_\nu^c(\alpha;\, \varphi) \subset \mathcal{K}_{\nu+1}^c(\alpha;\, \varphi) \qquad (\varphi \in \mathcal{N}).$$

Proof. Using (1.5) and Theorem 3, we have

$$\begin{aligned}
f(z) \in \mathcal{K}_\nu^c(\alpha;\, \varphi) &\Longleftrightarrow B_\nu^c f(z) \in \mathcal{K}(\alpha;\, \varphi) \\
&\Longleftrightarrow z\,(B_\nu^c f(z))' \in \mathcal{S}^*(\alpha;\, \varphi) \\
&\Longleftrightarrow B_\nu^c\,(zf'(z)) \in \mathcal{S}^*(\alpha;\, \varphi) \\
&\Longleftrightarrow zf'(z) \in \mathcal{S}_\nu^c(\alpha;\, \varphi) \\
&\Longrightarrow zf'(z) \in \mathcal{S}_{\nu+1}^c(\alpha;\, \varphi) \\
&\Longleftrightarrow B_{\nu+1}^c\,(zf'(z)) \in \mathcal{S}^*(\alpha;\, \varphi) \\
&\Longleftrightarrow z\left(B_{\nu+1}^c f(z)\right)' \in \mathcal{S}^*(\alpha;\, \varphi) \\
&\Longleftrightarrow B_{\nu+1}^c f(z) \in \mathcal{K}(\alpha;\, \varphi) \\
&\Longleftrightarrow f(z) \in \mathcal{K}_{\nu+1}^c(\alpha;\, \varphi).
\end{aligned}$$

□

If we take

$$\varphi(z) = \left(\frac{1+Az}{1+Bz}\right)^\delta \qquad (-1 \le B < A \le 1;\, 0 < \delta \le 1;\, z \in \mathcal{U})$$

in Theorems 3 and 4,then we obtain the corollaries as:

Corollary 5. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1 \quad (0 \leq \alpha < 1)$. Then*

$$\mathcal{S}_\nu^c(\alpha;\, A, B;\, \delta) \subset \mathcal{S}_{\nu+1}^c(\alpha;\, A, B;\, \delta) \qquad (-1 \leq B < A \leq 1;\, 0 < \delta \leq 1)$$

and

$$\mathcal{K}_\nu^c(\alpha;\, A, B;\, \delta) \subset \mathcal{K}_{\nu+1}^c(\alpha;\, A, B;\, \delta) \qquad (-1 \leq B < A \leq 1;\, 0 < \delta \leq 1).$$

Theorem 6. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1 \quad (0 \leq \alpha < 1)$. Then*

$$f \in \mathcal{C}_\nu^c(\alpha, \beta;\, \varphi, \psi) \Longrightarrow f \in \mathcal{C}_{\nu+1}^c(\alpha, \beta;\, \varphi, \psi)$$

or equivalently,

$$\mathcal{C}_\nu^c(\alpha, \beta;\, \varphi, \psi) \subset \mathcal{C}_{\nu+1}^c(\alpha, \beta;\, \varphi, \psi) \qquad (0 \leq \alpha, \beta < 1;\, \varphi, \psi \in \mathcal{N}).$$

Proof. Let $f \in \mathcal{C}_\nu^c(\alpha, \beta;\, \varphi, \psi)$. Then $\exists$ a function $g \in \mathcal{S}_\nu^c(\alpha;\, \varphi) \ni$

$$\frac{1}{1-\beta}\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} - \beta\right) \prec \psi(z), \quad (0 \leq \beta < 1,\ z \in \mathcal{U}).$$

Now let

$$\omega(z) = \frac{1}{1-\beta}\left(\frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c g(z)} - \beta\right), \tag{2.5}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ with $\omega(0) = 1$. Making use of (1.4) we also have

$$\begin{aligned}
\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} &= \frac{B_\nu^c\big(zf'(z)\big)}{B_\nu^c g(z)} \\
&= \frac{z\big(B_{\nu+1}^c(zf'(z))\big)' + (\nu-1)B_{\nu+1}^c(zf'(z))}{z\big(B_{\nu+1}^c g(z)\big)' + (\nu-1)B_{\nu+1}^c g(z)} \\
&= \left[\frac{z\big(B_{\nu+1}^c(zf'(z))\big)'}{B_{\nu+1}^c g(z)} + (\nu-1)\frac{B_{\nu+1}^c(zf'(z))}{B_{\nu+1}^c g(z)}\right] \cdot \left[\frac{z\big(B_{\nu+1}^c g(z)\big)'}{B_{\nu+1}^c g(z)} + \nu - 1\right]^{-1}
\end{aligned} \tag{2.6}$$

By Theorem 3,

$$g \in \mathcal{S}_\nu^c(\alpha;\, \varphi) \Longrightarrow g \in \mathcal{S}_{\nu+1}^c(\alpha;\, \varphi),$$

therefore, we set

$$\vartheta(z) = \frac{1}{1-\alpha}\left(\frac{z\big(B_{\nu+1}^c g(z)\big)'}{B_{\nu+1}^c g(z)} - \alpha\right), \tag{2.7}$$

where $\Re\big(\vartheta(z)\big) > 0 \ \ (z \in \mathcal{U})$. Applying (2.5) and (2.7) into (2.6) we have

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} = \frac{\Big[z\big(B_{\nu+1}^c(zf'(z))\big)'\Big] \cdot \big[B_{\nu+1}^c g(z)\big]^{-1} + (\nu-1)[(1-\beta)\omega(z) + \beta]}{(1-\alpha)\vartheta(z) + \alpha + \nu - 1} \tag{2.8}$$

The logarithmic differentiation of (2.5) gives

$$\frac{z\big(B_{\nu+1}^c(zf'(z))\big)'}{B_{\nu+1}^c g(z)} = (1-\beta)z\omega'(z) + [(1-\alpha)\vartheta(z) + \alpha] \cdot [(1-\beta)\omega(z) + \beta],$$

which, in conjunction with (2.8), yields

$$\frac{1}{1-\beta}\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)}-\beta\right)=\omega(z)+\frac{z\omega'(z)}{(1-\alpha)\vartheta(z)+\alpha+\nu-1}.$$

Since $\alpha+\nu>1$ and $\vartheta(z)\prec\varphi(z)$ in $\mathcal{U}$,

$$\Re\big\{(1-\alpha)\vartheta(z)+\alpha+\nu-1\big\}>0 \qquad (z\in\mathcal{U}).$$

Finally, by applying Lemma 2, we have $\omega(z)\prec\psi(z)$, so that $f\in\mathcal{C}_{\nu+1}^c(\alpha,\beta;\,\varphi,\psi)$. □

3. **Inclusion Relationships Involving the Integral Operator F_σ**

The generalized Bernardi-Libera-Livingston integral operator F_σ $(\sigma>-1)$ (cf. [3, 10, 13]) considered here and is defined by

$$F_\sigma(f):=F_\sigma(f)(z)=\frac{\sigma+1}{z^\sigma}\int_0^z t^{\sigma-1}f(t)dt \qquad (f\in\mathcal{A};\,\sigma>-1). \tag{3.1}$$

Theorem 7. *Let $f(z)\in\mathcal{A}$, $c\in\mathbb{C}$, $\nu\in\mathbb{R}\setminus\mathbb{Z}_0^-$ and $\sigma\geq 0$. If $f\in\mathcal{S}_\nu^c(\alpha;\,\varphi)$ $(0\leq\alpha<1;\,\varphi\in\mathcal{N})$, then $F_\sigma(f)\in\mathcal{S}_\nu^c(\alpha;\,\varphi)$ $(0\leq\alpha<1;\,\varphi\in\mathcal{N})$.*

Proof. Let $f\in\mathcal{S}_\nu^c(\alpha;\,\varphi)$ and set

$$q(z)=\frac{1}{1-\alpha}\left(\frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c f(z)}-\alpha\right), \tag{3.2}$$

such that $q(z)$ is analytic in $\mathcal{U}$ with $q(0)=1$. Utilizing (3.1) we obtain,

$$\nu\frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)}=\frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c f(z)}+(\nu-1). \tag{3.3}$$

By using (3.2) and (3.3), we conclude that

$$\frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)}=\frac{1}{\nu}\big[(1-\alpha)q(z)+\alpha+\nu-1\big]. \tag{3.4}$$

By applying logarithmic differentiation on (3.4), we get

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c f(z)}=\frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c f(z)}+\frac{(1-\alpha)zq'(z)}{(1-\alpha)q(z)+\alpha+\nu-1}$$

in view of (3.2), yields

$$\frac{1}{1-\alpha}\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c f(z)}-\alpha\right)=q(z)+\frac{zq'(z)}{(1-\alpha)q(z)+\alpha+\nu-1} \qquad (z\in\mathcal{U}). \tag{3.5}$$

Finally, by applying Lemma 1 and (3.5), we have $q(z)\prec\varphi(z)$, hence $f\in\mathcal{S}_{\nu+1}^c(\alpha;\,\varphi)$. □

Theorem 8. *Let $f\in\mathcal{A}$, $c\in\mathbb{C}$, $\nu\in\mathbb{R}\setminus\mathbb{Z}_0^-$ and $\sigma\geq 0$. If $f\in\mathcal{K}_\nu^c(\alpha;\,\varphi)$ $(0\leq\alpha<1;\,\varphi\in\mathcal{N})$, then $F_\sigma(f)\in\mathcal{K}_\nu^c(\alpha;\,\varphi)$ $(0\leq\alpha<1;\,\varphi\in\mathcal{N})$.*

Proof. Using (1.5) along with Theorem 7, we see that

$$\begin{aligned} f \in \mathcal{K}_\nu^c(\alpha;\,\varphi) &\Longleftrightarrow zf'(z) \in \mathcal{S}_\nu^c(\alpha;\,\varphi) \\ &\Longrightarrow F_\sigma(zf'(z)) \in \mathcal{S}_\nu^c(\alpha;\,\varphi) \\ &\Longleftrightarrow z\left(F_\sigma(f)(z)\right)' \in \mathcal{S}_\nu^c(\alpha;\,\varphi) \\ &\Longleftrightarrow F_\sigma(f)(z) \in \mathcal{K}_\nu^c(\alpha;\,\varphi). \end{aligned}$$

□

Following corollaries due to Theorem 7 and Theorem 8 given as:

Corollary 9. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{S}_\nu^c(\alpha;\,A,B;\,\delta)$ $(0 \leq \alpha < 1;\, -1 \leq B < A \leq 1;\, 0 < \delta \leq 1)$, then $F_\sigma(f) \in \mathcal{S}_\nu^c(\alpha;\,A,B;\,\delta)$ $(0 \leq \alpha < 1;\, -1 \leq B < A \leq 1;\, 0 < \delta \leq 1)$.*

Corollary 10. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{K}_\nu^c(\alpha;\,A,B;\,\delta)$ $(0 \leq \alpha < 1;\, -1 \leq B < A \leq 1;\, 0 < \delta \leq 1)$, then $F_\sigma(f) \in \mathcal{K}_\nu^c(\alpha;\,A,B;\,\delta)$ $(0 \leq \alpha < 1;\, -1 \leq B < A \leq 1;\, 0 < \delta \leq 1)$.*

Theorem 11. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{C}_\nu^c(\alpha,\beta;\,\varphi,\psi)$, then $F_\sigma(f) \in \mathcal{C}_\nu^c(\alpha,\beta;\,\varphi,\psi)$ $(0 \leq \alpha,\beta < 1;\, \varphi,\psi \in \mathcal{N})$.*

Proof. Let $f \in \mathcal{C}_\nu^c(\alpha,\beta;\,\varphi,\psi)$. Then, $\exists\ g \in \mathcal{S}_\nu^c(\alpha;\,\varphi) \ni$

$$\frac{1}{1-\beta}\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} - \beta\right) \prec \psi(z), \quad (0 \leq \beta < 1,\ z \in \mathcal{U}).$$

Thus, we set

$$Q(z) = \frac{1}{1-\beta}\left(\frac{z\big(B_\nu^c F_\sigma(f)(z)\big)'}{B_\nu^c F_\sigma(g)(z)} - \beta\right), \tag{3.6}$$

where $Q(z)$ is analytic in $\mathcal{U}$ and $Q(0) = 1$. The use of (3.3) results

$$\begin{aligned} \frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} &= \frac{B_\nu^c\big(zf'(z)\big)}{B_\nu^c g(z)} \\ &= \frac{z\big(B_\nu^c F_\sigma(zf'(z))\big)' + \sigma B_\nu^c F_\sigma(zf'(z))}{z\big(B_\nu^c F_\sigma(g)(z)\big)' + \sigma B_\nu^c F_\sigma(g)(z)} \\ &= \frac{\frac{z(B_\nu^c F_\sigma(zf'(z)))'}{B_\nu^c F_\sigma(g)(z)} + \sigma\frac{B_\nu^c F_\sigma(zf'(z))}{B_\nu^c F_\sigma(g)(z)}}{\frac{z(B_\nu^c F_\sigma(g)(z))'}{B_\nu^c F_\sigma(g)(z)} + \sigma}, \end{aligned} \tag{3.7}$$

By Theorem 7,

$$g \in \mathcal{S}_\nu^c(\alpha;\,\varphi) \Longrightarrow F_\sigma(g) \in \mathcal{S}_\nu^c(\alpha;\,\varphi),$$

so that we can set

$$H(z) = \frac{1}{1-\alpha}\left(\frac{z\big(B_\nu^c F_\sigma(g)(z)\big)'}{B_\nu^c F_\sigma(g)(z)} - \alpha\right),$$

where $Re\big(H(z)\big) > 0$ $(z \in \mathcal{U})$. As parallel as the method used to prove the Theorem 6, from (3.6) and (3.7), we have

$$\frac{1}{1-\beta}\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} - \beta\right) = Q(z) + \frac{zQ'(z)}{(1-\alpha)H(z) + \alpha + \sigma}.$$

Hence, by using Lemma 2, we notice that $Q(z) \prec \psi(z)$, so $F_\sigma(f) \in \mathcal{C}_\nu^c(\alpha, \beta;\, \varphi, \psi)$. □

4. **Remarks and Observations**

The study of inclusion relationships for some subclasses of analytic functions with $\mathcal{W}_{l,b,c}(z)$ permits the study of Bessel $(J_v(z))$, modified Bessel $(I_v(z))$ and spherical Bessel functions $(j_n(z))$ together. By specializing the parameters in the operator B_ν^c we obtain the following new operators associated with $J_v(z)$, $I_v(z)$ and $j_n(z)$ (see, [2] and [6]):

- Choosing $b = c = 1$ in (1.3) we get the operator $\mathcal{J}_l : \mathcal{A} \longrightarrow \mathcal{A}$ associated with $J_l(z)$ as:

$$\mathcal{J}_l f(z) = \phi_{l,1,1}(z) * f(z) = \left[2^l \Gamma(l+1) z^{1-l/2} J_l(z^{\frac{1}{2}})\right] * f(z)$$
$$= z + \sum_{k=1}^{\infty} \frac{(-1)^k a_{k+1}}{4^k (l+1)_k} \frac{z^{k+1}}{k!}.$$

- Taking $b = 1$ and $c = -1$ in (1.3) we have $\mathcal{I}_l : \mathcal{A} \longrightarrow \mathcal{A}$ associated with $I_l(z)$ as:

$$\mathcal{I}_l f(z) = \phi_{l,1,-1}(z) * f(z) = \left[2^l \Gamma(l+1) z^{1-l/2} I_l(z^{\frac{1}{2}})\right] * f(z)$$
$$= z + \sum_{k=1}^{\infty} \frac{a_{k+1}}{4^k (l+1)_k} \frac{z^{k+1}}{k!}.$$

- Letting $b = 2$ and $c = 1$ in (1.3) we get $\mathcal{Q}_{\,l} : \mathcal{A} \longrightarrow \mathcal{A}$ associated with $j_l(z)$ as:

$$\mathcal{Q}_{\,l} f(z) = \phi_{l,2,1}(z) * f(z) = \left[\pi^{-1/2} 2^{l+1/2} \Gamma(l+3/2) z^{1-l/2} j_l(z^{\frac{1}{2}})\right] * f(z)$$
$$= z + \sum_{k=1}^{\infty} \frac{(-1)^k a_{k+1}}{4^k (l+3/2)_k} \frac{z^{k+1}}{k!}.$$

Thus, our inclusion results (Theorems 3-6) can be applied with a view of deducing the following corollaries.

Corollary 12. *Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0 \quad (0 \le \alpha < 1)$. Then*

$$f \in \mathcal{S}_{p+1}^1(\alpha;\, \varphi) \Longrightarrow f \in \mathcal{S}_{p+2}^1(\alpha;\, \varphi) \quad (\varphi \in \mathcal{N})$$

or equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{S}^*(\alpha;\, \varphi) \text{ then } f(z) \in \mathcal{S}_{p+n}^1(\alpha;\, \varphi) \quad (n \in \mathbb{N} \setminus \{1\};\, \varphi \in \mathcal{N}).$$

Corollary 13. *Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0 \quad (0 \le \alpha < 1)$. Then*

$$f \in \mathcal{K}_{p+1}^1(\alpha;\, \varphi) \Longrightarrow f \in \mathcal{K}_{p+2}^1(\alpha;\, \varphi) \quad (\varphi \in \mathcal{N})$$

or equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{K}(\alpha;\, \varphi) \text{ then } f(z) \in \mathcal{K}_{p+n}^1(\alpha;\, \varphi) \quad (n \in \mathbb{N} \setminus \{1\};\, \varphi \in \mathcal{N}).$$

Corollary 14. *Let* $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ *and* $\alpha + p > 0 \quad (0 \leq \alpha < 1)$. *Then*

$$f \in \mathcal{C}^1_{p+1}(\alpha, \beta;\, \varphi, \psi) \Longrightarrow f \in \mathcal{C}^1_{p+2}(\alpha, \beta;\, \varphi, \psi) \quad (0 \leq \beta < 1;\, \varphi, \psi \in \mathcal{N})$$

or equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{C}(\alpha, \beta;\, \varphi, \psi) \text{ then } f(z) \in \mathcal{C}^1_{p+n}(\alpha, \beta;\, \varphi, \psi) \quad (n \in \mathbb{N} \setminus \{1\};\, 0 \leq \beta < 1;\, \varphi, \psi \in \mathcal{N}).$$

Finally, we remark that similar results can be obtained involving the operators $\mathcal{I}_p$ and $\mathcal{Q}_p$ by specializing the parameter in Theorems 3-6. We also remark that several other applications and corollaries of our main results (Theorems 3-6) can indeed be derived similarly.

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K.A. Selvakumaran: Department of Mathematics, R.M.K College of Engg. and Technology, Puduvoyal - 601206, Tamil Nadu, India.
E-mail address: selvaa1826@gmail.com

Huda A. Al-Kharsani: Department of mathematics, Girls College, 838, Dammam University, Saudi Arabia.
E-mail address: hakh73@hotmail.com

D. Baleanu: Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University-06530, Ankara, Turkey. Institute of Space Sciences, Magurele-Bucharest, Romania.
E-mail address: dumitru@cankaya.edu.tr

S.D. Purohit: Department of HEAS (Mathematics), Rajasthan Technical University, Kota-324010, Rajasthan, India
E-mail address: sunil_a_purohit@yahoo.com

K.S. Nisar: Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Al-Dawaser, Saudi Arabia.
E-mail address: ksnisar1@gmail.com

Fixed point properties of Suzuki generalized nonexpansive set-valued mappings in complete CAT(0) spaces ☆

Jing Zhou*,[a], Yunan Cui[b]

[a]*Department of Mathematics, Harbin Institute of Technology, Harbin 150080, P.R. China*
[b]*Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, P.R. China*

Abstract

In this paper, we introduce the notion of Suzuki generalized nonexpansive set-valued mappings from traditional Banach spaces to CAT(0) spaces. Moreover we discuss fixed point properties including the existence, $\triangle-$convergence and strong convergence for this kind of mappings in complete CAT(0) spaces. Our results extend the results of B. Nanjaras [12] for the Suzuki generalized nonexpansive single-valued mappings.

Key words: Suzuki generalized nonexpansive set-valued mapping, CAT(0) space, Existence theorem, Convergence theorem, Fixed point

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☆NFS of Hei Longjiang Provience (A2015018)
*Corresponding author
Email addresses: zhoujinggirl@126.com (Jing Zhou), cuiya@hrbust.edu.cn (Yunan Cui)

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1. Introduction

In recent years, more and more classical fixed point theorems for single-valued nonexpansive mappings are extended by set-valued nonexpansive mappings. As a result, fixed point theory for set-valued nonexpansive mappings get rapid development[see [6–10]].

In 2008, Suzuki [11] brought in a new kind of mappings which was called satisfying the condition (C) in Banach spaces. In 2010, B. Nanjaras, B. Panyanak and W. Phuengrattana [12] established fixed point theorems for the mappings satisfying the condition (C) in CAT(0) spaces. In 2011, Abkar and Eslamian [14] gave the definition of set-valued mappings for the condition (C) and then they proved the existence of fixed point in uniformly convex Banach spaces [13].

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $x' \in X$ is a map c from a closed interval $[0, h] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(h) = x'$ and $d(c(l), c(l')) = |l - l'|$, for all l, $l' \in [0, h]$. In particular, c is an isometry and $d(x, x') = h$. The image γ of c is called a geodesic (or metric) segment joining x and x' denoted by $[x, x']$ whenever it is unique. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and x' for each $x, x' \in X$.

A geodesic space is said to be a CAT(0) space if the following CAT(0) inequality

$$d\left(z, \frac{x_1 \oplus x_2}{2}\right)^2 \leq \frac{1}{2}d(z, x_1)^2 + \frac{1}{2}d(z, x_2)^2 - \frac{1}{4}d(x_1, x_2)^2$$

satisfies for all $x_1, x_2, z \in X$. This is the (CN) inequality of Bruhat and Tits [1](More details spaces see [2]).

Lemma 1.1. *Let (X, d) be a CAT(0) space.*
(i)[3, Lemma 2.1(iv)] For each x_1, $x_2 \in X$ and $\alpha \in [0, 1]$, there exists a unique point $y \in [x_1, x_2]$ such that

$$d(x_1, y) = \alpha d(x_1, x_2), \quad d(x_2, y) = (1 - \alpha)d(x_1, x_2).$$

Denote $y = (1 - \alpha)x_1 \oplus \alpha x_2$ in the above equations conveniently.
(ii)[3, Lemma 2.4] For each x_1, x_2, $y \in X$ and $\alpha \in [0, 1]$. We have

$$d((1 - \alpha)x_1 \oplus \alpha x_2, y) \leq (1 - \alpha)d(x_1, y) + \alpha d(x_2, y). \tag{1.1}$$

Obviously, Lemma 1.1. shows the convexness of CAT(0) spaces.

Lemma 1.2. *([4, Lemma 2.5]) Let (X, d) be a CAT(0) space. Then*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + t\,d(y,z)^2 - t(1-t)d(x,y)^2$$

for all $t \in [0,1]$ and x, y, $z \in X$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $z \in X$, we set

$$r(z, \{x_n\}) = \limsup_{n\to\infty} d(z, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(z, \{x_n\}) : \ z \in X\}.$$

The asymptotic radius $r_D(\{x_n\})$ of $\{x_n\}$ with respect to $D \subset X$ is given by

$$r_D(\{x_n\}) = \inf\{r(z, \{x_n\}) : z \in D\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{z \in X : \ r(z, \{x_n\}) = r(\{x_n\})\}.$$

And the asymptotic center $A_D(\{x_n\})$ of $\{x_n\}$ with respect to $D \subset X$ is the set

$$A_D(\{x_n\}) = \{z \in D : \ r(z, \{x_n\}) = r(\{x_n\})\}.$$

It follows from [5, Proposition 7]) that $A(\{x_n\})$ consists of exactly one point in a CAT(0) space. In 1976, Lim [1] introduced the concept of $\Delta-$ convergence in a general metric space. In 2008, Kirk and Panyanak [4] brought in $\Delta-$convergence to CAT(0) spaces and proved that there is an analogy between $\Delta-$convergence and weak convergence.

Definition 1.3. ([2]) A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 1.4. *([2]) If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic centerasymptotic center of $\{x_n\}$ is in D.*

Lemma 1.5. *([2]) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 1.6. *([4]) If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{p\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

The purpose of this paper is to bring in the concept of Suzuki generalized nonexpansive set-valued mappings in CAT(0) spaces. Then we shall prove a common fixed point theorem for commuting pairs consisting of a single-valued and a set-valued mapping both satisfying the condition (C) which is analogous to the results in Banach spaces [13]. Furthermore, we also establish $\triangle-$convergence and strong convergence of Mann iteration in CAT(0) spaces.

2. Preliminaries

Let D be a nonempty subset of a CAT(0) space X. We denote by $B(D)$ the collection of all nonempty bounded closed subsets of D and $C(D)$ the collection of all nonempty compact subsets of D. Suppose H is the Hausdorff metric with respect to d, that is,

$$H(U,V) := \max\left\{\sup_{u\in U} dist(u,V), \sup_{v\in V} dist(v,U)\right\}, \quad U,V \in B(X)$$

where $dist(u,V) = \inf_{v\in V} d(u,v)$ is the distance from the point u to the set V.

Let $T : X \to 2^X$ be a set-valued mapping. If an element $x \in X$ satisfies $x \in Tx$, then x is called a fixed point of T. The set of fixed points of T is denoted by $Fix(T)$.

Definition 2.1. A set-valued mapping $T : X \to B(X)$ is
(i) nonexpansive provided

$$H(Tx,Ty) \le d(x,y), \quad x,y \in X;$$

(ii) quasi nonexpansive if $Fix(T) \neq \emptyset$ and

$$H(Tx,p) \le d(x,p), \quad x \in X,$$

for all $p \in Fix(T)$.

Definition 2.2. A set-valued mapping $T : X \to B(X)$ is called to satisfy the condition (C) if

$$\frac{1}{2} dist(x, Tx) \leq d(x, y) \quad implies \quad H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

This kind of set-valued mappings satisfying the condition (C) can be called Suzuki generalized nonexpansive as well.

Proposition 2.3. *Suppose a set-valued mapping $T : X \to B(X)$ satisfies the condition (C) with the nonempty fixed point set. Then T is a quasi-nonexpansive set-valued mapping.*

Theorem 2.4. *([12, Theorem 4.1]) Suppose D is a nonempty bounded closed convex subset of a complete CAT(0) space X and $t : D \to D$ is a mapping which satisfies the condition (C). Then $F(t)$ is nonempty in D.*

Corollary 2.5. *([12, Corollary 4.2]) Suppose D is a nonempty bounded closed convex subset of a complete CAT(0) space X and $t : D \to D$ is a mapping which satisfies the condition (C). Then $F(t)$ is nonempty closed, convex and hence contractible.*

Definition 2.6. Suppose D is a nonempty bounded closed convex subset of a CAT(0) space X. Let $t : D \to D$ and $T : D \to B(D)$ be a single-valued mapping and a set-valued mapping respectively. Then t and T are said to be commuting mappings if for every $x, y \in D$ such that $x \in Ty$ and $ty \in D$, we have $tx \in Tty$.

Definition 2.7. A set-valued mapping $T : X \to B(X)$ is said to satisfy the condition (E_μ) provided that

$$dist(x, Ty) \leq \mu dist(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies the condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Proposition 2.8. *Let $T : X \to B(X)$ be a set-valued mapping which satisfies the condition (C). Then T satisfies the condition (E_3).*

PROOF. Given $x \in X$. Since for any $z \in Tx$,

$$dist(x, Tx) \leq d(x, z) \tag{2.1}$$

then

$$\frac{1}{2}dist(x, Tx) \leq \frac{1}{2}d(x, z) \leq d(x, z).$$

From our assumption, we have

$$H(Tx, Tz) \leq d(x, z) \tag{2.2}$$

for any $x \in X$ and $z \in Tx$. Then for $x, y \in X$ and $z \in Tx$ either $\frac{1}{2}dist(x, Tx) \leq d(x, y)$ or $\frac{1}{2}H(Tx, Tz) \leq d(y, z)$ holds. On the contrary, together with (2.1) and (2.2), we get

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< \frac{1}{2}dist(x, Tx) + \frac{1}{2}H(Tx, Tz) \\ &\leq \frac{1}{2}d(x, z) + \frac{1}{2}d(x, z) \\ &= d(x, z) \end{aligned}$$

which is a contradiction.

Case I. $\frac{1}{2}dist(x, Tx) \leq d(x, y)$holds.

From the above hypothesis, we can obtain $H(Tx, Ty) \leq d(x, y)$ by the condition (C). Hence,

$$\begin{aligned} dist(x, Ty) &\leq dist(x, Tx) + H(Tx, Ty) \\ &\leq dist(x, Tx) + d(x, y) \\ &\leq 3dist(x, Tx) + d(x, y). \end{aligned}$$

Case II. $\frac{1}{2}H(Tx, Tz) \leq d(y, z)$holds. Then,

$$\begin{aligned} \frac{1}{2}dist(z, Tz) &\leq \frac{1}{2}\sup_{u \in Tx} dist\,(u, Tz) \\ &\leq \frac{1}{2}max\{\sup_{u \in Tx} dist\,(u, Tz)\,, \sup_{v \in Tz} dist\,(v, Tx)\} \\ &= \frac{1}{2}H(Tx, Tz) \\ &\leq d(y, z). \end{aligned}$$

From the assumption, we have

$$H(Ty, Tz) \leq d(y, z). \tag{2.3}$$

Together with (2.1), (2.2) and (2.3), we can obtain

$$\begin{aligned} dist(x, Ty) &\leq dist(x, Tx) + H(Tx, Ty) \\ &\leq dist(x, Tx) + H(Tx, Tz) + H(Tz, Ty) \\ &\leq d(x, z) + d(x, z) + d(y, z) \\ &\leq 2d(x, z) + d(x, y) + d(x, z) \\ &= 3d(x, z) + d(x, y). \end{aligned}$$

Because this is applied for any $z \in Tx$, we get

$$dist(x, Ty) \leq 3dist(x, Tx) + d(x, y).$$

By overall consideration, T satisfies the condition (E_3). $\square$

Lemma 2.9. *([12, Lemma 2.5]) Let $\{x_n\}$, $\{y_n\}$ be bounded sequences in a CAT(0) space X and let $\{\alpha_n\} \in [0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n) x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in N$. Then*

$$\lim_{n \to \infty} d(x_n, y_n) = 0.$$

3. Fixed point properties of Suzuki generalized nonexpansive set-valued mappings

In this section, we denote by D a nonempty bounded closed convex subset of a complete CAT(0) space X. Firstly, we shall discuss the existence of a common fixed point.

Theorem 3.1. *Let $t : D \to D$ and $T : D \to C(D)$ be a single-valued mapping and a set-valued mapping respectively. If both t and T satisfy the condition (C) and in the meantime, they are commuting, then they have a common fixed point, that is, there exists a point $z \in D$ such that $z = tz \in Tz$.*

PROOF. By Theorem 2.4 and Corollary 2.5, we know that the mapping t has a fixed point set $Fix(t)$ which is a nonempty closed convex subset of X. Let $p \in Fix(t)$. As t and T are commuting, we have $tq \in Ttp = Tp$ for each

$q \in Tp$. Therefore, Tp is invariant under t for each $p \in Fix(t)$. Since Tp is a bounded closed convex subset of X, we can obtain that t has a fixed point in Tp. Hence, $Tp \bigcap Fix(t) \neq \emptyset$ for $p \in Fix(t)$.

Take $x_1 \in Fix(t)$. Because $Tx_1 \bigcap Fix(t) \neq \emptyset$, we are able to select $y_1 \in Tx_1 \bigcap Fix(t)$. Define $x_2 = \frac{1}{2}(x_1 + y_1)$. Due to the fact that $Fix(t)$ is a convex set, we have $x_2 \in Fix(t)$. Let $y_2 \in Tx_2$ be selected as follows

$$d(y_1, y_2) = dist(y_1, Tx_2).$$

Since $\frac{1}{2}d(y_1, ty_1) = 0 \leq d(y_1, y_2)$, from the assumption, we know

$$d(y_1, ty_2) = d(ty_1, ty_2) \leq d(y_1, y_2).$$

Note that $ty_2 \in Tx_2$, hence, there comes a contradiction. Therefore, $y_2 \in Fix(t)$. In such a way, we can find a sequence $\{x_n\}$ in $Fix(t)$ such that

$$x_{n+1} = \frac{1}{2}(x_n + y_n), \quad n \geq 1$$

where $y_n \in Tx_n \bigcap Fix(t)$ and $d(y_{n-1}, y_n) = dist(y_{n-1}, Tx_n)$. Therefore, by Lemma 1.1.(i) we get

$$\frac{1}{2}d(x_n, y_n) = d(x_n, x_{n+1})$$

from which we can conclude

$$\frac{1}{2}dist(x_n, Tx_n) \leq \frac{1}{2}d(x_n, y_n) = d(x_n, x_{n+1}), \quad n \geq 1.$$

Hence,

$$H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1$$

which implies

$$d(y_n, y_{n+1}) = dist(y_n, Tx_{n+1}) \leq H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}).$$

We now apply Lemma 2.9. to obtain that

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

where $y_n \in Tx_n$, i.e., we find an approximate fixed point sequence for T in $Fix(t)$.

Assume that $z = A(\{x_n\})$. By Lemma 1.4. we know that $z \in Fix(t)$. For each $n \geq 1$, we choose $z_n \in Tz$ such that

$$d(y_n, z_n) = dist(y_n, Tz).$$

Because $lim_{n\to\infty} d(x_n, y_n) = 0$, there exists n_0 such that

$$d(x_n, y_n) \leq d(x_n, z), \quad n \geq n_0.$$

This implies that

$$\frac{1}{2} dist(x_n, Tx_n) \leq d(x_n, z),$$

and hence,

$$H(Tx_n, Tz) \leq d(x_n, z), \quad n \geq n_0.$$

Therefore,

$$d(y_n, z_n) \leq H(Tx_n, Tz) \leq d(x_n, z), \quad n \geq n_0.$$

As $\frac{1}{2} d(y_n, Ty_n) = 0 \leq d(y_n, z_n)$ for each $n \geq 1$, we get

$$d(y_n, tz_n) = d(ty_n, tz_n) \leq d(y_n, z_n).$$

Since $z \in Fix(t)$ and $z_n \in Tz$, by the fact that the mappings t and T are commuting, we can obtain that $tz_n \in Ttz_n = Tz$. Now by the uniqueness of z_n as the nearest point to y_n, we get $tz_n = z_n \in Fix(t)$.

As Tz is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim\limits_{k\to\infty} z_{n_k} = z^* \in Tz$. Because $z_{n_k} \in Fix(t)$ for all n, and $Fix(t)$ is closed, we can obtain that $z^* \in Fix(t)$. Take the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$, we have

$$d(x_{n_k}, z^*) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, z^*) \tag{3.1}$$

and for all $n_k \geq n_0$,

$$d(y_{n_k}, z_{n_k}) \leq d(x_{n_k}, z).$$

Letting $k \to \infty$ and taking superior limit on the both sides of (3.1), we have

$$\limsup_{k\to\infty} d(x_{n_k}, z^*) \leq \limsup_{k\to\infty} d(x_{n_k}, z).$$

By the uniqueness of asymptotic centers, this shows $z = z^* \in Tz$ and hence $z = tz \in Tz$. $\square$

Corollary 3.2. *Let $t : D \to D$ and $T : D \to C(D)$ be a single-valued mapping and a set-valued mapping respectively. Assume that t and T are commuting mappings. Then there exists a point $z \in D$ such that $z = tz \in Tz$.*

Corollary 3.3. *Suppose $T : D \to C(D)$ is set-valued mapping which satisfies the condition (C). Then T has a fixed point.*

We will prove $\Delta-$convergence and strong convergence theorems in the following. Before that, we discuss some Lemmas which will be used in the main proofs.

Lemma 3.4. *Suppose $T : D \to C(D)$ is a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ by $x_1 \in D$ and*

$$x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n y_n, \quad \alpha_n \subset \left[\frac{1}{2}, 1\right), n \geq 1$$

where $y_n \in Tx_n$ such that $d(y_{n-1}, y_n) = dist(y_{n-1}, Tx_n)$. Then

$$\lim_{n\to\infty} dist(Tx_n, x_n) = 0.$$

PROOF. It follows from Lemma 1.1.(i) that for every natural number $n \geq 1$

$$\frac{1}{2} dist(x_n, Tx_n) \leq \alpha_n dist(x_n, Tx_n) \leq \alpha_n d(x_n, y_n) = d(x_n, x_{n+1})$$

From our assumption, we get

$$H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1.$$

Hence

$$d(y_n, y_{n+1}) = dist(y_n, Tx_{n+1}) \leq H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1.$$

We can conclude that $\lim_{n\to\infty} d(x_n, y_n) = 0$ where $y_n \in Tx_n$ by Lemma 2.9., i.e., $\lim_{n\to\infty} dist(Tx_n, x_n) \leq \lim_{n\to\infty} d(x_n, y_n) = 0$ $\square$

Lemma 3.5. *Suppose $T : D \to C(D)$ is a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ as in Lemma 3.4. Then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in Fix(T)$.*

PROOF. By Theorem 3.1. $F(T)$ is nonempty. Take $p \in Fix(t)$, by Lemma 1.2. and Proposition 2.3. we have

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n H^2(Tx_n, Tp) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &= d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n). \end{aligned}$$

This entails

$$d^2(x_{n+1}, p) \leq d^2(x_n, p).$$

Therefore, $d(x_{n+1}, p) \leq d(x_n, p)$ for all $n \geq 1$ which implies $\{d(x_n, p)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in Fix(T)$. □

Lemma 3.6. *Let $T : D \to C(D)$ be a set-valued mapping which satisfies the condition (C). If $\{x_n\}$ is a sequence in D such that $dist(Tx_n, x_n) \to 0$ and $\Delta-$ converges to some $\omega \in X$. Then $\omega \in D$ and $\omega \in T\omega$.*

PROOF. We first note that $\omega \in D$ by Lemma 1.4. For each $n \geq 1$, we select $\omega_n \in T\omega$ such that $d(x_n, \omega_n) = dist(x_n, T\omega)$. By the compactness of $T\omega$, there exists a subsequence $\{\omega_{n_k}\}$ of $\{\omega_n\}$ such that $\{\omega_{n_k}\} \to \omega^* \in T\omega$. Taking the subsequence $\{x_{n_k}\}$ of $\{x_n\}$, it follows from Proposition 2.8. that

$$dist(x_{n_k}, T\omega) \leq 3dist(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, \omega).$$

Note that

$$\begin{aligned} d(x_{n_k}, \omega^*) &\leq d(x_{n_k}, \omega_{n_k}) + d(\omega_{n_k}, \omega^*) \\ &\leq 3dist(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, \omega) + d(\omega_{n_k}, \omega^*). \end{aligned}$$

Letting $k \to \infty$ and taking superior limit on the both sides of the above inequation, we have

$$\limsup_{k\to\infty} d(x_{n_k}, \omega^*) \leq \limsup_{k\to\infty} d(x_{n_k}, \omega).$$

By the uniqueness of asymptotic centers, we have $\omega = \omega^* \in T\omega$. □

Theorem 3.7. *Let $T : D \to C(D)$ be a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ as in Lemma 3.4. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

PROOF. By Lemma 3.4., $\lim_{n\to\infty} dist\,(Tx_n, x_n) = 0$. Now we prove that

$$W_\omega(x_n) := \bigcup_{\{\mu_n\}\subset\{x_n\}} A(\{\mu_n\}) \subset Fix\,(T) \tag{3.2}$$

and $W_\omega(x_n)$ consists of exactly one point.

In fact, let $\mu \in W_\omega(x_n)$, then there exists a sequence $\{\mu_n\}$ of $\{x_n\}$ such that $A(\{\mu_n\}) = \{\mu\}$. By Lemma 1.4. and 1.5., there exists a subsequence $\{\nu_n\}$ of $\{\mu_n\}$ such that $\Delta-\lim_{n\to\infty}\nu_n = \nu \in D$. Since $\lim_{n\to\infty} dist\,(T\nu_n, \nu_n) = 0$, then $\nu \in Fix\,(T)$ by Lemma 3.6. and $\lim_{n\to\infty} d\,(x_n, \nu)$ exists by Lemma 3.5. By Lemma 1.6. $\mu = \nu$. This implies that $W_\omega(x_n) \subset Fix\,(T)$.

Next we prove that $W_\omega(x_n)$ consists of exactly one point. Let $\{\mu_n\}$ be a subsequence of $\{x_n\}$ with $A(\{\mu_n\}) = \{\mu\}$ and let $A(\{x_n\}) = \{x\}$. Since $\mu \in W_\omega(x_n) \subset Fix\,(T)$, from Lemma 3.5. we know that $\{d\,(x_n, \mu)\}$ is convergent. In view of Lemma 1.6, $x = \mu$. □

Finally, we shall give the strong convergence for Suzuki generalized nonexpansive set-valued mappings in complete CAT(0) spaces.

Theorem 3.8. *Suppose D is a nonempty compact convex subset of a complete CAT(0) space and $T : D \to C(D)$ is a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ as in Lemma 3.4. Then $\{x_n\}$ converges strongly to a fixed point of T.*

PROOF. By Lemma 3.4., $\lim_{n\to\infty} dist\,(Tx_n, x_n) = 0$. Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z$ for some $z \in D$. By Proposition 2.8., we have

$$\begin{aligned} dist(z, Tz) &\leq d(z, x_{n_k}) + dist(x_{n_k}, Tz) \\ &\leq d(z, x_{n_k}) + 3dist(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, z) \\ &= 2d(z, x_{n_k}) + 3dist(x_{n_k}, Tx_{n_k}). \end{aligned}$$

Letting $k \to \infty$, we have $z \in Fix(T)$. Since $\{x_{n_k}\}$ converges strongly to z and $\lim_{n\to\infty} d\,(x_n, z)$ exists, we can conclude that $\{x_{n_k}\}$ converges strongly to z. □

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WEIGHTED COMPOSITION FOLLOWED AND PROCEEDED BY DIFFERENTIATION OPERATORS FROM ZYGMUND SPACES TO BERS-TYPE SPACES

JIANREN LONG AND CONGLI YANG

ABSTRACT. In this paper, we investigate boundedness and compactness of the weighted composition followed and proceeded by differentiation operators from Zygmund spaces to Bers-type spaces and little Bers-type spaces. Some sufficient and necessary conditions for the boundedness and compactness of these operators are obtained.

1. INTRODUCTION

Let $\Delta = \{z : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, and let $H(\Delta)$ be the class of all analytic functions on Δ.

Assume that μ is a positive continuous function on $[0, 1)$, having the property that there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0, 1)$, such that

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta, 1), \lim_{r\to 1} \frac{\mu(r)}{(1-r)^s} = 0,$$
$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta, 1), \lim_{r\to 1} \frac{\mu(r)}{(1-r)^t} = \infty.$$

Then μ is called a normal function (see [6], [18]).

An analytic function f on Δ is said to belong to the Bers-type space, denoted by H^∞_μ, if

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\Delta} \mu(|z|)|f(z)| < \infty,$$

and it is said to belong to the little Bers-type space $H^\infty_{\mu,0}$ if

$$\lim_{|z|\to 1} \mu(|z|)|f(z)| = 0.$$

It is clear that both H^∞_μ and $H^\infty_{\mu,0}$ are Banach spaces with the norm $\|\cdot\|_{H^\infty_\mu}$, and $H^\infty_{\mu,0}$ is a closed subspace of H^∞_μ. When $\mu \equiv 1$, the space H^∞_μ is just H^∞, which is defined by

$$H^\infty = \{f \in H(\Delta) : \|f\|_\infty = \sup_{z\in\Delta} |f(z)| < \infty\}.$$

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See [25] for more about Bers-type space.

An f in $H(\Delta)$ is said to belong to the Zygmund space, denoted by $\mathcal{Z}$, if

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\Delta$ and $h > 0$. By Theorem 5.3 in [2], we see that $f \in \mathcal{Z}$ if and only if

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\Delta}(1 - |z|^2)|f''(z)| < \infty.$$

It is easy to check that $\mathcal{Z}$ is a Banach space under the above norm. For every $f \in \mathcal{Z}$, by using a result in [9], we have that

$$|f'(z)| \leq C\|f\|_{\mathcal{Z}} \ln \frac{e}{1 - |z|^2}.$$

Let $\mathcal{Z}_0$ denote the subspace of $\mathcal{Z}$ consisting of those $f \in \mathcal{Z}$ for which

$$\lim_{|z|\to 1}(1 - |z|^2)|f''(z)| = 0.$$

The space $\mathcal{Z}_0$ is called the little Zygmund space.

Let φ be a nonconstant analytic self-map of Δ, and let ϕ be an analytic function in Δ. For $f \in H(\Delta)$, we define the linear operators

$$\phi C_\varphi Df = \phi(f'o\varphi) = \phi f'(\varphi)$$

and

$$\phi DC_\varphi f = \phi(fo\varphi)' = \phi f'(\varphi)\varphi'.$$

They are called weighted composition followed and proceeded by differentiation operators respectively, where C_φ and D are composition and differentiation operators respectively. Associated with φ is the composition operator $C_\varphi f = f \circ \varphi$ and weighted composition operator $\phi C_\varphi f = \phi f \circ \varphi$ for $\phi \in H(\Delta)$ and $f \in H(\Delta)$. It is interesting to provide a function theoretic characterization for φ inducing a bounded or compact composition operator, weighted composition operator and related ones on various spaces (see, *e.g.*, [1, 3, 10, 15, 17, 19-21, 23-24, 26]). For example, it is well known that C_φ is bounded on the classical Hardy, Bloch and Bergman spaces. Operators DC_φ and $C_\varphi D$ as well as some other products of linear operators were studied, for example, in [5, 7-8, 11, 13, 16, 22] (see also the references therein). There has been some considerable recent interest in investigation various type of operators from or to Zygmund type spaces (see, [4, 9-12, 27]).

In this paper, we investigate the operators ϕDC_φ and $\phi C_\varphi D$ from Zygmund spaces to Bers-type spaces and little Bers-type spaces by using the

similar ways in [14]. Some sufficient and necessary conditions for the boundedness and compactness of these operators are given.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \approx B$ means that there is a positive constant C such that $\frac{B}{C} \leq$A$\leq$CB.

2. Main results and proofs

In this section, we state and prove our main results. In order to formulate our main results, we quote several lemmas which will be used in the proofs of the main results in this paper. The following lemma can be proved in a standard way (see,*e.g.*, Proposition 3.11 in [1]). Hence we omit the details.

Lemma 2.1. *Let φ be an analytic self-map of Δ, ϕ be an analytic function in Δ. Suppose that μ is normal. Then ϕDC_φ (or $\phi C_\varphi D$):$\mathcal{Z}$ (or $\mathcal{Z}_0$)$\to H^\infty_\mu$ is compact if and only if ϕDC_φ (or $\phi C_\varphi D$):$\mathcal{Z}$ (or $\mathcal{Z}_0$)$\to H^\infty_\mu$ is bounded and for any bounded sequence $\{f_n\}_{n\in N}$ in $\mathcal{Z}$ (or $\mathcal{Z}_0$) which converges to zero uniformly on compact subsets of Δ as $n \to \infty$, and $\|\phi DC_\varphi f_n\|_{H^\infty_\mu} \to$ 0 (or $\|\phi C_\varphi D f_n\|_{H^\infty_\mu} \to$ 0) as $n \to \infty$.*

Lemma 2.2. *A closed set $\mathbb{K}$ of $H^\infty_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in\mathbb{K}} \mu(|z|)|f(z)| = 0. \tag{2.1}$$

Proof. First of all, we suppose that $\mathbb{K}$ is compact and let $\varepsilon > 0$. By the definition of $H^\infty_{\mu,0}$, we can choose an $\frac{\varepsilon}{2}$-net which center at $f_1, f_2, \cdots, f_n$ in $\mathbb{K}$ respectively, and a positive number r ($0 < r < 1$), such that $\mu(|z|)|f_i(z)| < \frac{\varepsilon}{2}$, for $1 \leq i \leq n$ and $|z| > r$. If $f \in \mathbb{K}$, $\|f - f_i\|_{H^\infty_\mu} < \frac{\varepsilon}{2}$ for some f_i, so we have

$$\mu(|z|)|f(z)| \leq \|f - f_i\|_{H^\infty_\mu} + \mu(|z|)|f_i(z)| < \varepsilon,$$

for $|z| > r$. This establishes (2.1).

On the other hand, if $\mathbb{K}$ is a closed bounded set which satisfies (2.1) and $\{f_n\}$ is a sequence in $\mathbb{K}$, then by the Montel's theorem, there is a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of Δ to some analytic function f. According to (2.1), for every $\varepsilon > 0$, there is an r, $0 < r < 1$, such that for all $g \in \mathbb{K}$, $\mu(|z|)|g(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. It follows that $\mu(|z|)|f(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. Since $\{f_{n_k}\}$ converges uniformly to f on $|z| \leq r$, it follows that $\lim_{k\to\infty} \sup \|f_{n_k} - f\|_{H^\infty_\mu} \leq \varepsilon$, i.e $\lim_{k\to\infty} \|f_{n_k} - f\|_{H^\infty_\mu} = 0$, so that $\mathbb{K}$ is compact. □

Theorem 2.3. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then the following statements are equivalent.*

(i) *$\phi DC_\varphi : \mathcal{Z} \to H^\infty_\mu$ is bounded;*
(ii) *$\phi DC_\varphi : \mathcal{Z}_0 \to H^\infty_\mu$ is bounded;*
(iii)

$$\sup_{z\in\Delta} \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty. \tag{2.2}$$

Proof. (i)⇒(ii). This implication is obvious.

(ii)⇒(iii). Assume that ϕDC_φ:$\mathcal{Z}_0 \to H^\infty_\mu$ is bounded, i.e., there exists a constant C such that

$$\|\phi DC_\varphi f\|_{H^\infty_\mu} \le C\|f\|_{\mathcal{Z}}$$

for all $f \in \mathcal{Z}_0$. Taking the function $f(z) = z \in \mathcal{Z}_0$, we get

$$\sup_{z\in\Delta} \mu(|z|)|\phi(z)\varphi'(z)| < \infty. \tag{2.3}$$

Set

$$h(z) = (z-1)[(1+\ln\frac{1}{1-z})^2 + 1]$$

and

$$h_a(z) = \frac{h(\bar{a}z)}{\bar{a}}(\ln\frac{1}{1-|a|^2})^{-1} \tag{2.4}$$

for $a \in \Delta \setminus \{0\}$. It is known that $h_a \in \mathcal{Z}_0$ (see [9]). Since

$$h_a'(z) = (\ln\frac{1}{1-\bar{a}z})^2(\ln\frac{1}{1-|a|^2})^{-1}, \tag{2.5}$$

for $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$C\|\phi DC_\varphi\|_{\mathcal{Z}_0\to H^\infty_\mu} \ge \|\phi DC_\varphi h_{\varphi(\lambda)}\|_{H^\infty_\mu} \ge \mu(|\lambda|)|\phi(\lambda)\varphi'(\lambda)| \ln\frac{1}{1-|\varphi(\lambda)|^2}.$$

Hence, we have that

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}} \mu(|\lambda|)|\phi(\lambda)\varphi'(\lambda)| \ln\frac{1}{1-|\varphi(\lambda)|^2} < \infty. \tag{2.6}$$

On the other hand, from the inequality (2.3) we have that

(2.7)

$$\sup_{|\varphi(\lambda)|\le\frac{1}{2}} \mu(|\lambda|)|\phi(\lambda)\varphi'(\lambda)| \ln\frac{1}{1-|\varphi(\lambda)|^2} \le \sup_{\lambda\in\Delta} \mu(|\lambda|)|\phi(\lambda)\varphi'(\lambda)| \ln\frac{4}{3} < \infty.$$

Hence, from (2.3), (2.6) and (2.7), we obtain (2.2).

(iii)⇒(i). Assume that (2.2) holds. Then, for every $f \in \mathcal{Z}$, we have

$$\begin{aligned}\mu(|z|)|(\phi DC_\varphi f)(z)| &= \mu(|z|)|\phi(z)\varphi'(z)f'(\varphi(z))| \\ &\leq C\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}}. \end{aligned} \tag{2.8}$$

Taking the supremum in (2.8) for $z \in \Delta$, and employing (2.2), we deduce that $\phi DC_\varphi : \mathcal{Z} \to H^\infty_\mu$ is bounded. The proof of Theorem 2.3 is completed.□

Theorem 2.4. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then the following statements are equivalent.*

(i) *$\phi DC_\varphi : \mathcal{Z} \to H^\infty_\mu$ is compact;*
(ii) *$\phi DC_\varphi : \mathcal{Z}_0 \to H^\infty_\mu$ is compact;*
(iii) *$\phi DC_\varphi : \mathcal{Z} \to H^\infty_\mu$ is bounded, and*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0. \tag{2.9}$$

Proof. (i)⇒(ii). This implication is clear.

(ii)⇒(iii). Assume that ϕDC_φ:$\mathcal{Z}_0 \to H^\infty_\mu$ is compact. Then it is clear that ϕDC_φ:$\mathcal{Z}_0 \to H^\infty_\mu$ is bounded. By Theorem 2.3 we know that ϕDC_φ:$\mathcal{Z}$ $\to H^\infty_\mu$ is bounded. Let $(z_n)_{n\in N}$ be a sequence in Δ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$ and $\varphi(z_n) \neq 0$, $n \in N$ (if such a sequence does not exist then (2.9) is vacuously satisfied). Set

$$h_n(z) = \frac{h(\overline{\varphi(z_n)}z)}{\overline{\varphi(z_n)}}(\ln \frac{1}{1-|\varphi(z_n)|^2})^{-1},\ n \in N. \tag{2.10}$$

Then from the proof of Theorem 2.3, we see that $h_n \in \mathcal{Z}_0$ for each $n \in N$. Moreover $h_n \to 0$ uniformly on compact subsets of Δ as $n \to \infty$ and

$$h_n'(\varphi(z_n)) = \ln \frac{1}{1-|\varphi(z_n)|^2}.$$

Since ϕDC_φ:$\mathcal{Z}_0 \to H^\infty_\mu$ is compact, by Lemma 2.1, we have

$$\lim_{n\to\infty} \|\phi DC_\varphi h_n\|_{H^\infty_\mu} = 0.$$

Hence,

$$\lim_{n\to\infty} \mu(|z_n|)|\phi(z_n)\varphi'(z_n)| \ln \frac{1}{1-|\varphi(z_n)|^2} = 0. \tag{2.11}$$

From (2.11) easily follows that $\lim_{n\to\infty} \mu(|z_n|)|\phi(z_n)\varphi'(z_n)| = 0$, which altogether imply (2.9).

(iii)⇒(i). Suppose that ϕDC_φ:$\mathcal{Z} \to H^\infty_\mu$ is bounded and that conditions (2.9) holds. From Theorem 2.3, we know that

$$C = \sup_{z\in\Delta} \mu(|z|)|\phi(z)\varphi'(z)| < \infty. \tag{2.12}$$

By the assumption, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \varepsilon, \tag{2.13}$$

whenever $\delta < |\varphi(z)| < 1$.

Assume that $(f_k)_{k\in N}$ is a sequence in $\mathcal{Z}$ such that $\sup_{k\in N} \|f_k\|_{\mathcal{Z}} \le L$ and f_k converges to 0 uniformly on compact subsets of Δ as $k \to \infty$. Let $K = \{z \in \Delta : |\varphi(z)| \le \delta\}$. Then by (2.12) and (2.13), we have that

$$\begin{aligned}
&\sup_{z\in\Delta} \mu(|z|)|(\phi DC_\varphi f_k)(z)| = \sup_{z\in\Delta} \mu(|z|)|\phi(z)\varphi'(z) f_k'(\varphi(z))| \\
&\le \sup_{z\in K} \mu(|z|)|\phi(z)\varphi'(z) f_k'(\varphi(z))| + \sup_{z\in\Delta\setminus K} \mu(|z|)|\phi(z)\varphi'(z) f_k'(\varphi(z))| \\
&\le \sup_{z\in K} \mu(|z|)|\phi(z)\varphi'(z) f_k'(\varphi(z))| \\
&+ C \sup_{z\in\Delta\setminus K} \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} \|f_k\|_{\mathcal{Z}} \\
&\le C \sup_{|\omega|\le\delta} |f_k'(\omega)| + C\varepsilon \|f_k\|_{\mathcal{Z}},
\end{aligned}$$

i.e., we obtain

$$\|\phi DC_\varphi f_k\|_{H^\infty_\mu} \le C \sup_{|\omega|\le\delta} |f_k'(\omega)| + C\varepsilon \|f_k\|_{\mathbb{Z}} + |\phi(0)||f_k'(\varphi(0))||\varphi'(0)|. \tag{2.14}$$

Since f_k converges to 0 uniformly on compact subsets of Δ as $k \to \infty$, Cauchy′s estimate gives that $f_k' \to 0$ as $k \to \infty$ on compact subsets of Δ. Hence, letting $k \to \infty$ in (2.14), and using the fact that ε is an arbitrary positive number, we obtain

$$\lim_{k\to\infty} \|\phi DC_\varphi f_k\|_{H^\infty_\mu} = 0.$$

Combining this with Lemma 2.1 the result easily follows. The proof of Theorem 2.4 is completed. □

Theorem 2.5. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then $\phi DC_\varphi : \mathcal{Z}_0 \to H^\infty_{\mu,0}$ is bounded if and only if $\phi DC_\varphi : \mathcal{Z}_0 \to H^\infty_\mu$ is bounded and*

$$\lim_{|z|\to 1} \mu(|z|)|\phi(z)\varphi'(z)| = 0. \tag{2.15}$$

Proof. Assume that $\phi DC_\varphi{:}\mathcal{Z}_0 \to H^\infty_{\mu,0}$ is bounded. Then, it is clear that $\phi DC_\varphi{:}\mathcal{Z}_0 \to H^\infty_\mu$ is bounded. Taking the test function $f(z) = z$, we obtain (2.15).

Conversely, assume that $\phi DC_\varphi{:}\mathcal{Z}_0 \to H^\infty_\mu$ is bounded and (2.15) holds. Then for each polynomial p, we have that

$$\mu(|z|)|(\phi DC_\varphi p)(z)| \le \mu(|z|)|\phi(z)\varphi'(z)p'(\varphi(z))|. \tag{2.16}$$

In view of the facts that

$$\sup_{\omega\in\Delta}|p'(\omega)|<\infty,$$

from (2.15) and (2.16), it follows that $\phi DC_\varphi p\in H^\infty_{\mu,0}$. Since the set of all polynomials is dense in $\mathcal{Z}_0$ (see [11]), we have that for every $f\in\mathcal{Z}_0$, there is a sequence of polynomials $(p_n)_{n\in N}$ such that $\|f-p_n\|_{\mathcal{Z}}\to 0$ as $n\to\infty$. Hence

$$\|\phi DC_\varphi f-\phi DC_\varphi p_n\|_{H^\infty_\mu}\le\|\phi DC_\varphi\|_{\mathcal{Z}_0\to H^\infty_\mu}\|f-p_n\|_{\mathcal{Z}}\to 0$$

as $n\to\infty$. Since the operator $\phi DC_\varphi{:}\mathcal{Z}_0\to H^\infty_\mu$ is bounded, so $\phi DC_\varphi(\mathcal{Z}_0)\subseteq H^\infty_{\mu,0}$, which implies the boundedness of $\phi DC_\varphi{:}\mathcal{Z}_0\to H^\infty_{\mu,0}$.

Theorem 2.6. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then the following statements are equivalent.*

(i) $\phi DC_\varphi:\mathcal{Z}\to H^\infty_{\mu,0}$ *is compact;*
(ii) $\phi DC_\varphi:\mathcal{Z}_0\to H^\infty_{\mu,0}$ *is compact;*
(iii)

$$\lim_{|z|\to 1}\mu(|z|)|\phi(z)\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|^2}=0. \tag{2.17}$$

Proof. (i)⇒(ii). This implication is trivial.

(ii)⇒(iii). Assume that $\phi DC_\varphi:\mathcal{Z}_0\to H^\infty_{\mu,0}$ is compact. Then $\phi DC_\varphi:\mathcal{Z}_0\to H^\infty_{\mu,0}$ is bounded. From the proof of Theorem 2.5, we know that

$$\lim_{|z|\to 1}\mu(|z|)|\phi(z)\varphi'(z)|=0. \tag{2.18}$$

Hence, if $\|\varphi\|_\infty<1$, from (2.18), we obtain that

$$\lim_{|z|\to 1}\mu(|z|)|\phi(z)\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|^2}\le\ln\frac{e}{1-\|\varphi\|^2_\infty}\lim_{|z|\to 1}\mu(|z|)|\phi(z)\varphi'(z)|=0,$$

from which the result follows in this case.

Now assume that $\|\varphi\|_\infty=1$. Let $(z_k)_{k\in N}$ be a sequence such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Since $\phi DC_\varphi:\mathcal{Z}_0\to H^\infty_\mu$ is compact, by Theorem 2.4,

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)|\phi(z)\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|^2}=0. \tag{2.19}$$

From (2.18) and (2.19), we have that for every $\varepsilon>0$, there exists an $r\in(0,1)$ such that

$$\mu(|z|)|\phi(z)\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|^2}<\varepsilon,$$

when $r < |\varphi(z)| < 1$, and there exists a $\sigma \in (0,1)$ such that

$$\mu(|z|)|\phi(z)\varphi'(z)| \leq \frac{\varepsilon}{\ln \frac{e}{1-r^2}},$$

when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \varepsilon. \tag{2.20}$$

On the other hand, if $\sigma < |z| < 1$ and $|\varphi(z)| \leq r$, we obtain

$$\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-r^2} < \varepsilon. \tag{2.21}$$

Inequality (2.20) together with (2.21) gives the (2.17).

(iii)⇒(i). Let $f \in \mathcal{Z}$. we have

$$\mu(|z|)|(\phi DC_\varphi f)(z)| \leq C\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}}.$$

Taking the supremum in this inequality over all $f \in \mathcal{Z}$ such that $\|f\|_{\mathcal{Z}} \leq 1$, then letting $|z| \to 1$, and using (2.17), we obtain that

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{Z}}\leq 1} \mu(|z|)|(\phi DC_\varphi f)(z)| = 0.$$

Using Lemma 2.2 we obtain that the operator ϕDC_φ:$\mathcal{Z} \to H^\infty_{\mu,0}$ is compact.□

Similarly to the proofs of Theorems 2.3-2.6, we can get the following results, we omit the proof.

Theorem 2.7. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then the following statements are equivalent.*

(i) *$\phi C_\varphi D : \mathcal{Z} \to H^\infty_\mu$ is bounded;*
(ii) *$\phi C_\varphi D : \mathcal{Z}_0 \to H^\infty_\mu$ is bounded;*
(iii)

$$\sup_{z\in\Delta} \mu(|z|)|\phi(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty.$$

Theorem 2.8. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then the following statements are equivalent.*

(i) *$\phi C_\varphi D : \mathcal{Z} \to H^\infty_\mu$ is compact;*
(ii) *$\phi C_\varphi D : \mathcal{Z}_0 \to H^\infty_\mu$ is compact;*
(iii) *$\phi C_\varphi D : \mathcal{Z} \to H^\infty_\mu$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|\phi(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0.$$

Theorem 2.9. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then $\phi C_\varphi D : \mathcal{Z}_0 \to H^\infty_{\mu,0}$ is bounded if and only if $\phi C_\varphi D : \mathcal{Z}_0 \to H^\infty_\mu$ is bounded and $\phi(z) \in H^\infty_{\mu,0}$.*

Theorem 2.10. *Let φ be an analytic self-map of Δ, and ϕ be an analytic function in Δ. Suppose that μ is normal. Then the following statements are equivalent.*

(i) *$\phi C_\varphi D : \mathcal{Z} \to H^\infty_{\mu,0}$ is compact;*
(ii) *$\phi C_\varphi D : \mathcal{Z}_0 \to H^\infty_{\mu,0}$ is compact;*
(iii)

$$\lim_{|z|\to 1} \mu(|z|)|\phi(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0.$$

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Jianren Long
Department of Mathematical Science, Guizhou Normal University, 550001, Guiyang, P.R. China.
E-mail address: longjianren2004@163.com

Congli Yang
Department of Mathematical Science, Guizhou Normal University, 550001, Guiyang, P.R. China.
E-mail address: yangcongli@gznu.edu.cn

Global stability in n-dimensional stochastic difference equations for predator-prey models

Sang-Mok Choo[a], Young-Hee Kim[*,b]

[a]*Department of Mathematics, University of Ulsan, Ulsan 44610, Korea.*
[b] *Ingenium College of Liberal Arts-Mathematics, Kwangwoon University, Seoul 01897, Korea*

Abstract

There are relatively few theoretical papers to consider the positivity of solutions of discrete time stochastic difference equations (DSDEs), compared to many publications on theoretical analysis of solutions of deterministic difference equations and stochastic differential equations. Additionally, no papers theoretically investigate the global stability of nontrivial solutions of n-dimensional DSDEs. In this paper, we consider the Euler-Maruyama scheme for n-dimensional stochastic difference equations that are a generalization of a two-dimensional model of stochastic predator-prey interactions, and show the positivity and the global stability of nontrivial solutions of the scheme by applying a new discretized version of the Itô formula. Numerical simulations are introduced to support the results.

Key words: Euler-Maruyama scheme, Positivity, Global stability, Stochastic difference equations.

1. Introduction

Stochastic differential equation (SDE) models have been increasingly used in a range of application areas, including biology, chemistry, mechanics, economics, and finance. In general, the exact solutions of SDEs are not known, so one has to numerically solve these SDEs. This leads us to consider and analyze discrete time stochastic difference equations (DSDEs), which can be also viewed as stochastically perturbed versions of deterministic difference equations (DDEs). There are many publications on estimations of the difference between solutions of SDEs and DSDEs. The global asymptotic stability of the trivial solution of DSDEs has been also widely addressed (see [1], [2], [3] and references therein). However, relatively few studies theoretically consider the positivity of solutions of DSDEs that are scalar equations on a finite time interval (see [4] references therein). In particular, to the best of our knowledge, there is no paper that theoretically deals with the global stability of nontrivial solutions of DSDEs, except [5], in which two-dimensional DSDEs are treated with a new discretized version of the Itô formula. Therefore, the aim of this paper is to extend the method used in [5] for investigating the positivity and the global stability of nontrivial solutions of n-dimensional DSDEs on an infinite time interval with stochastic predator-prey models.

*Corresponding author
Email addresses: smchoo@ulsan.ac.kr (Sang-Mok Choo), yhkim@kw.ac.kr (Young-Hee Kim)

We generalize the two dimensional predator-prey model in [6]

$$\begin{aligned} dx(t) &= x(t)\{r_1 - a_{11}x(t) - a_{12}y(t)\}dt + \sigma_1 x(t)dW_1(t), \\ dy(t) &= y(t)\{-r_2 + a_{21}x(t) - a_{22}y(t)\}dt + \sigma_2 y(t)dW_2(t), \end{aligned} \tag{1}$$

to the n-dimensional stochastic differential equations

$$dx^i(t) = x^i(t)\left(r_i + \sum_{j=1}^{i-1} a_{ij}x^j(t) - \sum_{j=i}^{n} a_{ij}x^j(t)\right)dt + \sigma_i x^i(t)dW_i(t), \tag{2}$$

where W_i are independent and real valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Although all the parameters r_i, a_{ij} and σ_i in (1) are positive, we weaken the conditions on the signs in (2) such that

$$r_i \in \mathbb{R}, a_{ii} > 0, a_{ij} \geq 0, \sigma_i > 0 \ (1 \leq i, j \leq n, i \neq j).$$

Consider the Euler-Maruyama scheme for (2)

$$x_{k+1}^i = x_k^i\left\{1 + \Delta t\left(r_i + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{n} a_{ij}x_k^j\right) + \sqrt{\Delta t}\sigma_i \xi_{k+1}^i\right\}, \tag{3}$$

where $1 \leq i \leq n$, $k \geq 0$, $x_0^i > 0$, $\Delta t = \frac{1}{N}$ for $N \in \mathbb{N}$, $t_k = k\Delta t$, and discrete Wiener processes $W_i(t_{k+1}) - W_i(t_k)$ are $\sqrt{\Delta t}\xi_{k+1}^i$ with a mutually independent and identically distributed sequence $(\xi_k^1, \cdots, \xi_k^n)_{k=1}^{\infty}$ of the standard normal random variables. The solutions of (3) are defined with respect to a complete, filtered probability space $(\Omega_N, \mathcal{F}_N, \{\mathcal{F}_k\}_{k=1}^{\infty}, \mathbb{P}_N)$, where $\{\mathcal{F}_k\}_{k=1}^{\infty}$ is the natural filtration generated by the stochastic sequence $(\xi_k^1, \cdots, \xi_k^n)_{k=1}^{\infty}$.

The positivity of the solutions of continuous time SDEs (1) is obtained in the infinite time interval $[0, \infty)$ without the assumption of boundedness of the noises $W_i(t)$ by using the concept of explosion time (see [7] and [6]). However, for obtaining the positivity of the solutions of discrete time DSDEs (3) in the infinite time interval, we restrict noises to bounded noises, which means that ξ_k^i are assumed to be truncated standard normal random variables with support $[-\varsigma, \varsigma]$ for a positive constant ς that satisfies

$$E(\xi_k^i) = 0, \ E\left[(\xi_k^i)^2\right] = 1 - \eta_\varsigma. \tag{4}$$

The positive value $\eta_\varsigma = \frac{2\varsigma\phi(\varsigma)}{\Phi(\varsigma) - \Phi(-\varsigma)}$ can be assumed to be sufficiently close to 0, where ϕ and Φ are the probability density and the cumulative distribution functions of the standard normal random variable, respectively. For example, when $\varsigma = 20$, we have $0 < \eta_\varsigma < 10^{-85}$.

The paper is organized as follows. Section 2 gives the positivity and the boundedness of the solutions of (3). In Section 3, we introduce the new discrete Itô formula developed in [5] by using a known discrete Itô formula for stochastic difference equations (see [8] [9] and [10]), which is the main tool for finding conditions for the global stability of the solutions of (3). Section 4 introduces auxiliary equations, the solutions of which are used for upper bounds of the solutions of (3). We show the global stability of the solutions of the two-dimensional model (3) in Section 5. The properties of the solutions of the auxiliary equations are used in Section 6 to find conditions for at least one of the solutions of (3) to converge to zero. In addition, the approaches in Section 5 are extended into the n-dimensional model (3) for finding conditions for the global stability the solutions of (3). Section 7 gives simulation results to confirm the results obtained in this paper.

2. Positivity and boundedness of solutions of DSDEs

In this section, we show the positivity and boundedness of solutions of the n-dimensional model (3) by applying the approach used in the DDE model (3) with $\sigma_1 = \sigma_2 = 0$ (see [11]). For simplicity, we use the notations for every constant a,

$$\hat{a} = a \cdot \Delta t,\ \tilde{a} = a \cdot \sqrt{\Delta t}$$

and

$$\mathbb{x}_k^i = (x_k^1, \cdots, x_k^{i-1}, x_k^{i+1}, \cdots, x_k^n).$$

Then the n-dimensional discrete model (3) can be written as

$$x_{k+1}^i = F_{\mathbb{x}_k^i}(x_k^i),$$

where

$$F_{\mathbb{x}_k^i}(x_k^i) = x_k^i \left(1 + \hat{r}_i + \sum\nolimits_{1\le j\le i-1} \hat{a}_{ij} x_k^j - \sum\nolimits_{i\le j\le n} \hat{a}_{ij} x_k^j + \tilde{\sigma}_i \xi_{k+1}^i\right). \tag{5}$$

For a vector $\boldsymbol{\zeta}_k^i = (\zeta_k^1, \cdots, \zeta_k^{i-1}, \zeta_k^{i+1}, \cdots, \zeta_k^n)$ of real numbers, define

$$V(\boldsymbol{\zeta}_k^i) = (2\hat{a}_{ii})^{-1} \left(1 + \hat{r}_i + \sum\nolimits_{j=1}^{i-1} \hat{a}_{ij} \zeta_k^j - \sum\nolimits_{j=i+1}^{n} \hat{a}_{ij} \zeta_k^j + \tilde{\sigma}_i \xi_{k+1}^i\right). \tag{6}$$

If $V(\boldsymbol{\zeta}_k^i) > 0$, then (5) gives

$$F_{\boldsymbol{\zeta}_k^i}(x) \text{ is increasing on } 0 \le x < V(\boldsymbol{\zeta}_k^i). \tag{7}$$

For $1 \le i \le n$, denote

$$\chi_i = \tilde{a}_{ii}^{-1} \left(\tilde{r}_i + \sum\nolimits_{1\le j\le i-1} \tilde{a}_{ij} \chi_j + \sigma_i \varsigma\right) \tag{8}$$

and assume that

$$\chi_i \le (2\hat{a}_{ii})^{-1} \left(1 + \hat{r}_i - \sum\nolimits_{i+1\le j\le n} \hat{a}_{ij} \chi_j - \tilde{\sigma}_i \varsigma\right), \tag{9}$$

$$\hat{r}_i + \sum\nolimits_{1\le j\le i-1} \hat{a}_{ij} \chi_j + \tilde{\sigma}_i \varsigma \le 1. \tag{10}$$

The model (3) is also assumed to satisfy the initial condition

$$(x_0^1, \cdots, x_0^n) \in \prod\nolimits_{1\le i\le n} (0, \chi_i). \tag{11}$$

Theorem 1. *Let x_k^i be the solutions of (3) and χ_i be defined in (8). Then*

$$(x_k^1, \cdots, x_k^n) \in \prod\nolimits_{1\le i\le n} (0, \chi_i),\ k \ge 0.$$

Proof. It follows from (11), (9) and (6) that for $1 \le i \le n$

$$0 < x_0^i < \chi_i < (2\hat{a}_{ii})^{-1} \left(1 + \hat{r}_i - \sum\nolimits_{i+1\le j\le n} \hat{a}_{ij} \chi_j - \tilde{\sigma}_i \varsigma\right) < V(\mathbb{x}_0^i).$$

Then letting $\boldsymbol{\zeta}_0^i = \mathbb{x}_0^i$ in (7) gives the positivity

$$x_1^i = F_{\mathbb{x}_0^i}(x_0^i) > F_{\mathbb{x}_0^i}(0) = 0,\ 1 \le i \le n.$$

Let $\omega \in \Omega_N$. If $\hat{r}_i + \sum_{1\le j\le i-1} \hat{a}_{ij}x_0^j(\omega) - \sum_{i\le j\le n} \hat{a}_{ij}x_0^j(\omega) + \tilde{\sigma}_i\xi_1^i(\omega) \le 0$, then

$$x_1^i(\omega) = F_{\aleph_0^i}(x_0^i)(\omega) \le x_0^i(\omega) < \chi_i,$$

and otherwise, we have $0 < x_0^i(\omega) < f(\aleph_0^i)(\omega)$ with

$$f(\aleph_0^i) = \tilde{a}_{ii}^{-1}\left(\tilde{r}_i + \sum\nolimits_{1\le j\le i-1} \tilde{a}_{ij}x_0^j - \sum\nolimits_{i+1\le j\le n} \tilde{a}_{ij}x_0^j + \sigma_i\xi_1^i\right).$$

Since $0 < f(\aleph_0^i) < V(\aleph_0^i)$ by (10), using both (7) with $\boldsymbol{\zeta}_k^i = \aleph_0^i$ and (8) with (11), we have

$$x_1^i(\omega) = F_{\aleph_0^i}(x_0^i)(\omega) < F_{\aleph_0^i}(f(\aleph_0^i))(\omega) = f(\aleph_0^i)(\omega) < \chi_i \ , 1 \le i \le n.$$

Therefore if $(x_0^1, \cdots, x_0^n) \in \prod_{1\le i\le n}(0, \chi_i)$, then

$$(x_1^1, \cdots, x_1^n) \in \prod_{1\le i\le n}(0, \chi_i),$$

and hence applying mathematical induction, we can complete the proof. □

Remark 1. Since (8) and (9) can be written as

$$\chi_i = (\Delta t)^{-0.5}{a_{ii}}^{-1}\left(\tilde{r}_i + \sum\nolimits_{1\le j\le i-1} \tilde{a}_{ij}\chi_j + \sigma_i\varsigma\right),$$
$$\chi_i \le (\Delta t)^{-1}(2a_{ii})^{-1}\left(1 + \hat{r}_i - \sum\nolimits_{i+1\le j\le n} \hat{a}_{ij}\chi_j - \tilde{\sigma}_i\varsigma\right),$$

the conditions (9) and (10) can be satisfied when taking small values of Δt. For example, take $n = 3$ in (3). The definition (8) gives $\chi_1 = \tilde{a}_{11}^{-1}(\tilde{r}_1 + \sigma_1\varsigma)$, $\chi_2 = \tilde{a}_{22}^{-1}(\tilde{r}_2 + \tilde{a}_{21}\chi_1 + \sigma_2\varsigma)$ and $\chi_3 = \tilde{a}_{33}^{-1}(\tilde{r}_3 + \tilde{a}_{31}\chi_1 + \tilde{a}_{32}\chi_2 + \sigma_3\varsigma)$. Let $\Delta t = 0.0001$, $\varsigma = 20$, $r_i = a_{ij} = \sigma_i = 1$ for $1 \le i, j \le 3$. Then the conditions (9) and (10) are satisfied.

3. A new discretized version of the Itô formula

In order to find conditions for the stability of the solutions of (3), we need a discretized form of the Itô formula. Although there are discretized versions of the Itô formula (see [8], [9] and [10]), we developed a variant which is suitable for our purpose in [5]. For the completeness of this paper, we include the proof of the new discretized version of the Itô formula in Appendix below.

We write $q_1(h) = O(q_2(h))$ (or $q_1(h) = O(q_2(h))$ for $h \to 0$ to be more precise) if there exist positive constants C and h_0 such that $|q_1(h)| \le C|q_2(h)|$ for all h with $0 < h \le h_0$. We make the following assumptions about the noise ξ:

(a) The noise ξ satisfies that for some C and μ with $0 < \mu < 1$

$$E(\xi) = 0,\ E\left(\xi^2\right) = 1 - \mu,\ E\left(|\xi|^\ell\right) \le C\ (\ell = 1, 3).$$

(b) The probability density function p exists and satisfies that for some C and all sufficiently large $|x|$

$$|x|^3 p(x) \le \frac{C}{|x|}.$$

The truncated Normal random variables satisfy both (a) and (b) with $\mu = \eta_\varsigma$ in (4). By $C^3(\mathbb{R})$, we denote the set of all functions defined on $\mathbb{R}$ that are continuously differentiable up to the order 3.

Lemma 1. *Consider functions* $\phi,\ \varphi : \mathbb{R} \to \mathbb{R}$ *satisfying for some* $\delta > 0$,

(i) $\varphi = \phi$ *on* $[1-\delta, 1+\delta]$.
(ii) $\varphi \in C^3(\mathbb{R})$ *and* $|\varphi'''(x)| \le C$ *for some* C *and all* $x \in \mathbb{R}$.
(iii) $\int_{\mathbb{R}} |\varphi(x) - \phi(x)| dx < \infty$.

Let ξ *be an* $\mathcal{F}$*-independent random variable satisfying (a) and (b). Let* f *and* g *be* $\mathcal{F}$*-measurable random variables satisfying that for some positive constants* ε *and* C,

$$\max\{h|f|, \sqrt{h}|g|\} \le Ch^{\varepsilon} \tag{12}$$

Then

$$E\left[\phi\left(1 + hf + \sqrt{h}g\xi\right)\middle|\mathcal{F}\right] = \phi(1) + \phi'(1)hf + \frac{1}{2}\phi''(1)hg^2 \cdot (1-\mu) + hfO\left(h^{\varepsilon}\right) + hg^2 O\left(h^{\varepsilon}\right).$$

Remark 2. For the solutions x_k^i of (3), let $f = r_i + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{n} a_{ij}x_k^j$ and $g = \sigma_i$ Then f and g satisfy (12) with $\varepsilon = 0.5$ due to $0 < x_k^i \le \chi_i = O(h^{-0.5})$ for $1 \le i \le n$ and $k \ge 0$.

Remark 3. In order to construct φ corresponding to the function

$$\phi(x) = \begin{cases} \ln|x| & (|x| > 0) \\ 0 & (x = 0) \end{cases},$$

we modify the function φ in [2]. Define the function φ as follows:

$$\varphi(x) = \begin{cases} \ln|x| & (|x| \ge e^{-1}) \\ -\frac{1}{4}e^4x^4 + e^2x^2 - \frac{7}{4} + \frac{e^6}{6}(x-e^{-1})^3(x+e^{-1})^3 & (|x| \le e^{-1}) \end{cases}.$$

Then φ satisfies all the conditions in Lemma 1.

Notation 1. For simplicity, we use the notations for $1 \le i \le n$ and $k > 0$,

$$\overline{E}(x_k^i) = \frac{1}{k}\sum_{s=0}^{k-1} E\left(x_s^i\right),$$

and for every constant a and η_ς in (4)

$$\mathring{a} = a \cdot \left\{1 + O(h^{0.5})\right\},\ a_\eta = a \cdot (1 - \eta_\varsigma),\ r_{i\sigma} = r_i - 0.5\sigma_{i\eta}^2.$$

Remark 4. Since the solutions x_{k+1}^i of (3) are positive, we can take logarithm of (3). Then applying Lemma 1 with (ϕ, φ) in Remark 3, the $\mathcal{F}_k$-independent and normally truncated random variable ξ_{k+1} and (f, g) in Remark 2, we have

$$E\left[\ln x_{k+1}^i \middle| \mathcal{F}_k\right] = E\left[\ln x_k^i \middle| \mathcal{F}_k\right] + E\left[\phi\left(1 + hf + \sqrt{h}g\xi_{k+1}\right)\middle|\mathcal{F}_k\right],$$

which gives

$$\begin{aligned} E(\ln x_{k+1}^i) &= \ln x_k^i + hf + \frac{1}{2}hg^2 \cdot (1 - \eta_\varsigma u) + hfO\left(h^{0.5}\right) + hg^2 O\left(h^{0.5}\right) \\ &= \ln x_k^i + \mathring{h}\left(r_i - \frac{1}{2}\sigma_{i\eta}^2 + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{n} a_{ij}x_k^j\right). \end{aligned} \tag{13}$$

Taking expectation of (13) and adding the result, we can obtain

$$E(\ln x_k^i) = E(\ln x_0^i) + k\mathring{h}\left(r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\overline{E}(x_k^j) - \sum_{j=i}^{n} a_{ij}\overline{E}(x_k^j)\right). \tag{14}$$

4. Auxiliary equations

In order to find upper bounds of x_k^i, we consider the auxiliary equations for $1 \le i \le n$ and $k \ge 0$

$$z_{k+1}^i = z_k^i\left(1 + \hat{r}_i + \sum\nolimits_{j=1}^{i-1} \hat{a}_{ij}z_k^j - \hat{a}_{ii}z_k^i + \tilde{\sigma}_i\xi_{k+1}^i\right),\ z_0^i = x_0^i. \tag{15}$$

Since (15) is the system (3) with $a_{ij} = 0$ for $1 \le i < j \le n$, Theorem 1 gives

$$(z_k^1, \cdots, z_k^n) \in \prod\nolimits_{1\le i\le n}(0, \chi_i),\ k \ge 0. \tag{16}$$

Let β_i be the solutions of the equations

$$r_{i\sigma} + \sum_{1\le j\le i-1} a_{ij}\beta_j - a_{ii}\beta_i = 0,\ 1 \le i \le n. \tag{17}$$

Note that (13) and (14) with $a_{1j} = 0$ $(2 \le j \le n)$ become

$$E\left(\ln z_{k+1}^1\right) = \ln z_k^1 + \mathring{\Delta}t\left(r_{1\sigma} - a_{11}z_k^1\right), \tag{18}$$

$$\begin{aligned} E\left(\ln z_k^1\right) &= E\left(\ln z_0^1\right) + k\mathring{\Delta}t\left\{r_{1\sigma} - a_{11}\overline{E}\left(z_s^1\right)\right\} \\ &= E\left(\ln z_0^1\right) + k\mathring{\Delta}ta_{11}\left\{\beta_1 - k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right)\right\}, \end{aligned} \tag{19}$$

due to $\beta_1 = r_{1\sigma}a_{11}^{-1}$ in (17).

The proofs of Lemma 2 and 3 were given in [5]. For the completeness of this paper, we write the proofs again.

Lemma 2. *Let z_k^1 and β_1 be the solutions of (15) and (17), respectively. If $\beta_1 > 0$, then for every $\epsilon > 0$ and some integer $N_\epsilon > 0$*

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right) \le \beta_1 + \epsilon,\ k \ge N_\epsilon.$$

Proof. Suppose that the theorem is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1} \sum\nolimits_{s=0}^{k_m-1} E\left(z_s^1\right) \geq \beta_1 + \varepsilon_0, \tag{20}$$

and for all k with $k \neq k_m$

$$k^{-1} \sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right) < \beta_1 + \varepsilon_0. \tag{21}$$

Combining (20) and (19), we have

$$\lim_{m\to\infty} E\left(\ln z_{k_m}^1\right) = -\infty. \tag{22}$$

Substituting (22) and the boundedness of z_k^1 into (18) gives

$$\lim_{m\to\infty} \ln z_{k_m-1}^1 = -\infty, a.s.$$

and then

$$\lim_{m\to\infty} z_{k_m-1}^1 = 0, \ a.s. \tag{23}$$

Thus the dominated convergence theorem with (16) leads to

$$\lim_{m\to\infty} E(z_{k_m-1}^1) = 0. \tag{24}$$

In order to obtain a contraction we show the following Claim 1 and 2.
Claim 1: For all sufficiently large k

$$k^{-1} \sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right) \geq \beta_1 + \varepsilon_0. \tag{25}$$

Assume that there exists $k = k_m - 1$ satisfying (21). The system of (20) and (21) becomes

$$\begin{aligned} \sum\nolimits_{s=0}^{k_m-1} E\left(z_s^1\right) &\geq k_m\left(\beta_1 + \varepsilon_0\right), \\ \sum\nolimits_{s=0}^{k_m-2} E\left(z_s^1\right) &< \left(k_m - 1\right)\left(\beta_1 + \varepsilon_0\right), \end{aligned}$$

which gives

$$E(z_{k_m-1}^1) > \beta_1 + \varepsilon_0. \tag{26}$$

Hence there exist finitely many k satisfying (21) due to (24) and (26).
Claim 2: As (20) implies (24), the equation (25) implies

$$\lim_{k\to\infty} E(z_k^1) = 0,$$

which is contradictive to (25) due to $\beta_1 + \varepsilon_0 > 0$. This contradiction completes the proof. □

Lemma 3. *Let (z_k^1, z_k^2) and (β_1, β_2) be the solutions of (15) and (17), respectively.*

(a) *If $r_{i\sigma} < 0$ $(i = 1, 2)$, then $\lim_{k\to\infty} z_k^i = 0$ $(i = 1, 2)$, a.s.*

(b) *Assume $r_{1\sigma} > 0$. Then $\lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$.*
 (i) *If $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} z_k^2 = 0$, a.s.*

(ii) *If* $r_{2\sigma} + a_{21}\beta_1 > 0$, *then* $\lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(z_s^2\right) = \beta_2$.

Proof. (a) Since $r_{1\sigma} < 0$ is equivalent to $\beta_1 < 0$, the equation (19) with the positivity of z_k^1 gives that if $r_{1\sigma} < 0$, then $\lim_{k\to\infty} E\left(\ln z_k^1\right) = -\infty$ and so we have $\lim\limits_{k\to\infty} z_k^1 = 0,\ a.s.$ Hence the dominated convergence theorem yields $\lim\limits_{k\to\infty} E(z_k^1) = 0$, which implies

$$\lim_{k\to\infty} \overline{E}(z_k^1) = 0. \tag{27}$$

It remains to show that $\lim_{k\to\infty} z_k^2 = 0,\ a.s.$ Using (13) and (14) with $a_{2j} = 0$ $(3 \le j \le n)$, we have

$$\begin{aligned} E\left(\ln z_{k+1}^2\right) &= \ln z_k^2 + \mathring{\Delta}t\left(r_{2\sigma} + a_{21}z_k^1 - a_{22}z_k^2\right), & (28)\\ E\left(\ln z_k^2\right) &= E\left(\ln z_0^2\right) + k\mathring{\Delta}t\left\{r_{2\sigma} + a_{21}\overline{E}(z_k^1) - a_{22}\overline{E}(z_k^2)\right\} \\ &= E\left(\ln z_0^2\right) + k\mathring{\Delta}t a_{22}\left[a_{22}^{-1}\left\{r_{2\sigma} + a_{21}\overline{E}(z_k^1)\right\} - \overline{E}(z_k^2)\right] & (29) \end{aligned}$$

Combining (27) and (29) with $r_{2\sigma} < 0$ and the positivity of z_k^2, we have

$$\lim_{k\to\infty} E\left(\ln z_k^2\right) = -\infty$$

Therefore, as (22) implies (23), we can obtain that if $r_{2\sigma} < 0$, then $\lim_{k\to\infty} z_k^2 = 0,\ a.s.$
(b) Assume $r_{1\sigma} > 0$, which gives $\beta_1 = r_{1\sigma}a_{11}^{-1} > 0$.
Due to Lemma 2, it is enough to prove that for all $\epsilon > 0$ there exists an integer N_ϵ satisfying

$$\beta_1 - \epsilon \le k^{-1} \sum_{s=0}^{k-1} E\left(z_s^1\right),\ k \ge N_\epsilon. \tag{30}$$

Suppose that (30) is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ such that

$$\beta_1 - \varepsilon_0 > k_m^{-1} \sum_{s=0}^{k_m-1} E\left(z_s^1\right).$$

Then the boundedness of z_k^1 and (19) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^1\right) > E\left(\ln z_0^1\right) + k_m\mathring{\Delta}t a_{11}\varepsilon_0,$$

which is a contradiction. Therefore (30) is true and then Lemma 2 gives

$$\lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1. \tag{31}$$

(b)-(i) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.
Applying (31) to (29) with $r_{2\sigma} + a_{21}\beta_1 < 0$ and the positivity of z_k^2, we have

$$\lim_{k\to\infty} E(\ln z_k^2) = -\infty.$$

Therefore, as (22) implies (23), we can obtain $\lim_{k\to\infty} z_k^2 = 0,\ a.s.$.
(b)-(ii) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 > 0$ and then $\beta_2 = a_{22}^{-1}\left(r_{2\sigma} + a_{21}\beta_1\right) > 0$.

Following the proof of Lemma 2, we can obtain that for every $\epsilon > 0$ and some integer $N_\epsilon > 0$

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^2\right) \le \beta_2 + \epsilon,\ k \ge N_\epsilon, \tag{32}$$

by replacing z_k^1, β_1, (18) and (19) in Lemma 2 with z_k^2, β_2, (28) and (29), respectively, and using (31).
On the other hand, following the proof of (30) with (29) instead of (19), we can obtain that for all $\epsilon > 0$ there exists an integer N_ϵ satisfying

$$\beta_2 - \epsilon \le k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^2\right),\ k \ge N_\epsilon. \tag{33}$$

Combining (32) and (33), we obtain $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = \beta_2$. □

The proofs of Remark 5 and 6 are given in Appendix below.

Remark 5. Using the idea in Lemma 3, we can find conditions under which the solution z_k^3 of (15) converges. Let $\tilde{\beta}_3 = a_{33}^{-1}(r_{3\sigma} + a_{31}\beta_1)$, which is equal to β_3 when $\beta_2 = 0$.

(a) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.
 (i) If $r_{3\sigma} + a_{31}\beta_1 < 0$, then $\lim\limits_{k\to\infty} z_k^3 = 0,\ a.s.$
 (ii) If $r_{3\sigma} + a_{31}\beta_1 > 0$, then $\lim\limits_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^3\right) = \tilde{\beta}_3$.

(b) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 > 0$.
 (i) If $r_{3\sigma} + \sum_{j=1}^{2} a_{3j}\beta_j < 0$, then $\lim\limits_{k\to\infty} z_k^3 = 0,\ a.s.$
 (ii) If $r_{3\sigma} + \sum_{j=1}^{2} a_{3j}\beta_j > 0$, then $\lim\limits_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^3\right) = \beta_3$.

Remark 6. Replacing (3) with (15), the equation (14) becomes

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{\Delta}t\left\{r_{i\sigma} + \sum\nolimits_{j=1}^{i-1} a_{ij}\overline{E}(z_k^j) - a_{ii}\overline{E}(z_k^i)\right\}. \tag{34}$$

Substituting (17) to (34) yields

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{\Delta}t\left[\sum\nolimits_{j=1}^{i-1} a_{ij}\left\{\overline{E}(z_k^j) - \beta_j\right\} - a_{ii}\left\{\overline{E}(z_k^i) - \beta_i\right\}\right], \tag{35}$$

so that we can extend (b)-(ii) in both Lemma 3 and Remark 5 to the n-dimensional case: If $r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0$ $(1 \le i \le n)$, then $\lim_{k\to\infty} \overline{E}(z_k^i) = \beta_i$ $(1 \le i \le n)$ and, as a result, (35) yields

$$\lim_{k\to\infty} \frac{1}{k} E(\ln z_k^i) = 0,\ 1 \le i \le n. \tag{36}$$

Lemma 4. *Let x_k^i and z_k^i be the solutions of (3) and (15), respectively. Then*

$$0 < x_k^i \le z_k^i,\ 1 \le i \le n,\ k \ge 0.$$

Proof. Let $\mathbf{0}_0^k$ be the zero vector of $n-1$ entries for $k \ge 0$. Since $\varkappa_0^1 > 0$ and $F_{\boldsymbol{\zeta}_0^1}(x)$ in (5) is decreasing in $\boldsymbol{\zeta}_0^1$, we have

$$x_1^1 = F_{\varkappa_0^1}(x_0^1) \le F_{\mathbf{0}_0^1}(x_0^1). \tag{37}$$

Using (6) and (9), we have

$$0 < x_0^1 \le z_0^1 \le \chi_1 \le V(\mathbf{0}_0^1)$$

and then (7) yields

$$F_{\mathbf{0}_0^1}(x_0^1) \le F_{\mathbf{0}_0^1}(z_0^1) = z_1^1. \tag{38}$$

Hence combining (37) and (38) gives

$$x_1^1 \le z_1^1. \tag{39}$$

Note that for $k \ge 1$

$$\not{x}_k^1 > 0,\ z_k^1 \le \chi_1 \le V(\mathbf{0}_0^k) \tag{40}$$

by Theorem 1, (16), (6), (9). Following the proof of (39), we can obtain

$$x_k^1 \le z_k^1\ (k \ge 0)$$

by using mathematical induction with (40) and $\mathbf{0}_0^k$.
Similarly, letting $\boldsymbol{z}_0^2 = (z_0^1, 0, \cdots, 0)$ and using $0 < x_0^2 \le z_0^2 \le \chi_2 \le V(\boldsymbol{z}_0^2)$, we have

$$x_1^2 = F_{\not{x}_0^2}(x_0^2) \le F_{\mathbf{0}_0^2}(x_0^2) \le F_{\mathbf{0}_0^2}(z_0^2) = z_1^2. \tag{41}$$

Hence mathematical induction with (41) and $\boldsymbol{z}_{k-1}^2 = (z_{k-1}^1, 0, \cdots, 0)$ gives

$$x_k^2 \le z_k^2\ (k \ge 0).$$

Therefore, using mathematical induction with $\boldsymbol{z}_k^i = (z_k^1, \cdots, z_k^{i-1}, 0, \cdots, 0)$, we can complete the proof. □

Remark 7. If $r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0$ $(1 \le i \le n)$, then both Lemma 4 and (36) imply that for every $\epsilon > 0$ there exists an integer N_ϵ such that

$$\frac{1}{k} E(\ln x_k^i) \le \epsilon,\ k \ge N_\epsilon, \tag{42}$$

which will be used in Section 5 and 6 to show that all the solutions of (3) converge to positive values.

5. Global stability of the n-dimensional DSDEs

In this section, we first find conditions under which at least one of the solutions of (3) converges to zero, a.s. and then another conditions under which all $\overline{E}(x_k^i)$ $(1 \le i \le n)$ converge to positive values. The extension of the approach used for the global stability of the two-dimensional DSDEs can lead us to the global stability of the n-dimensional DSDEs, so that we again treat the two-dimensional DSDEs in [5].

5.1. Global stability of the two-dimensional DSDEs

In this section, we consider the two-dimensional model (3) with $n = 2$

$$\begin{aligned} x_{k+1}^1 &= x_k^1(1 + \hat{r}_1 - \hat{a}_{11}x_k^1 - \hat{a}_{12}x_k^2 + \tilde{\sigma}_1\xi_{k+1}^1), \\ x_{k+1}^2 &= x_k^2(1 + \hat{r}_2 + \hat{a}_{21}x_k^1 - \hat{a}_{22}x_k^2 + \tilde{\sigma}_2\xi_{k+1}^2). \end{aligned} \tag{43}$$

Define A_2 and D_2^i $(i = 1, 2)$ by

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix} \text{ and } \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = A_2 \begin{pmatrix} D_2^1 \\ D_2^2 \end{pmatrix}. \tag{44}$$

Theorem 2. *Let x_k^i and β_i be the solutions of (43) and (17), respectively.*

(a) *If $r_{i\sigma} < 0$ $(i = 1, 2)$, then $\lim_{k\to\infty} x_k^i = 0$ $(i = 1, 2), a.s.$*
(b) *If $r_{1\sigma} > 0,\ r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim\limits_{k\to\infty} \overline{E}(x_k^1) = \beta_1$ and $\lim\limits_{k\to\infty} x_k^2 = 0,\ a.s.$*
(c) *If $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 > 0$, then $\lim_{k\to\infty} \overline{E}(x_k^i) = D_2^i$ $(i = 1, 2)$, where D_2^i is defined in (44).*

Proof. (a) The proof is followed by applying Lemma 3-(a) and Lemma 4.
(b) From Lemma 3-(b)-(i) and Lemma 4, we have $\lim_{k\to\infty} x_k^2 = 0,\ a.s.$ which gives that for every $\varepsilon > 0$, there exists an integer N_ε satisfying

$$0 < x_k^2 \le \varepsilon,\ k \ge N_\varepsilon. \tag{45}$$

Consider the system

$$\begin{aligned} u_{k+1}^L &= u_k^L(1 + \hat{r}_1 - \hat{a}_{11}u_k^L - \hat{a}_{12}\varepsilon + \tilde{\sigma}_1\xi_{k+1}^1),\ u_0^L = x_0^1, \\ u_{k+1}^U &= u_k^U(1 + \hat{r}_1 - \hat{a}_{11}u_k^U + \hat{a}_{12}\varepsilon + \tilde{\sigma}_1\xi_{k+1}^1),\ u_0^U = x_0^1. \end{aligned}$$

Following the proofs of both Lemma 4 and (45), we can have

$$0 < u_k^L \le x_k^1 \le u_k^U \ \text{ for } k \ge N_\varepsilon,$$

and then due to (31), we can obtain

$$\lim_{k\to\infty} \frac{1}{k} \sum_{s=0}^{k-1} E\left(u_s^L\right) = \frac{r_{1\sigma} - a_{12}\varepsilon}{a_{11}},\ \lim_{k\to\infty} \frac{1}{k} \sum_{s=0}^{k-1} E\left(u_s^U\right) = \frac{r_{1\sigma} + a_{12}\varepsilon}{a_{11}},$$

which gives the desired result.
(c) Let $|A_2|$ be the determinant of A_2 in (44), which is positive. Then

$$\begin{pmatrix} D_2^1 \\ D_2^2 \end{pmatrix} = A_2^{-1} \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = |A_2|^{-1} \begin{pmatrix} a_{22}r_{1\sigma} - a_{12}r_{2\sigma} \\ a_{11}\left(a_{21}\beta_1 + r_{2\sigma}\right) \end{pmatrix} \tag{46}$$

and hence the conditions in (c) imply $D_2^i > 0$ for $i = 1, 2$.
Applying (44) to (14) with $n = 2$ gives

$$\begin{pmatrix} E\left(\ln x_k^1\right) \\ E\left(\ln x_k^2\right) \end{pmatrix} = \begin{pmatrix} E\left(\ln x_0^1\right) \\ E\left(\ln x_0^2\right) \end{pmatrix} + k\mathring{\Delta}tA_2 \begin{pmatrix} D_2^1 - \overline{E}\left(x_k^1\right) \\ D_2^2 - \overline{E}\left(x_k^2\right) \end{pmatrix}, \tag{47}$$

and multiplying the matrix $|A_2|A_2^{-1}$ to (47), we have

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) = C_1 + k\mathring{\Delta}t|A_2|\left(D_2^1 - \overline{E}\left(x_k^1\right)\right), \tag{48}$$

$$a_{21}E(\ln x_k^1) + a_{11}E(\ln x_k^2) = C_2 + k\mathring{\Delta}t|A_2|\left(D_2^2 - \overline{E}\left(x_k^2\right)\right), \tag{49}$$

where $C_1 = a_{22}E(\ln x_0^1) - a_{12}E(\ln x_0^2)$ and $C_2 = a_{21}E(\ln x_0^1) + a_{11}E(\ln x_0^2)$.
Since $|A_2| > 0,\ D_2^1 > 0$ in (48), and $-a_{12}E(\ln x_k^2) > -\infty$ from Theorem 1, we can follow the proof of Lemma 2 with (48) instead (19) and, as a result, obtain that for every $\epsilon_2^1 > 0$ there exists an integer $N_{\epsilon_2^1}$ such that

$$\overline{E}(x_k^1) \le D_2^1 + \epsilon_2^1,\ k \ge N_{\epsilon_2^1}. \tag{50}$$

Substituting (42) into (49) gives that for every $\epsilon_2^2 > 0$ there exists an integer $N_{\epsilon_2^2}$ satisfying

$$\overline{E}(x_k^2) \ge D_2^2 - \epsilon_2^2,\ k \ge N_{\epsilon_2^2}. \tag{51}$$

Applying (50) to the second equation in (47), we have for $k \ge N_{\epsilon_2^1}$

$$E(\ln x_k^2) \le E(\ln x_0^2) + k\mathring{\Delta}t a_{22}\left\{a_{22}^{-1}a_{21}\epsilon_2^1 + D_2^2 - \overline{E}(x_k^2)\right\}. \tag{52}$$

Instead of (19) and (18), using (52) and (14) with $n = i = 2$ and $D_2^2 > 0$, and following the proof of Lemma 2, we can obtain that for every $\epsilon_2'^2 > 0$ there exists an integer $N_{\epsilon_2'^2}$ such that

$$\overline{E}(x_k^2) \le D_2^2 + \epsilon_2'^2,\ k \ge N_{\epsilon_2'^2}. \tag{53}$$

Hence (51) and (53) imply

$$\lim_{k\to\infty} \overline{E}(x_k^2) = D_2^2. \tag{54}$$

It remains to show $\lim_{k\to\infty} \overline{E}(x_k^1) = D_2^1$. Applying (54) to (49) with (42) yields

$$\lim_{k\to\infty} \frac{1}{k}E(\ln x_k^1) = \lim_{k\to\infty} \frac{1}{k}E(\ln x_k^2) = 0,$$

with which (48) gives the desired result. □

Remark 8. Assume $r_{1\sigma} < 0$. Then it follows from Lemma 3-(a) and Lemma 4 that $\lim_{k\to\infty} x_k^1 = 0, a.s.$ In addition, if $r_{2\sigma} > 0$, then we can have

$$\lim_{k\to\infty} \overline{E}(x_k^2) = a_{22}^{-1}r_{2\sigma}$$

by following the proof of Theorem 2-(b) with $\lim_{k\to\infty} x_k^1 = 0, a.s.$, instead of $\lim_{k\to\infty} x_k^2 = 0, a.s.$

Remark 9. Since $|A_2| > 0$, the identity (46) gives that the two conditions in Theorem 2-(c) are equivalent to $D_2^1 > 0$ and $D_2^2 > 0$. This equivalence with $|A_2| > 0$ is used to find conditions for the global stability of solutions of the n-dimensional model (3) in the next section.

5.2. Model reduction

Using Lemma 3, Remark 5 and Lemma 4, we can find conditions under which at least one of the solutions of (3) converges to zero, a.s. For example, the three-dimensional model (3) with $n = 3$ can be reduced to the two-dimensional model (3) and hence we can apply Theorem 2 as follows.

Theorem 3. *Let x_k^i and β_i be the solutions of (3) and (17) with $n = 3$.*

(a) *If $r_{1\sigma} < 0$, then $\lim_{k\to\infty} x_k^1 = 0$, a.s.*

(b) *Assume $r_{1\sigma} > 0$.*

(i) *Let $r_{i\sigma}+a_{i1}\beta_i < 0$ $(i = 2, 3)$. Then $\lim\limits_{k\to\infty} x_k^i = 0$, a.s. $(i = 2, 3)$ and $\lim_{k\to\infty} \overline{E}(x_k^1) = \beta_1$.*

(ii) *Let $r_{2\sigma} + a_{21}\beta_1 < 0$ and $r_{3\sigma} + a_{31}\beta_1 > 0$. Then $\lim\limits_{k\to\infty} x_k^2 = 0$, a.s.*
Furthermore, if $r_{1\sigma} > a_{33}^{-1}a_{13}r_{3\sigma}$, then $\lim\limits_{k\to\infty} \left(\overline{E}(x_k^1), \overline{E}(x_k^3)\right) = (\check{D}_2^1, \check{D}_2^3)$.

(iii) *Let $r_{2\sigma} + a_{21}\beta_1 > 0$ and $r_{3\sigma} + \sum\limits_{i=1}^{2} a_{3i}\beta_i < 0$. Then $\lim\limits_{k\to\infty} x_k^3 = 0$, a.s.*
Furthermore, if $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$, then $\lim\limits_{k\to\infty} \left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right) = (D_2^1, D_2^2)$.

Here $(\check{D}_2^1, \check{D}_2^3)$ is equal to (D_2^1, D_2^2) in (44) when a_{12}, a_{21} and a_{22} are replaced with a_{13}, a_{31} and a_{33}, respectively.

Remark 10. Assume $r_{1\sigma} < 0$, which gives $\lim_{k\to\infty} x_k^1 = 0, a.s.$ by Lemma 3-(a) and Lemma 4. Then the three-dimensional model (3) can be considered as the two-dimensional model (3) with two state variables x_k^2, x_k^3 and $a_{21} = a_{31} = 0$. Therefore, we can apply Theorem 2 and Remark 8.

Remark 11. Assume that $r_3 < 0$ and $a_{31} > 0$ in Theorem 3-(ii). Then $r_{3\sigma} < 0$ and hence $r_{3\sigma} + a_{31}\beta_1 = r_{3\sigma} + a_{31}a_{11}^{-1}r_{1\sigma} > 0$ gives $r_{1\sigma} > 0 \geq a_{33}^{-1}a_{13}r_{3\sigma}$. Therefore Theorem 3-(ii) is satisfied without using the condition $r_{1\sigma} > a_{33}^{-1}a_{13}r_{3\sigma}$ if $r_3 < 0$. Similarly, if $r_2 < 0$ and $a_{21} > 0$, then Theorem 3-(iii) is satisfied without the condition $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$.

5.3. Convergence of all state variables to nonzero values

Now, we find conditions under which $\overline{E}(x_k^i)$ $(1 \leq i \leq n)$ converge to positive values D_n^i. The conditions are extensions of the following three conditions used in the proof of Theorem 2-(c):

(A1) $|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} + a_{12}a_{21} > 0.$

(A2) $D_2^i > 0,\ i = 1, 2$ (see Remark 9).

(A3) The system of equations (48) and (49) has the sign-pattern matrix

$$\operatorname{sgn}\begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} + & -_0 \\ +_0 & + \end{pmatrix},$$

which means that the signs of $a_{22}, -a_{12}, a_{21}$, and a_{11} are positive, non-positive, non-negative, and positive, respectively.

Define A_n and D_n^i, extensions of (44), by

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ -a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n(n-1)} & a_{nn} \end{pmatrix}, \quad \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \\ \vdots \\ r_{n\sigma} \end{pmatrix} = A_n \begin{pmatrix} D_n^1 \\ D_n^2 \\ \vdots \\ D_n^n \end{pmatrix}, \tag{55}$$

with the assumptions corresponding to (A1) and (A2):

(A1′) $|A_n| > 0$.
(A2′) $D_n^i > 0$ for $1 \le i \le n$.

Using (55), we can write (14) as

$$E(\ln x_k^i) = E(\ln x_0^i) + k\mathring{\Delta}t \left\{ -\sum_{j=1}^{i-1} a_{ij} \left(D_n^j - \overline{E}(x_k^j) \right) + \sum_{j=i}^{n} a_{ij} \left(D_n^j - \overline{E}(x_k^j) \right) \right\},$$

which is corresponding to (47). Hence (48) and (49) are extended as

$$\sum\nolimits_{1\le j\le n} C_{ji} E(\ln x_k^j) = \sum\nolimits_{1\le j\le n} C_{ji} E(\ln x_0^j) + k\mathring{\Delta}t |A_n| \left\{ D_n^i - \overline{E}(x_k^i) \right\}, \tag{56}$$

where cofactors C_{ij} of A_n satisfy the assumption corresponding to $(A3)$:

$$\text{(A3′)} \ \operatorname{sgn} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix} = \begin{pmatrix} + & -_0 & \cdots & -_0 & -_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -_0 & -_0 & \cdots & + & -_0 \\ +_0 & +_0 & \cdots & +_0 & + \end{pmatrix}.$$

Remark 12. It follows from (17) and (55) that (A2′) gives

$$\begin{aligned} a_{11}\beta_1 &= r_{1\sigma} = \sum\nolimits_{1\le j\le n} a_{1j} D_n^j \ge a_{11} D_n^1 > 0, \\ a_{22}\beta_2 &= r_{2\sigma} + a_{21}\beta_1 \ge r_{2\sigma} + a_{21} D_n^1 = \sum\nolimits_{2\le j\le n} a_{2j} D_n^j \ge a_{22} D_n^2. \end{aligned}$$

Repeating this process, we can conclude that (A2′) implies $r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0$ $(1 \le i \le n)$, and then we can use (42) in the proof of the following theorem.

Now we can extend the proof of Theorem 2-(c) to the n-dimensional model (3).

Theorem 4. *Let x_k^i be the solutions of (3) and D_n^i be defined in (55). Assume that (A1′)–(A3′) are satisfied. Then*

$$\lim_{k\to\infty} \frac{1}{k} \sum\nolimits_{s=0}^{k-1} E\left(x_s^i\right) = D_n^i, \ 1 \le i \le n.$$

Proof. Instead of (48), using (56) for $1 \le i \le n-1$ and following the proof of (50) with (A1′) and (A2′), we can obtain that for every $\epsilon_n^i > 0$ there exists an integer $N_{\epsilon_n^i}$ such that

$$\overline{E}(x_k^i) \le D_n^i + \epsilon_n^i, \ 1 \le i \le n-1, \ k \ge N_{\epsilon_n^i}. \tag{57}$$

Substituting (42) into (56) with $i = n$ yields that for every $\epsilon_n^n > 0$ there exists an integer $N_{\epsilon_n^n}$ such that

$$\overline{E}(x_k^n) \geq D_n^n - \epsilon_n^n,\ k \geq N_{\epsilon_n^n}. \tag{58}$$

As in the proof of (53), we can have that for every $\epsilon_n'^n > 0$ there exists an integer $N_{\epsilon_n'^n}$ such that

$$\overline{E}(x_k^n) \leq D_n^n + \epsilon_n'^n,\ k \geq N_{\epsilon_n'^n} \tag{59}$$

by replacing both (50) and the second equation in (47) with (57) and

$$E(\ln x_k^n) = E(\ln x_0^n) + k\mathring{\Delta}t \left\{ \sum_{j=1}^{n-1} a_{nj} \left(\overline{E}(x_k^j) - D_n^j \right) - a_{nn} \left(\overline{E}(x_k^n) - D_n^n \right) \right\},$$

respectively. Hence (58) and (59) give

$$\lim_{k\to\infty} \overline{E}(x_k^n) = D_n^n. \tag{60}$$

Now it remains to show $\lim_{k\to\infty} \overline{E}(x_k^i) = D_n^i$ $(1 \leq i \leq n-1)$. Substituting (60) to (56) with $i = n$ and using both $C_{in} \geq 0$ $(1 \leq i \leq n)$ in (A3′) and (42), we can obtain

$$\lim_{k\to\infty} \frac{1}{k} E(\ln x_k^i) = 0,\ 1 \leq i \leq n-1,$$

with which (56) gives the desired result. □

Remark 13. Following the proof of Lemma 1 under the assumptions about the noise ξ, we can obtain the new discretized Itô formula for the Milstein method

$$\begin{aligned} &E\left[\phi\left(1 + hf + \sqrt{h}g\xi + \frac{1}{2}hg^2 \cdot (\xi^2 - 1) \right) \middle| \mathcal{F} \right] \\ &= \phi(1) + \phi'(1)\left(hf - \frac{1}{2}hg^2\mu \right) + \frac{1}{2}\phi''(1)hg^2 \cdot (1-\mu) + h(f+g^2)O\left(h^{\varepsilon}\right), \end{aligned}$$

which gives

$$\begin{aligned} &E\left[\ln\left(1 + hf + \sqrt{h}g\xi + 0.5hg^2 \cdot (\xi^2 - 1) \right) \middle| \mathcal{F} \right] \\ &= h\left\{ \left(f - 0.5g^2 \right) + (f+g^2)O\left(h^{\varepsilon}\right) \right\}. \end{aligned}$$

Therefore, replacing $r_{i\sigma} = r_i - 0.5\sigma_{i\eta}^2$ and $\sqrt{\Delta t}\sigma_i\xi_k^i$ with $r_{i\sigma} = r_i - 0.5\sigma_i^2$ and $\sqrt{\Delta t}\sigma_i\xi_k^i + 0.5\Delta t\sigma_i^2\left\{(\xi_k^i)^2 - 1\right\}$ for $1 \leq i \leq n$, respectively, we can conclude that the solutions of the Milstein scheme for (2) satisfy all the results in this paper.

6. Numerical examples

In this section, we provide some simulations that illustrate our results in Theorem 3 and 4 for the three-dimensional model (3) with $\Delta t = 0.001$ and $\varsigma = 20$ in (4). Let $r_1 = 1.1, a_{11} = 2.1, a_{12} = 0.2, a_{13} = 0.1, a_{21} = 0.5, a_{22} = 1.1, a_{23} = 0.1, a_{31} = 3.1, a_{32} = 5.1, a_{33} = 0.5$ and $\sigma_2 = \sigma_3 = 0.1$ in the model (3) with $n = 3$. The model is simulated 1000 and 5000 times in Figure 1 and 2, respectively, for calculating the values of expectations.

Example 1. Figure 1-(a) denotes three plots in the first column of Figure 1. Each figure in the ith row of Figure 1 is the sequence of the 1000 realizations of x_k^i $(i = 1, 2, 3)$. In Figure 1-(a), we let $(-r_2, -r_3) = (0.1, 2.1)$ and $\sigma_1 = 2$. The values $(-r_2, -r_3)$ in Figure 1-(b), (c) and (d) are $(5.1, 5.1), (5.1, 1.1), (0.1, 5.1)$, respectively, with $\sigma_1 = 0.1$. Then the values of parameters in Figure 1-(a), (b), (c) and (d) satisfy the conditions in Theorems 3-(a), (b)-(i), (ii) and (iii), respectively, and Figure 1 demonstrates Theorem 3: For example, all x_k^i $(i = 1, 2, 3)$ in Figure 1-(a) converge to zero. At $k = 2 \cdot 10^6$, which means time 2000, the errors $|\overline{E}(x_k^1) - a_{11}^{-1} r_{1\sigma}|$ in Figure 1-(b), $|\overline{E}(x_k^i) - \check{D}_2^i|$ $(i = 1, 3)$ in Figure 1-(c), and $|\overline{E}(x_k^i) - D_2^i|$ $(i = 1, 2)$ in Figure 1-(d) are $0.00012, 0.000075, 0.0019, 0.000038$ and 0.00068, respectively.

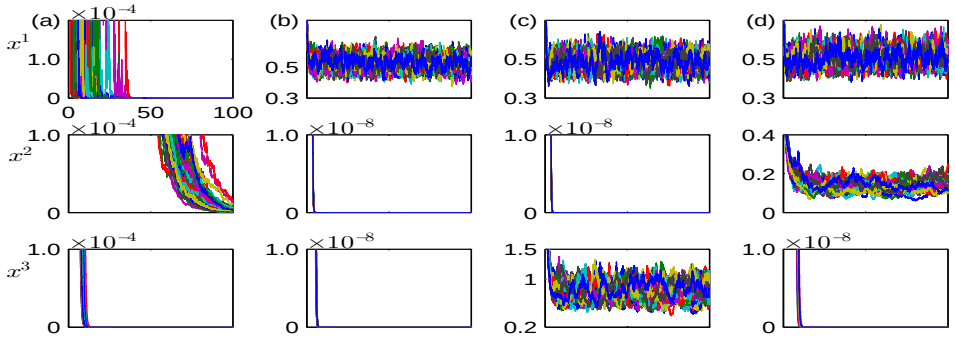


Figure 1: All the x-axes denote time $k\Delta t$ from 0 to 100. The curves in each figure are 1000 realizations of one of the solutions x_k^i $(1 \le i \le 3)$ of DSDEs. The solutions x_k^i $(1 \le i \le 3)$ in (a), x_k^i $(i = 2, 3)$ in (b), x_k^2 in (c), x_k^3 in (d) are all convergent to zero.

Example 2. Let $r_2 = -0.1, r_3 = -2.1$ and $\sigma_1 = 0.1$. Then all the conditions in Theorem 4 are satisfied. Consequently, the solutions x_k^i $(1 \le i \le 3)$ are positive, which is demonstrated in Figure 2-(a), and $\overline{E}(x_k^i)$ are convergent to positive values D_3^i, which is demonstrated in Figure 2-(b). At $k = 5 \cdot 10^6$, which means time 5000, the error vector $(|\overline{E}(x_k^1) - D_3^1|, |\overline{E}(x_k^2) - D_3^2|, |\overline{E}(x_k^3) - D_3^3|)$ is marked with the larger star in Figure 2-(b) and equal to (0.000028, 0.00011, 0.0018).

7. Conclusion

In this paper, dealing with the n-dimensional stochastic difference model, we have extended the new approach to obtain the global stability of the fixed point of the two-dimensional stochastic and discrete predator-prey system. As in the two-dimensional model, we have found the conditions under which at least one discrete solution converges to

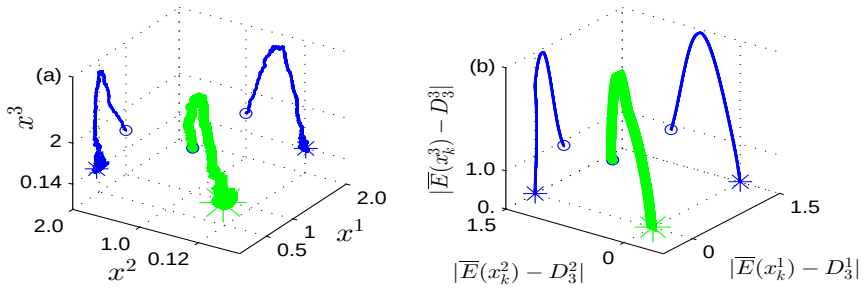


Figure 2: The start and the final points of each curve are denoted by the circle and the star symbols, respectively. Two thicker curves in (a) and (b) show one realization of (x_k^1, x_k^2, x_k^3) $(0 \le k \le 10^6)$ and error vectors $\left(|\overline{E}(x_k^1) - D_3^1|, |\overline{E}(x_k^2) - D_3^2|, |\overline{E}(x_k^3) - D_3^3|\right)$ $(0 \le k \le 5 \cdot 10^6)$, respectively. Each of the other curves is the projection of one of the thicker curves onto one plane: The two planes in Figure 2-(a) are $x^1 = 2$ and $x^2 = 2$.

zero by using a model reduction method. Additionally, we have also found the conditions under which all expectations of solutions globally converge to non-zero fixed values. While proving the global stability of solutions of the two- and n-dimensional stochastic difference models, we have used our new discretized form of the Itô formula and therefore found the possibility that the new discrete Itô formula will be applied to other stochastic difference models.

Appendix

A.1. The proof of Lemma 1

By Taylor expansion,

$$\varphi(1+x) = \varphi(1) + \varphi'(1)x + \frac{\varphi''(1)}{2}x^2 + \frac{\varphi'''(\theta)}{6}x^3$$

with θ lying between 1 and x. We substitute $x = hf + \sqrt{h}g\xi$ and take expectations. Since ξ is $\mathcal{F}$-independent with $E(\xi) = 0$ and $E(\xi^2) = 1-\mu$, and f, g are $\mathcal{F}$-measurable, we have

$$E\left(x \middle| \mathcal{F}\right) = hf, \;\; E\left(x^2 \middle| \mathcal{F}\right) = (hf)^2 + hg^2 \cdot (1-\mu).$$

Note that

$$\left| E\left(\frac{\varphi'''(\theta)}{6}x^3 \middle| \mathcal{F} \right) \right| \leqslant \frac{C}{6} E\left(\left|x^3\right| \;\middle|\; \mathcal{F} \right) = hfO\left(h^{\varepsilon}\right) + hg^2 O\left(h^{\varepsilon}\right)$$

by expanding x^3 and using $E\left(|\xi^i|\right) < \infty$ $(i = 1, 2, 3)$ and (12). Therefore

$$E\left(\varphi(1+x)\,|\mathcal{F}\right) \tag{61}$$

$$= \varphi(1) + \varphi'(1)hf + \frac{\varphi''(1)}{2}hg^2 \cdot (1-\mu) + hfO\left(h^{\varepsilon}\right) + hg^2 O\left(h^{\varepsilon}\right). \tag{62}$$

Now it is enough to show

$$E\left(\phi\left(1 + hf + \sqrt{h}g\xi\right) - \varphi\left(1 + hf + \sqrt{h}g\xi\right) \;\middle|\; \mathcal{F} \right) = hg^2 O\left(h^{\varepsilon}\right).$$

Since f, g are $\mathcal{F}$-measurable and ξ is $\mathcal{F}$ -independent, letting $c_1 = 1 + hf,\ c_2 = \sqrt{h}g$ and $U_\delta = [1-\delta, 1+\delta]$ gives

$$\begin{aligned} & E\left(\phi\left(1+hf+\sqrt{h}g\xi\right)-\varphi\left(1+hf+\sqrt{h}g\xi\right)\ \middle|\ \mathcal{F}\right) \\ &= \int_{\mathbb{R}} \left\{\phi\left(c_1+c_2x\right)-\varphi\left(c_1+c_2x\right)\right\} p(x)\ dx \\ &= \int_{\mathbb{R}-U_\delta} \left\{\phi\left(s\right)-\varphi\left(s\right)\right\} p\left(\frac{s-c_1}{c_2}\right)\ \frac{ds}{|c_2|} \end{aligned} \tag{63}$$

where p is the probabilty density function of ξ.
Note that

$$\begin{aligned} & \left|\int_{\mathbb{R}-U_\delta} \left\{\phi\left(s\right)-\varphi\left(s\right)\right\} p\left(\frac{s-c_1}{c_2}\right)\ \frac{ds}{|c_2|}\right| \\ &\leqslant \left\{\int_{\mathbb{R}-U_\delta} |\phi(s)-\varphi(s)|\ \frac{ds}{|c_2|}\right\} \sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right)\ \frac{1}{|c_2|}\right\} \\ &\leqslant C\cdot |c_2|^2 \sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right)\ \frac{1}{|c_2|^3}\right\} \\ &= C\cdot hg^2 \sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{\sqrt{h}g}\right)\ \frac{1}{\left|\sqrt{h}g\right|^3}\right\}. \end{aligned}$$

Here letting $y = \frac{s-1-hf}{\sqrt{h}g}$, we have

$$\sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{\sqrt{h}g}\right)\ \frac{1}{\left|\sqrt{h}g\right|^3}\right\} = \sup_{s\notin U_\delta}\left\{\ \frac{p\left(y\right)|y|^3}{|s-1-hf|^3}\right\}$$

For all $s \notin U_\delta$, there exists some δ_0 such that for all sufficiently small $h > 0$

$$|s-1-hf| > |s-1| - h|f| > \delta - Ch^\epsilon > \delta_0 > 0 \tag{64}$$

and then we have

$$|y| \equiv \frac{|s-1-hf|}{\sqrt{h}\,|g|} \geqslant \frac{\delta_0}{O\left(h^\varepsilon\right)}. \tag{65}$$

Hence it follows from (64), (65) and the assumption (b) that

$$\sup_{s\notin U_\delta}\left\{\ \frac{p\left(y\right)|y|^3}{|s-1-hf|^3}\right\} = O\left(h^\varepsilon\right),$$

which gives

$$\left|\int_{\mathbb{R}-U_\delta} \left\{\phi\left(s\right)-\varphi\left(s\right)\right\} p\left(\frac{s-c_1}{c_2}\right)\ \frac{ds}{|c_2|}\right| = hg^2 O\left(h^\varepsilon\right). \tag{66}$$

Therefore using (61), (63) and (66), we obtain the desired result.

A.2. The proof of Remark 5

Using (13) and (14) with $a_{3j} = 0$ $(4 \leq j \leq n)$, we have

$$E\left(\ln z_{k+1}^3\right) = \ln z_k^3 + \mathring{\Delta}t\left(r_{3\sigma} + a_{31}z_k^1 + a_{32}z_k^2 - a_{33}z_k^3\right), \quad (67)$$

$$E\left(\ln z_k^3\right) = E\left(\ln z_0^3\right) + k\mathring{\Delta}t a_{33}\left[a_{33}^{-1}\left\{r_{3\sigma} + a_{31}\overline{E}(z_k^1) + a_{32}\overline{E}(z_k^2)\right\} - \overline{E}(z_k^3)\right]. \quad (68)$$

A.2.1. The proof of Remark 5-(a)-(i)

Assume that

$$r_{1\sigma} > 0,\ r_{2\sigma} + a_{21}\beta_1 < 0,\ r_{3\sigma} + a_{31}\beta_1 < 0.$$

Lemma 3-(b) and (b)-(i) give that

$$\lim_{k\to\infty} \overline{E}(z_k^1) = \beta_1 \quad (69)$$

and

$$\lim_{k\to\infty} z_k^2 = 0, \quad (70)$$

respectively. Applying the dominated convergence theorem with (70), we have

$$\lim_{k\to\infty} E(z_k^2) = 0$$

and then

$$\lim_{k\to\infty} \overline{E}(z_k^2) = 0. \quad (71)$$

Substituting (69) and (71) into (68) and using the positivity of z_k^3, we obtain

$$\lim_{k\to\infty} E\left(\ln z_k^3\right) = 0. \quad (72)$$

Therefore combining (72) and (67) with the boundedness of z_k^i $(i = 1, 2, 3)$ in (16), we can obtain the desired result: $\lim_{k\to\infty} z_k^3 = 0,\ a.s.$

A.2.2. The proof of Remark 5-(a)-(ii)

Assume that

$$r_{1\sigma} > 0,\ r_{2\sigma} + a_{21}\beta_1 < 0,\ r_{3\sigma} + a_{31}\beta_1 > 0$$

and then

$$\lim_{k\to\infty} \overline{E}(z_k^1) = \beta_1,\ \lim_{k\to\infty} \overline{E}(z_k^2) = 0,\ \tilde{\beta}_3 = a_{33}^{-1}\left(r_{3\sigma} + a_{31}\beta_1\right) > 0. \quad (73)$$

Following the proof of Lemma 2, we can obtain that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^3\right) \leq \tilde{\beta}_3 + \epsilon,\ k \geq N_\epsilon, \quad (74)$$

by replacing z_k^1, β_1,(18) and (19) in Lemma 2 with z_k^3, $\tilde{\beta}_3$, (67) and (68), respectively and using (73). Hence it remains to show that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$\tilde{\beta}_3 - \epsilon \leq k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^3\right),\ k \geq N_\epsilon. \quad (75)$$

Following the proof of (30) with (68) instead of (31), we can obtain (75), which gives the desired result : $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^3\right) = \tilde{\beta}_3$.

A.2.3. The proof of Remark 5-(b)-(i)
Assume that

$$r_{1\sigma} > 0,\ r_{2\sigma} + a_{21}\beta_1 > 0,\ r_{3\sigma} + \sum\nolimits_{j=1}^{2} a_{3j}\beta_j < 0$$

and then

$$\lim_{k\to\infty} \overline{E}(z_k^i) = \beta_i\ (i = 1, 2),\ \beta_3 = a_{33}^{-1}\left(r_{3\sigma} + \sum\nolimits_{j=1}^{2} a_{3j}\beta_j\right) < 0, \tag{76}$$

due to Lemma 3-(b) and (b)-(ii).
Combining (76) and (68) with the positivity of z_k^3, we can have

$$\lim_{k\to\infty} E\left(\ln z_k^3\right) = -\infty.$$

Therefore, as (22) implies (23), we can obtain the desired result: $\lim\limits_{k\to\infty} z_k^3 = 0,\ a.s.$

A.3. The proofs of Remark 5-(b)-(ii) and Remark 6
Using mathematical induction, we prove that

$$\text{if } r_{i\sigma} + \textstyle\sum_{j=1}^{i-1} a_{ij}\beta_j > 0\ (1 \le i \le n), \text{ then } \lim_{k\to\infty} \overline{E}(z_k^i) = \beta_i\ (1 \le i \le n).$$

Lemma 3-(a) gives that

$$\text{if } r_{1\sigma} > 0, \text{ then } \lim_{k\to\infty} \overline{E}(z_k^1) = \beta_1.$$

Assume that for some $\ell \in \{2, \cdots, n\}$

$$r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0\ (1 \le i \le \ell) \text{ and } \lim_{k\to\infty} \overline{E}(z_k^i) = \beta_i\ (1 \le i \le \ell - 1). \tag{77}$$

Using (15) with $i = \ell$ instead of (3), the equations (13) and (14) become

$$E(\ln z_{k+1}^\ell) = \ln z_k^\ell + \mathring{\Delta}t\left(r_{\ell\sigma} + \sum\nolimits_{j=1}^{\ell-1} a_{\ell j} z_k^j - a_{\ell\ell} z_k^\ell\right), \tag{78}$$

$$E(\ln z_k^\ell) = E(\ln z_0^\ell) + k\mathring{\Delta}t a_{\ell\ell}\left\{a_{\ell\ell}^{-1}\sum\nolimits_{j=1}^{\ell-1} a_{\ell j}\overline{E}(z_k^j) - \overline{E}(z_k^\ell)\right\}. \tag{79}$$

Following the proof of Lemma 2, we can obtain that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$\lim_{k\to\infty} \overline{E}(z_k^\ell) \le \beta_\ell + \epsilon,\ k \ge N_\epsilon, \tag{80}$$

by replacing z_k^1, β_1,(18) and (19) in Lemma 2 with z_k^ℓ, β_ℓ, (78) and (79), respectively and using (77). Hence it remains to show that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$\beta_\ell - \epsilon \le \lim_{k\to\infty} \overline{E}(z_k^\ell),\ k \ge N_\epsilon. \tag{81}$$

Following the proof of (30) with (79) instead of (31), we can obtain (81), which gives

$$\lim_{k\to\infty} \overline{E}(z_k^\ell) = \beta_\ell.$$

Therefore the desired result is obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Entire solutions of certain type of nonlinear differential equations and differential-difference equations *

Min Feng Chen† and Zong Sheng Gao
LMIB and School of Mathematics and Systems Science, Beihang University,
Beijing, 100191, P. R. China

Abstract. In this paper, we investigate the transcendental entire solutions of finite order of the differential equations or differential-difference equations

$$f^2(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $p_1, p_2, \alpha_1, \alpha_2$ are nonzero constants with $\alpha_1 \neq \alpha_2$, and $P(f)$ denotes a differential polynomial or differential-difference polynomial in $f(z)$ with degree 1. And we partially answer a question proposed by Li [10] (P. Li, Entire solutions of certain type of differential equations II, J. Math. Anal. Appl. 375(2011), 310 − 319).
Mathematics Subject Classification (2010). 39B32, 34M05, 30D35.
Keywords. Entire solutions, Differential equations, Differential-difference equations, finite order.

1 Introduction and Results

In this paper, we assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [1, 2]. In addition, we denote by $S(r, f)$ any quantify satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, outside of a possible exceptional set of finite logarithmic measure.

Recently, a number of papers (including [3 − 12]) have focused on solvability and existence of meromorphic solutions of differential equations, difference equations or differential-difference equations in complex plane. Specifically, it shows in [4] that the equation $4f^3 + 3f'' = -\sin 3z$ has exactly three nonconstant entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ and $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. The following two Theorems obtained more general results.

Theorem A. (See [5]) *Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in $f(z)$ of degree at most $n-2$, and λ, p_1, p_2 be three nonzero constants. If $f(z)$ is an entire solution of the following equation*

$$f^n(z) + P(f) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{1.1}$$

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†Corresponding author. E-mail: chenminfeng198710@126.com.

then

$$f(z) = c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n},$$

where c_1 *and* c_2 *are constants and* $c_i^2 = p_i, i = 1, 2.$

Theorem B. (See [10]) *Let* $n \geq 2$ *be an integer,* $P(f)$ *be a differential polynomial in* $f(z)$ *of degree at most* $n-2$*, and* $p_1, p_2, \alpha_1, \alpha_2$ *be nonzero constants and* $\alpha_1 \neq \alpha_2$*. If* $f(z)$ *is a transcendental meromorphic solution of the following equation*

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.2}$$

and satisfying $N(r, f) = S(r, f)$*, then one of the following holds:*
(i) $f(z) = c_0(z) + c_1 e^{\alpha_1 z/n}$*;*
(ii) $f(z) = c_0(z) + c_2 e^{\alpha_2 z/n}$*;*
(iii) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$ *and* $\alpha_1 + \alpha_2 = 0$*,*
where $c_0(z)$ *is a small function of* $f(z)$ *and* c_1, c_2 *are constants satisfying* $c_i^2 = p_i, i = 1, 2.$

For further study, Li [10] proposed the following question.

Question. How to find the solutions of (1.2) under the condition $\deg P(f) = n - 1$?

In this paper, we study this question and partially answer this question, and obtain the following result.

Theorem 1.1. *Let* $p_1, p_2, \alpha_1, \alpha_2$ *be nonzero constants such that* $\alpha_1 \neq \alpha_2$*,* $a(z)$ *be a nonzero polynomial. If* $f(z)$ *is a transcendental entire solution of finite order of the differential equation*

$$f^2(z) + a(z) f'(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.3}$$

and satisfying $N\left(r, \frac{1}{f}\right) = S(r, f)$*, then* $a(z)$ *must be a constant, and one of the following holds:*
(i) $f(z) = c_1 e^{\alpha_1 z/2},\ ac_1\alpha_1 = 2p_2,\ \alpha_1 = 2\alpha_2$*;*
(ii) $f(z) = c_2 e^{\alpha_2 z/2},\ \ ac_2\alpha_2 = 2p_1,\ \alpha_2 = 2\alpha_1$*;*
where c_1, c_2 *are constants satisfying* $c_i^2 = p_i,\ i = 1, 2.$

Remark 1.1. *We conjecture that the condition* $N\left(r, \frac{1}{f}\right) = S(r, f)$ *in Theorem* 1.1 *can be omitted using another method although we have not found a suitable one yet.*

In [11], Wang and Li investigated the existence of entire solutions of differential-difference equation

$$f^n(z) + q(z) f^{(k)}(z + c) = a e^{ibz} + d e^{-ibz}, \tag{1.4}$$

and obtained the following Theorem.

Theorem C. (See [11]) *For two integers* $n \geq 3,\ k \geq 0$ *and a nonlinear differential-difference equation* (1.4)*, where* $q(z)$ *is a polynomial and* a, b, c, d *are constants such that* $|a| + |d| \neq 0,\ bc \neq 0.$
(i) Let $n = 3$*. If* $q(z)$ *is nonconstant, then the equation* (1.4) *does not admit entire solutions of finite order. If* $q := q(z)$ *is a constant, then equation* (1.4) *admits three distinct transcendental entire solutions of finite order, provided that*

$$bc = 3m\pi\ (m \neq 0,\ if\ q \neq 0),\ q^3 = (-1)^{m+1} \left(\frac{3i}{b}\right)^{3k} 27ad,$$

when k is an even, or

$$bc = \frac{3\pi}{2} + 3m\pi \ (if\ q \neq 0),\ q^3 = i(-1)^m \left(\frac{3i}{b}\right)^{3k} 27ad,$$

when k is an odd, for an integer m.
(ii) Let $n \geq 3$. If $ad \neq 0$, then the equation (1.4) does not admit entire solutions of finite order. If $ad = 0$, then equation (1.4) admits n distinct transcendental entire solutions of finite order, provided that $q := q(z) \equiv 0$.

In this paper, we shall tackle differential-difference equations in the form (1.4) with $n = 2$, and obtain the following result.

Theorem 1.2. *Let p_1, p_2, λ, c be nonzero constants, $k \geq 0$ be an integer and $a(z)$ be a nonzero polynomial. If $f(z)$ is a transcendental entire solution of finite order of the differential-difference equation*

$$f^2(z) + a(z)f^{(k)}(z + c) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{1.5}$$

then $a(z)$ must be a constant, and satisfying one of the following relations:
(i) $f(z) = \pm\frac{i}{2}a(\frac{\lambda}{2})^k + c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}$, $e^{\lambda c} = -1$, when k is an odd;
(ii) $f(z) = \pm\frac{1}{2}a(\frac{\lambda}{2})^k + c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}$, $e^{\lambda c} = 1$, when k is an even and $k > 0$,
where a, c_1, c_2 are constants with $\frac{1}{64}a^4(\frac{\lambda}{2})^{4k} = p_1 p_2$ and $c_i^2 = p_i$, $i = 1, 2$;
(iii) $f(z) = \pm\frac{1}{2}a + c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}$, $e^{\lambda c} = 1$, when $k = 0$,
where a, c_1, c_2 are constants with $\frac{1}{64}a^4 = p_1 p_2$ or $\frac{9}{64}a^4 = p_1 p_2$ and $c_i^2 = p_i, i = 1, 2$.

Example 1.1. *$f(z) = e^z$ is a transcendental entire solution of finite order of the differential equation*

$$f^2(z) + f'(z) = e^{2z} + e^z.$$

Example 1.2. *The differential-difference equation*

$$f^2(z) + f(z + \pi i/2) = \frac{1}{4}e^{4z} + \frac{9}{16}e^{-4z}.$$

has exactly two entire solutions, namely $f_1(z) = \frac{1}{2} + \frac{1}{2}e^{2z} - \frac{3}{4}e^{-2z}$ and $f_2(z) = \frac{1}{2} - \frac{1}{2}e^{2z} + \frac{3}{4}e^{-2z}$.

Example 1.3. *The differential-difference equation*

$$f^2(z) - f'(z + \pi i/2) = \frac{1}{4}e^{2z} + \frac{1}{16}e^{-2z}.$$

has exactly two entire solutions, namely $f_1(z) = \frac{i}{2} + \frac{1}{2}e^z + \frac{1}{4}e^{-z}$ and $f_2(z) = \frac{i}{2} - \frac{1}{2}e^z - \frac{1}{4}e^{-z}$.

Example 1.4. *The differential-difference equation*

$$f^2(z) - 2f''(z + \pi i) = \frac{1}{4}e^{2z} + e^{-2z}.$$

has exactly two entire solutions, namely $f_1(z) = -1 + \frac{1}{2}e^z - e^{-z}$ and $f_2(z) = -1 - \frac{1}{2}e^z + e^{-z}$.

2 Lemmas for the Proof of Theorems

Lemma 2.1. *(Clunie's Theorem)*(See [2, 13]) *Let $f(z)$ be a transcendental meromorphic solution of*

$$f^n(z)P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in $f(z)$ and its derivatives with meromorphic coefficients, say $\{a_\lambda|\lambda \in I\}$, such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree of $Q(z,f)$ as a polynomial in $f(z)$ and its derivatives is $\leq n$, then

$$m(r,P(z,f)) = S(r,f). \tag{2.1}$$

Lemma 2.2. (See [14]) *Let $f(z)$ be a nonconstant finite order meromorphic solution of*

$$f^n(z)P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in $f(z)$ and its shifts with small meromorphic coefficients, and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts is $\leq n$, then

$$m(r,P(z,f)) = o\left(\frac{T(r+|c|,f)}{r^\delta}\right) + o(T(r,f)) \tag{2.2}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Remark 2.1. *In Lemma 2.2, if $f(z)$ is transcendental with finite order, and $P(z,f), Q(z,f)$ are differential-difference polynomials in $f(z)$, then by using a similar method as in the proof of Lemma 2.1, we see that a similar conclusion of Lemma 2.2 holds. Moreover, we know that if the coefficients of $P(z,f)$ and $Q(z,f)$ are polynomials $a_j(z)$, $j = 1,\ldots,k$, then (2.2) can be replaced by*

$$m(r,P(z,f)) = S(r,f) + O\left(\sum_{j=1}^{k} m(r,a_j(z))\right), \tag{2.3}$$

where r is sufficiently large.

Lemma 2.3. (See [1, 2]) *Let $f(z)$ be a transcendental meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r,\frac{f^{(k)}(z)}{f(z)}\right) = S(r,f). \tag{2.4}$$

Lemma 2.4. *Let α be a nonzero constant, and $H(z)$ be a nonvanishing polynomial. Then the differential equation*

$$\alpha f(z) - 2f'(z) = H(z) \tag{2.5}$$

has a special solution $c_0(z)$ which is a nonzero polynomial.

Proof. Similarly to the proof of [Lemma 2.3, 8]. If $H(z)$ is a nonzero constant, then $c_0(z) = \frac{H(z)}{\alpha}$ is a special solution of (2.5). Now suppose that

$$H(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

where $n \geq 1$ is an integer, $a_n \neq 0, a_{n-1}, \ldots, a_1, a_0$ are constants.
We use the method of undetermined coefficients to derive the polynomial solution $c_0(z)$ satisfying (2.5) by $\alpha, a_n, a_{n-1}, \ldots, a_1, a_0$. By (2.5), we see that $\deg c_0(z) = \deg H(z)$. For $n = 1$, equation (2.5) has a polynomial solution $c_0(z) = \frac{1}{\alpha}a_1 z + \frac{1}{\alpha}(a_0 + \frac{2}{\alpha}a_1)$.
In a general case, for $n \geq 2$, (2.5) has a polynomial solution

$$c_0(z) = b_n z^n + \cdots + b_j z^j + \cdots + b_1 z + b_0,$$

where

$$b_n = \frac{1}{\alpha}a_n, \quad b_j = \frac{1}{\alpha}a_j + \frac{2(j+1)}{\alpha}b_{j+1},\ j = n-1, \ldots, 0.$$

Therefore, (2.5) has a nonzero polynomial solution $c_0(z)$.

Lemma 2.5. (See [8]) *Let λ be a nonzero constant, and $H(z)$ be a nonvanishing polynomial. Then the differential equation*

$$4f''(z) - \lambda^2 f(z) = H(z) \tag{2.6}$$

has a special solution $c_0(z)$ which is a nonzero polynomial.

Lemma 2.6. (See [15]) *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions,*
(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
(3) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j(z)) = o(T(r, e^{g_h(z)-g_k(z)}))(r \to \infty, r \notin E)$, where $E \subset [1, \infty)$ is finite linear measure or finite logarithmic measure.
Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Lemma 2.7. *Let $P(z)$ be a nonzero polynomial, A, B, c be nonzero constants and $A \neq 1$. If*

$$P^2(z) \equiv AP(z)P(z+c) + B, \tag{2.7}$$

then $P(z)$ must be a constant, $P(z) := p$(constant).

Proof. If $P(z)$ is a nonconstant polynomial, then $\deg P(z) \geq 1$. Now suppose that

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $n \geq 1$ is an integer, $a_n \neq 0, a_{n-1}, \ldots, a_0$ are constants, then

$$P(z+c) = a_n(z+c)^n + a_{n-1}(z+c)^{n-1} + \cdots + a_0,$$

$$P^2(z) = a_n^2 z^{2n} + 2a_n a_{n-1} z^{2n-1} + \cdots,$$

$$P(z)P(z+c) = a_n^2 z^{2n} + (na_n^2 c + 2a_n a_{n-1}) z^{2n-1} + \cdots,$$

$$P^2(z) - AP(z)P(z+c) = (1-A)a_n^2 z^{2n} + \cdots.$$

From $A \neq 1$, we see that $\deg[P^2(z) - AP(z)P(z+c)] = 2n \geq 2$, it contradicts with (2.7). Thus $P(z)$ must be a nonzero constant, set $P(z) := p(\neq 0)$.

3 Proof of Theorems

Proof of Theorem 1.1
Denote $P = a(z)f'(z)$. Suppose that $f(z)$ is a transcendental entire solution of finite order of equation (1.3). By differentiating (1.3), we have

$$2ff' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{3.1}$$

Eliminating $e^{\alpha_1 z}, e^{\alpha_2 z}$ from (1.3) and (3.1), respectively, we have

$$\alpha_1 f^2 - 2ff' + \alpha_1 P - P' = (\alpha_1 - \alpha_2)p_2 e^{\alpha_2 z}, \tag{3.2}$$

$$\alpha_2 f^2 - 2ff' + \alpha_2 P - P' = (\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}. \tag{3.3}$$

Differentiating (3.3) yields

$$2\alpha_2 ff' - 2(f')^2 - 2ff'' + \alpha_2 P' - P'' = \alpha_1(\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\varphi = -Q, \tag{3.5}$$

where

$$\varphi = \alpha_1\alpha_2 f^2 - 2(\alpha_1 + \alpha_2)ff' + 2(f')^2 + 2ff'', \tag{3.6}$$

and

$$Q = \alpha_1\alpha_2 P - (\alpha_1 + \alpha_2)P' + P''. \tag{3.7}$$

If $\varphi \not\equiv 0$, by (3.5) – (3.7) and Lemma 2.3, we see that

$$m\left(r, \frac{\varphi}{f}\right) = S(r, f) \quad and \quad m\left(r, \frac{\varphi}{f^2}\right) = S(r, f). \tag{3.8}$$

From $N\left(r, \frac{1}{f}\right) = S(r, f)$ and (3.8), we see that

$$\begin{aligned} T(r, f) + S(r, f) = m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{\varphi}{f}\right) + m\left(r, \frac{1}{\varphi}\right) \\ &\leq T(r, \varphi) + S(r, f) = m(r, \varphi) + S(r, f) \\ &\leq m\left(r, \frac{\varphi}{f}\right) + m(r, f) + S(r, f) \\ &= T(r, f) + S(r, f), \end{aligned}$$

that is,

$$T(r, \varphi) = T(r, f) + S(r, f). \tag{3.9}$$

From (3.8) and (3.9), we see that

$$\begin{aligned} 2T(r, f) + S(r, f) = 2m\left(r, \frac{1}{f}\right) &= m\left(r, \frac{1}{f^2}\right) \\ &\leq m\left(r, \frac{\varphi}{f^2}\right) + m\left(r, \frac{1}{\varphi}\right) \\ &\leq T(r, \varphi) + S(r, f) \\ &= T(r, f) + S(r, f), \end{aligned}$$

that is $T(r,f) \leq S(r,f)$, a contradiction. Then we have $\varphi \equiv 0$, that is $Q \equiv 0$. By (3.7), we have

$$\alpha_1\alpha_2 P - (\alpha_1+\alpha_2)P' + P'' \equiv 0. \tag{3.10}$$

Since $P = a(z)f'(z)$, it is obvious that $P \not\equiv 0$. Since $\alpha_1 \neq \alpha_2$, we see that $\alpha_1 P - P' \equiv 0$ and $\alpha_2 P - P' \equiv 0$ cannot hold simultaneously. Suppose $\alpha_1 P - P' \not\equiv 0$. By (3.10), we get

$$\alpha_1 P - P' = Ae^{\alpha_2 z}, \tag{3.11}$$

where A is a nonzero constant. Substituting (3.11) into (3.2), we have

$$f(\alpha_1 f - 2f') = \frac{[(\alpha_1-\alpha_2)p_2 - A]\alpha_1}{A}P - \frac{(\alpha_1-\alpha_2)p_2 - A}{A}P'. \tag{3.12}$$

Since the right-hand side of (3.12) is a differential polynomial in $f(z)$ of degree ≤ 1. By Lemma 2.1, we have

$$m(r, \alpha_1 f - 2f') = S(r,f). \tag{3.13}$$

If $\alpha_1 f - 2f' \not\equiv 0$, since $f(z)$ is a transcendental entire function of finite order, by (3.13), we see that

$$m(r, \alpha_1 f - 2f') = S(r,f) = O(\log r),$$

which implies that $\alpha_1 f - 2f'$ is a nonzero polynomial. Then we have

$$\alpha_1 f - 2f' = H(z), \tag{3.14}$$

where $H(z)$ is a nonvanishing polynomial, but $H(z)$ may be a nonzero constant. By Lemma 2.4, we know that (3.14) has a nonzero polynomial solution, say, $c_0(z)$.
Since the differential equation

$$\alpha_1 f - 2f' = 0 \tag{3.15}$$

has a fundamental solution $f(z) = e^{\alpha_1 z/2}$. Then the general entire solution $f(z)$ of (3.14) can be expressed as

$$f(z) = c_0(z) + c_1 e^{\alpha_1 z/2}, \tag{3.16}$$

where c_1 is a constant, $c_0(z)$ is a nonzero polynomial.
Substituting (3.16) into (1.3), we have

$$(c_1^2 - p_1)e^{\alpha_1 z} - p_2 e^{\alpha_2 z} + \left(2c_0(z) + \frac{\alpha_1}{2}a(z)\right)c_1 e^{\alpha_1 z/2} + c_0^2(z) + a(z)c_0'(z) = 0. \tag{3.17}$$

By $\alpha_1 \neq \alpha_2$, if $\alpha_2 \neq \frac{\alpha_1}{2}$, by Lemma 2.6, we see that $p_2 = 0$, a contradiction.
If $\alpha_2 = \frac{\alpha_1}{2}$, then (3.17) can be rewritten as

$$(c_1^2 - p_1)e^{\alpha_1 z} + \left[\left(2c_0(z) + \frac{\alpha_1}{2}a(z)\right)c_1 - p_2\right]e^{\alpha_2 z} + c_0^2(z) + a(z)c_0'(z) = 0. \tag{3.18}$$

By $\alpha_1 \neq \alpha_2$ and Lemma 2.6, we have

$$\left(2c_0(z) + \frac{\alpha_1}{2}a(z)\right)c_1 - p_2 \equiv 0,\ c_0^2(z) + a(z)c_0'(z) \equiv 0, \tag{3.19}$$

then $c_0(z) \equiv 0$, a contradiction.
Therefore, we have $\alpha_1 f - 2f' \equiv 0$, which yields

$$f^2 = Be^{\alpha_1 z}, \tag{3.20}$$

where B is a nonzero constant. By (1.3), (3.11) and (3.20), we have

$$\left(1-\frac{p_1}{B}\right)f^2=\frac{\alpha_1p_2-A}{A}P-\frac{p_2}{A}P'. \tag{3.21}$$

If $p_1\neq B$, then by (3.21) and Lemma 2.1, we get $T(r,f)=S(r,f)$, which is impossible. Therefore, $p_1=B$ and $f(z)=c_1e^{\alpha_1z/2}$, where c_1 is a nonzero constant satisfying $c_1^2=p_1$. Substituting $f(z)=c_1e^{\alpha_1z/2}$ and $c_1^2=p_1$ into (1.3), we have

$$\frac{c_1\alpha_1}{2}a(z)e^{\alpha_1z/2}-p_2e^{\alpha_2z}=0. \tag{3.22}$$

By $\alpha_1\neq\alpha_2$ and Lemma 2.6, we see that $\alpha_1=2\alpha_2$ and $c_1\alpha_1a(z)\equiv 2p_2$, then $a(z)$ must be a constant, set $a(z):=a$.

If $\alpha_2P-P'\not\equiv 0$, then by a similar method, we can deduce that $f(z)=c_2e^{\alpha_2z/2},\ \alpha_2=2\alpha_1,\ c_2\alpha_2a=2p_1,\ c_2^2=p_2$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2

Denote $P_1(f)=a(z)f^{(k)}(z+c)$. Suppose that $f(z)$ be a transcendental entire solution of finite order of equation (1.5). By differentiating both sides of equation (1.5), we have

$$2ff'+P_1'(f)=\lambda(p_1e^{\lambda z}-p_2e^{-\lambda z}). \tag{3.23}$$

Differentiating (3.23), we obtain

$$2(f')^2+2ff''+P_1''(f)=\lambda^2(p_1e^{\lambda z}+p_2e^{-\lambda z}). \tag{3.24}$$

Combining (1.5) with (3.24), we have

$$(f')^2=\frac{1}{2}\lambda^2f^2-ff''+Q_1(f), \tag{3.25}$$

where $Q_1(f)=\frac{1}{2}(\lambda^2P_1(f)-P_1''(f))$.

Eliminating $e^{\lambda z}$, $e^{-\lambda z}$ from (1.5) and (3.23), we obtain

$$\lambda^2(f^2+P_1(f))^2-(2ff'+P_1'(f))^2=4p_1p_2\lambda^2,$$

which implies that

$$\lambda^2f^4-4f^2(f')^2=R_3(f), \tag{3.26}$$

where

$$R_3(f)=-[2\lambda^2f^2P_1(f)+\lambda^2(P_1(f))^2-4ff'P_1'(f)-(P_1'(f))^2-4p_1p_2\lambda^2].$$

Substituting (3.25) into (3.26), we see that

$$f^3(4f''-\lambda^2f)=T_3(f), \tag{3.27}$$

where $T_3(f)=4f^2Q_1(f)+R_3(f)$.

Now, we consider two cases, case 1: $T_3(f)\equiv 0$ and case 2: $T_3(f)\not\equiv 0$.

Case 1: $T_3(f)\equiv 0$. By (3.27), we have

$$4f''-\lambda^2f\equiv 0. \tag{3.28}$$

Every entire solution $f(z)(\not\equiv 0)$ of (3.28) can be expressed as

$$f(z) = c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}, \tag{3.29}$$

where c_1, c_2 are constants, and at least one of them being not equal to zero.
Then

$$f^{(k)}(z+c) = c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} e^{\lambda z/2} + (-1)^{k+1} c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} e^{-\lambda z/2}. \tag{3.30}$$

Substituting (3.29) and (3.30) into (1.5), we obtain

$$(c_1^2-p_1)e^{\lambda z}+(c_2^2-p_2)e^{-\lambda z}+a(z)c_1(\frac{\lambda}{2})^k e^{\lambda c/2}e^{\lambda z/2}+(-1)^{k+1}a(z)c_2(\frac{\lambda}{2})^k e^{-\lambda c/2}e^{-\lambda z/2}+2c_1c_2 = 0. \tag{3.31}$$

By Lemma 2.6, we see that

$$a(z)c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} \equiv 0,\ (-1)^{k+1} a(z) c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} \equiv 0. \tag{3.32}$$

From $\lambda \neq 0$ and $a(z) \not\equiv 0$, then $c_1 = c_2 = 0$, a contradiction.
Case 2: $T_3(f) \not\equiv 0$. Since $f(z)$ is a transcendental entire function of finite order, we see that (3.27) satisfies the conditions of Lemma 2.2 and Remark 2.1. Thus we have

$$m(r, 4f'' - \lambda^2 f) = S(r,f) + O(m(r, a(z))) = O(\log r),$$

which implies that $4f'' - \lambda^2 f$ is a polynomial. Thus, from (3.27) and $T_3(f) \not\equiv 0$, we have

$$4f'' - \lambda^2 f = H(z), \tag{3.33}$$

where $H(z)$ is a nonvanishing polynomial. By Lemma 2.5, we see that (3.33) must have a nonzero polynomial solution, say, $c_0(z)$.
Since the differential equation

$$4f'' - \lambda^2 f = 0, \tag{3.34}$$

has two fundamental solutions

$$f_1(z) = e^{\frac{\lambda}{2}z}, \quad f_2(z) = e^{-\frac{\lambda}{2}z}.$$

Then the general entire solution $f(z)(\not\equiv 0)$ of (3.33) can be expressed as

$$f(z) = c_0(z) + c_1 e^{\frac{\lambda}{2}z} + c_2 e^{-\frac{\lambda}{2}z}, \tag{3.35}$$

where c_1, c_2 are constants, $c_0(z)$ is a nonzero polynomial.
If k is an odd, then

$$f^{(k)}(z+c) = c_0^{(k)}(z+c) + c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} e^{\lambda z/2} - c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} e^{-\lambda z/2}. \tag{3.36}$$

Substituting (3.35) and (3.36) into (1.5), we obtain

$$(c_1^2 - p_1)e^{\lambda z} + (c_2^2 - p_2)e^{-\lambda z} + \left(2c_1c_0(z) + a(z)c_1\left(\frac{\lambda}{2}\right)^k e^{\lambda c/2}\right)e^{\lambda z/2}$$

$$+ \left(2c_2c_0(z) - a(z)c_2\left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2}\right)e^{-\lambda z/2} + c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 = 0. \tag{3.37}$$

By Lemma 2.6, we see that

$$\begin{cases} c_1^2 = p_1,\ c_2^2 = p_2; \\ 2c_1c_0(z) + a(z)c_1(\frac{\lambda}{2})^k e^{\lambda c/2} \equiv 0; \\ 2c_2c_0(z) - a(z)c_2(\frac{\lambda}{2})^k e^{-\lambda c/2} \equiv 0; \\ c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 \equiv 0. \end{cases} \tag{3.38}$$

From (3.38), we know that

$$a(z) \equiv a(constant), \quad c_0(z) \equiv c_0(constant),$$

and then

$$c_0 = \pm\frac{i}{2}a\left(\frac{\lambda}{2}\right)^k,\ \frac{1}{64}a^4\left(\frac{\lambda}{2}\right)^{4k} = p_1p_2,\ e^{\lambda c} = -1,\ c_i^2 = p_i, i = 1, 2.$$

If k is an even, then

$$f^{(k)}(z+c) = c_0^{(k)}(z+c) + c_1\left(\frac{\lambda}{2}\right)^k e^{\lambda c/2}e^{\lambda z/2} + c_2\left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2}e^{-\lambda z/2}. \tag{3.39}$$

Substituting (3.35) and (3.39) into (1.5), we obtain

$$(c_1^2 - p_1)e^{\lambda z} + (c_2^2 - p_2)e^{-\lambda z} + \left(2c_1c_0(z) + a(z)c_1\left(\frac{\lambda}{2}\right)^k e^{\lambda c/2}\right)e^{\lambda z/2}$$

$$+\left(2c_2c_0(z) + a(z)c_2\left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2}\right)e^{-\lambda z/2} + c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 = 0. \tag{3.40}$$

By Lemma 2.6, we see that

$$\begin{cases} c_1^2 = p_1,\ c_2^2 = p_2; \\ 2c_1c_0(z) + a(z)c_1(\frac{\lambda}{2})^k e^{\lambda c/2} \equiv 0; \\ 2c_2c_0(z) + a(z)c_2(\frac{\lambda}{2})^k e^{-\lambda c/2} \equiv 0; \\ c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 \equiv 0. \end{cases} \tag{3.41}$$

If k is even and $k > 0$, from (3.41), we know that

$$a(z) \equiv a(constant), \quad c_0(z) \equiv c_0(constant),$$

and then

$$c_0 = \pm\frac{1}{2}a\left(\frac{\lambda}{2}\right)^k,\ \frac{1}{64}a^4\left(\frac{\lambda}{2}\right)^{4k} = p_1p_2,\ e^{\lambda c} = 1,\ c_i^2 = p_i, i = 1, 2.$$

If $k = 0$, from (3.41), we see that

$$c_0^2(z) \equiv \pm 2c_0(z)c_0(z+c) - 2c_1c_2. \tag{3.42}$$

By (3.42) and Lemma 2.7, we know that $c_0(z) \equiv c_0$(constant). By (3.41), we have $a(z) \equiv a$(constant), and

$$c_0 = \pm\frac{1}{2}a,\ \frac{1}{64}a^4 = p_1p_2\ or\ \frac{9}{64}a^4 = p_1p_2,\ e^{\lambda c} = 1,\ c_i^2 = p_i, i = 1, 2.$$

This completes the proof of Theorem 1.2.

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Min-Feng Chen
LMIB & School of Mathematics and Systems Science
Beihang University
Beijing 100191
P. R. China
Email: chenminfeng198710@126.com

Zong-Sheng Gao
LMIB & School of Mathematics and Systems Science
Beihang University
Beijing 100191
P. R. China
E-mail: zshgao@buaa.edu.cn

APPROXIMATE GENERALIZED QUADRATIC MAPPINGS IN (β, p)-BANACH SPACES

HARK-MAHN KIM AND HONG-MEI LIANG

Abstract. In this article, we present general solution of a generalized quadratic functional equation with several variables, and then obtain its generalized Hyers–Ulam stability results for approximate generalized quadratic mappings in (β, p)-Banach spaces.

1. Introduction

In 1940, S. M. Ulam [30] gave the following question associated with the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$?

The question of Ulam was first solved by D. H. Hyers [12] for approximate Cauchy additive mappings on Banach spaces. Th. M. Rassias [22] provided a generalized Hyers–Ulam stability for the unbounded Cauchy difference controlled by a sum of unbounded function $\varepsilon(\|x\|^p + \|y\|^p)$ for case $0 \leq p < 1$. And then Z. Gajda [8] provided a generalized Hyers–Ulam stability for the Cauchy difference controlled by the same unbounded function $\varepsilon(\|x\|^p + \|y\|^p)$ for case $p > 1$. In 1984, J. M. Rassias [23] gave a similar stability for the unbounded Cauchy difference controlled by a product of unbounded function $\varepsilon\|x\|^p\|y\|^q$, $p+q \neq 1$. More generally, Găvruta [10] established a generalized Hyers–Ulam stability under replacing the bound of Cauchy difference controlled by an integrated general control function with regular condition.

The following functional equation

$$f(x + y) + f(x - y) = 2[f(x) + f(y)]$$

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† Corresponding author. hmliang124@126.com.

is called a quadratic functional equation since it may be originated from the important parallelogram equality

$$\|x+y\|^2+\|x-y\|^2=2[\|x\|^2+\|y\|^2]$$

in inner product spaces. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers–Ulam stability problem for the quadratic functional equation was first proved by F. Skof [29] for a mapping $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1992, S. Czerwik [6] demonstrated the Hyers–Ulam stability of the quadratic functional equation with the sum of powers of norms in the sense of Th. M. Rassias approach using the direct method. In the same year, J. M. Rassias [24] certified the Hyers–Ulam stability of the quadratic functional equation with the product of powers of norms using the direct method. In 1995, C. Borelli and G. L. Forti [2] have verified the generalized Hyers–Ulam stability theorem of the quadratic functional equation. V. Radu [21], L. Cădariu and V. Radu [3, 4] have proposed to investigate the stability of functional equations using the fixed point method which is based on the alternative fixed point theorem. Since then, the stability of several functional equations using the fixed point method has been extensively investigated by several mathematicians [9, 11, 14, 16, 17].

Recently, A. Zivari-Kazempour and M. Eshaghi Gordji [32] have determined the general solution of the quadratic functional equation

$$f(x+2y)+f(y+2z)+f(z+2x)=2f(x+y+z)+3\big[f(x)+f(y)+f(z)\big],$$

and then have investigated its generalized Hyers–Ulam stability. Motivated from this quadratic functional equation, we now consider a generalized functional equation

$$\begin{aligned}(1.1)\qquad &\sum_{i=1}^{n}\sum_{1\le i_1\le\cdots\le i_n\le n} f\Big(kx_i+\sum_{j=1,i_j\neq i}^{n-k}x_{i_j}\Big)\\ &=(n+k-1)\begin{pmatrix} n-2\\ n-k-1\end{pmatrix} f\Big(\sum_{i=1}^{n}x_i\Big)+\frac{nk(k-1)}{n-k}\begin{pmatrix} n-2\\ n-k-1\end{pmatrix}\sum_{i=1}^{n}f(x_i),\end{aligned}$$

where n, k are fixed integers with $n \ge 3$ and $2 \le k \le n-1$. Kim and Liang [15] have presented the classical stability results of quadratic functional equation (1.1) by using the fixed point approach in normed spaces. In this article, we give the general solution of the functional equation (1.1), and then investigate generalized Hyers–Ulam stability results for approximate generalized quadratic mappings in (β, p)-Banach spaces.

2. The general solution of Equation (1.1)

First of all, we introduce basic definitions, notations and preliminary theorems in the sequel. Let $\mathbb{N}$ be the set of all natural numbers, $n \in \mathbb{N}$ and let X and Y be vector spaces. A

mapping $A_n : X^n \to Y$ is called n-additive if it is additive in each of its variables. A mapping A_n is called symmetric if $A_n(x_1, \cdots, x_n) = A_n(x_{\pi(1)}, \cdots, x_{\pi(n)})$ for every permutation $\{\pi(1), \cdots, \pi(n)\}$ of $\{1, \cdots, n\}$ and all $x_1, \cdots, x_n \in X$. If $A_n(x_1, \cdots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, \cdots, x)$ for all $x \in X$, which will be called a monomial mapping of degree n. Note that $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all rational number $r \in \mathbb{Q}$. A mapping $p : X \to Y$ is called a generalized polynomial of degree $n \in \mathbb{N}$ provided that there exist an i-additive symmetric mappings $A_i : X^i \to Y$ $(1 \leq i \leq n)$ such that $p(x) = \sum_{i=0}^{n} A^i(x)$ for all $x \in X$, where $A^n \not\equiv 0$ and $A^0(x) = A^0 \in Y$ is a constant.

For $f : X \to Y$, let Δ_h be the difference operator defined as follows: $\Delta_h f(x) = f(x + h) - f(x)$ for all $x, h \in X$. We notice that these difference operators satisfy commutativity: $\Delta_{h_1} \circ \Delta_{h_2} f(x) = \Delta_{h_2} \circ \Delta_{h_1} f(x)$ for all $x, h_1, h_2 \in X$, where $\Delta_{h_1} \circ \Delta_{h_2}$ denotes the composition of the operators Δ_{h_1} and Δ_{h_2}. Furthermore, let $\Delta_h^0 f(x) = f(x), \Delta_h^1 f(x) = \Delta_h f(x)$ and $\Delta_h^{n+1} f(x) = \Delta_h \circ \Delta_h^n f(x)$ for all $n \in \mathbb{N}$ and all $x, h \in X$. In explicit form, the functional equation $\Delta_h^{n+1} f(x) = 0$ can be written as

$$\Delta_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \begin{pmatrix} n+1 \\ j \end{pmatrix} f(x + jh) = 0$$

for all $x, h \in X$. For any given $n \in \mathbb{N}$, the following is well-known Fréchet functional equation

$$\Delta_{h_{n+1}} \circ \cdots \circ \Delta_{h_1} f(x) = 0$$

for all $x, h_1, \cdots, h_{n+1} \in X$.

The following theorem was proved by Mazur and Orlicz [19, 20] and by Djoković [7] in greater generality.

Theorem 2.1. *Let X and Y be vector spaces, $n \in \mathbb{N}$ and $f : X \to Y$, then the followings are equivalent.*

(1) $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in X$.

(2) $\Delta_{h_{n+1}} \circ \cdots \circ \Delta_{h_1} f(x) = 0$ for all $x, h_1, \cdots, h_{n+1} \in X$.

(3) $f(x) = A^n(x) + \cdots + A^0(x)$ for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal of an i-additive symmetric mapping $A_i : X^i \to Y$ $(i = 1, \cdots, n)$.

The following two theorems ([27]) are need for us to establish general solution of the functional equation (1.1) (see also [28, 31] for details).

Theorem 2.2. *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Let the multiplication by $n!$ be bijective in S. The function $f : G \to S$ is a solution of Fréchet functional equation $\Delta_{h_{n+1}} \circ \cdots \circ \Delta_{h_1} f(x) = 0$ for all $x, h_1, \cdots, h_{n+1} \in G$ if and only if f is a generalized polynomial of degree at most n.*

Theorem 2.3. *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G)$ $(i = 1, \cdots, n+1)$. If the functions $f, f_i : G \to S$ $(i = 1, \cdots, n+1)$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0$$

for all $x, y \in G$, then f satisfies Fréchet functional equation $\Delta_{h_{n+1}} \circ \cdots \circ \Delta_{h_1} f(x) = 0$ for all $x, h_1, \cdots, h_{n+1} \in G$.

Before taking up our subject, we note that a mapping f is quadratic if and only if f satisfies the functional equation

$$f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y),$$

which is equivalent to the functional equation

$$f(3x + y) + f(x + 3y) = 6f(x + y) + 4f(x) + 4f(y)$$

for all $x, y \in X$ [5]. Moreover, it is very meaningful and elementary to see that the functional equation

$$f(kx + y) + f(x + ky) = 2kf(x + y) + (k - 1)^2[f(x) + f(y)] \tag{2.1}$$

between vector spaces, where k is a fixed positive integer with $k \geq 2$, is equivalent to the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.2}$$

for all $x, y \in X$ using Theorem 2.2 and Theorem 2.3 together with $f(kx) = k^2 f(x)$ for all $x \in X$, and so the mapping f satisfying the equation (2.1) is quadratic.

Now, we are ready to present the general solution of the functional equation (1.1).

Theorem 2.4. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ is solution of the functional equation (1.1), then f satisfies the functional equation (2.2) and so it is quadratic.*

Proof. Assume that a mapping $f : X \to Y$ satisfies the functional equation (1.1). Letting $x_1 = \cdots = x_n := 0$ in (1.1), we obtain

$$n \begin{pmatrix} n-1 \\ n-k \end{pmatrix} f(0) = (n + k - 1) \begin{pmatrix} n-2 \\ n-k-1 \end{pmatrix} f(0) + \frac{n^2 k(k-1)}{n-k} \begin{pmatrix} n-2 \\ n-k-1 \end{pmatrix} f(0),$$

which yields $f(0) = 0$.

On the other hand, one has the following identity by setting $x_1 := x, x_2 = \cdots = x_n := 0$ in (1.1)

$$\begin{aligned}&\binom{n-1}{n-k} f(kx) + (n-1)\binom{n-2}{n-k-1} f(x)\\ &= (n+k-1)\binom{n-2}{n-k-1} f(x) + \frac{nk(k-1)}{n-k}\binom{n-2}{n-k-1} f(x)\end{aligned}$$

for all $x \in X$. Thus $f(kx) = k^2 f(x)$ for all $x \in X$.

Putting $x_1 := x, x_2 := y, x_3 = \cdots = x_n := 0$ in (1.1), we get

$$\begin{aligned}&\binom{n-2}{n-k-1}[f(kx+y) + f(x+ky)] + \binom{n-2}{n-k}[f(kx) + f(ky)]\\ &+(n-2)\binom{n-3}{n-k-2} f(x+y) + (n-2)\binom{n-3}{n-k-1}[f(x) + f(y)]\\ &= (n+k-1)\binom{n-2}{n-k-1} f(x+y) + \frac{nk(k-1)}{n-k}\binom{n-2}{n-k-1}[f(x) + f(y)],\end{aligned}$$

from which it follows that

$$f(kx+y) + f(x+ky) = 2kf(x+y) + (k-1)^2[f(x) + f(y)]$$

for all $x, y \in X$. Hence f is a quadratic mapping. □

3. The generalized Hyers–Ulam stability of Equation (1.1)

We recall some basic facts concerning the (β, p)-normed spaces [25].

Let β be a fixed real number with $0 < \beta \leq 1$ and X a linear space over $\mathbb{K}$, where $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A quasi-β-norm is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;
(3) There is a constant $M \geq 1$ such that $\|x+y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a quasi-β-normed space. A quasi-β-Banach space is a complete quasi-β-normed space. Let p be a real number with $0 < p \leq 1$. The quasi-β-norm $\|\cdot\|$ on X is called a (β, p)-norm if, moreover, $\|\cdot\|^p$ satisfies the following triangle inequality

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space. We notice that quasi-1-normed spaces are equivalent to quasi-normed spaces and that $(1, p)$-Banach spaces with $(1, p)$-norm are equivalent to p-Banach spaces with p-norm. We may

refer to [1, 26] for the concept of quasi-normed spaces and p-Banach spaces. Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [26], each quasi-norm is equivalent to some p-norm [1].

Now, we study the generalized Hyers–Ulam stability of the functional equation (1.1) by using the direct method. For convenience, we use the following abbreviations:

$$\begin{aligned} &Df(x_1,\cdots,x_n)\\ &:= \sum_{i=1}^{n} \sum_{1\le i_1\le\cdots\le i_n\le n} f\Big(kx_i + \sum_{j=1, i_j\neq i}^{n-k} x_{i_j}\Big) - (n+k-1)\begin{pmatrix} n-2 \\ n-k-1 \end{pmatrix} f\Big(\sum_{i=1}^{n} x_i\Big)\\ &\qquad -\frac{nk(k-1)}{n-k}\begin{pmatrix} n-2 \\ n-k-1 \end{pmatrix}\sum_{i=1}^{n} f(x_i), \quad (x_1,\cdots,x_n)\in X^n := X\times\cdots\times X;\\ &M := \frac{n-k}{k(k-1)\begin{pmatrix} n-2 \\ n-k-1 \end{pmatrix}}. \end{aligned}$$

Theorem 3.1. *Let X be a vector space and Y a (β,p)-Banach space. Assume that $\varphi : X^n \to [0,\infty)$ is a function satisfying*

$$\Phi_1(x_1,\cdots,x_n) := \sum_{j=0}^{\infty} \frac{\varphi\big(n^j x_1,\cdots,n^j x_n\big)^p}{n^{2j\beta p}} < \infty \tag{3.1}$$

for all $x_1,\cdots,x_n \in X$. If a mapping $f : X \to Y$ satisfies the functional inequality

$$\|Df(x_1,\cdots,x_n)\| \le \varphi(x_1,\cdots,x_n) \tag{3.2}$$

for all $x_1,\cdots,x_n \in X$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ such that

$$\|f(x) - Q_1(x)\| \le \frac{M^\beta}{n^{2\beta}}\Phi_1(x,\cdots,x)^{\frac{1}{p}} \tag{3.3}$$

for all $x \in X$.

Proof. Letting $x_1 = \cdots = x_n := x$ in (3.2), we obtain

$$\|f(nx) - n^2 f(x)\| \le M^\beta \varphi(x,\cdots,x)$$

for all $x \in X$. Dividing the above inequality by $n^{2\beta}$, one has

$$\Big\|\frac{f(nx)}{n^2} - f(x)\Big\| \le \frac{M^\beta}{n^{2\beta}}\varphi(x,\cdots,x)$$

for all $x \in X$. Replacing x by $n^l x$ and dividing by $n^{2l\beta}$, we get

$$\Big\|\frac{f(n^{l+1}x)}{n^{2(l+1)}} - \frac{f(n^l x)}{n^{2l}}\Big\|^p \le \frac{M^{\beta p}}{n^{2(l+1)\beta p}}\varphi(n^l x,\cdots,n^l x)^p$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. Thus, for any $m \in \mathbb{N}$, one deduces that

$$\|\frac{f(n^{l+m}x)}{n^{2(l+m)}} - \frac{f(n^l x)}{n^{2l}}\|^p \leq \frac{M^{\beta p}}{n^{2\beta p}} \sum_{i=l}^{l+m-1} \frac{\varphi(n^i x, \cdots, n^i x)^p}{n^{2i\beta p}} \tag{3.4}$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. In view of (3.4), it is easily checked that the sequence $\{\frac{f(n^l x)}{n^{2l}}\}$ is Cauchy in Y. Since Y is complete, the sequence is convergent in Y. Hence, we may define a mapping $Q_1 : X \to Y$ by

$$Q_1(x) := \lim_{l\to\infty} \frac{f(n^l x)}{n^{2l}} \tag{3.5}$$

for all $x \in X$. Moreover, if we take $l = 0$ and $m \to \infty$ in (3.4), then we arrive at the approximation (3.3). By (3.2) and (3.5), we see that

$$\|DQ_1(x_1, \cdots, x_n)\|^p = \lim_{l\to\infty} \|\frac{Df(n^l x_1, \cdots, n^l x_n)}{n^{2l}}\|^p \leq \lim_{l\to\infty} \frac{\varphi(n^l x_1, \cdots, n^l x_n)^p}{n^{2l\beta p}} = 0$$

for all $x_1, \cdots, x_n \in X$. Hence Q_1 satisfies the functional equation (1.1), and so it is quadratic.

To show the uniqueness of Q_1, we suppose there exists another quadratic mapping $Q_1^{'} : X \to Y$ which satisfies the functional inequality

$$\|f(x) - Q_1^{'}(x)\| \leq \frac{M^\beta}{n^{2^\beta}} \Phi_1(x, \cdots, x)^{\frac{1}{p}}$$

for all $x \in X$. Since Q_1 and $Q_1^{'}$ are quadratic mappings, we see that

$$\begin{aligned}
\|Q_1(x) - Q_1^{'}(x)\|^p &= \frac{1}{n^{2l\beta p}} \|Q_1(n^l x) - Q_1^{'}(n^l x)\|^p \\
&\leq \frac{1}{n^{2l\beta p}} [\|Q_1(n^l x) - f(n^l x)\|^p + \|f(n^l x) - Q_1^{'}(n^l x)\|^p] \\
&\leq \frac{2M^{\beta p}}{n^{2(l+1)\beta p}} \sum_{j=0}^{\infty} \frac{\varphi(n^{j+l}x, \cdots, n^{j+l}x)^p}{n^{2j\beta p}} = \frac{2M^{\beta p}}{n^{2\beta p}} \sum_{j=l}^{\infty} \frac{\varphi(n^j x, \cdots, n^j x)^p}{n^{2j\beta p}} = 0
\end{aligned}$$

for all $x \in X$. Hence Q_1 is a unique quadratic mapping satisfying (3.3). □

Theorem 3.2. *Let X be a vector space and Y a (β, p)-Banach space. Suppose there exists a function $\varphi : X^n \to [0, \infty)$ satisfying*

$$\Phi_2(x_1, \cdots, x_n) := \sum_{i=1}^{\infty} n^{2i\beta p} \varphi(\frac{x_1}{n^i}, \cdots, \frac{x_n}{n^i})^p < \infty$$

for all $x_1, \cdots, x_n \in X$. If a mapping $f : X \to Y$ satisfies the functional inequality (3.2), then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\|f(x) - Q_2(x)\| \leq \frac{M^\beta}{n^{2\beta}} \Phi_2(x, \cdots, x)^{\frac{1}{p}}$$

for all $x \in X$.

Proof. We see from (3.2) that

$$\|f(nx) - n^2 f(x)\|^p \leq M^{\beta p}\varphi(x, \cdots, x)^p$$

for all $x \in X$. Thus, it follows that

$$\|n^{2l} f(\frac{x}{n^l}) - f(x)\|^p \leq \frac{M^{\beta p}}{n^{2\beta p}} \sum_{i=1}^{l} n^{2i\beta p}\varphi(\frac{x}{n^i}, \cdots, \frac{x}{n^i})^p$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. The remaining assertion goes in a similar way as the corresponding part of Theorem 3.1. □

For the moment, we recall the fixed point alternative theorem from [18].

Theorem 3.3. *Let (X, d) be a generalized complete metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n, or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set

$$Y = \{y \in X | d(J^{n_0}x, y) < \infty\};$$

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In the following, we are going to apply the fixed point method to investigate the generalized Hyers–Ulam stability of the functional equation (1.1).

Theorem 3.4. *Let X be a vector space and Y a (β, p)-Banach space. Suppose there exists a positive number L_1 with $0 < L_1 < 1$ for which a function $\varphi : X^n \to [0, \infty)$ satisfies*

$$\varphi(nx_1, \cdots, nx_n) \leq n^{2\beta} L_1 \varphi(x_1, \cdots, x_n) \tag{3.6}$$

for all $x_1, \cdots, x_n \in X$. If a mapping $f : X \to Y$ satisfies the functional inequality (3.2), then there exists a unique quadratic mapping $Q_1 : X \to Y$ such that

$$\|f(x) - Q_1(x)\| \leq \frac{M^\beta}{n^{2\beta}(1 - L_1)}\varphi(x, \cdots, x) \tag{3.7}$$

for all $x \in X$.

Proof. First, we denote a set of mappings from X to Y by

$$Y^X := \{g : X \to Y \text{ a mapping}\},$$

and define a generalized metric d on Y^X as follows:

$$d(g,h) := \inf\{\alpha \in [0,\infty] : \|g(x) - h(x)\| \leq \alpha\varphi(x,\cdots,x)\,\forall x \in X\}$$

for all $g, h \in Y^X$. Then we may show that (Y^X, d) is a complete generalized metric space (see [13]).

Now, we define a mapping $J_1 : Y^X \to Y^X$ by

$$J_1 g(x) := \frac{g(nx)}{n^2}$$

for all $g \in Y^X$ and all $x \in X$. Given $g, h \in Y^X$, letting $c_{gh} \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq c_{gh}$, we can write

$$\|J_1 g(x) - J_1 h(x)\| = \|\frac{g(nx)}{n^2} - \frac{h(nx)}{n^2}\| \leq \frac{1}{n^{2\beta}} c_{gh}\varphi(nx,\cdots,nx) \leq L_1 c_{gh}\varphi(x,\cdots,x),$$

which yields $d(J_1 g, J_1 h) \leq L_1 c_{gh}$ and so $d(J_1 g, J_1 h) \leq L_1 d(g,h)$ by letting $c_{gh} \to d(g,h)^+$. Hence J_1 is a strictly contractive mapping with Lipschitz constant L_1 on Y^X.

Second, if we set $x_1 = \cdots = x_n := x$ in the hypothesis (3.2) and divide both sides by $n^{2\beta}$, then we have

$$\|f(x) - \frac{f(nx)}{n^2}\| \leq \frac{M^\beta}{n^{2\beta}}\varphi(x,\cdots,x)$$

for all $x \in X$, which implies

$$(f, J_1 f) \leq \frac{M^\beta}{n^{2\beta}} < \infty,$$

and so

$$d(J_1^k f, J_1^{k+1} f) \leq d(f, J_1 f) \leq \frac{M^\beta}{n^{2\beta}} < \infty$$

for all $k \in \mathbb{N}$. Thus according to Theorem 3.3, J_1 has a unique fixed point $Q_1 : X \to Y$ in the set

$$\Delta = \{g \in Y^X : d(f,g) < \infty\},$$

where Q_1 is defined by

$$Q_1(x) := \lim_{k\to\infty} J_1^k f(x) = \lim_{k\to\infty} \frac{f(n^k x)}{n^{2k}} \tag{3.8}$$

for all $x \in X$.

Moreover,

$$d(f, Q_1) \leq \frac{1}{1 - L_1} d(f, J_1 f) \leq \frac{M^\beta}{n^{2\beta}(1 - L_1)},$$

this means that (3.7) holds.

Finally, we prove that the mapping Q_1 is quadratic. It is follows from (3.2) and (3.8) that

$$\|DQ_1(x_1,\cdots,x_n)\| = \lim_{k\to\infty}\frac{1}{n^{2k\beta}}\|Df(n^kx_1,\cdots,n^kx_n)\|$$
$$\leq \lim_{k\to\infty}\frac{1}{n^{2k\beta}}\varphi(n^kx_1,\cdots,n^kx_n) \leq \lim_{k\to\infty}L_1^k\varphi(x_1,\cdots,x_n) = 0$$

for all $x_1,\cdots,x_n \in X$. By Theorem 2.4, the mapping $Q_1 : X \to Y$ is quadratic, as desired. □

Theorem 3.5. *Let X be a vector space and Y a (β,p)–Banach space. Assume that $\varphi : X^n \to [0,\infty)$ is a function satisfying*

$$\varphi(\frac{x_1}{n},\cdots,\frac{x_n}{n}) \leq \frac{L_2}{n^{2\beta}}\varphi_(x_1,\cdots,x_n)$$

for some real number L_2 with $0 < L_2 < 1$ and all $x_1,\cdots,x_n \in X$. If a mapping $f : X \to Y$ satisfies the functional inequality (3.2), then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\|f(x) - Q_2(x)\| \leq \frac{M^\beta L_2}{n^{2\beta}(1-L_2)}\varphi(x,\cdots,x)$$

for all $x \in X$.

Proof. Now, we define a mapping $J_2 : Y^X \to Y^X$ on the function space (Y^X, d) by

$$J_2g(x) := n^2g(\frac{x}{n})$$

for all $g \in Y^X$ and all $x \in X$. Then we see from (3.2) that

$$\|f(x) - n^2f(\frac{x}{n})\| \leq M^\beta\varphi(\frac{x}{n},\cdots,\frac{x}{n}) \leq \frac{M^\beta L_2}{n^{2\beta}}\varphi(x,\cdots,x)$$

for all $x \in X$, which induces

$$d(f,J_2f) \leq \frac{M^\beta L_2}{n^{2\beta}} < \infty$$

by definition.

The rest of proof follows from the similar argument to the corresponding part of Theorem 3.4. □

Corollary 3.6. *Let X be a quasi-α-normed space and Y a (β,p)-Banach space. If a mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x_1,\cdots,x_n)\| \leq \theta(\sum_{i=1}^n\|x_i\|^r)$$

for all $x_1, \cdots, x_n \in X$, *where* $r, \theta > 0, r\alpha \neq 2\beta$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{M^{\beta} n\theta}{|n^{2\beta p} - n^{\alpha r p}|^{\frac{1}{p}}} \|x\|^r$$

for all $x \in X$.

Corollary 3.7. *Let* X *be a quasi-*α*-normed space and* Y *a* (β, p)*-Banach space, and assume that* θ *is a given positive real number and* $r_1, \cdots, r_n$ *are real numbers with* $\sum_{i=1}^{n} r_i := r > 0$ *and* $r\alpha \neq 2\beta$. *If a mapping* $f : X \to Y$ *satisfies the functional inequality*

$$\|Df(x_1, \cdots, x_n)\| \leq \theta(\prod_{i=1}^{n} \|x_i\|^{r_i})$$

for all $x_1, \cdots, x_n \in X$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{M^{\beta} \theta}{|n^{2\beta p} - n^{\alpha r p}|^{\frac{1}{p}}} \|x\|^r$$

for all $x \in X$.

Corollary 3.8. *Let* X *be a vector space and* Y *a* (β, p)*-Banach space, and assume* $\varepsilon \geq 0$ *is any given real number. If a mapping* $f : X \to Y$ *satisfies the functional inequality*

$$\|Df(x_1, \cdots, x_n)\| \leq \varepsilon$$

for all $x_1, \cdots, x_n \in X$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{M^{\beta}}{(n^{2\beta p} - 1)^{\frac{1}{p}}} \varepsilon$$

for all $x \in X$.

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Hark-Mahn Kim, Department of Mathematics, Chungnam National University, 99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hmkim@cnu.ac.kr

Hong-Mei Liang, Department of Mathematics , Qiqihar University, Qiqihar 161-006, China; Department of Mathematics, Chungnam National University, 99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hmliang124@126.com

FIXED POINT THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN FUZZY HILBERT SPACES

AFSHAN BATOOL, TAYYAB KAMRAN, CHOONKIL PARK*, AND JUNG RYE LEE*

Abstract. The aim of the study is to investigate nonexpansive, nonspreading, hybrid and contractive mappings in fuzzy Hilbert spaces and to prove some fixed point theorems.

1. Introduction and preliminaries

It was Katsaras [17], who while studying fuzzy topological vector spaces, was the first to introduce the idea of fuzzy norm on a linear space in 1984. Later on many other mathematicians like Felbin [13], Cheng & Mordeson [11], Bag & Samanta [1] etc. introduced definitions of fuzzy normed linear spaces in different approach. A large number of paper have been published in fuzzy normed linear spaces, for reference please see [2, 3, 4, 5, 6, 7, 14, 15]. On the other hand studies on fuzzy inner product spaces are relatively recent and few works have been done in fuzzy inner product spaces. Biswas [8], El-Abyed & Hamouly [12] were among the first who gave a meaningful definition of fuzzy inner product space and associated fuzzy norm function. Later on, Kohli & Kumar [20] modified the definition of inner product space introduced by Biswas. We introduce a broad class of nonlinear mappings in fuzzy Hilbert spaces and then we prove some fixed point theorems for the class of such mappings.

Definition 1.1. Let U be a real linear space. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if, for all $x, u \in U$ and $c \in F$, the following conditions are satisfied:

(N1) $\forall t \in \mathbb{R}\ , t \leq 0; N(x,t) = 0$;

(N2) $\forall t \in \mathbb{R}, t > 0; N(x,t) = 1$ if and and only if $x = 0$;

(N3) $\forall t \in \mathbb{R}, t > 0; N(x,t) = N(x, \frac{t}{|c|})\ if\ c \neq 0$;

(N4) $\forall s, t \in \mathbb{R}, x, u \in U$;

$$N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$$

(N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim\limits_{t\to\infty} N(x,t) = 1$.

The pair (U, N) will be referred to as a fuzzy normed linear space.

Theorem 1.2. *Let (U, N) be a fuzzy normed linear space. Assume further that,*

(N6) $\forall t > 0, N(x,t) > 0$ implies $x = 0$.

Define $\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\}, \alpha \in (0,1)$.

Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U and they are called α-norms on U corresponding to the fuzzy norm N on U.

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*Corresponding authors.

Theorem 1.3. *Let (U, N) be a fuzzy normed linear space satisfying (N6). Assume further that*

(N7) for $x \neq 0$, $N(x, \cdot)$ is a continuous function of $\mathbb{R}$.

Let $\|x\|_\alpha = \wedge \{t > 0 : N(x,t) \geq \alpha\}, \alpha \in (0,1)$ and $N' : U \times \mathbb{R} \to [0,1]$ be a function defined by

$$N'(x,t) = \begin{cases} \wedge \{\alpha \in (0,1) : \|x\|_\alpha \leq t\}, & \text{if } (x,t) \neq (0,0) \\ 0 & \text{if } (x,t) = (0,0). \end{cases}$$

Then (i) $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U.

(ii) N' is a fuzzy norm on U.

(iii) $N' = N$

In a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$ (see [31]). Furthermore, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{1.2}$$

for all $x, y, z, w \in H$. Using means and the Riesz theorem, we can obtain the following result (see [22, 25, 26, 27]).

Lemma 1.4. *Let H be a Hilbert space, $\{x_n\}$ a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n | n \in N\}$ such that*

$$\mu_n \langle x_n, y\rangle = \langle z_0, y\rangle, \forall y \in H.$$

We can define the following nonlinear mappings (see [9, 16, 18, 19, 21, 28, 29]) in fuzzy Hilbert spaces.

Let H be a fuzzy Hilbert space with inner product $\{\langle ., . \rangle_\alpha : \alpha \in (0,1)\}$ and norm $\{\|.\|_\alpha : \alpha \in (0,1)\}$. Let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive*, *nonspreading*, and *hybrid* if

$$\|Tx - Ty\|_\alpha \leq \|x-y\|_\alpha,$$

$$2\|Tx-Ty\|_\alpha^2 \leq \|Tx-y\|_\alpha^2 + \|Ty-x\|_\alpha^2$$

and

$$3\|Tx-Ty\|_\alpha^2 \leq \|x-y\|_\alpha^2 + \|Tx-y\|_\alpha^2 + \|Ty-x\|_\alpha^2$$

for all $x, y \in C$, respectively. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx-Fy\|_\alpha^2 \leq \langle x-y, Fx-Fy\rangle_\alpha$$

for all $x, y \in C$. A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|_\alpha^2 + \beta\|x-Ty\|_\alpha^2 + \gamma\|Tx-y\|_\alpha^2 + \delta\|x-y\|_\alpha^2$$

$$+max\left\{\varepsilon\|x-Tx\|_\alpha^2, \zeta\|y-Ty\|_\alpha^2\right\} \leq 0$$

for all $x, y \in C$ and T is called symmetric generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2)$$

$$+\gamma \|x - y\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) \leq 0 \tag{1.3}$$

for all $x, y \in C$. Such a mapping T is also called $(\alpha, \beta, \gamma, \delta)$-*symmetric generalized hybrid*. If $\alpha = 1, \beta = \delta = 0$ and $\gamma = -1$ in (1.3), then the mapping T is *nonexpansive*. If $\alpha = 2, \beta = -1$ and $\gamma = \delta = 0$ in (1.3), then the mapping T is *nonspreading*. Furthermore, if $\alpha = 3, \beta = \gamma = -1$ and $\delta = 0$ in (1.3), then the mapping T is *hybrid*.

Let H be a fuzzy Hilbert space and let C be a nonempty subset of H. Then $T : C \to H$ is called a *widely strict pseudo-contraction* if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + r \|(I - T)x - (I - T)y\|_\alpha^2, \quad \forall\ x, y \in C.$$

We call such T a *widely r-strict pseudo-contraction*. If $0 \leq r < 1$, then T is a *strict pseudo-contraction* (see [10, 23, 24]). Furthermore, if $r = 0$, then T is nonexpansive. Conversely, let $S : C \to H$ be a nonexpansive mapping and define $T : C \to H$ by $T = \frac{1}{1+n}S + \frac{n}{1+n}I$ for all $x \in C$ and $n \in \mathbb{N}$. Then T is a widely (-n)-strict pseudocontraction. In fact, from the definition of T, it follows that $S = (1 + n)T - nI$. Since S is nonexpansive, we have that for any $x, y \in C$,

$$\|(1 + n)Tx - nx - ((1 + n)Ty - ny)\|_\alpha^2 \leq \|x - y\|_\alpha^2$$

and hence

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + n \|(I - T)x - (I - T)y\|_\alpha^2.$$

We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. Let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T.

2. Fixed point theorems

Theorem 2.1. *Let H be a fuzzy Hilbert space, C a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

Proof. Suppose that T has a fixed point z. Then $\{T^n z : n = 0, 1, ...\} = \{z\}$ and hence $\{T^n z : n = 0, 1, ...\}$ is bounded. Conversely, suppose that there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. Since T is an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping of C into itself, we have that

$$\alpha \left\|Tx - T^{n+1}z\right\|_\alpha^2 + \beta(\left\|x - T^{n+1}z\right\|_\alpha^2 + \|Tx - T^n z\|_\alpha^2)$$

$$+\gamma \|x - T^n z\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \left\|T^n z - T^{n+1}z\right\|_\alpha^2) \leq 0$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Since $\mu_n \|Tx - T^n z\|_\alpha^2 = \mu_n \|Tx - T^{n+1}z\|_\alpha^2$ and $\mu_n \|x - T^{n+1}z\|_\alpha^2 = \mu_n \|x - T^n z\|_\alpha^2$, we have that

$$(\alpha+\beta)\mu_n \|Tx - T^n z\|_\alpha^2 + (\beta+\gamma)\mu_n \|x - T^n z\|_\alpha^2$$

$$+\delta(\|x - Tx\|_\alpha^2 + \mu_n \|T^n z - T^{n+1}z\|_\alpha^2) \leq 0.$$

Furthermore, since

$$\mu_n \|Tx - T^n z\|_\alpha^2 = \|Tx - x\|_\alpha^2 + 2\mu_n \langle Tx - x, x - T^n z\rangle_\alpha + \mu_n \|x - T^n z\|_\alpha^2,$$

we have that

$$(\alpha+\beta+\delta)\|Tx - x\|_\alpha^2 + 2(\alpha+\beta)\mu_n \langle Tx - x, x - T^n z\rangle_\alpha$$

$$+(\alpha+2\beta+\gamma)\mu_n \|x - T^n z\|_\alpha^2 + \delta\mu_n \|T^n z - T^{n+1}z\|_\alpha^2 \leq 0.$$

From (1) $\alpha + 2\beta + \gamma \geq 0$ and (3) $\delta \geq 0$, we have that

$$(\alpha+\beta+\delta)\|Tx - x\|_\alpha^2 + 2(\alpha+\beta)\mu_n \langle Tx - x, x - T^n z\rangle_\alpha \leq 0. \tag{2.1}$$

Since there exists $p \in H$ from Lemma 1.4 such that

$$\mu_n \langle y, T^n z\rangle_\alpha = \langle y, p\rangle_\alpha$$

for all $y \in H$, we have from (2.1) that

$$(\alpha+\beta+\delta)\|Tx - x\|_\alpha^2 + 2(\alpha+\beta)\langle Tx - x, x - p\rangle_\alpha \leq 0. \tag{2.2}$$

Since C is closed and convex, we have that

$$p \in \overline{co}\{T^n x : n \in N\} \subset C.$$

Putting $x = p$, we obtain from (2.2) that

$$(\alpha+\beta+\delta)\|Tp - p\|_\alpha^2 \leq 0. \tag{2.3}$$

We have from (2) $\alpha+\beta+\delta > 0$ that $\|Tp - p\|_\alpha^2 \leq 0$. This implies that p is a fixed point in T.

Next suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\alpha \|Tp_1 - Tp_2\|_\alpha^2 + \beta(\|p_1 - Tp_2\|_\alpha^2 + \|Tp_1 - p_2\|_\alpha^2)$$

$$+\gamma \|p_1 - p_2\|_\alpha^2 + \delta(\|p_1 - Tp_1\|_\alpha^2 + \|p_2 - Tp_2\|_\alpha^2) \leq 0$$

and hence$(\alpha + 2\beta + \gamma)\|p_1 - p_2\|_\alpha^2 \leq 0$. We have from $\alpha + 2\beta + \gamma > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. □

We can derive the following theorem from Theorem 2.1.

Theorem 2.2. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

A mapping T from C into H is called $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2) + \gamma \|x - y\|_\alpha^2$$

$$+\delta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) + \zeta \|x - y - (Tx - Ty)\|_\alpha^2 \leq 0 \tag{2.4}$$

for all $x, y \in C$.

Theorem 2.3. *Let H be a fuzzy Hilbert space, C a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

Proof. Since $T : C \to C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ satisfying (2.4). We also have that

$$\|x - y - (Tx - Ty)\|_\alpha^2 = \|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2$$

$$- \|x - Ty\|_\alpha^2 - \|y - Tx\|_\alpha^2 + \|x - y\|_\alpha^2 + \|Tx - Ty\|_\alpha^2 \tag{2.5}$$

for all $x, y \in C$. Thus we obtain from (2.4) that

$$(\alpha + \zeta) \|Tx - Ty\|_\alpha^2 + (\beta - \zeta)(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2)$$

$$+(\gamma + \zeta) \|x - y\|_\alpha^2 + (\delta + \zeta)(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) \leq 0. \tag{2.6}$$

The conditions (1) $\alpha + 2\beta + \gamma \geq 0$ and (2) $\alpha + \beta + \delta + \zeta > 0$ are equivalent to $(\alpha + \zeta) + 2(\beta - \zeta) + (\gamma + \zeta) \geq 0$ and $(\alpha + \zeta) + (\beta - \zeta) + (\delta + \zeta) > 0$, respectively. Furthermore, since (3) $\delta + \zeta \geq 0$ holds, we have the desired result from Theorem 2.1. □

As a direct consequence of Theorem 2.3, we obtain the following.

Theorem 2.4. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

We can extend the above theorem as follows.

Theorem 2.5. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) *there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \geq 0$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

Proof. Since $T : C \to C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping, we obtain that

$$\alpha \left\|Tx - T^{n+1}z\right\|_\alpha^2 + \beta(\left\|x - T^{n+1}z\right\|_\alpha^2 + \|Tx - T^n z\|_\alpha^2) + \gamma \|x - T^n z\|_\alpha^2$$

$$+\delta(\|x - Tx\|_\alpha^2 + \left\|T^n z - T^{n+1}z\right\|_\alpha^2) + \zeta \left\|(x - Tx) - \left(T^n z - T^{n+1}z\right)\right\|_\alpha^2 \leq 0$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in C$.

Let $\lambda \in [0, 1) \cap \{\lambda : (\alpha + \beta)\lambda + \zeta + \eta \geq 0\}$ and define $S = (1 - \lambda)T + \lambda I$. Since C is convex, S is a mapping from C into itself. Since C is bounded, $\{S^n z : n = 0, 1, ...\}$ is bounded for all $z \in C$. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$\begin{aligned}
& \alpha \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|_\alpha^2 \\
& +\beta \left\| x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|_\alpha^2 + \beta \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) - y \right\|_\alpha^2 + \gamma \|x - y\|_\alpha^2 \\
& +\delta \left\| x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) \right\|_\alpha^2 + \delta \left\| y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|_\alpha^2 \\
& +\zeta \left\| \left(x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) \right) - \left(y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right) \right\|_\alpha^2 \\
= \; & \alpha \left\| \left(\frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x - y) \right) \right\|_\alpha^2 \\
& +\beta \left\| x - \left(\frac{1}{1-\lambda}(x - Sy) - \frac{\lambda}{1-\lambda}(x - y) \right) \right\|_\alpha^2 \\
& +\beta \left\| x - \left(\frac{1}{1-\lambda}(Sx - y) - \frac{\lambda}{1-\lambda}(x - y) \right) \right\|_\alpha^2 + \gamma \|x - y\|_\alpha^2 \\
& +\delta \left\| \frac{1}{1-\lambda}(x - Sx) \right\|_\alpha^2 + \delta \left\| \frac{1}{1-\lambda}(y - Sy) \right\|_\alpha^2 \\
& +\zeta \left\| \frac{1}{1-\lambda}(x - Sx) - \frac{1}{1-\lambda}(y - Sy) \right\|_\alpha^2 \\
= \; & \frac{\alpha}{1-\lambda}\|Sx - Sy\|_\alpha^2 + \frac{\beta}{1-\lambda}\|x - Sy\|_\alpha^2 \\
& +\frac{\beta}{1-\lambda}\|Sx - y\|_\alpha^2 + \left(-\frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma \right) \|x - y\|_\alpha^2 \\
& +\frac{\delta + \beta\lambda}{(1-\lambda)^2}\|x - Sx\|_\alpha^2 + \frac{\delta + \beta\lambda}{(1-\lambda)^2}\|y - Sy\|_\alpha^2
\end{aligned}$$

$$+\frac{\zeta + \alpha\lambda}{(1-\lambda)^2}\|(x - Sx) - (y - Sy)\|_\alpha^2 \leq 0$$

Therefore, S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma, \frac{\delta+\beta\lambda}{(1-\lambda)^2}, \frac{\zeta+\alpha\lambda}{(1-\lambda)^2}\right)$-symmetric more generalized hybrid mapping. Furthermore, we obtain that

$$\frac{\alpha}{1-\lambda}+\frac{2\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma=\alpha+2\beta+\gamma\geq 0,$$

$$\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\delta+\beta\lambda}{(1-\lambda)^2}+\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}=\frac{\alpha+\beta+\delta+\zeta}{(1-\lambda)^2}>0,$$

$$\frac{\delta+\beta\lambda}{(1-\lambda)^2}+\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}=\frac{(\alpha+\beta)\lambda+\delta+\zeta}{(1-\lambda)^2}\geq 0.$$

Therefore, by Theorem 2.4, we obtain $F(S)\neq\phi$.

Next, suppose that $\alpha+2\beta+\gamma>0$. Let p_1 and p_2 be fixed points of T. Then

$$\alpha\|Tp_1-Tp_2\|_\alpha^2+\beta(\|p_1-Tp_2\|_\alpha^2+\|Tp_1-p_2\|_\alpha^2)+\gamma\|p_1-p_2\|_\alpha^2$$

$$\delta(\|p_1-Tp_1\|_\alpha^2+\|p_2-Tp_2\|_\alpha^2)+\zeta\|(p_1-Tp_1)+(p_2-Tp_2)\|_\alpha^2$$

$$=(\alpha+2\beta+\gamma)\|p_1-p_2\|_\alpha^2\leq 0$$

and hence $p_1=p_2$. Therefore a fixed point of T is unique. □

For the case $\beta+\delta=0$ in Theorem 2.5, we have the following theorem.

Theorem 2.6. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha,-\beta,\gamma,\beta,\zeta)$-symmetric more generalized hybrid mapping from C into itself, i.e., there exist α, β, γ, $\zeta\in\mathbb{R}$ such that*

$$\alpha\|Tx-Ty\|_\alpha^2+\beta(\|x-Ty\|_\alpha^2+\|Tx-y\|_\alpha^2)+\gamma\|x-y\|_\alpha^2$$

$$-\beta(\|x-Tx\|_\alpha^2+\|y-Ty\|_\alpha^2)+\zeta\|x-y-(Tx-Ty)\|_\alpha^2\leq 0 \tag{2.7}$$

for all $x,y\in C$. Furthermore, suppose that T satisfies the following conditions (1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\zeta>0$ *and* (3) *there exists $\lambda\in[0,1)$ such that $(\alpha+\beta)\lambda+\delta+\zeta\geq 0$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+2\beta+\gamma>0$ on the condition (1).*

Using Theorem 2.3, we prove the following fixed point theorem.

Theorem 2.7. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself, i.e., there exists $r\in\mathbb{R}$ with $r<1$ such that*

$$\|Tx-Ty\|_\alpha^2\leq\|x-y\|_\alpha^2+r\|(I-T)x-(I-T)y\|_\alpha^2,\forall x,y\in C. \tag{2.8}$$

Then T has a fixed point in C.

Proof. We first assume that $r\leq 0$. We have from (2.8) that for all $x,y\in C$,

$$\|Tx-Ty\|_\alpha^2-\|x-y\|_\alpha^2-r\|(I-T)x-(I-T)y\|_\alpha^2\leq 0 \tag{2.9}$$

Then T is a $(1,0,-1,0,-r)$-symmetric more generalized hybrid mapping. Furthermore, (1) $\alpha+2\beta+\gamma=1-1\geq 0$, (2) $\alpha+\beta+\delta+\zeta=1-r>0$ and (3) $\delta+\zeta=-r\geq 0$ in Theorem 1.4

are satisfied. Thus T has a fixed point from Theorem 2.3. Assume that $0 \leq r < 1$ and define a mapping T as follows:

$$Sx = \lambda x + (1-\lambda)Tx, \forall x \in C,$$

where $r \leq \lambda < 1$. Then S is a mapping from C into itself and $F(S) = F(T)$. From $Sx = \lambda x + (1-\lambda)Tx$, we also have that

$$Tx = \frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x.$$

Thus we have

$$\begin{aligned}
0 \geq & \left\| \frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|_\alpha^2 \\
& - \|x-y\|_\alpha^2 - r\left\|x-y-\left\{\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\}\right\|_\alpha^2 \\
= & \left\| \frac{1}{1-\lambda}(Sx-Sy) - \frac{\lambda}{1-\lambda}(x-y)\right\|_\alpha^2 \\
& - \|x-y\|_\alpha^2 - r\left\|\frac{1}{1-\lambda}(x-y) - \frac{1}{1-\lambda}(Sx-Sy)\right\|_\alpha^2 \\
= & \frac{1}{1-\lambda}\|Sx-Sy\|_\alpha^2 - \frac{\lambda}{1-\lambda}\|x-y\|_\alpha^2 \\
& + \frac{1}{1-\lambda}.\frac{\lambda}{1-\lambda}\|x-y-(Sx-Sy)\|_\alpha^2 - \|x-y\|_\alpha^2 \\
& - \frac{r}{(1-\lambda)^2}\|x-y-(Sx-Sy)\|_\alpha^2 \\
= & \frac{1}{1-\lambda}\|Sx-Sy\|_\alpha^2 - \frac{1}{1-\lambda}\|x-y\|_\alpha^2 + \frac{\lambda-r}{(1-\lambda)^2}\|x-y-(Sx-Sy)\|_\alpha^2.
\end{aligned}$$

Then S is a $\left(\frac{1}{1-\lambda}, 0, -\frac{1}{1-\lambda}, 0, \frac{\lambda-r}{(1-\lambda)^2}\right)$-symmetric more generalized hybrid. From

$$\begin{aligned}
\frac{1}{1-\lambda} + 2.0 - \frac{1}{1-\lambda} &= 0, \\
\frac{1}{1-\lambda} + \frac{\lambda-r}{(1-\lambda)^2} &> 0 \qquad \text{and} \\
\frac{\lambda-r}{(1-\lambda)^2} &\geq 0,
\end{aligned}$$

(1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ in Theorem 1.4 are satisfied.Thus S has a fixed point in C from Theorem 2.3 and hence T has a fixed point. This completes the proof. □

Let H be a fuzzy Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. For $u \in H$ and $s, t \in (0,1)$, we define the following mapping:

$$Sx = tx + (1-t)(su + (1-s)Tx)$$

for all $x \in C$. We call such S a TWY mapping generated by u, T, s, t. Since $Sx = tx + s(1-t)u + (1-t)(1-s)Tx$, we have that for all $x, y \in C$,

$$\begin{aligned} \|Sx - Sy\|_\alpha^2 &= \|t(x-y) + (1-t)(1-s)(Tx - Ty)\|_\alpha^2 \\ &= t^2\|x-y\|_\alpha^2 + (1-t)^2(1-s)^2\|Tx-Ty\|_\alpha^2 \\ &\quad +2t(1-t)(1-s)\langle x-y, Tx-Ty\rangle_\alpha \\ &= t^2\|x-y\|_\alpha^2 + (1-t)^2(1-s)^2\|Tx-Ty\|_\alpha^2 \\ &\quad +t(1-t)(1-s)(\|x-Ty\|_\alpha^2 + \|y-Tx\|_\alpha^2 \\ &\quad -\|x-Tx\|_\alpha^2 - \|y-Ty\|_\alpha^2) \\ &= t^2\|x-y\|_\alpha^2 + (1-t)^2(1-s)^2\|Tx-Ty\|_\alpha^2 \\ &\quad +t(1-t)(1-s)(\|x-Ty\|_\alpha^2 + \|y-Tx\|_\alpha^2) \\ &\quad -t(1-t)(1-s)(\|x-Tx\|_\alpha^2 - \|y-Ty\|_\alpha^2). \end{aligned} \tag{2.10}$$

Similarly, we have that

$$\begin{aligned} &\|x-Sy\|_\alpha^2 + \|y-Sx\|_\alpha^2 \\ =\ & s(1-t)^2\left(\|u-x\|_\alpha^2 + \|u-y\|_\alpha^2\right) \\ &-s(1-s)(1-t)^2(\|u-Tx\|_\alpha^2 + \|u-Ty\|_\alpha^2) \\ &-t(1-t)(1-s)(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2) \\ &+(1-t)(1-s)(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2) + 2t\|x-y\|_\alpha^2, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\|x-Sx\|_\alpha^2 + \|y-Sy\|_\alpha^2 \\ =\ & s(1-t)^2\left(\|u-x\|_\alpha^2 + \|u-y\|_\alpha^2\right) \\ &-s(1-s)(1-t)^2(\|u-Tx\|_\alpha^2 + \|u-Ty\|_\alpha^2) \\ &+s(1-s)(1-t)^2(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2). \end{aligned}$$

We also have that

$$\begin{aligned} &\|x-y-Sx-Sy\|_\alpha^2 \\ =\ & (1-s)(1-t)^2(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2) \\ &-(1-s)(1-t)^2(\|x-Ty\|_\alpha^2 + \|y-Tx\|_\alpha^2) \\ &+(1-t)^2\|x-y\|_\alpha^2 + (1-t)^2(1-s)^2\|Tx-Ty\|_\alpha^2. \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12), we have that

$$\begin{aligned} &\|x-Sx\|_\alpha^2 + \|y-Sy\|_\alpha^2 - \|x-Sy\|_\alpha^2 - \|y-Sx\|_\alpha^2 \\ =\ & (1-s)(1-t)(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2 \\ &-\|x-Ty\|_\alpha^2 + \|y-Tx\|_\alpha^2 - 2t\|x-y\|_\alpha^2). \end{aligned} \tag{2.13}$$

Using (2.10) and (2.13), we have the following theorem.

Theorem 2.8. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $r < 1$ such that*

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + r \|(I - T)x - (I - T)y\|_\alpha^2 \quad \forall x, y \in C. \tag{2.14}$$

Let $u \in C$ and $s \in (0, 1)$. Define a mapping $U : C \to C$ as follows:

$$Ux = su + (1 - s)Tx, \forall x \in C.$$

Then U has a unique fixed point in C.

Proof. Since T is a widely r-strict pseudo-contraction from C into itself, we have that for all $x, y \in C$,

$$\|Tx - Ty\|_\alpha^2 - \|x - y\|_\alpha^2 - r \|(I - T)x - (I - T)y\|_\alpha^2 \leq 0.$$

If $r \leq 0$, then T is a nonexpansive mapping. Therefore U is a contractive mapping. Using the fixed point theorem for contractive mappings, we have that U has a unique fixed point in C. Let $0 < r < 1$. Since

$$\begin{aligned} \|x - y - (Tx - Ty)\|_\alpha^2 &= \|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2 \\ &\quad - \|x - Ty\|_\alpha^2 - \|y - Tx\|_\alpha^2 + \|x - y\|_\alpha^2 + \|Tx - Ty\|_\alpha^2, \end{aligned}$$

we have that

$$(1 - r) \|Tx - Ty\|_\alpha^2 - (1 + r) \|x - y\|_\alpha^2$$

$$-r \|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2 - \|x - Ty\|_\alpha^2 - \|y - Tx\|_\alpha^2 \leq 0.$$

For u, T and $s, r \in (0, 1)$, define a TWY mapping S as follows:

$$Sx = rx + (1 - r) (su + (1 - s)Tx), \forall x \in C.$$

Then we have from (2.10) that

$$\begin{aligned} &\frac{1}{(1 - r)(1 - s)^2} \|Sx - Sy\|_\alpha^2 - \frac{r^2}{(1 - r)(1 - s)^2} \|x - y\|_\alpha^2 \\ &+ \frac{r}{(1 - s)} \left(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2 - \|x - Ty\|_\alpha^2 - \|y - Tx\|_\alpha^2 \right) \\ &- (1 + r) \|x - y\|_\alpha^2 \\ &-r(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2 - \|x - Ty\|_\alpha^2 - \|y - Tx\|_\alpha^2) \leq 0. \end{aligned}$$

We have from (2.13) that

$$\begin{aligned} &\frac{1}{(1 - r)(1 - s)^2} \|Sx - Sy\|_\alpha^2 - \frac{r^2}{(1 - r)(1 - s)^2} \|x - y\|_\alpha^2 \\ &+ \frac{r}{(1 - r)(1 - s)^2} \left(\|x - Sx\|_\alpha^2 + \|y - Sy\|_\alpha^2 - \|x - Sy\|_\alpha^2 - \|y - Sx\|_\alpha^2 \right) \\ &+ \frac{2r^2}{(1 - r)(1 - s)^2} \|x - y\|_\alpha^2 - - (1 + r) \|x - y\|_\alpha^2 \\ &- \frac{r}{(1 - r)(1 - s)} \left(\|x - Sx\|_\alpha^2 + \|y - Sy\|_\alpha^2 - \|x - Sy\|_\alpha^2 - \|y - Sx\|_\alpha^2 \right) \end{aligned}$$

$$-\frac{2r^2}{(1-r)(1-s)}\|x-y\|_\alpha^2 \leq 0$$

and hence

$$\begin{aligned}
&\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|_\alpha^2 \\
&-\frac{rs}{(1-r)(1-s)^2}\left(\|x-Sy\|_\alpha^2+\|y-Sx\|_\alpha^2\right) \\
&+\left(\frac{r^2}{(1-r)(1-s)^2}-\frac{1-s+r^2(1+s)}{(1-r)(1-s)}\right)\|x-y\|_\alpha^2 \\
&+\frac{rs}{(1-r)(1-s)^2}\left(\|x-Sx\|_\alpha^2+\|y-Sy\|_\alpha^2\right)\leq 0.
\end{aligned}$$

For this inequality, we apply Theorem 2.2. We first obtain that

$$\begin{aligned}
&\frac{1}{(1-r)(1-s)^2}-\frac{2rs}{(1-r)(1-s)^2}+\frac{r^2}{(1-r)(1-s)^2}-\frac{1-s+r^2(1+s)}{(1-r)(1-s)} \\
=\ &\frac{s(1+r)(2-s(1-r))}{(1-r)(1-s)^2}>0.
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
&\frac{1}{(1-r)(1-s)^2}-\frac{rs}{(1-r)(1-s)^2}+\frac{rs}{(1-r)(1-s)^2} \\
=\ &\frac{1}{(1-r)(1-s)^2}>0,
\end{aligned}$$

$$\frac{rs}{(1-r)(1-s)^2}\geq 0.$$

Thus S has a unique fixed point z in C from Theorem 1.3. Since z is a fixed point of S, we have $z=rz+(1-r)(su+(1-s)Tz$. From $1-r\ \neq 0$, we have that

$$z=su+(1-s)Tz.$$

This completes the proof. □

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Afshan Batool
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mail address: afshan.batoolqua@gmail.com

Tayyab Kamran
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mail address: TayyabKamran@gmail.com

Choonkil Park
Research Institute for Natural Sciences Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Jung Rye Lee
Department of Mathematics, Daejin University, Kyunggi 11159, Korea
E-mail address: jrlee@daejin.ac.kr

Some properties on Dirichlet-Hadamard product of Dirichlet series *

Yong-Qin Cui[a] and Hong-Yan Xu[b†]

[a] Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi, 333403, China
e-mail: cuicui285@163.com

[b] Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi, 333403, China
e-mail: xhyhhh@126.com

Abstract

By constructing a new form Dirichlet-Hadamard product of Dirichlet series, we investigate the relation about the growth of Dirichlet series and obtain some estimates on the upper and the lower bounds of the (lower) q-order and the (lower) q-type of Dirichlet-Hadamard product of Dirichlet series. We also study the growth on scalar multiplication and shift of Dirichlet series. Our results of this paper are improvements of the previous theorems given by Kong and Deng.

Key words: Dirichlet-Hadamard product, growth, scalar multiplication, Dirichlet series.

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1 Introduction and basic notes

Consider Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_n e^{\lambda_n s},\qquad s=\sigma+it, \tag{1}$$

where

$$0\le\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty,\quad as\quad n\to\infty;$$

$s=\sigma+it$ (σ,t are real variables); a_n are nonzero complex numbers. Let $f(s)$ satisfy

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0, \tag{2}$$

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\lambda_n}=-\infty, \tag{3}$$

then we have the abscissas of convergence and absolute convergence are $+\infty$ by applying the Valion's formula (see [4]), that is, $f(s)$ is an analytic function in the whole plane $\mathbb{C}$. We denote D to be the class of all functions $f(s)$ satisfying (2),(3).

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†Corresponding author.

Definition 1.1 *(see [16]). Let $f(s) \in D$, the order of $f(s)$ is defined by*

$$\rho = \limsup_{\sigma \to +\infty} \frac{\log\log M(\sigma, f)}{\sigma},$$

where

$$M(\sigma, f) = \sup_{-\infty < t \in < +\infty} \{|f(\sigma + it)|\}, \quad for \quad \sigma \in \mathbb{R}.$$

For $\rho = 0, 0 < \rho < \infty, \rho = \infty$, $f(s)$ can be called zero order, finite order, infinite order Dirichlet series, respectively. For infinite order Dirichlet series, we will introduce the definition of q-order as follows.

Definition 1.2 *(see [16]). Let $f(s) \in D$, we define the q-order ρ and lower q-order χ of $f(s)$ as follows*

$$\rho = \rho_{[q]} = \limsup_{\sigma \to +\infty} \frac{\log^{[q]} M(\sigma, f)}{\sigma}, \quad \chi = \chi_{[q]} = \liminf_{\sigma \to +\infty} \frac{\log^{[q]} M(\sigma, f)}{\sigma}.$$

In addition, if $\rho \in (0, +\infty)$, the q-type T and lower q-type τ can be defined as follows:

$$T = \limsup_{\sigma \to +\infty} \frac{\log^{[q-1]} M(\sigma, f)}{e^{\sigma\rho}}, \quad \tau = \liminf_{\sigma \to +\infty} \frac{\log^{[q-1]} M(\sigma, f)}{e^{\sigma\rho}},$$

where $q = 2, 3, \cdots, \log^{[0]} x = x, \log^{[k]} = \log(\log^{[k-1]} x)$.

Definition 1.3 *(see [16]). If $\rho = \chi$, $f(s)$ is called $\rho_{[q]}$-regular growth, and if $\tau = T$, $f(s)$ is called $\rho_{[q]}$-perfectly regular growth.*

In the past several decades, considerable attention has been paid to the growth and the value distribution of Dirichlet series; see [4, 16] for some results. For examples, J. R. Yu, D. C. Sun and Z. S. Gao investigated the growth and value distribution of entire functions defined by Dirichlet series (see [1, 2, 3, 5, 9, 12, 15]); M. N. Sheremeta, A. Nautiyal, and H. Y. Xu studied the problem on the approximation of Dirichlet series (see [8, 10, 13, 14]); Y. Y. Kong, K. A. M. Sayyed, M. S. Metwally and M. T. Mohamed studied the growth of Hadamard-product of Dirichlet series (see [6, 7, 11]), and so on. We list several classical results as follows.

Theorem 1.1 *Let $f(s) \in D$ be of order ρ, then*

$$\rho = \limsup_{n \to +\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |a_n|^{-1}}, \tag{4}$$

$$T = \limsup_{n \to +\infty} |a_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}). \tag{5}$$

Theorem 1.2 *Let $f(s) \in D$ be of lower order χ, then*

$$\chi \leq \liminf_{n \to +\infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |a_n|^{-1}}, \tag{6}$$

where $q = 2, 3, \cdots$,the equality holds if and only if

$$\varphi(n) = \frac{\log |a_n| - \log |a_{n+1}|}{\lambda_{n+1} - \lambda_n}$$

is a non-decreasing function with n.

Theorem 1.3 *Let $f(s) \in D$ be of order $\rho(0 < \rho < +\infty)$ and of type τ, then*

$$\tau \leq \liminf_{n\to+\infty} |a_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_{n-1}}{e\rho}), \tag{7}$$

where $q = 2, 3, \cdots$, the equality holds if and only if $\varphi(n)$ is a non-decreasing function with n, and $\log^{[q-2]} \lambda_{n-1} \sim \log^{[q-2]} \lambda_n, n \to +\infty$.

In 2009 and 2014, Y. Y. Kong and G. T. Deng investigated their growth of Dirichlet series by defining the Dirichlet-Hadamard product (see [5, 6]) and obtained some interesting results.

Definition 1.4 *(see [5, 6]). Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{\gamma_n s}$, $f_2(s) = \sum_{n=1}^{\infty} b_n e^{\xi_n s}$ and $f_1(s)$, $f_2(s) \in D$, the Dirichlet-Hadamard product function $F_1(s)$ of $f_1(s), f_2(s)$ is*

$$F_1(s) = (f_1 \triangle f_2)(\mu, \nu; s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s},$$

where $c_n = a_n^{\mu} b_n^{\nu}$, a_n, b_n are nonzero complex numbers; $\lambda_n = \frac{\gamma_n + \xi_n}{2}$, μ, ν are positive numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$.

Theorem 1.4 *(see [6, Theorem 2.1]). Let $f_1(s), f_2(s) \in D$ be of q-order ρ_1, ρ_2, respectively, and satisfy*

$$\gamma_n = \eta \xi_n, \tag{8}$$

then the q-order ρ of the Dirichlet-Hadamard product $F_1(s)$ satisfies

$$\rho \leq \frac{\rho_1 \rho_1 (2 + \frac{1}{\eta} + \eta)}{2\nu\rho_1(\frac{1}{\eta} + 1) + 2\mu\rho_2(1 + \eta)}, \qquad \rho_1, \rho_2 \in [0, +\infty).$$

Theorem 1.5 *(see [6, Theorem 2.2]). Let $f_1(s), f_2(s) \in D$ be of lower q-order χ_1, χ_2, respectively, and satisfy the condition of Lemma 2.2, then the lower q-order χ of the Dirichlet-Hadamard product $F_1(s)$ satisfies*

$$\chi \geq \frac{(2 + \frac{1}{\eta} + \eta)\chi_1\chi_2}{2\nu\chi_1(\frac{1}{\eta} + 1) + 2\mu\chi_2(1 + \eta)}.$$

In this paper, we will introduce a more general form Dirichlet-Hadamard product of Dirichlet series, which is improvement of Kong's definition [6].

Definition 1.5 *Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{\gamma_n s}$, $f_2(s) = \sum_{n=1}^{\infty} b_n e^{\xi_n s}$ and $f_1(s), f_2(s) \in D$, the generalized Dirichlet-Hadamard product can be defined by*

$$F(s) = (f_1 \triangle f_2)(\mu, \nu; \alpha, \beta; s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s}, \ c_n = a_n^{\mu} b_n^{\nu}, \quad \lambda_n = \alpha\gamma_n + \beta\xi_n,$$

where α, β, μ, ν are positive numbers, a_n, b_n are nonzero complex numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$.

Remark 1.1 *When $\alpha = \beta = \frac{1}{2}$, the generalized Dirichlet-Hadamard product is the Dirichlet-Hadamard product by Kong.*

2 Some Lemmas and the (lower) q-order of the generalized Dirichlet-Hadamard product

In this paper, we prove some theorems about the upper and the lower bounds of the (lower) q-order and the (lower) q-type of generalized Drichlet-Hadamard product.

Lemma 2.1 *Let $f_1(s), f_2(s) \in D$ and satisfy (8), then $F(s)$ is analytic in the whole complex plane, that is, $F(s)$ is an entire function.*

Proof:

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=\limsup_{n\to\infty}\frac{\log n}{\alpha\gamma_n+\beta\xi_n}\leq\frac{1}{\alpha}\limsup_{n\to\infty}\frac{\log n}{\gamma_n}=0,$$

and

$$\limsup_{n\to\infty}\frac{\log|c_n|}{\lambda_n}=\limsup_{n\to\infty}\frac{\mu\log|a_n|+\nu\log|b_n|}{\alpha\gamma_n+\beta\xi_n}\leq\limsup_{n\to\infty}\frac{\mu\log|a_n|}{\alpha\gamma_n+\beta\xi_n}=-\infty.$$

This completes the proof of Lemma 2.1. □

Lemma 2.2 *Let γ_n, ξ_n satisfy (8), and*

$$\varphi_1(n)=(\gamma_{n+1}-\gamma_n)^{-1}\log|\frac{a_n}{a_{n+1}}|,\quad \varphi_2(n)=(\xi_{n+1}-\xi_n)^{-1}\log|\frac{b_n}{b_{n+1}}|$$

be non-decreasing functions with n, then

$$\varphi(n)=(\lambda_{n+1}-\lambda_n)^{-1}\log|\frac{c_n}{c_{n+1}}|$$

is a non-decreasing function with n.

Proof:

$$\begin{aligned}\varphi(n)&=\frac{1}{(\alpha\gamma_{n+1}+\beta\xi_{n+1})-(\alpha\gamma_n+\beta\xi_n)}\log|\frac{a_n^\mu b_n^\nu}{a_{n+1}^\mu b_{n+1}^\nu}|\\&=\frac{\mu\log|\frac{a_n}{a_{n+1}}|}{(\gamma_{n+1}-\gamma_n)(\alpha+\beta\frac{\xi_{n+1}-\xi_n}{\gamma_{n+1}-\gamma_n})}+\frac{\nu\log|\frac{b_n}{b_{n+1}}|}{(\xi_{n+1}-\xi_n)(\alpha\frac{\gamma_{n+1}-\gamma_n}{\xi_{n+1}-\xi_n}+\beta)}\\&=\frac{\mu\varphi_1(n)}{\alpha+\frac{\beta}{\eta}}+\frac{\nu\varphi_2(n)}{\beta+\alpha\eta}.\end{aligned}\tag{9}$$

Since $\varphi_1(n), \varphi_2(n)$ are non-decreasing functions, thus it follows from (9) that $\varphi(n)$ is a non-decreasing function. □

Theorem 2.1 *Let $f_1(s), f_2(s)(\in D)$ be of q-order ρ_1, ρ_2, respectively. If $f_1(s), f_2(s)$ satisfy (8), then $F(s)$ is of q-order ρ satisfying*

$$\rho\leq\frac{(\alpha\eta+\beta)\rho_1\rho_2}{\mu\eta\rho_2+\nu\rho_1}.$$

Proof: We only prove the case $\rho_1, \rho_2 \in (0, +\infty)$. From (4), for any $\varepsilon > 0$, there exist two positive integers N_1, N_2 such that $n > N = \max\{N_1, N_2\}$, we have

$$\frac{\gamma_n \log^{[q-1]} \gamma_n}{\log |a_n|^{-1}} < \rho_1 + \varepsilon, \quad \frac{\xi_n \log^{[q-1]} \xi_n}{\log |b_n|^{-1}} < \rho_2 + \varepsilon. \tag{10}$$

Since $c_n = a_n^{\mu} b_n^{\nu}$, it follows from (10) that

$$\log |c_n|^{-1} = \mu \log |a_n|^{-1} + \nu \log |b_n|^{-1} > \frac{\mu \gamma_n \log^{[q-1]} \gamma_n}{\rho_1 + \varepsilon} + \frac{\nu \xi_n \log^{[q-1]} \xi_n}{\rho_2 + \varepsilon}, \tag{11}$$

then from (11) we have

$$\frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} < \frac{\lambda_n \log^{[q-1]} \lambda_n}{\frac{\mu}{\rho_1+\varepsilon} \gamma_n \log^{[q-1]} \gamma_n + \frac{\nu}{\rho_2+\varepsilon} \xi_n \log^{[q-1]} \xi_n}. \tag{12}$$

Since $q = 2, 3, \cdots$ and $\xi_n = \frac{1}{\alpha\eta+\beta} \lambda_n, \gamma_n = \frac{1}{\alpha+\frac{\beta}{\eta}} \lambda_n$, it follows

$$\log^{[q-1]} \gamma_n \sim \log^{[q-1]} \xi_n \sim \log^{[q-1]} \lambda_n. \tag{13}$$

Since ε is arbitrary, it follows from (12) and (13) that

$$\rho = \limsup_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \leq \frac{1}{\frac{\mu}{\rho_1} \frac{1}{\alpha+\frac{\beta}{\eta}} + \frac{\nu}{\rho_2} \frac{1}{\alpha\eta+\beta}} = \frac{(\alpha\eta+\beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}.$$

□

Theorem 2.2 *Let $f_1(s), f_2(s)(\in D)$ be of lower q-order χ_1, χ_2, respectively. If $f_1(s)$, $f_2(s)$ satisfy the conditions of Lemma 2.2, then the lower q-order χ of $F(s)$ satisfies*

$$\chi \geq \frac{(\alpha\eta+\beta)\chi_1\chi_2}{\eta\mu\chi_2 + \nu\chi_1}.$$

Proof: Suppose that $\chi_1, \chi_2 > 0$. From Theorem 1.2, for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$, we have

$$\frac{\gamma_n \log^{[q-1]} \gamma_{n-1}}{\log |a_n|^{-1}} > \chi_1 - \varepsilon, \quad \frac{\xi_n \log^{[q-1]} \xi_{n-1}}{\log |b_n|^{-1}} > \chi_2 - \varepsilon. \tag{14}$$

Since $c_n = a_n^{\mu} b_n^{\nu}$, it follows from (14) that

$$\begin{aligned} \log |c_n|^{-1} &= \mu \log |a_n|^{-1} + \nu \log |b_n|^{-1} \\ &< \frac{\mu}{\chi_1 - \varepsilon} (\gamma_n \log^{[q-1]} \gamma_{n-1}) + \frac{\nu}{\chi_2 - \varepsilon} (\xi_n \log^{[q-1]} \xi_{n-1}). \end{aligned} \tag{15}$$

Thus, from (13) and (15) we have

$$\frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |c_n|^{-1}} > \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\frac{\mu}{\chi_1-\varepsilon} (\gamma_n \log^{[q-1]} \gamma_{n-1}) + \frac{\nu}{\chi_2-\varepsilon} (\xi_n \log^{[q-1]} \xi_{n-1})}. \tag{16}$$

By Lemma 2.2, we have $\varphi(n)$ is a non-decreasing function. And since ε is arbitrary, it follows from (16) that

$$\chi = \liminf_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |c_n|^{-1}} \geq \left(\frac{1}{\alpha + \frac{\beta}{\eta}} \frac{\mu}{\chi_1} + \frac{1}{\alpha\eta+\beta} \frac{\nu}{\chi_2} \right)^{-1} = \frac{(\alpha\eta+\beta)\chi_1\chi_2}{\mu\eta\chi_2 + \nu\chi_1}.$$

This completes the proof of Theorem 2.2. □

Theorem 2.3 *Suppose that $f_1(s), f_2(s)$ are two $\rho_{[q]}$-regular growth functions and of q-order ρ_1, ρ_2, respectively, and if $f_1(s), f_2(s)$ satisfy the conditions of Lemma 2.2.*

(i) Then $F(s)$ is also $\rho_{[q]}$-regular growth function, and of q-order ρ satisfying

$$\rho = \frac{(\alpha+\beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}, \quad \rho_1, \rho_2 \in [0, +\infty).$$

(ii) If $\rho_1, \rho_2 \in (0, +\infty)$ and $f_1(s), f_2(s)$ are of q-type T_1, T_2, respectively, then q-type T of $F(s)$ satisfy

$$T \leq \begin{cases} T_1^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}} T_2^{\frac{v\rho}{(\alpha\eta+\beta)\rho_2}}, & q = 3, 4, 5, \ldots, \\ \frac{\alpha\eta+\beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2 T_2)^{\frac{v\rho}{\rho_2(\alpha\eta+\beta)}}, & q = 2. \end{cases}$$

Proof: (i) Since $f_1(s), f_2(s)$ are two $\rho_{[q]}$-regular growth functions, thus $\chi_1 = \rho_1, \chi_2 = \rho_2$, where χ_1, χ_2 are the lower q-order of $f_1(s), f_2(s)$, respectively. Thus, it follows by Theorem 2.1 and Theorem 2.2 that

$$\rho = \frac{(\alpha\eta+\beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}, \quad \rho_1, \rho_2 \in [0, +\infty).$$

This proves (i).

(ii) From Theorem 1.1, we have

$$T_1 = \limsup_{n\to\infty} |a_n|^{\frac{\rho_1}{\gamma_n}} \log^{[q-2]}(\frac{\gamma_n}{e\rho_1}), \quad T_2 = \limsup_{n\to\infty} |b_n|^{\frac{\rho_2}{\xi_n}} \log^{[q-2]}(\frac{\xi_n}{e\rho_2}).$$

So for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$, we have

$$|a_n|^{\frac{\rho_1}{\gamma_n}} \leq \frac{T_1+\varepsilon}{\log^{[q-2]}(\frac{\gamma_n}{e\rho_1})}, |b_n|^{\frac{\rho_2}{\xi_n}} \leq \frac{T_2+\varepsilon}{\log^{[q-2]}(\frac{\xi_n}{e\rho_2})}.$$

If $q = 3, 4, 5, \cdots$, we have

$$\log^{[q-2]}(\frac{\gamma_n}{e\rho_1}) \sim \log^{[q-2]}\gamma_n, \log^{[q-2]}(\frac{\xi_n}{e\rho_2}) \sim \log^{[q-2]}\xi_n. \tag{17}$$

And $c_n = a_n^\mu b_n^\nu$, then it follows from (17) that

$$\begin{aligned} |c_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}) &= (|a_n|^\mu |b_n|^\nu)^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}) \\ &= (|a_n|^{\frac{\rho_1}{\gamma_n}})^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} (|b_n|^{\frac{\rho_2}{\xi_n}})^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}) \\ &\leq (\frac{T_1+\varepsilon}{\log^{[q-2]}(\frac{\gamma_n}{e\rho_1})})^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} (\frac{T_2+\varepsilon}{\log^{[q-2]}(\frac{\xi_n}{e\rho_2})})^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}) \\ &= (\frac{T_1+\varepsilon}{\log^{[q-2]}(\frac{\gamma_n}{e\rho_1})})^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} (\frac{T_2+\varepsilon}{\log^{[q-2]}(\frac{\xi_n}{e\rho_2})})^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}). \end{aligned} \tag{18}$$

Thus, from (18) we have

$$T = \limsup_{n\to\infty} |c_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}) \leq (T_1+\varepsilon)^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} (T_2+\varepsilon)^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}}. \tag{19}$$

If $q = 2$, from (5), we have for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$

$$|a_n|^{\frac{\rho_1}{\gamma_n}} \leq \frac{T_1 + \varepsilon}{\frac{\gamma_n}{e\rho_1}}, \quad |b_n|^{\frac{\rho_2}{\xi_n}} \leq \frac{T_2 + \varepsilon}{\frac{\xi_n}{e\rho_2}}, \tag{20}$$

Since $c_n = a_n^{\mu} b_n^{\nu}$, it follows from (20) that

$$\begin{aligned} |c_n|^{\frac{\rho}{\lambda_n}} \frac{\lambda_n}{e\rho} &= (|a_n|^{\frac{\rho_1}{\gamma_n}})^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} (|b_n|^{\frac{\rho_2}{\xi_n}})^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \frac{\lambda_n}{e\rho} \\ &\leq (\frac{T_1 + \varepsilon}{\frac{\gamma_n}{e\rho_1}})^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} (\frac{T_2 + \varepsilon}{\frac{\xi_n}{e\rho_2}})^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}} (\frac{\lambda_n}{e\rho}). \end{aligned} \tag{21}$$

Since ε is arbitrary, it follows from (19) and (21) that

$$T \leq \begin{cases} T_1^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}} T_2^{\frac{v\rho}{(\alpha\eta+\beta)\rho_2}}, & q = 3, 4, 5, \ldots, \\ \frac{\alpha\eta+\beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2 T_2)^{\frac{v\rho}{\rho_2(\alpha\eta+\beta)}}, & q = 2. \end{cases}$$

Thus, this completes the proof of Theorem 2.3. □

Theorem 2.4 *Let $f_1(s), f_2(s)$ be two $\rho_{[q]}$-perfectly regular growth functions, and satisfy (8),the condition of lemma2.2 and*

$$\log^{[q-2]} \gamma_{n-1} \sim \log^{[q-2]} \gamma_n, \quad \log^{[q-2]} \xi_{n-1} \sim \log^{[q-2]} \xi_n, n \to \infty. \tag{22}$$

Set $\rho_1, \rho_2, T_1, T_2, \tau_1$ and τ_2 be the q-order,q-type and lower q-type of $f_1(s), f_2(s)$, then $F(s)$ is of $\rho_{[q]}$-perfectly regular growth ρ and its q-type T satisfies

$$T = \begin{cases} T_1^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}} T_2^{\frac{v\rho}{(\alpha\eta+\beta)\rho_2}}, & q = 3, 4, 5, \ldots, \\ \frac{\alpha\eta+\beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2 T_2)^{\frac{v\rho}{\rho_2(\alpha\eta+\beta)}}, & q = 2. \end{cases}$$

Proof: Suppose that $\rho_1, \rho_2 \in (0, +\infty)$,$\tau_1, \tau_2 < +\infty$. From Theorem 1.3, for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$, we have

$$|a_n|^{\frac{\rho_1}{\gamma_n}} \log^{[q-2]}(\frac{\gamma_{n-1}}{e\rho_1}) \geq \tau_1 - \varepsilon, \quad |b_n|^{\frac{\rho_2}{\xi_n}} \log^{[q-2]}(\frac{\xi_{n-1}}{e\rho_2}) \geq \tau_2 - \varepsilon.$$

If $q \geq 3$, it follows

$$\begin{aligned} \tau &= \liminf_{n\to\infty} |c_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_{n-1}}{e\rho}) \\ &= \liminf_{n\to\infty} (|a_n|^{\mu}|b_n|^{\nu})^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_{n-1}}{e\rho}) \\ &\geq \liminf_{n\to\infty} \{[\frac{\tau_1 - \varepsilon}{\log^{[q-2]}(\frac{\gamma_{n-1}}{e\rho_1})}]^{\frac{\mu\gamma_n}{\rho_1}} [\frac{\tau_2 - \varepsilon}{\log^{[q-2]}(\frac{\xi_{n-1}}{e\rho_2})}]^{\frac{\nu\xi_n}{\rho_2}}\}^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_{n-1}}{e\rho}) \\ &\geq (\tau_1 - \varepsilon)^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} (\tau_2 - \varepsilon)^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}}. \end{aligned} \tag{23}$$

From Theorem 2.3 and since $\varepsilon >$ is arbitrary, it follows from (23) that

$$T_1^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}} T_2^{\frac{v\rho}{(\alpha\eta+\beta)\rho_2}} \geq T \geq \tau \geq (\tau_1)^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} (\tau_2)^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}}. \tag{24}$$

If $q=2$, from (22) we have

$$\begin{aligned}\tau &= \liminf_{n\to\infty} |c_n|^{\frac{\rho}{\lambda_n}} \frac{\lambda_{n-1}}{e\rho} \\ &\geq \liminf_{n\to\infty} \left(\frac{\tau_1-\varepsilon}{\frac{\gamma_{n-1}}{e\rho_1}}\right)^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} \left(\frac{\tau_2-\varepsilon}{\frac{\xi_{n-1}}{e\rho_2}}\right)^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \\ &\geq \frac{1}{\rho}\left[\frac{(\alpha\eta+\beta)\rho_1\tau_1}{\eta}\right]^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} \left[(\alpha\eta+\beta)\rho_2\tau_2\right]^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}} \\ &= \frac{\alpha\eta+\beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1\tau_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2\tau_2)^{\frac{v\rho}{\rho_2(\alpha\eta+\beta)}} \end{aligned} \tag{25}$$

From Theorem 2.3 and since $\varepsilon >$ is arbitrary, it follows from (25) that

$$\begin{aligned}\frac{\alpha\eta+\beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2 T_2)^{\frac{v\rho}{\rho_2(\alpha\eta+\beta)}} &\geq T \geq \tau \\ &\geq \frac{\alpha\eta+\beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1\tau_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2\tau_2)^{\frac{v\rho}{\rho_2(\alpha\eta+\beta)}} \end{aligned} \tag{26}$$

Since $f_1(s), f_2(s)$ are $\rho_{[q]}$-perfectly regular growth and $\tau_j = T_j, j=1,2$, from (24) and (26), it is easy to get the conclusions of Theorem 2.4.

Thus, we complete the proof of Theorem 2.4. □

3 The linear substitution of Dirichlet series

Next, we define the scalar multiplication of Dirichlet series as follows

Definition 3.1 *Let k be a positive number, we define the scalar multiplication of Dirichlet series as follows*

$$H(s) = f(ks) = \sum_{n=1}^{\infty} a_n e^{\lambda_n(ks)} = \sum_{n=1}^{\infty} a_n e^{\zeta_n s}, \zeta_n = k\lambda_n.$$

Theorem 3.1 *Let $f(s) \in D$, then $H(s) \in D$. Furthermore, if $\varphi(n)$ is a non-decreasing function with n, then the q-order ρ^* and the lower q-order χ^* of $H(s)$ satisfy $\rho^* = k\rho$ and $\chi^* = k\chi$.*

Proof: Since

$$\limsup_{n\to\infty} \frac{\log|a_n|}{\zeta_n} = \limsup_{n\to\infty} \frac{\log|a_n|}{k\lambda_n} = -\infty,$$

thus, we have $H(s) \in D$.

Furthermore, we have

$$\begin{aligned}\rho^* &= \limsup_{n\to\infty} \frac{\zeta_n \log^{[q-1]} \zeta_n}{\log|a_n|^{-1}} = \limsup_{n\to\infty} \frac{k\lambda_n \log^{[q-1]} k\lambda_n}{\log|a_n|^{-1}} \\ &= \limsup_{n\to\infty} k\frac{\lambda_n \log^{[q-1]} \lambda_n}{\log|a_n|^{-1}} = k\rho,\end{aligned}$$

and

$$\begin{aligned}\chi^* &= \liminf_{n\to\infty} \frac{\zeta_n \log^{[q-1]} \zeta_{n-1}}{\log|a_n|^{-1}} = \liminf_{n\to\infty} \frac{k\lambda_n \log^{[q-1]} k\lambda_{n-1}}{\log|a_n|^{-1}} \\ &= \liminf_{n\to\infty} k\frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log|a_n|^{-1}} = k\chi.\end{aligned}$$

This completes the proof of Theorem 3.1. □

From Theorem 3.1, we can obtain the following result easily.

Theorem 3.2 *Let $f_1(s), f_2(s) \in D$ and satisfy the conditions of Lemma 2.2. Set ρ, χ be the q-order and the lower q-order of $F(s)$, then the q-order ρ^* and the lower q-order χ^* of $H^*(s) = F(ks)$ satisfy $\rho^* = k\rho$, $\chi^* = k\chi$.*

Let $f_1(s), f_2(s) \in D$ and k, m be positive numbers. Set $H_1(s) = f_1(ks), H_2(s) = f_2(ms)$ and

$$H^{**}(s) = (H_1 \triangle H_2)(\mu, \nu, s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s},\ c_n = a_n^{\mu} b_n^{\nu},\quad \lambda_n = \alpha k \gamma_n + \beta m \xi_n,$$

where α, β, μ, ν are positive numbers, a_n, b_n are nonzero complex numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$. The following result is about the growth of $H^{**}(s)$.

Theorem 3.3 *Let $f_1(s), f_2(s) \in D$ satisfy (8) and the conditions of Lemma 2.2. Let ρ, χ be the q-order and lower q-order of $F(s)$, then the q-order ρ^{**} and the lower q-order χ^{**} of $H^{**}(s)$ satisfy*

$$\rho^{**} = \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta}\rho, \chi^{**} = \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta}\chi.$$

Proof: Since

$$\begin{aligned}
\rho^{**} &= \limsup_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \\
&= \limsup_{n\to\infty} \frac{(k\alpha\gamma_n + m\beta\xi_n) \log^{[q-1]}(k\alpha\gamma_n + m\beta\xi_n)}{\log |c_n|^{-1}} \\
&= \limsup_{n\to\infty} \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \frac{\lambda_n \log^{[q-1]}(k\alpha\gamma_n + m\beta\xi_n)}{\log |c_n|^{-1}} \\
&= \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta}\rho, \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
\chi^{**} &= \liminf_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |c_n|^{-1}} \\
&= \liminf_{n\to\infty} \frac{(k\alpha\gamma_n + m\beta\xi_n) \log^{[q-1]}(k\alpha\gamma_{n-1} + m\beta\xi_{n-1})}{\log |c_n|^{-1}} \\
&= \liminf_{n\to\infty} \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \frac{\lambda_n \log^{[q-1]}(k\alpha\gamma_{n-1} + m\beta\xi_{n-1})}{\log |c_n|^{-1}} \\
&= \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta}\chi, \qquad (28)
\end{aligned}$$

thus from (27) and (28) we can prove the conclusions of Theorem 3.3. □

Remark 3.1 *From Theorem 3.3, we can get that $\rho^* = \rho^{**}, \chi^* = \chi^{**}$ if $k = m$.*

In 2008, Kong defined the shift of Dirichlet series (see [6]).

Definition 3.2 *(see [6]). Let α be a real number, the shift of Dirichlet series can be defined as follows*

$$G(s) = f(s + \alpha) = \sum_{n=1}^{\infty} a_n e^{\lambda_n(s+\alpha)} = \sum_{n=1}^{\infty} a_n' e^{\lambda_n s},$$

where $a_n^{'} = a_n e^{\lambda_n \alpha}$.

Let $f_1(s), f_2(s) \in D$, α_1, α_2 be two real numbers and k be a positive number. Set $G_1(s) = f_1(s+\alpha_1), G_2(s) = f_2(s+\alpha_2)$, $H_1(s) = f_1(ks)$, and

$$G^*(s) = (G_1 \triangle G_2)(\mu, \nu, s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s},\ c_n = (a_n')^{\mu}(b_n')^{\nu},\quad \lambda_n = \alpha\gamma_n + \beta\xi_n,$$

and

$$G^{**}(s) = (H_1 \triangle G_2)(\mu, \nu, s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s},\ c_n = a_n^{\mu}(b_n')^{\nu},\quad \lambda_n = \alpha k\gamma_n + \beta\xi_n,$$

where α, β, μ, ν are positive numbers, a_n, b_n are nonzero complex numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$. We investigate the growth of $G^*(s), G^{**}(s)$ and obtain the following results

Theorem 3.4 *Let $f_1(s), f_2(s) \in D$ satisfy (8) and the conditions of Lemma 2.2. Let ρ, χ be the q-order and lower q-order of $F(s)$, then the q-order ρ_1^{**} and the lower q-order χ_1^{**} of $G^*(s)$ satisfy $\rho_1^{**} = \rho, \chi_1^{**} = \chi$.*

Proof: Since

$$\rho_1^{**} = \limsup_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \tag{29}$$

$$= \limsup_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1} - (\mu\gamma_n\alpha_1 + v\xi_n\alpha_2)} \tag{30}$$

$$= \limsup_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}(1 + \frac{\mu\gamma_n\alpha_1 + v\xi_n\alpha_2}{\log |c_n|})} \tag{31}$$

$$= \rho, \tag{32}$$

that is, $\rho_1^{**} = \rho$.

Similarly, we have $\chi_1^{**} = \chi$. □

Theorem 3.5 *Let $f_1(s), f_2(s) \in D$ satisfy (8) and the conditions of Lemma 2.2. Let ρ, χ be the q-order and lower q-order of $F(s)$, then the q-order ρ_2^{**} and the lower q-order χ_2^{**} of $G^{**}(s)$ satisfy*

$$\rho_2^{**} = \frac{k\alpha\eta + \beta}{\alpha\eta + \beta}\rho, \chi_2^{**} = \frac{k\alpha\eta + \beta}{\alpha\eta + \beta}\chi.$$

Proof: Since

$$\rho_2^{**} = \limsup_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \tag{33}$$

$$= \limsup_{n\to\infty} \frac{(k\alpha\gamma_n + \beta\xi_n) \log^{[n-1]}(k\alpha\gamma_n + \beta\xi_n)}{\log |c_n|^{-1} - (v\xi_n\alpha_2)} \tag{34}$$

$$= \limsup_{n\to\infty} \frac{\frac{k\alpha\eta+\beta}{\alpha\eta+\beta}\lambda_n \log^{[n-1]}(\alpha k\gamma_n + \beta\xi_n)}{\log |c_n|^{-1}(1 + \frac{v\xi_n\alpha_2}{\log |c_n|})} \tag{35}$$

$$= \frac{k\alpha\eta + \beta}{\alpha\eta + \beta}\rho. \tag{36}$$

that is, $\rho_2^{**} = \frac{k\alpha\eta+\beta}{\alpha\eta+\beta}\rho$.

Similarly, we have $\chi_2^{**} = \frac{k\alpha\eta+\beta}{\alpha\eta+\beta}\chi$. □

Remark 3.2 *From Theorem 3.5, we can get that $\rho_2^{**} = \rho_1^{**}, \chi_2^{**} = \chi_1^{**}$ if $k = 1$.*

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The differential and subdifferential for fuzzy mappings based on the generalized difference of n-cell fuzzy-numbers

Shexiang Hai[a], Zengtai Gong[b*]
[a] *School of Science, Lanzhou University of Technology, Lanzhou, 730050, P.R. China*
[b] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*

Abstract We use the concept of generalized difference for fuzzy n-cell numbers which is presented in this paper to introduce and study the differential and gradient of fuzzy n-cell mappings. At the same time, some connections between gradient, *boundary-function-wise* gradient and *level-wise* gradient of fuzzy n-cell mappings are established. Furthermore, the subdifferential for fuzzy n-cell mappings based on the ordering $\preceq_c$ is discussed.
Keywords: Fuzzy numbers; fuzzy n-cell mappings; gradient; subdifferential.

1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh [1], enormous researchers have been dedicated on development of various aspects of the theory and applications of fuzzy sets. Soon after, Zadeh proposed the notion of fuzzy numbers in [2,3,4]. Since then, fuzzy numbers have been extensively investigated by many authors.

The importance of the derivative of a function in the study of mathematical programming and fuzzy differential equations is well-known. It is necessary to introduce a concept of differentiability for fuzzy mappings. Toward this end, in fuzzy analysis, there are a variety of notions of derivative for fuzzy mappings. The concept of fuzzy derivative first introduced by Chang and Zadeh [5] in 1972. Since then, numerous definitions of the differentiability of fuzzy mappings have been presented. In 1983, Puri and Ralescu [6] defined the derivative and G-derivative of fuzzy mappings from an open subset of a normed space into n-dimension fuzzy number space E^n by using embedding theorem (which shows how to isometrically embed E^n into a Banach space as a closed convex cone of vertex zero) and Hukuhara difference. In 1987, Kaleva [7] discussed the G-derivative, obtained a sufficient condition of the H-differentiability of the fuzzy mappings from $[a,b]$ into E^n and a necessary condition for the H-differentiability of fuzzy mapping from $[a,b]$ into E^1. In 2003, Wang and Wu [8] put forward the concepts of directional derivative, differential and sub-differential of fuzzy mappings from R^n into E^1 by using Hukuhara difference. However, the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [7] and the H-difference of two fuzzy numbers does not always exist [9]. The g-difference proposed in [9] overcomes these shortcomings of the above discussed concepts and the g-difference of two fuzzy numbers always exists. Based on the novel generalizations of the Hukuhara difference for fuzzy sets, Bede [9] introduced and studied new generalized differentiability concepts for fuzzy valued functions in 2013, in particular, a new very general fuzzy differentiability concept was defined and studied, the so-called g-derivative, and it was shown that the g-derivative is the most general among all similar definitions.

Motivated both by [9] and the importance of the concept of differential for fuzzy analysis, the concept of differential and gradient for fuzzy n-cell mappings is introduced, which is based on the novel generalizations difference of fuzzy n-cell numbers presented in this paper.

The remainder of the paper is organised as follows: First of all, we give the preliminary terminology used in the present paper. And then, in Section 3, we present the concept of generalized difference for fuzzy n-cell numbers and discuss several properties for it. We use the generalized difference for fuzzy n-cell numbers to introduce and study differential and gradient for fuzzy n-cell mappings in Section 4. At

[†]Supported by the Natural Scientific Fund of China (11461062, 61262022).
[*]Corresponding Author:Zeng-Tai Gong. Tel.: +869317971430. E-mail addresses: zt-gong@163.com

last, using the concept of the ordering $\preceq_c$ for fuzzy n-cell numbers, section 5 deals with the subdifferential for fuzzy n-cell mappings.

2. Preliminaries

Throughout this paper, R^n denote the n-dimensional Euclidean space and $F(R^n)$ denote the set of all fuzzy subsets on R^n. A fuzzy subset of (in short, a fuzzy set) R^n is a function $u : R^n \to [0,1]$. For each fuzzy sets u, we denote by $[u]^r = \{x \in R^n : u(x) \geq r\}$, for any $r \in (0,1]$, its r-level set. By $\text{supp}u = \{x \in R^n : u(x) > 0\}$ we represent the support of u. Suppose $u \in F(R^n)$, satisfies the following conditions:

(1) u is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is a convex fuzzy set, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$,

(3) u is upper semicontinuous ,

(4) $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}} = \overline{\bigcup_{r\in(0,1]}[u]^r}$ is compact, here $\overline{A}$ denotes the closure of A.

Then u is called a fuzzy number. We use E^n to denote the fuzzy number space [10,11,12,13].

It is clear that each $u \in R^n$ can be considered as a fuzzy number u defined by

$$\widetilde{u}(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the fuzzy number $\widetilde{0}$ is defined as $\widetilde{0}(x) = 1$ if $x = 0$, and $\widetilde{0}(x) = 0$ otherwise.

Definition 2.1. [14] If $u \in E^n$, and $[u]^r$ is a cell, i.e., for any $r \in [0,1]$,

$$[u]^r = \prod_{i=1}^{n}[u_i^-(r), u_i^+(r)] = [u_1^-(r), u_1^+(r)] \times [u_2^-(r), u_2^+(r)] \times \cdots \times [u_n^-(r), u_n^+(r)],$$

where $u_i^-(r), u_i^+(r) \in R$ with $u_i^-(r) \leq u_i^+(r)$ $(i = 1, 2, \cdots, n)$, then we call u a fuzzy n-cell number. Denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

For any $r \in [0,1]$, $l_i[u]^r = u_i^+(r) - u_i^-(r)$ $(i = 1, 2, \cdots, n)$ is called the r-level length of a fuzzy n-cell number u with respect to the ith component.

Theorem 2.1. [14] (Representation theorem). If $u \in L(E^n)$, then for $i = 1, 2, \cdots, n$, $u_i^-(r), u_i^+(r)$ are real-valued functions on $[0,1]$, and satisfy

(1) $u_i^-(r)$ are non-decreasing, left continuous at $r \in (0,1]$ and right continuous at $r = 0$,

(2) $u_i^+(r)$ are non-increasing, left continuous at $r \in (0,1]$ and right continuous at $r = 0$,

(3) $u_i^-(r) \leq u_i^+(r)$ (it is equivalent to $u_i^-(1) \leq u_i^+(1)$).

Conversely if $a_i(r), b_i(r)$ $(i = 1, 2, \cdots, n)$ are real-valued functions on $[0,1]$ which satisfy conditions (1)-(3), then there exists a unique $u \in L(E^n)$ such that $[u]^r = \prod_{i=1}^{n}[a_i(r), b_i(r)]$ for any $r \in [0,1]$.

Theorem 2.2. [14] Let $u, v \in L(E^n)$ and $k \in R$. Then for any $r \in [0,1]$,

(1) $[u+v]^r = [u]^r + [v]^r = \prod_{i=1}^{n}[u_i^-(r) + v_i^-(r), u_i^+(r) + v_i^+(r)]$,

(2) $[ku]^r = k[u]^r = \begin{cases} \prod_{i=1}^{n}[ku_i^-(r), ku_i^+(r)], & k \geq 0, \\ \prod_{i=1}^{n}[ku_i^+(r), ku_i^-(r)], & k < 0, \end{cases}$

(3) $[uv]^r = \prod_{i=1}^{n}[\min\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\},$

$$\max\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\}].$$

Given $u, v \in L(E^n)$, the distance $D : L(E^n) \times L(E^n) \to [0, +\infty)$ between u and v is defined by the equation

$$\begin{aligned} D(u,v) &= \sup_{r\in[0,1]} d([u]^r, [v]^r) \\ &= \sup_{r\in[0,1]} \max_{1\leq i\leq n}\{\mid u_i^-(r) - v_i^-(r) \mid, \mid u_i^+(r) - v_i^+(r) \mid\}. \end{aligned}$$

Then $(L(E^n), D)$ is a complete metric space, and satisfies $D(u+w, v+w) = D(u,v)$, $D(ku, kv) = |k|D(u,v)$ for any $u, v, w \in L(E^n)$, $k \in R$.

In recent years, several authors have discussed different ordering relation of fuzzy numbers [15]. To the best of our knowledge, very few investigations have been appeared to study ordering relation of fuzzy n-cell numbers. For this reason, an ordering $\preceq_c$ of fuzzy n-cell numbers will be introduced and be applied to solve fuzzy constrained minimization problem.

Definition 2.2. Let $\tau : L(E^n) \to R^n$ be a vector-valued function defined by

$$\begin{aligned}\tau(u) &= (2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_1dx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr, 2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_2dx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr, \cdots, 2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_ndx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr)\\ &= (\int_0^1 r(u_1^+(r)+u_1^-(r))dr, \int_0^1 r(u_2^+(r)+u_2^-(r))dr, \cdots, \int_0^1 r(u_n^+(r)+u_n^-(r))dr),\end{aligned}$$

where $\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_idx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr$ $(i = 1, 2, \cdots, n)$ is the Lebesque integral of $r\frac{\int\cdots\int_{[u]^r} x_idx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}$ $(i = 1, 2, \cdots, n)$ on $[0, 1]$. The vector-valued function τ is called a ranking value function defined on $L(E^n)$.

In this case $\tau(u)$ represents a centroid of the fuzzy n-cell number u. From the ranking value function $\tau(u)$, we consider the following ordering relation $\preceq_c$ on $L(E^n)$.

Definition 2.3. Let $u, v \in L(E^n)$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. We say that $u \preceq_c v$ (u precedes v) if

$$\tau(v) \in \tau(u) + C \ \ (\tau(v) - \tau(u) \in C).$$

Obviously the order relation $\preceq_c$ is reflexive and transitive, and $\preceq_c$ is a partial order relation on $L(E^n)$. If $u, v \in E^1$, $C = [0, +\infty) \subseteq R$, then Definition 2.3 coincides with Definition 2.5 of reference [15].

We say that $u \prec_c v$ if $u \preceq_c v$ and $\tau(u) \neq \tau(v)$. Sometimes we may write $v \succeq_c u$ (resp. $v \succ_c u$) instead of $u \preceq_c v$ (resp. $u \prec_c v$).

Remark 2.1. Let $u, v \in L(E^n)$, $k_1, k_2 \in R$. According to Theorem 2.2 and Definition 2.2, it is easy to verify that $\tau(k_1u + k_2v) = k_1\tau(u) + k_2\tau(v)$.

Theorem 2.3. Let $u_1, u_2, v_1, v_2, \in L(E^n)$, $k_1, k_2 \in [0, +\infty]$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. If $u_1 \preceq_c v_1$ and $u_2 \preceq_c v_2$, then $k_1u_1 + k_2u_2 \preceq_c k_1v_1 + k_2v_2$.

Proof. It is follows from Definition 2.3 that $\tau(v_1) - \tau(u_1) \in C$ and $\tau(v_2) - \tau(u_2) \in C$. On the other hand, closed convex cone C is closed under addition and positive scalar multiplication, thus

$$k_1(\tau(v_1) - \tau(u_1)) + k_2(\tau(v_2) - \tau(u_2)) \in C,$$

which implies that $k_1\tau(v_1) + k_2\tau(v_2) \in k_1\tau(u_1) + k_2\tau(u_2) + C$. It is obvious from Remark 2.1 that $\tau(k_1v_1 + k_2v_2) \in \tau(k_1u_1 + k_2u_2) + C$, then $k_1u_1 + k_2u_2 \preceq_c k_1v_1 + k_2v_2$.

3. Generalized difference for fuzzy n-cell numbers

We denote by $\mathcal{K}_C^n$ the family of all nonempty compact convex subsets of R^n, that is $\{\mathcal{K}_C^n \subset R^n : A \neq \emptyset$ is compact and convex$\}$. Stefanini [16] defined the generalized Hukuhara difference of two sets $A \in \mathcal{K}_C^n$ and $B \in \mathcal{K}_C^n$ as follows:

$$A \ominus_{gH} B = C \Longleftrightarrow \left\{ \begin{array}{c} (1)\ A = B + C, \\ \text{or } (2)\ B = A + (-1)C. \end{array} \right.$$

For any $A \in \mathcal{K}_C^n$ and $B \in \mathcal{K}_C^n$, if the generalized Hukuhara difference $C = A \ominus_{gH} B$ exists, it is unique.

Lemma 3.1. [16] Let $A = \prod_{i=1}^n A_i$, $B = \prod_{i=1}^n B_i$, where $A_i = [a_i^-, a_i^+]$ and $B_i = [b_i^-, b_i^+]$ are real compact intervals ($\prod_{i=1}^n$ denotes the cartesian product). If $A \ominus_{gH} B$ exists, then

$$A \ominus_{gH} B = \prod_{i=1}^n (A_i \ominus_{gH} B_i) = \prod_{i=1}^n [\min\{a_i^- - b_i^-, a_i^+ - b_i^+\}, \max\{a_i^- - b_i^-, a_i^+ - b_i^+\}].$$

Lemma 3.2. [16] The gH-difference $A \ominus_{gH} B$ exists if and only if one of the two conditions is satisfied:

(i) $a_i^- - b_i^- \leq a_i^+ - b_i^+$, $i = 1, 2, \cdots, n$

or

(ii)$a_i^- - b_i^- \geq a_i^+ - b_i^+$, $i = 1, 2, \cdots, n$.

According to Lemma 3.2, the definition of generalized Hukuhara difference for real compact intervals is extended to the fuzzy case.

Definition 3.1. Let $u, v \in L(E^n)$. If $l_i[u]^r \leq l_i[v]^r$ or $l_i[u]^r \geq l_i[v]^r$ for any $r \in [0, 1]$ and $i = 1, 2, \cdots, n$, then the generalized difference (g-difference for short) is given by its level sets as

$$[u \ominus_g v]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \ \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}],$$

where $\beta \in [r, 1]$.

Remark 3.1. If $u, v \in E^1$, we have

$$[u \ominus_g v]^r = [\inf_{\beta \geq r} \min\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}, \sup_{\beta \geq r} \max\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}],$$

which coincides with Definition 7 of reference [9].

Theorem 3.1. Let $u, v \in L(E^n)$. If $l_i[u]^r \leq l_i[v]^r$ or $l_i[u]^r \geq l_i[v]^r$ for any $r \in [0, 1]$ and $i = 1, 2, \cdots, n$, then the g-difference $u \ominus_g v$ exists and $u \ominus_g v \in L(E^n)$.

Proof. Assume that

$$\begin{aligned} [w]^r &= [u \ominus_g v]^r \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \\ &\qquad \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}], \end{aligned}$$

for any $r \in [0, 1]$. We can prove that the class of sets $[w]^r$ determines a fuzzy n-cell number.

For any $r \in [0, 1]$, we have

$$\begin{aligned} w_i^-(r) &= \inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\} \\ &\leq \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\} \\ &= w_i^+(r). \end{aligned}$$

It can be easily seen that $w_i^-(r)$ are non-decreasing while $w_i^+(r)$ are non-increasing, $w_i^-(r)$ and $w_i^+(r)$ are left continuous on $(0, 1]$ and right continuous at 0. It follows from Theorem 2.1 that the g-difference $u \ominus_g v$ exists and $u \ominus_g v = w \in L(E^n)$.

From now on, throughout this paper, we will assume that the g-difference $u \ominus_g v$ for any fuzzy n-cell numbers u and v exists.

Theorem 3.2. For any $u,\ v,\ w \in L(E^n)$, we have

(1) $u \ominus_g u = \widetilde{0}$, $u \ominus_g \widetilde{0} = u$, $\widetilde{0} \ominus_g u = -u$,
(2) $u \ominus_g v = -(v \ominus_g u)$,
(3) $(u + v) \ominus_g (u + w) = v \ominus_g w$,
(4) $k(u \ominus_g v) = ku \ominus_g kv$, $k \in R$,
(5) $(u + v) \ominus_g v = u$,
(6) $\widetilde{0} \ominus_g (u \ominus_g v) = v \ominus_g u = (-u) \ominus_g (-v)$,
(7) $u \ominus_g v = v \ominus_g u = w$ if and only if $w = -w$, furthermore, $w = \widetilde{0}$ if and only if $u = v$.

Proof. The proof of (1), (3) and (4) are immediate.

(2) According to Definition 3.1, we have

$$\begin{aligned} & -[v \ominus_g u]^r \\ = & -\prod_{i=1}^n [\inf_{\beta \geq r} \min\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}, \\ & \qquad \sup_{\beta \geq r} \max\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}] \\ = & \prod_{i=1}^n [-\sup_{\beta \geq r} \max\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}, \\ & \qquad -\inf_{\beta \geq r} \min\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}] \\ = & \prod_{i=1}^n [-\sup_{\beta \geq r}(-\min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}), \\ & \qquad -\inf_{\beta \geq r}(-\max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\})] \\ = & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \\ & \qquad \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}] \\ = & [u \ominus_g v]^r, \end{aligned}$$

for any $r \in [0,1]$. It follows from Theorem 2.2 that $u \ominus_g v = -(v \ominus_g u)$.

(5) We have from Theorem 2.2 that

$$\begin{aligned}
& [(u+v) \ominus_g v]^r \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}, \\
& \qquad \sup_{\beta \geq r} \max\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}] \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta), u_i^+(\beta)\},\ \sup_{\beta \geq r} \max\{u_i^-(\beta), u_i^+(\beta)\}] \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} u_i^-(\beta),\ \sup_{\beta \geq r} u_i^+(\beta)] \\
= & \prod_{i=1}^n [u_i^-(r),\ u_i^+(r)] \\
= & [u]^r,
\end{aligned}$$

for any $r \in [0,1]$. Then $(u+v) \ominus_g v = u$.

(6) It follows from (1), (2) and (3) that the proof of (6) is immediate.

(7) We have from (2) that the proof of (7) is immediate.

4. The differential and gradient for fuzzy n-cell mappings

In this work, let M be a convex set of m-dimensional Euclidean space R^m. We consider mappings F from M into $L(E^n)$. Such a mapping is called a fuzzy n-cell mapping. For the sake of brevity, F is called a fuzzy mapping. Let $\widetilde{F} : M \to L(E^n)$, for any $r \in [0,1]$, we denote $F_r(t) = \prod_{i=1}^n [F_i^-(r,t), F_i^+(r,t)]$.

Definition 4.1. Let $\widetilde{F} : M \to L(E^n)$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$, $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If g-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there exist $u_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \sum_{j=1}^m u_j(t_j - t_j^0))}{d(t, t_0)} = 0,$$

then we say that $\widetilde{F}$ is differentiable at t_0 and $(u_1,\ u_2, \cdots, u_m)$ is the gradient of $\widetilde{F}$ at t_0, denoted by $\nabla \widetilde{F}(t_0)$, i.e., $\nabla \widetilde{F}(t_0) = (u_1, u_2, \cdots, u_m)$.

Remark 4.1. Let $\widetilde{F} : M \to L(E^n)$, $t_0 \in M$. Then the gradient $\nabla \widetilde{F}(t_0)$ exists at t_0 if and only if $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there are $u_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$u_j = \lim_{h \to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h},$$

where $h \in R$ and $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$.

Here the limit is taken in the metric space $(L(E^n), D)$.

Definition 4.2. Let $\widetilde{F} : M \to L(E^n)$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$, $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If there exists $u_{ij}^-(r),\ u_{ij}^+(r) \in R$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{|F_i^-(r,t) - F_i^-(r,t_0) - \sum_{j=1}^m u_{ij}^-(r)(t_j - t_j^0)|}{d(t, t_0)} = 0 \ (i = 1, 2, \cdots, n),$$

and

$$\lim_{t \to t_0} \frac{|F_i^+(r,t) - F_i^+(r,t_0) - \sum_{j=1}^m u_{ij}^+(r)(t_j - t_j^0)|}{d(t, t_0)} = 0 \ (i = 1, 2, \cdots, n),$$

uniformly for $r \in [0,1]$, then we say that $\widetilde{F}$ is *boundary-function-wise* differentiable (b-differentiable for short) at t_0.

Theorem 4.1. Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping. If $\widetilde{F}$ is b-differentiable at $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$, then there exist $u_j \in L(E^n)$, such that for any $r \in [0,1]$,

$$[u_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \ (j = 1, 2, \cdots, m).$$

Proof. For any $r \in [0,1]$, we can show that the class of sets

$$\prod_{i=1}^{n}[\inf_{\beta\geq r}\min\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\},\sup_{\beta\geq r}\max\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}]\ (j=1,2,\cdots,m)$$

satisfies the conditions of Theorem 2.1. According to Definition 4.2, if there exists $\delta>0$, such that for any $|h|<\delta$ with $t=(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\in \mathrm{int}M$, we have

$$u_{ij}^{-}(r)=\lim_{h\to 0}\frac{F_i^{-}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{-}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h},$$

$$u_{ij}^{+}(r)=\lim_{h\to 0}\frac{F_i^{+}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{+}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h},$$

for all $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$. Since $F_i^{-}(r,t)$ and $F_i^{+}(r,t)$ are left continuous with respect $r\in(0,1]$ and right continuous at $r=0$,

$$\frac{F_i^{-}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{-}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}$$

and

$$\frac{F_i^{+}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{+}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}$$

are left continuous at $r\in(0,1]$ and right continuous at $r=0$. Thus for any $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$,

$$\inf_{\beta\geq r}\min\{\frac{F_i^{-}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{-}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h},\frac{F_i^{+}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{+}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\},$$

$$\sup_{\beta\geq r}\max\{\frac{F_i^{-}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{-}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h},\frac{F_i^{+}(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^{+}(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\}$$

are left continuous at $r\in(0,1]$ and right continuous at $r=0$. Therefore, for any $r\in[0,1]$, $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$, we have

(1) $\inf_{\beta\geq r}\min\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}$ are non-decreasing and left continuous at $r\in(0,1]$ and right continuous at $r=0$,

(2) $\sup_{\beta\geq r}\max\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}$ are non-increasing and left continuous at $r\in(0,1]$ and right continuous at $r=0$,

(3) $\inf_{\beta\geq r}\min\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}\leq\sup_{\beta\geq r}\max\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}$.

Consequently, there exist $u_j\in L(E^n)$ $(j=1,2,\cdots,m)$, such that for any $r\in[0,1]$,

$$[u_j]^r=\prod_{i=1}^{n}[\inf_{\beta\geq r}\min\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\},\sup_{\beta\geq r}\max\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}]\ (j=1,2,\cdots,m).$$

Definition 4.3. Let $\widetilde{F}:M\to L(E^n)$ is b-differentiable at t_0. For any $r\in[0,1]$, we denote

$$[u_j]^r=\prod_{i=1}^{n}[\inf_{\beta\geq r}\min\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\},\sup_{\beta\geq r}\max\{u_{ij}^{-}(\beta),u_{ij}^{+}(\beta)\}]\ (j=1,2,\cdots,m),$$

then we say that $(u_1,\ u_2,\cdots,\ u_m)$ is the *boundary-function-wise* gradient (b-gradient for short) of $\widetilde{F}$ at t_0, denoted by $\nabla_b\widetilde{F}(t_0)$, i.e.,

$$\nabla_b\widetilde{F}(t_0)=(u_1,\ u_2,\cdots,\ u_m).$$

Remark 4.2. Let $\widetilde{F}: M \to L(E^n)$, $t_0 \in M$. Then the b-gradient $\nabla_b \widetilde{F}(t_0)$ exists at t_0 if and only if there are $u_{ij}^-(r)$, $u_{ij}^+(r) \in R$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$, such that

$$u_{ij}^-(r) = \lim_{h\to 0} \frac{F_i^-(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^-(r, t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}$$

and

$$u_{ij}^+(r) = \lim_{h\to 0} \frac{F_i^+(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^+(r, t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}$$

uniformly for $r \in [0, 1]$, where $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$ and

$$[u_j]^r = \prod_{i=1}^{n} [\inf_{\beta\geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta\geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}]\ (j = 1, 2, \cdots, m).$$

Theorem 4.2. Let the b-gradient $\nabla_b \widetilde{F}(t_0)$ of fuzzy mapping $\widetilde{F}: M \to L(E^n)$ be exist at $t_0 \in \text{int}M$. If $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla \widetilde{F}(t_0)$ of $\widetilde{F}$ at t_0 exists and we have $u_j = v_j$ $(j = 1, 2, \cdots, m)$, where $\nabla \widetilde{F}(t_0) = (u_1, u_2, \cdots, u_m)$, $\nabla_b \widetilde{F}(t_0) = (v_1, v_2, \cdots, v_m)$.

Proof. Let $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$, $h \in R$ and $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. According to Theorem 2.2 and Definition 3.1, for any $r \in [0, 1]$, we have

$$\begin{aligned}
& [\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}]^r \\
= \ & \tfrac{1}{h}[\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)]^r \\
= \ & \tfrac{1}{h} \textstyle\prod_{i=1}^{n} [\inf_{\beta\geq r} \min\{F_i^-(\beta, t) - F_i^-(\beta, t_0), F_i^+(\beta, t) - F_i^+(\beta, t_0)\}, \\
& \qquad \sup_{\beta\geq r} \max\{F_i^-(\beta, t) - F_i^-(\beta, t_0), F_i^+(\beta, t) - F_i^+(\beta, t_0)\}] \\
= \ & \textstyle\prod_{i=1}^{n} [\inf_{\beta\geq r} \min\{\frac{F_i^-(\beta,t)-F_i^-(\beta,t_0)}{h}, \frac{F_i^+(\beta,t)-F_i^+(\beta,t_0)}{h}\}, \\
& \qquad \sup_{\beta\geq r} \max\{\frac{F_i^-(\beta,t)-F_i^-(\beta,t_0)}{h}, \frac{F_i^+(\beta,t)-F_i^+(\beta,t_0)}{h}\}].
\end{aligned}$$

Because the b-gradient $\nabla_b \widetilde{F}(t_0)$ of $\widetilde{F}$ be exist at $t_0 \in \text{int}M$, for any $r \in [0, 1]$, we have

$$\begin{aligned}
& \lim_{h\to 0}[\tfrac{\widetilde{F}(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\ominus_g \widetilde{F}(t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}]^r \\
= \ & \lim_{h\to 0} \textstyle\prod_{i=1}^{n} [\inf_{\beta\geq r} \min\{\frac{F_i^-(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(\beta,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}, \\
& \qquad \frac{F_i^+(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(\beta,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\}, \\
& \qquad \sup_{\beta\geq r} \max\{\frac{F_i^-(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(\beta,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}, \\
& \qquad \frac{F_i^+(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(\beta,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\}] \\
= \ & \textstyle\prod_{i=1}^{n} [\inf_{\beta\geq r} \min\{\lim_{h\to 0} \frac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}, \\
& \qquad \lim_{h\to 0} \frac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\}, \\
& \qquad \sup_{\beta\geq r} \max\{\lim_{h\to 0} \frac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}, \\
& \qquad \lim_{h\to 0} \frac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\}] \\
= \ & \textstyle\prod_{i=1}^{n} [\inf_{\beta\geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta\geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \\
= \ & [v_j]^r.
\end{aligned}$$

Therefore, there exist $u_j = v_j \in L(E^n)$, such that

$$\lim_{h\to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, \ t_j^0, \cdots, \ t_m^0)}{h} = u_j \ (j = 1, 2, \cdots, m),$$

which implies that the gradient $\nabla\widetilde{F}(t_0)$ of $\widetilde{F}$ at t_0 exists and $u_j = v_j$ $(j = 1, 2, \cdots, m)$.

Definition 4.4. Let $\widetilde{F} : M \to L(E^n)$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If the gH-difference $F_r(t) \ominus_{gH} F_r(t_0)$ exist for all $r \in [0, 1]$, and there exist $[u_j]^r = \prod_{i=1}^n [u_{ij}^-(r), u_{ij}^+(r)] \subseteq R^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t\to t_0} \frac{d(F_r(t) \ominus_{gH} F_r(t_0), \sum_{j=1}^m [u_j]^r (t_j - t_j^0))}{d(t, t_0)} = 0$$

uniformly for $r \in [0, 1]$, then we say that $\widetilde{F}$ is *level-wise* differentiable (l-differentiable for short) at t_0.

Theorem 4.3. Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping. If $\widetilde{F}$ is l-differentiable at $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$, then there exist $u_j \in L(E^n)$, such that for any $r \in [0, 1]$,

$$[v_j]^r = \prod_{i=1}^n [\inf_{\beta\geq r} u_{ij}^-(\beta), \sup_{\beta\geq r} u_{ij}^+(\beta)] \ (j = 1, 2, \cdots, m).$$

Proof. For any $r \in [0, 1]$, We can show that the class of sets

$$\prod_{i=1}^n [\inf_{\beta\geq r} u_{ij}^-(\beta), \sup_{\beta\geq r} u_{ij}^+(\beta)] \ (j = 1, 2, \cdots, m)$$

satisfies the conditions of Theorem 2.1. According to Lemma 3.1 and Definition 4.4, if there exists $\delta > 0$, such that for any $|h| < \delta$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$, we have

$$\begin{aligned} u_{ij}^-(r) &= \min\{\lim_{h\to 0} \tfrac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^-(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}, \\ &\qquad \lim_{h\to 0} \tfrac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^+(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}, \\ u_{ij}^+(r) &= \max\{\lim_{h\to 0} \tfrac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^-(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}, \\ &\qquad \lim_{h\to 0} \tfrac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^+(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}, \end{aligned}$$

for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Since $F_i^-(r, t)$ and $F_i^+(r, t)$ are left continuous with respect $r \in (0, 1]$ and right continuous at $r = 0$,

$$\frac{F_i^-(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^-(r, t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}$$

and

$$\frac{F_i^+(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^+(r, t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}$$

are left continuous at $r \in (0, 1]$ and right continuous at $r = 0$. Thus, for any $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$,

$$\begin{aligned} &\inf_{\beta\geq r} \min\{\tfrac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^-(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}, \\ &\qquad \tfrac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^+(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\}, \end{aligned}$$

and

$$\begin{aligned} &\sup_{\beta\geq r} \max\{\tfrac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^-(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}, \\ &\qquad \tfrac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) - F_i^+(r,t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}\} \end{aligned}$$

are left continuous at $r \in (0,1]$ and right continuous at $r=0$. Therefore, for any $r \in [0,1]$, $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$, we have

(1)

$$\inf_{\beta \geq r} \min\{\lim_{h\to 0} \frac{F_i^-(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h},$$
$$\lim_{h\to 0} \frac{F_i^+(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}$$

are non-decreasing and left continuous at $r \in (0,1]$ and right continuous at $r=0$,

(2)

$$\sup_{\beta \geq r} \max\{\lim_{h\to 0} \frac{F_i^-(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h},$$
$$\lim_{h\to 0} \frac{F_i^+(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}$$

are non-increasing and left continuous at $r \in (0,1]$ and right continuous at $r=0$,

(3)

$$\inf_{\beta \geq r} \min\{\lim_{h\to 0} \frac{F_i^-(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h},$$
$$\lim_{h\to 0} \frac{F_i^+(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}$$
$$\leq \sup_{\beta \geq r} \max\{\lim_{h\to 0} \frac{F_i^-(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h},$$
$$\lim_{h\to 0} \frac{F_i^+(\beta,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(\beta,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}.$$

Consequently, there exist $v_j \in L(E^n)$ $(j=1,2,\cdots,m)$, such that for any $r \in [0,1]$,

$$[v_j]^r = \prod_{i=1}^{n} [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)] \ (j=1,2,\cdots,m).$$

Definition 4.5. Let $\widetilde{F}: M \to L(E^n)$ is l-differentiable at t_0, for any $r \in [0,1]$, we denote

$$[v_j]^r = \prod_{i=1}^{n} [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)] \ (j=1,2,\cdots,m),$$

then we say that $(v_1,\ v_2,\cdots,\ v_m)$ is the *level-wise* gradient (l-gradient for short) of $\widetilde{F}$ at t_0, denoted by $\nabla_l \widetilde{F}(t_0)$, i.e.,

$$\nabla_l \widetilde{F}(t_0) = (v_1,\ v_2,\cdots,\ v_m).$$

Remark 4.3. Let $\widetilde{F}: M \to L(E^n)$, $t_0 \in M$. Then the l-gradient $\nabla_l \widetilde{F}(t_0)$ exists at t_0 if and only if $F_r(t) \ominus_{gH} F_r(t_0)$ exist and there are $[u_j]^r = \prod_{i=1}^{n} [u_{ij}^-(r), u_{ij}^+(r)] \subseteq R^n$ $(j=1,2,\cdots,m)$, such that

$$u_{ij}^-(r) = \min\{\lim_{h\to 0} \frac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h},$$
$$\lim_{h\to 0} \frac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}$$

and

$$u_{ij}^+(r) = \max\{\lim_{h\to 0} \frac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h},$$
$$\lim_{h\to 0} \frac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}$$

uniformly for $r \in [0,1]$, where $h \in R$ with $t=(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0) \in \mathrm{int}M$ and

$$[u_j]^r = \prod_{i=1}^{n} [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \ (j=1,2,\cdots,m).$$

Theorem 4.4. Let the l-gradient $\nabla_l\widetilde{F}(t_0)$ of fuzzy mapping $\widetilde{F}: M \to L(E^n)$ be exist at $t_0 \in \text{int}M$. If $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla\widetilde{F}(t_0)$ of $\widetilde{F}$ at t_0 exists and we have $u_j = v_j$ $(j = 1, 2, \cdots, m)$, where $\nabla\widetilde{F}(t_0) = (u_1, u_2, \cdots, u_m)$, $\nabla_l\widetilde{F}(t_0) = (v_1, v_2, \cdots, v_m)$.

Proof. Let $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$, $h \in R$ and $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. We denote $[u_j]^r = \prod_{i=1}^n [\inf_{\beta\geq r} u_{ij}^-(\beta), \sup_{\beta\geq r} u_{ij}^+(\beta)]$, then $u_j \in L(E^n)$ and

$$\begin{aligned}
u_{ij}^-(r) &= \min\{\lim_{h\to 0} \tfrac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}, \\
&\qquad \lim_{h\to 0} \tfrac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\}, \\
u_{ij}^+(r) &= \max\{\lim_{h\to 0} \tfrac{F_i^-(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^-(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}, \\
&\qquad \lim_{h\to 0} \tfrac{F_i^+(r,t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)-F_i^+(r,t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}\},
\end{aligned}$$

for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. It follows from Theorem 2.2 and Lemma 3.1 that

$$\begin{aligned}
& D(\tfrac{\widetilde{F}(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\ominus_g\widetilde{F}(t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}, u_j) \\
= \ & \sup_{r\in[0,1]} d(\tfrac{1}{h}\prod_{i=1}^n [\inf_{\beta\geq r} \min\{F_i^-(\beta, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^-(\beta, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), \\
& \qquad F_i^+(\beta, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^+(\beta, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)\}, \\
& \qquad \sup_{\beta\geq r} \max\{F_i^-(\beta, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^-(\beta, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), \\
& \qquad F_i^+(\beta, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^+(\beta, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)\}], \\
& \qquad \prod_{i=1}^n [\inf_{\beta\geq r} u_{ij}^-(\beta), \sup_{\beta\geq r} u_{ij}^+(\beta)]) \\
\leq \ & \sup_{r\in[0,1]} d(\tfrac{1}{h}\prod_{i=1}^n [\min\{F_i^-(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^-(r, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), \\
& \qquad F_i^+(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^+(r, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)\}, \\
& \qquad \max\{F_i^-(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^-(r, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), \\
& \qquad F_i^+(r, t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - F_i^+(r, t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)\}], \\
& \qquad \prod_{i=1}^n [\inf_{\beta\geq r} u_{ij}^-(\beta), \sup_{\beta\geq r} u_{ij}^+(\beta)]) \\
= \ & \sup_{r\in[0,1]} d(\tfrac{F_r(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\ominus_{gH}F_r(t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}, [u_j]^r).
\end{aligned}$$

Because

$$\lim_{h\to 0} \frac{F_r(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_{gH} F_r(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)}{h} = [u_j]^r$$

uniformly for $r \in [0, 1]$, for any $\varepsilon > 0$, there exists $\delta > 0$, when $|h| < \delta$, we have

$$D(\frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)}{h}, u_j) \leq \sup_{r\in[0,1]} d(\frac{F_r(t_0 + h) \ominus_{gH} F_r(t_0)}{h}, [u_j]^r) < \varepsilon.$$

Therefore, the gradient $\nabla\widetilde{F}(t_0)$ of $\widetilde{F}$ at t_0 exists and $\nabla\widetilde{F}(t_0) = (u_1, u_2, \cdots, u_m) = \nabla_l\widetilde{F}(t_0)$.

5. The subdifferential for fuzzy n-cell mappings

In recent years, nonsmooth analysis has increasingly come to play a role in functional analysis, optimization, optimal design, differential equations and control theory. The subdifferential is an important tool, used widely in nonsmooth analysis and optimization, thus we will discuss subdifferential concept for fuzzy n-cell mappings based on the ordering $\preceq_c$.

Definition 5.1. [17] Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy n-cell mapping. $\widetilde{F}$ is said to be convex (c.) on M if

$$\widetilde{F}(\lambda t + (1 - \lambda)t') \preceq_c \lambda\widetilde{F}(t) + (1 - \lambda)\widetilde{F}(t')$$

for any $t, t \in M$ and $\lambda \in [0,1]$.

The convex fuzzy n-cell mappings in the following arguments are assumed to be comparable.

Definition 5.2. Let $\widetilde{F} : M \to L(E^n)$ be a convex fuzzy n-cell mapping on M, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$, $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. if there exist $u_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0) \succeq_C hu_j,$$

then we call $(u_1, u_2, \cdots, u_m)$ a subgradient of $\widetilde{F}$ at t_0, and say the set of all subgradients of $\widetilde{F}$ at t_0 to be subdifferential of $\widetilde{F}$ at t_0, denoted by $\partial\widetilde{F}(t_0)$, i.e.,

$$\partial\widetilde{F}(t_0) = \{(u_1, u_2, \cdots, u_m) : \widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0) \succeq_C hu_j,\ u_j \in L(E^n)\}.$$

According to Theorem 2.3, it is easy to verify the following conclusion.

Theorem 5.1. Let $\widetilde{F} : M \to L(E^n)$ be a convex fuzzy n-cell mapping on M. Then, we have

$$\partial(\alpha\widetilde{F}(t) + \beta\widetilde{G}(t)) = \alpha\partial\widetilde{F}(t) + \beta\partial\widetilde{G}(t),$$

for any $t \in \text{int}M$ and $\alpha, \beta \geq 0$.

Theorem 5.2. Let $\widetilde{F} : M \to L(E^n)$ be a convex fuzzy n-cell mapping on M. Then the subdifferential $\partial\widetilde{F}(t)$ is a convex set in $L(E^n)$.

Proof. For an empty subdifferential, the assertion is trivial. Take two arbitrary subgradients

$$(u_1, u_2, \cdots, u_m),\ (v_1, v_2, \cdots, v_m) \in \partial\widetilde{F}(t_1, \cdots, t_j, \cdots, t_m).$$

When $(t_1, \cdots, t_j + h, \cdots, t_m) \in \text{int}M$, we have

$$\lambda(\widetilde{F}(t_1, \cdots, t_j + h, \cdots, t_m) \ominus_g \widetilde{F}(t_1, \cdots, t_j, \cdots, t_m)) \succeq_C \lambda hu_j,$$

$$(1-\lambda)(\widetilde{F}(t_1, \cdots, t_j + h, \cdots, t_m) \ominus_g \widetilde{F}(t_1, \cdots, t_j, \cdots, t_m)) \succeq_C (1-\lambda)hv_j,$$

for any $\lambda \in [0,1]$. It follows from Theorem 2.3 that

$$\widetilde{F}(t_1, \cdots, t_j + h, \cdots, t_m) \ominus_g \widetilde{F}(t_1, \cdots, t_j, \cdots, t_m) \succeq_C h(\lambda u_j + (1-\lambda)v_j)$$

for any $j = 1, 2, \cdots, m$. Therefore,

$$\lambda(u_1, u_2, \cdots, u_m) + (1-\lambda)(v_1, v_2, \cdots, v_m) \in \partial\widetilde{F}(t),$$

which implies that the subdifferential $\partial\widetilde{F}(t)$ is a convex set in E^n.

Next, We study the problems of minimizing and maximizing a convex fuzzy n-cell mapping and discuss the necessary and sufficient conditions for optimality.

Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy n-cell mapping. We consider an unconstrained fuzzy minimization problem (FMP):

Minimize $\widetilde{F}(t)$,

Subject to $t \in \text{int}M$.

A point $t_0 \in \text{int}M$ is called a feasible solution to the problem, if for no $t \in \text{int}M$ such that $\widetilde{F}(t) \preceq_c \widetilde{F}(t_0)$, then t_0 is called an optimal solution, or a global minimum point.

With the aid of definitions for subdifferential and optimal solution we can immediately present a necessary and sufficient optimality condition. This theorem is formulated without proof because it is an obvious consequence of the definition of the subdifferential.

Theorem 5.3. Let $\widetilde{F} : M \to L(E^n)$ be a convex fuzzy n-cell mapping on M. Then t_0 is a global minimum point if and only if $(\widetilde{0}, \widetilde{0}, \cdots, \widetilde{0}) \in \partial\widetilde{F}(t_0)$.

6. Conclusion

In this paper, we introduce the concept of generalized difference of n-cell fuzzy-numbers and an ordering relation on the fuzzy n-cell number space is considered. Using the generalized difference of n-cell fuzzy-numbers the generalized differential and gradient concepts for fuzzy n-cell mappings are discussed. Furthermore, we have used the ordering relation $\preceq_c$ to obtain the subdifferential for fuzzy n-cell mappings based on the generalized difference of n-cell fuzzy-numbers.

Future research includes studying optimality conditions for fuzzy constrained minimization problem. One alternative is to define the concept of invex function using g-differentiability and the ordering relation $\preceq_c$ for fuzzy n-cell mappings.

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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Associate Editors of Journal of Computational Analysis and Applications

Francesco Altomare
Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
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THE EXTENSION OF A MODIFIED INTEGRAL OPERATOR TO A CLASS OF GENERALIZED FUNCTIONS

S. K. Q. Al-Omari[1] and Dumitru Baleanu[2]

ABSTRACT. In this paper, we investigate a class of modified G-transforms having G-functions as kernels on a generalized space of sequences. We derive certain spaces of generalized functions named as Boehmians to legitimate the existence of the described integral. The modified G-transform is partially sharing the classical transform with some general properties. An inversion formula is also discussed on the generalized sense.

1. Introduction

H-functions being related to most of known special functions are defined by integrals of the Mellin-Barnes type with integrands involving products of Euler gamma functions. Being an intemperate generalization of the generalized hypergeometric functions ${}_pF_q$, H-functions are utilized for applications in a large variety of problems connected with statistical distributions, versatile integrals, reaction, diffusion, reaction diffusion, engineering, communications, fractional differential and integral equations and many areas of theoretical physics and statistical distribution theory as well.

Through a special case of H-integral transforms, the G-integral transform enfolds various integrals related to Laplace, Hankel, Hilbert and Riemann-Liouville fractional integral transforms and, that integrals of Gauss hypergeometric function kernel type. However, despite a variety of integral transforms may not be reduced to G-transform integral type they are indeed given in the form of H-transform integral type.

With the interest to study integral, dual and tripple equations, integral transforms of special kernel functions were motivated to include many mathematical problems and engineering applications.

Integral transforms having kernels of H-function type were frequently presented as [4]

$$(H\varphi)(\eta) = \int_0^\infty H_{p,q}^{m,n}\left[\eta\zeta \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\varphi(\zeta)\,\mathrm{d}\zeta,\ \eta>0, \tag{1}$$

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where $H_{p,q}^{m,n}$ are functions given in terms of the Mellin-Barnes type contour integral [6]

$$H_{p,q}^{m,n}(w)=\frac{1}{2\pi i}\int_{\mathcal{L}}X(\eta)\,w^{\eta}\mathtt{d}\eta, \tag{2}$$

where

$$X(\eta)=\frac{\prod_1^m\Gamma\left(b_j,\beta_j\eta\right)\prod_1^n\Gamma\left(1-a_j,\alpha_j\eta\right)}{\prod_{m+1}^q\Gamma\left(1-b_j,\beta_j\eta\right)\prod_{n+1}^p\Gamma\left(a_j,\alpha_j\eta\right)}. \tag{3}$$

The particular case of H-transforms where $\alpha_1=...=\alpha_p=1$ and $\beta_1=...=\beta_q=1$, gives the G-transform integral

$$(G\varphi)(\eta)=\int_0^{\infty}G_{p,q}^{m,n}\left[\eta\zeta\,\middle|\,\begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right]\varphi(\zeta)\,\mathtt{d}\zeta, \tag{4}$$

and that the amendment

$$G_{p,q}^{m,n}\left[\eta\,\middle|\,\begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right] \tag{5}$$

of the H-function is the so-called G-function.
For a somehow much more detailed account of G and H-functions we refer to [1, 8].
The numbers $a^*,\Delta^*,a_1^*,a_2^*,\alpha$ and β when they appear are given as follows [4, (6.1.5) − (6.1.11)]

$$\left.\begin{array}{l}a^*=2m+2n-p-q,\\ \Delta^*=q-p,\\ a_1^*=(m+n)-p,\\ a_2^*=(m+n)-q,\\ \alpha=\begin{cases}-\min\limits_{1\le j\le m}\mathrm{Re}\,(b_j) & ,m>0\\ -\infty & ,m=0\end{cases}\\ \beta=\begin{cases}1-\max\limits_{1\le i\le n}\mathrm{Re}\,(a_i)\,,n>0\\ \infty & ,n=0\end{cases}\\ \mu=\sum_{j=1}^q b_j-\sum_{i=1}^p a_i+\dfrac{p-q}{2}\end{array}\right\}. \tag{6}$$

By $\boldsymbol{l}_{v,r},v\in\mathbb{R},1\le r<\infty$, we denote the summable space of those Lebesgue measurable complex valued functions such that

$$\left.\begin{array}{l}\|\varphi\|_{v,r}=\left(\int\limits_0^{\infty}\left|\zeta^v\varphi(\zeta)\right|^r\dfrac{\mathtt{d}\zeta}{\zeta}\right)^{\frac{1}{r}}<\infty\\ \text{and}\\ \|\varphi\|_{v,\infty}=\operatorname{ess\,sup}\limits_{\zeta>0}\left(\zeta^v\left|\varphi(\zeta)\right|\right),v\in\mathbb{R}\end{array}\right\}. \tag{7}$$

The modified G-transform we consider in this note is given by the integral equation [4, (6.2.4)]

$$\left(G_{\sigma,\kappa}^1\varphi\right)(\eta)=\eta^{\sigma}\int_0^{\infty}G_{p,q}^{m,n}\left[\frac{\zeta}{\eta}\,\middle|\,\begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right]\zeta^{\kappa}\varphi(\zeta)\,\frac{\mathtt{d}\zeta}{\zeta}. \tag{8}$$

It is associated with the the radical integral transform (4) by the equation

$$\left(G_{\sigma,\kappa}^1\varphi\right)(\eta)=M_{\sigma}\left(G\left(RM_{\kappa}\varphi\right)\right)(\eta),$$

where R and M_κ are operators defined , respectively, by [4, (3.3.13) and (3.3.11)]

$$(R\varphi)(\zeta) = \frac{1}{\zeta}\varphi\left(\frac{1}{\zeta}\right) \text{ and } (M_\kappa\varphi)(\zeta) = \zeta^\kappa\varphi(\zeta), \kappa \in \mathbb{C}.$$

PARSEVAL FORMULA 1 The Parseval's formula for the modified G-transform is derived as

$$\int_0^\infty \varphi(\eta)\left(G^1_{\sigma,\kappa}g\right)(\eta)\,\mathtt{d}\eta = \int_0^\infty \left(G^2_{\kappa,\sigma}\varphi\right)(\eta)\,g(\eta)\,\mathtt{d}\eta,$$

where $G^2_{\kappa,\sigma}\varphi$ is the modified G-transform

$$\left(G^2_{\kappa,\sigma}\varphi\right)(\eta) = \eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{\zeta}{\eta}\,\middle|\,\begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right]\zeta^\kappa\varphi(\zeta)\,\frac{\mathtt{d}\zeta}{\eta}.$$

The following remark is of great importance to our investigation .

REMARK 2 ([4, Theorem 6.50 (i)]) Let κ and σ be real numbers and that numbers a^*, Δ, μ, be defined as in (6). Suppose the following are satisfied :

(i) $\alpha < v - \kappa < \beta$;

(ii) Either of (a) $a^* > 0$ or (b) $a^* = 0$ and $\Delta^*[v-\kappa] + \mathrm{Re}\,\mu \leqq 0$ holds. Then the transform $G^1_{\sigma,\kappa}\varphi$ is a one-one mapping from $\boldsymbol{l}_{v,2}$ into $\boldsymbol{l}_{v-\kappa-\sigma,2}$.

For a somehow much more detailed account of several significant results on the modified G-transforms, we refer the reader to [4].

Boehmians are motivations of regular operators with algebraic character of Mikusinski operators and do not have restriction on the support. With different function spaces various spaces of Boehmians can be obtained. Distributions, ultradistributions, regular operators are indeed contained in some well established spaces of Boehmians.

In a Boehmian context, various generalizations of various integral transforms were given once the topic was started. A complete account of the theory of Boehmian spaces was given in [2, 3, 5, 7] , [10]-[17].

However, the existed results in this theory are classical and none were discussed in the space of Boehmians. In this article, we develope the classical theory of the modified $G^1_{\sigma,\kappa}$ transform to the theory of Boehmians. In the following section we discuss the construction of the spaces of Boehmians. In Section 3, we give the representative of the modified $G^1_{\sigma,\kappa}$ transform and its inverse in the defined spaces of Boehmians. We further discuss certain results related to the proposed integrals.

2. Construction of Spaces of Boehmians

Let us first agree for the products we demand for our investigation.

The first product we should use here is the so-called Mellin type convolution product of first kind defined as [9]

$$(\varphi \curlyvee g)(\xi) = \int_0^\infty y^{-1}\varphi\left(\xi y^{-1}\right) g(y)\,\mathtt{d}y, \tag{9}$$

provided the integral exists.

A number of the properties of this integral that we find it worthwhile to be described here :

(i) $g_1 \curlyvee g_2 = g_2 \curlyvee g_1$;

(ii) $(g_1 \curlyvee g_2) \curlyvee g_3 = g_1 \curlyvee (g_2 \curlyvee g_3)$;

(iii) $(\alpha g_1) \curlyvee g_2 = \alpha(g_1 \curlyvee g_2)$;

(iv) $g_1 \curlyvee (g_2 + g_3) = g_1 \curlyvee g_2 + g_1 \curlyvee g_3$.

It is of great importance to introduce the following convolution product that will be worthy of attention

$$(\varphi \bullet g)(\xi) = \int_0^\infty \varphi\left(\xi y^{-1}\right) y^{\kappa-1-\sigma} g(y)\, \mathtt{d}y, \tag{10}$$

where κ and σ are real numbers.

Properties of this integral are to be provided in the text of the paper.

The relation between the convolution products are given by the following theorem.

THEOREM 3 Let φ and g be integrable functions on $(0,\infty)$. Then, we have

$$G^1_{\sigma,\kappa}(\varphi \curlyvee g)(\eta) = \left(G^1_{\sigma,\kappa}\varphi \bullet g\right)(\eta).$$

PROOF Under the hypothesis of the theorem and by using (8) for (9) we get

$$\begin{aligned} G^1_{\sigma,\kappa}(\varphi \curlyvee g)(\eta) &= \eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{\zeta}{\eta}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right] \zeta^\kappa \int_0^\infty y^{-1}\varphi\left(\zeta y^{-1}\right) g(y) \\ &\quad \times \mathtt{d}y \frac{\mathtt{d}\zeta}{\zeta} \\ &= \int_0^\infty g(y)\, y^{-1}\eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{\zeta}{\eta}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right] \zeta^\kappa \varphi\left(\zeta y^{-1}\right) \\ &\quad \times \frac{\mathtt{d}\zeta}{\zeta}\mathtt{d}y. \end{aligned} \tag{11}$$

On setting variables and using Fubini's theorem, (11) produce

$$\begin{aligned} G^1_{\sigma,\kappa}(\varphi \curlyvee g)(\eta) &= \int_0^\infty g(y)\, y^{-1}\eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{yw}{\sigma}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right] (yw)^\kappa \varphi(w) \\ &\quad \times \frac{y\mathtt{d}w}{yw}\mathtt{d}y \\ &= \int_0^\infty g(y)\, y^{-1}\eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{w}{\eta y^{-1}}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right] y^\kappa w^\kappa \varphi(w) \\ &\quad \times \frac{\mathtt{d}w}{w}\mathtt{d}y \\ &= \int_0^\infty g(y)\, y^{\kappa-1-\sigma}\left(\eta y^{-1}\right)^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{w}{\eta y^{-1}}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right] \\ &\quad \times w^\kappa \varphi(w)\, \frac{\mathtt{d}w}{w}\mathtt{d}y \\ &= \int_0^\infty G^1_{\sigma,\kappa}\varphi\left(\eta y^{-1}\right) y^{\kappa-1-\sigma} g(y)\, \mathtt{d}y \end{aligned}$$

The proof of the theorem is completely finished.

LEMMA 4 Let φ, g and ψ be integrable functions on the open interval $(0,\infty)$. We get

$$\varphi \bullet (g \curlyvee \psi) = (\varphi \bullet g) \bullet \psi\ .$$

PROOF On account of (10) we write

$$\begin{aligned}\varphi \bullet (g \curlyvee \psi)(\xi) &= \int_0^\infty \varphi\left(\xi y^{-1}\right) y^{\kappa-1-\sigma}\left(\int_0^\infty \zeta^{-1} g\left(y\zeta^{-1}\right)\psi(\zeta)\,\mathtt{d}\zeta\right)\mathtt{d}y \\ &= \int_0^\infty \psi(\zeta)\,\zeta^{-1}\left(\int_0^\infty \varphi\left(\xi y^{-1}\right) y^{\kappa-1-\sigma} g\left(y\zeta^{-1}\right)\mathtt{d}y\right)\mathtt{d}\zeta. \quad (12)\end{aligned}$$

By change of variables, (12) yields

$$\begin{aligned}\varphi \bullet (g \curlyvee \psi)(\xi) &= \int_0^\infty \psi(\zeta)\int_0^\infty \varphi\left(\xi(\zeta w)^{-1}\right)(\zeta w)^{\kappa-1-\sigma} g(w)\,\zeta \mathtt{d}w\mathtt{d}\zeta \\ &= \int_0^\infty \psi(\zeta)\,\zeta^{\kappa-1-\sigma}\int_0^\infty \varphi\left(\xi\zeta^{-1}w^{-1}\right) \\ &\quad \times w^{\kappa-1-\sigma} g(w)\,\mathtt{d}w\mathtt{d}\zeta \\ &= \int_0^\infty \psi(\zeta)\,\zeta^{\kappa-1-\sigma}(\varphi \bullet g)\left(\xi\zeta^{-1}\right)\mathtt{d}\zeta.\end{aligned}$$

The proof is completely finished.

Let $\mathcal{D}$ denote the standard notation of the space of test functions of compact supports in $(0,\infty)$. Then we have the following results.

THEOREM 5 Let $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$ and $g \in \mathcal{D}$ be given. Then, we have

$$\varphi \bullet g \in \boldsymbol{l}_{v-\kappa-\sigma,2}.$$

PROOF By appealing to (7) and the integral equation (10), we get

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} = \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\int_0^\infty \varphi\left(\zeta y^{-1}\right)^{\kappa-1-\sigma} g(y)\,\mathtt{d}y\right|^2 \frac{\mathtt{d}\zeta}{\zeta}.$$

Applying Jensen's inquality yields

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} \le \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\right|\int_0^\infty \left|\varphi\left(\zeta y^{-1}\right)\right|^2 \left|y^{\kappa-1-\sigma} g(y)\right|\mathtt{d}y\frac{\mathtt{d}\zeta}{\zeta}.$$

Using the Fubini's theorem implies

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} \le \int_0^\infty \left|g(y)\,y^{\kappa-1-\sigma}\right|\left(\int_0^\infty \left|\zeta^{v-\kappa-\sigma}\varphi\left(\zeta y^{-1}\right)\right|^2 \frac{\mathtt{d}\zeta}{\zeta}\right)\mathtt{d}y.$$

Now, let $[a,b]$, $0<a<b$, be an interval containing the support of g. Then, the hypothesis of the theorem $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$, reveals

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} \le M\,\|\varphi\|^2_{v-\kappa-\sigma,2}\int_a^b \left|y^{\kappa-1-\sigma} g(y)\right|\mathtt{d}y,$$

where $M = \int_a^b \left|y^{\kappa-1-\sigma} g(y)\right|\mathtt{d}y$.
Thus, the above equation further reveals

$$\|\varphi \bullet g\|_{v-\kappa-\sigma,2} < \infty.$$

The proof is completely finished.

THEOREM 6 There hold the following identities.

(i) Let $\{\varphi_n\}$, $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$ be such that $\varphi_n \to \varphi$ as $n \to \infty$. We have

$$\varphi_n \bullet g \to \varphi \bullet g \text{ as } n \to \infty$$

for every $g \in \mathcal{D}$.

(ii) Let $\varphi_1, \varphi_2 \in \boldsymbol{l}_{v-\kappa-\sigma,2}$ and $g \in \mathcal{D}$. Then we have the following identities satisfy

$$(\varphi_1 + \varphi_2) \bullet g = \varphi_1 \bullet g + \varphi_2 \bullet g \text{ and } \rho(\varphi_1 \bullet g) = (\rho\varphi_1) \bullet g,$$

for arbitrary complex number ρ.

The proof of this theorem can be followed by using simple integral calculus. Hence, we avoid adding more details.

DEFINITION 7 Let $\{\delta_n\} \in \mathcal{D}$ such that

(i) $\int_0^\infty \delta_n(\xi)\,\mathtt{d}\xi = 1$ $(n \in \mathbb{N})$.

(ii) $\int_0^\infty |\delta_n(\xi)|\,\mathtt{d}\xi < A$ ($A \in \mathbb{R}$ being positive).

(iii) $\operatorname{supp}\delta_n = \{\xi : \delta_n(\xi) \neq 0\} \to 0$ as $n \to \infty$.

The set of all sequences $\{\delta_n\}$ are denoted by Δ. Every $\{\delta_n\}$ in Δ is said to be a delta sequence which corresponds to the delta distribution.

THEOREM 8 Let $\{\delta_n\} \in \Delta$ and $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$. Then, we have

$$\varphi \bullet \delta_n \to \varphi \text{ in } \boldsymbol{l}_{v-\kappa-\sigma,2} \text{ as } n \to \infty. \tag{14}$$

PROOF By the first part of Definition 7 and Jensen's inequality we have

$$\begin{aligned} \|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \quad &\leq \quad \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\right|^2 \int_0^\infty \left|y^{\kappa-1-\sigma}\varphi\left(\zeta y^{-1}\right) - \varphi(\zeta)\right|^2 \\ &\quad \times |\delta_n(y)|\,\mathtt{d}y\frac{\mathtt{d}\zeta}{\zeta}. \end{aligned} \tag{15}$$

Therefore, by making use of Fubini's theorem, (15) gives

$$\begin{aligned} \|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \quad &\leq \quad \int_{a_n}^{b_n} |\delta_n(y)| \times \\ &\quad \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\left(\psi_y(\zeta) - \varphi(\zeta)\right)\right|^2 \frac{\mathtt{d}\zeta}{\zeta}\mathtt{d}y, \end{aligned} \tag{16}$$

where $\operatorname{supp}\delta_n(y) \subseteq [a_n, b_n], 0 < a_n < b_n, \forall n \in \mathbb{N}$.

Taking into account the fact that $\varphi(\zeta), \psi_y(\zeta) = \varphi\left(\zeta y^{-1}\right) y^{\kappa-1-\sigma} \in \boldsymbol{l}_{v-\kappa-\sigma,2}$, it follows from (16) that

$$\|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \leq M^* \int_{a_n}^{b_n} |\delta_n(y)|\,\mathtt{d}y.$$

for some positive constant M^*.

Therefore,

$$\|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \leq M^* M_1(a_n, b_n),$$

$M_1 > 0$.

The last inequality follows from the identity (iii) of Definition 7.

The proof of the theorem is completely finished.

The space $\mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2}, (\mathcal{D}, \Upsilon), \bullet, \Delta)$ is therefore generated and regarded as a space of Boehmians.

Construction of the space $\mathcal{B}(\boldsymbol{l}_{v,2}, (\mathcal{D}, \Upsilon), \Upsilon, \Delta)$ can be obtained by that technique similar to that of $\mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2}, (\mathcal{D}, \Upsilon), \bullet, \Delta)$ and the properties of Υ we have already cited above.

Sum of two Boehmians $\left[\frac{\varphi_n}{\delta_n}\right]$ and $\left[\frac{g_n}{\varepsilon_n}\right]$ in $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ can be expressed as

$$\left[\frac{\varphi_n}{\delta_n}\right]+\left[\frac{g_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\,\Upsilon\,\delta_n+g_n\,\Upsilon\,\delta_n}{\delta_n\,\Upsilon\,\varepsilon_n}\right].$$

Multiplication in $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ by $\alpha\in\mathbb{C}$ is defined as $\gamma\left[\frac{\varphi_n}{\delta_n}\right]=\left[\gamma\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\gamma\varphi_n}{\delta_n}\right]$.

The extensions of Υ and $\mathcal{D}^\alpha$ to $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ are introduced as

$$\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\left[\frac{g_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\,\Upsilon\,g_n}{\delta_n\,\Upsilon\,\varepsilon_n}\right]\text{ and }\mathcal{D}^\alpha\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\mathcal{D}^\alpha\varphi_n}{\delta_n}\right],\ \alpha\in\mathbb{R}.$$

Let $\left[\frac{\varphi_n}{\delta_n}\right]$ belong to $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ and ω be in $\boldsymbol{l}_{v,2}$. The operation Υ can be extended to $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)\times\boldsymbol{l}_{v,2}$ by

$$\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\,\omega=\left[\frac{\varphi_n\,\Upsilon\,\omega}{\delta_n}\right].$$

Let the sequence $\{\beta_n\}$ be in $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$. Then $\beta_n\xrightarrow{\delta}\beta$ in $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$, if there can be found a delta sequence $\{\delta_n\}$ such that for $(\beta_n\,\Upsilon\,\delta_k)$ and $(\beta\,\Upsilon\,\delta_k)\in\boldsymbol{l}_{v,2}$, $n,k\in\mathbb{N}$, we have

$$\lim_{n\to\infty}\beta_n\,\Upsilon\,\delta_k\to\beta\,\Upsilon\,\delta_k\text{ in }\boldsymbol{l}_{v,2}\text{ for every }k\in\mathbb{N}.$$

This can be expressed in $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ as :

$\beta_n\xrightarrow{\delta}\beta$ (as $n\to\infty$) if and only if there are $\varphi_{n,k}$, $\varphi_k\in\boldsymbol{l}_{v,2}$ and $\{\delta_k\}\in\Delta$, $\beta_n=\left[\frac{\varphi_{n,k}}{\delta_k}\right]$, $\beta=\left[\frac{\varphi_k}{\delta_k}\right]$ and to every $k\in\mathbb{N}$ we have $\lim_{n\to\infty}\varphi_{n,k}=\varphi_k$ in $\boldsymbol{l}_{v,2}$.

$\beta_n\xrightarrow{\Delta}\beta$ (as $n\to\infty$) if there can be found a $\{\delta_n\}\in\Delta$ such that $(\beta_n-\beta)\Upsilon\delta_n\in\boldsymbol{l}_{v,2}$ $(\forall n\in\mathbb{N})$ and $\lim_{n\to\infty}(\beta_n-\beta)\,\Upsilon\,\delta_n=0$ in $\boldsymbol{l}_{v,2}$.

On the other hand, addition of two Boehmians in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$ is defined as

$$\left[\frac{\varphi_n}{\delta_n}\right]+\left[\frac{g_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\bullet\delta_n+g_n\bullet\delta_n}{\delta_n\,\Upsilon\,\varepsilon_n}\right].$$

Multiplication and convergence in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$ can be defined similarly as in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$.

3. $G^1_{\sigma,\kappa}$ of Boehmians

In view of Remark 2 and Theorem 3, we extend the transform $G^1_{\sigma,\kappa}$ to the space $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ as

$$\widetilde{G^1_{\sigma,\kappa}}\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\tag{17}$$

in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$.

We recite some properties of the transform $\widetilde{G^1_{\sigma,\kappa}}$ in the course of the following theorems.

THEOREM 9 (i) The operator $\widehat{G^1_{\sigma,\kappa}}$ is well - defined and linear .

(ii) The operator $\widehat{G^1_{\sigma,\kappa}}$ is an isomorphism from $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ onto the space $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$.

(iii) The operator $\widehat{G^1_{\sigma,\kappa}}$ is continuous with respect to δ and Δ - convergence.

(iv) The operator $\widehat{G^1_{\sigma,\kappa}}:\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)\rightarrow\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$ transform is compatible with $G^1_{\sigma,\kappa}:\boldsymbol{l}_{v,2}\rightarrow\boldsymbol{l}_{v-\kappa-\sigma,2}$.

PROOF We prove Part (iv) since similar proofs for Part(i) - Part(iii) are available in many cited papers of the same author and of Roopkumar in [20].

To prove the last part of the theorem, let $\sigma\in\boldsymbol{l}_{v,2}$ and $\left[\frac{\sigma\ \Upsilon\ \delta_n}{\delta_n}\right]$ be its representative in $\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$ where $\{\delta_n\}\in\Delta\ (\forall n\in\mathbb{N})$. Clearly, for all $n\in\mathbb{N}$, $\{\delta_n\}$ is independent from the representative. Hence, from (17) and Theorem 3, we get

$$\widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\sigma\ \Upsilon\ \delta_n}{\delta_n}\right]\right)=\widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\sigma\ \Upsilon\ \delta_n}{\delta_n}\right]\right)=\left[\frac{G^1_{\sigma,\kappa}\left(\sigma\ \Upsilon\ \delta_n\right)}{\delta_n}\right]=\left[\frac{G^1_{\sigma,\kappa}\sigma\bullet\delta_n}{\delta_n}\right].$$

Thus $\left[\frac{G^1_{\sigma,\kappa}\sigma\bullet\delta_n}{\delta_n}\right]$ is the representative of $G^1_{\sigma,\kappa}\sigma$ in the space $\boldsymbol{l}_{v-\kappa-\sigma,2}$.

The proof is therefore finished.

In view of Theorem 9, we introduce the inverse transform of $\widehat{G^1_{\sigma,\kappa}}$ as follows.

DEFINITION 10 Let $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\boldsymbol{l}_{v,2},(\mathcal{D},\Upsilon),\Upsilon,\Delta\right)$. We define the inverse $\widehat{G^1_{\sigma,\kappa}}$ integral operator of $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]$ as

$$\left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\right)=\left[\frac{\varphi_n}{\delta_n}\right],$$

for each $\{\delta_n\}\in\Delta$.

THEOREM 11 Let $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$ and $\varphi\in\mathcal{D}$. We have

$$\left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi\right)=\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\ \varphi$$

and

$$\widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\ \varphi\right)=\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi.$$

PROOF Assume $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right] \in \mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\Upsilon),\bullet,\Delta\right)$. For every $\phi \in \mathcal{D}$, we have

$$\begin{aligned}\left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi\right) &= \left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n\bullet\varphi}{\delta_n}\right]\right)\\ \text{i.e.} &= \left[\frac{\left(G^1_{\sigma,\kappa}\right)^{-1}\left(G^1_{\sigma,\kappa}\varphi_n\bullet\varphi\right)}{\delta_n}\right].\end{aligned}$$

Using Theorem 3 and Definition 10 we obtain

$$\begin{aligned}\left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi\right) &= \left[\frac{\varphi_n\,\Upsilon\,\varphi}{\delta_n}\right]\\ &= \left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\,\varphi.\end{aligned}$$

Proof of the part $\widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\,\varphi\right) = \left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi$ is almost similar. We prefer to omit the details of the proof.
This completely finishes the proof of the theorem.

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Current address: [1]Department of Applied Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan., [2]Department of Mathematics and Computer Sciences, Cankaya University, Eskisehir Yolu 29.km, 06810 Ankara, Turkey and,, Institute of Space Sciences, Magurele-Bucharest, Romania.,

E-mail address: s.k.q.alomari@fet.edu.jo,dumitru@cankaya.edu.tr, Correspondence should be addressed to the first author; s.k.q.alomari@fet.edu.jo

HARMONIC QUASICONFORMAL MAPPINGS OF THE UNIT DISK ONTO THE HORIZONTAL STRIP AND HALF PLANE

JIAN-FENG ZHU

ABSTRACT. In this paper we consider two types of harmonic mappings $w(z) \in S_H(\mathbb{D}, \Omega_1)$ and $w(z) \in S_H(\mathbb{D}, R)$, where $\mathbb{D}$ is the unit disk and Ω_1, R are the domain defined by (2) and (3). Using the representation of harmonic mappings, we find the sufficient and necessary conditions to make $w(z)$ be a harmonic quasiconformal mapping. Furthermore, we obtain some estimates of $w(z)$.

1. INTRODUCTION

A real-valued function $u(x, y)$ on an open set $D \subseteq \mathbb{C}$ is harmonic if it is C^2 on D and satisfies Laplace's equation: $\triangle u = u_{xx} + u_{yy} = 0$. Assume that $z = x + iy$, $w(z) = u(x, y) + iv(x, y)$. Then a complex-valued function $w(z)$ is harmonic if and only if $u(x, y)$ and $v(x, y)$ are both harmonic. This has an equivalent form $w_{z\bar{z}} = 0$.

Let $w(z)$ be a harmonic mapping defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Then there exist two analytic functions $h(z)$ and $g(z)$ such that $w(z) = h(z) + \overline{g(z)}$.

For $z = re^{i\varphi} \in \mathbb{D}$, denote by

$$P(r, t - \varphi) = \frac{1}{2\pi} \frac{1 - r^2}{(1 - 2r\cos(t - \varphi) + r^2)}$$

the Poisson kernel. Then every bounded harmonic mapping $w(z)$ defined on $\mathbb{D}$ has the following representation

$$w(z) = P[f](z) = \int_0^{2\pi} P(r, t - \varphi) f(e^{it})\, dt, \tag{1}$$

where $z = re^{i\varphi} \in \mathbb{D}$ and f is a bounded integrable function defined on the unit circle $S^1 = \partial\mathbb{D}$.

For $z \in \mathbb{D}$, let

$$\Lambda_w(z) = \max_{0 \le \alpha \le 2\pi} |w_z(z) + e^{-2i\alpha} w_{\bar{z}}(z)| = |w_z(z)| + |w_{\bar{z}}(z)|$$

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and

$$\lambda_w(z) = \min_{0\le\alpha\le 2\pi} |w_z(z) + e^{-2i\alpha} w_{\bar{z}}(z)| = ||w_z(z)| - |w_{\bar{z}}(z)||.$$

According to Lewy's Theorem we know that $w(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies

$$J_w(z) = |w_z(z)|^2 - |w_{\bar{z}}(z)|^2 > 0 \quad \text{for every} \quad z \in \mathbb{D}.$$

Suppose that $w(z)$ is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a domain $\Omega \subseteq \mathbb{C}$. Then $w(z)$ is a harmonic K-quasiconformal mapping if and only if

$$K(w) := \sup_{z\in\mathbf{U}} \frac{|w_z(z)| + |w_{\bar{z}}(z)|}{|w_z(z)| - |w_{\bar{z}}(z)|} \le K.$$

It is interesting to consider such a question: under what conditions on f is $w = P[f](z)$ a harmonic quasiconformal mapping? Several authors have studied such a problem (see [5], [6], [8],[10], [11], [12], [13]). However, a univalent sense-preserving harmonic mapping defined on $\mathbb{D}$ doesn't determined by its image domain. In this paper we consider two types of harmonic mappings of $\mathbb{D}$ onto a unbounded convex domain. One maps the unit disk onto the horizontal strip and the other maps the unit disk onto the half plane.

Let S_H denote the class of all complex valued, sense-preserving univalent harmonic mappings $w(z)$ in $\mathbb{D}$, with the normalization $w(0) = w_z(0) - 1 = 0$. Let S_H^0 be the subclass of S_H with $w_{\bar{z}}(0) = 0$ and $w_z(0) > 0$. For a domain $\Omega \subseteq \mathbb{C}$ containing the origin, $S_H(\mathbb{D}, \Omega)$ will denote the class of all sense-preserving univalent harmonic mappings of $\mathbb{D}$ onto Ω normalized by $w(0) = w_{\bar{z}}(0) = 0$ and $w_z(0) > 0$.

Considering the following domains: the horizontal strip

$$\Omega_1 = \{w : -1 < \mathbf{Im} w < 1\}, \tag{2}$$

and the right half plane

$$R = \{w : \mathbf{Re} w > \frac{-1}{2}\}. \tag{3}$$

A conformal mapping φ from $\mathbb{D}$ onto Ω_1 normalized by $\varphi(0) = 0 < \varphi'(0)$ has the form

$$\varphi(z) = \frac{2}{\pi} \ln \frac{1+z}{1-z}. \tag{4}$$

For $z \in \mathbb{D}$ and $|\eta| = 1$, define the kernel

$$k(z,\eta) = \int_0^z \varphi'(\zeta) \frac{1+\bar{\eta}\zeta}{1-\bar{\eta}\zeta}\, d\zeta = \frac{4}{\pi} \int_0^z \frac{1+\bar{\eta}\zeta}{(1-\bar{\eta}\zeta)(1-\zeta^2)}\, d\zeta, \tag{5}$$

and the family

$$F = \{w : w(z) = \mathbf{Re} \int_{|\eta|=1} k(z,\eta) d\mu(\eta) + i\mathbf{Im}\varphi(z), \mu \in \mathbb{P}\}, \tag{6}$$

where $\mathbb{P}$ is the set of probability measures on the Borel sets of the unit circle S^1.

According to [4] we have the following theorem.

Theorem A. $\overline{S_H(\mathbb{D},\Omega_1)} = F$. *Here* $\overline{S_H(\mathbb{D},\Omega_1)}$ *is the closure of* $S_H(\mathbb{D},\Omega_1)$.

Similarly, define the kernel

$$F(z,\eta) = \begin{cases} \mathbf{Re}\frac{z}{1-z} + i\mathbf{Im}\frac{z}{(1-z)^2} & if\ \eta = 1 \\ \mathbf{Re}\frac{z}{1-z} + i\mathbf{Im}\left(\frac{2\eta}{(1-\eta)^2}\ln\frac{1-z}{1-\eta z} + \frac{1+\eta}{1-\eta}\frac{z}{1-z}\right) & if\ \eta \neq 1. \end{cases}$$

According to [1], we have the following theorem.

Theorem B. *Each harmonic mapping* $w(z) \in \overline{S_H(\mathbb{D},R)}$ *if and only if there is a probability measure* μ *on the unit circle such that*

$$w(z) = \int_{|\eta|=1} F(z,\eta)d\mu(\eta). \tag{7}$$

In this paper, we find the sufficient and necessary conditions on the kernel $k(z,\eta)$ and $F(z,\eta)$ which make harmonic mappings $w(z)$ of $S_H(\mathbb{D},\Omega_1)$ and $S_H(\mathbb{D},R)$ to be quasiconformal.

2. Necessary and sufficient conditions

Theorem 1. *Suppose that* $w \in S_H(\mathbb{D},\Omega_1)$, *which has the representation* (6). *Then* w *is a harmonic quasiconformal mapping if and only if it's kernel satisfies the following conditions:*

(i) $c := \operatorname{ess\,inf}_{z\in\mathbb{D}} \mathbf{Re}\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta) > 0$,

(ii) $M_1 := \operatorname{ess\,sup}_{z\in\mathbb{D}}\left|\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)\right| < \infty$,

where c *and* M_1 *are positive constant.*

Proof. Let $w(z) = h(z) + \overline{g(z)}$ be a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto Ω_1. According to Theorem A we have

$$w(z) = \mathbf{Re}\int_{|\eta|=1} k(z,\eta)d\mu(\eta) + i\mathbf{Im}\varphi(z),$$

where $k(z,\eta)$ and μ are defined by (5) and (6). This implies that

$$w(z) = h(z)+\overline{g(z)} = \frac{1}{2}\left(\int_{|\eta|=1} k(z,\eta)d\mu(\eta) + \varphi(z)\right)+\frac{1}{2}\overline{\left(\int_{|\eta|=1} k(z,\eta)d\mu(\eta) - \varphi(z)\right)}.$$

Then

$$h'(z) = \frac{\varphi'(z)}{2}\left(\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta) + 1\right) \tag{8}$$

and

$$g'(z) = \frac{\varphi'(z)}{2}\left(\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta) - 1\right). \tag{9}$$

It follows from (8) and (9) that

$$\left|\frac{g'(z)}{h'(z)}\right| = \left|\frac{\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)-1}{\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)+1}\right|.$$

Let $A_1 = \mathbf{Re}\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)$ and $A_2 = \mathbf{Im}\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)$.

The proof of 'if' part: Since $A_1 \geq c > 0$ applying condition (ii) we see that

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right|^2 = \operatorname*{ess\,sup}_{z\in\mathbb{D}}\frac{(A_1-1)^2+A_2^2}{(A_1+1)^2+A_2^2} = \operatorname*{ess\,sup}_{z\in\mathbb{D}}\left(1-\frac{4A_1}{(A_1+1)^2+A_2^2}\right) < 1.$$

This shows that $w(z)$ is a harmonic quasiconformal mapping.

The proof of 'only if' part: Assume that $w(z) \in S_H(\mathbb{D},\Omega_1)$ is a harmonic quasiconformal mapping. Then the following inequality

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right| = \operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)-1}{\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)+1}\right| \leq k$$

holds for some constant $k < 1$. Hence $\left|\int\limits_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)\right| < \infty$ and $(A_1-1)^2+A_2^2 \leq k^2(A_1+1)^2+k^2A_2^2$. This implies that $2A_1(1+k^2) \geq 1-k^2$. Thus $A_1 \geq \frac{1-k^2}{2(1+k^2)} > 0$.

The proof is completed. □

Remark: For any $z = re^{i\theta} \in \mathbb{D}$ and $\eta = e^{it} \in \partial\mathbb{D}$, we have $\mathbf{Re}\frac{\eta+z}{\eta-z} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} > 0$. If we additional assume that $\mu'(e^{it}) > 0$, then $\mathbf{Re}\int_{|\eta|=1}\frac{\eta+z}{\eta-z}\,d\mu(\eta) > 0$. According to (8) and (9) we see that

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right| < 1,$$

which implies that $w(z)$ is a harmonic quasiconformal mapping.

Theorem 2. *Let $w \in S_H(\mathbb{D},R)$ be a sense-preserving harmonic mapping which has the representation* (7)*. Then w is a quasiconformal mapping if and only if its kernel satisfies the following conditions:*

(i) $d := \operatorname*{ess\,inf}_{z\in\mathbb{D}}\left(1+2\mathbf{Re}\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right) > 0,$

(ii) $M_2 := \operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right| < \infty,$

where d and M_2 are positive constant.

Proof. According to [4] we know that $w(z)$ has the following representation:

$$\begin{aligned} w(z) &= \int_{|\eta|=1} F(z,\eta)d\mu(\eta) \\ &= \frac{1}{2}\int_{|\eta|=1}\left(\frac{z}{1-z}+\frac{2\eta}{(1-\eta)^2}\ln\frac{1-z}{1-\eta z}+\frac{1+\eta}{1-\eta}\frac{z}{1-z}\right)d\mu(\eta) \\ &+ \overline{\frac{1}{2}\int_{|\eta|=1}\left(\frac{z}{1-z}-\frac{2\eta}{(1-\eta)^2}\ln\frac{1-z}{1-\eta z}-\frac{1+\eta}{1-\eta}\frac{z}{1-z}\right)d\mu(\eta)} \\ &= h(z)+\overline{g(z)}. \end{aligned}$$

Then

$$h'(z) = \frac{1}{(1-z)^2}\int_{|\eta|=1}\frac{1}{1-\eta z}d\mu(\eta) \tag{10}$$

and

$$g'(z) = \frac{1}{(1-z)^2}\int_{|\eta|=1}\frac{-\eta z}{1-\eta z}d\mu(\eta). \tag{11}$$

This implies that

$$\left|\frac{g'(z)}{h'(z)}\right| = \frac{\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|}{\left|\int_{|\eta|=1}\left(1+\frac{\eta z}{1-\eta z}\right)d\mu(\eta)\right|} = \frac{\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|}{\left|1+\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|}.$$

The proof of 'if' part: Let

$$B = \int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta). \tag{12}$$

Then

$$\left|\frac{g'(z)}{h'(z)}\right|^2 = \frac{|B|^2}{|1+B|^2} = \frac{|B|^2}{1+|B|^2+2ReB}.$$

Applying condition(i) and (ii) we have

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right|^2 \le \frac{|B|^2}{|B|^2+d} < 1.$$

The proof of 'only if' part: Assume that $w(z)$ is a harmonic quasiconformal mapping of $\mathbb{D}$ onto R. Then the following inequality

$$\left|\frac{g'(z)}{h'(z)}\right|^2 \le k^2$$

holds for some constant $k<1$. This is equivalent to $|B|<\infty$ and $|B|^2 \le k^2(|B|^2 + 1 + 2\mathbf{Re}B)$. Hence

$$\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right| < \infty,$$

and

$$1+2\mathbf{Re}B \ge \frac{(1-k^2)|B|^2}{k^2},$$

where B is defined by (12).

This completes the proof. □

Remark: According to (6) we know that $\mu \in \mathbb{P}$ is a probability measure on the Borel set of $\mathbb{S}^1$. This implies that $\int_{|\eta|=1} d\mu(\eta) = 1$.

Theorem 3. *Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, \Omega_1)$ is a harmonic K-quasiconformal mapping. Then its Jacobian satisfies*

$$J_w \geq \frac{c}{\pi^2},$$

where c is a positive constant depends on K.

Proof. According to (8) and (9) we have $h'(z) = \frac{\varphi'(z)}{2}\left(\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1\right)$ and $g'(z) = \frac{\varphi'(z)}{2}\left(\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) - 1\right)$, where $\varphi(z) = \frac{2}{\pi} \ln \frac{1+z}{1-z}$. Let $A_1 = \mathbf{Re} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)$ and $A_2 = \mathbf{Im} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)$. Since $w(z)$ is a harmonic quasiconformal mapping, by using condition (i) in Theorem 1 we see that

$$J_w(z) = |h'(z)|^2 - |g'(z)|^2 = A_1 |\varphi'(z)|^2 \geq \frac{16c}{\pi^2(1+|z|)^4} \geq \frac{c}{\pi^2},$$

where c is a positive constant depends on K.

This completes the proof. □

Theorem 4. *Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, R)$ is a harmonic K-quasiconformal mapping. Then its Jacobian satisfies*

$$J_w(z) \geq \frac{d}{16},$$

where d is is a positive constant depends on K.

Proof. According to (10) and (11) we have

$$h'(z) = \frac{1}{(1-z)^2} \int_{|\eta|=1} \frac{1}{1-\eta z} d\mu(\eta) = \frac{1}{(1-z)^2} \left(1 + \int_{|\eta|=1} \frac{\eta z}{1-\eta z} d\mu(\eta)\right)$$

and $g'(z) = \frac{1}{(1-z)^2} \int_{|\eta|=1} \frac{-\eta z}{1-\eta z} d\mu(\eta)$. Using (12) and condition (i) of Theorem2 we obtain that

$$J_w = |h'|^2 - |g'|^2 = \frac{1}{|1-z|^4}(1 + 2\mathbf{Re}B) \geq \frac{d}{(1+|z|)^4} \geq \frac{d}{16},$$

where d is a positive constant depends on K.

This completes the proof. □

3. co-Lipschitz condition of $w(z)$

A complex-valued mapping $w(z)$ in $\mathbb{D}$ is said to be a co-Lipschitz (Lipschitz) mapping if there exists a constant $L > 0$ such that the following inequality

$$|w(z_1) - w(z_2)| \geq \frac{|z_1 - z_2|}{L}$$

$$(|w(z_1) - w(z_2)| \leq L|z_1 - z_2|)$$

holds for any $z_1, z_2 \in \mathbb{D}$. Suppose that $w(z)$ is a harmonic quasiconformal mapping of $\mathbb{D}$ onto a bounded convex domain. Many mathematicians have discussed about the bi-Lipschitz property of $w(z)$ (cf.[6],[9] and [11]). We point out that if $w(z)$ is a harmonic quasiconformal mapping defined by (6) and (7), then it would be a co-Lipschitz mapping.

Theorem 5. *Given $K \geq 1$. Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, \Omega_1)$ is a harmonic K-quasiconformal mapping. Then the following inequalities*

$$\frac{1+c}{2\pi} \leq |h'(z)| \leq \frac{2(M_1+1)}{\pi(1-|z|)^2}$$

hold for every $z \in \mathbb{D}$, where c and M_1 are positive constant depend on K. Furthermore, the inequality

$$|w(z_1) - w(z_2)| \geq \frac{(1-k)(1+c)}{2\pi}|z_1 - z_2|,$$

holds for any $z_1, z_2 \in \mathbb{D}$, where $k = \frac{K-1}{K+1}$.

Proof. According to (8), we have

$$|h'(z)| = \left|\frac{\varphi'(z)}{2}\right| \left|\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1\right| \leq \frac{2(1+M_1)}{\pi(1-|z|^2)}$$

and

$$|h'(z)| = \left|\frac{\varphi'(z)}{2}\right| \left|\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1\right| \geq \frac{2(1+A_1)}{\pi(1+|z|)^2} \geq \frac{(1+c)}{2\pi},$$

where c and M_1 are positive constant depend on K. Since $w(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping, we see that $\operatorname{ess\,sup}_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| \leq k < 1$, where $k = \frac{K-1}{K+1}$. Then

$$\Lambda_w(z) = |h'(z)| + |g'(z)| \leq |h'(z)|(1+k) \leq \frac{2(1+k)(1+M_1)}{\pi(1-|z|^2)}$$

and

$$\lambda_w(z) = ||h'(z)| - |g'(z)|| \geq \frac{2(1-k)(1+c)}{\pi(1+|z|)^2} \geq \frac{(1-k)(1+c)}{2\pi}.$$

Take $\zeta_1, \zeta_2 \in \Omega_1$ satisfying $z_1 = w^{-1}(\zeta_1)$, $z_2 = w^{-1}(\zeta_2)$. Let $\varphi(t) = w^{-1}(\zeta_1 + t(\zeta_2 - \zeta_1))$, $t \in [0,1]$. Then $\frac{d}{dt}w(\varphi(t)) = \zeta_2 - \zeta_1$. Since Ω_1 is a convex domain, we see that

$$\begin{aligned} |\zeta_1 - \zeta_2| &= \int_0^1 \left|\frac{d}{dt}w(\varphi(t))\right| dt = \int_0^1 \left|w_z(\varphi(t))\varphi'(t) + w_{\bar{z}}(\varphi(t))\overline{\varphi'(t)}\right| dt \\ &\geq \int_0^1 \left(|w_z(\varphi(t))\varphi'(t)| - |w_{\bar{z}}(\varphi(t))\overline{\varphi'(t)}|\right) dt \\ &\geq \inf_{u\in\mathbb{D}}(|w_z(u)| - |w_{\bar{z}}(u)|)\int_0^1 |\varphi'(t)|\, dt \qquad (13) \\ &\geq \frac{(1-k)(1+c)}{2\pi}|z_1 - z_2|. \end{aligned}$$

This completes the proof. □

Theorem 6. *Given $K \geq 1$. Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, R)$ is a harmonic K-quasiconformal mapping. Then the following inequalities*

$$\frac{d}{4} \leq |h'(z)| \leq \frac{(M_2+1)}{(1-|z|)^2},$$

hold for any $z \in \mathbb{D}$, where d and M_2 are positive constant depend on K. Furthermore, the following inequality

$$|w(z_1) - w(z_2)| \geq \frac{(1-k)d}{4}|z_1 - z_2|,$$

holds for any $z_1, z_2 \in \mathbb{D}$, where $k = \frac{K-1}{K+1}$.

Proof. According to (10) and (12), we have

$$|h'(z)| = \left|\frac{1}{(1-z)^2}\right|\left|1 + \int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right| = \left|\frac{1}{(1-z)^2}\right|\sqrt{1 + 2ReB + B^2} \geq \frac{d}{4},$$

and

$$|h'(z)| = \left|\frac{1}{(1-z)^2}\right|\left|1 + \int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right| \leq \frac{1}{(1-|z|)^2}(M_2+1),$$

where d and M_2 are positive constant depend on K. Since $w(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping, we see that $k = \operatorname{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right| < 1$, where $k = \frac{K-1}{K+1}$. Hence

$$\Lambda_w(z) = |h'(z)| + |g'(z)| \leq |h'(z)|(1+k) \leq \frac{(1+k)(1+M_2)}{(1-|z|)^2},$$

and

$$\lambda_w(z) = ||h'(z)| - |g'(z)|| \geq |h'(z)|(1-k) \geq \frac{(1-k)d}{4}.$$

For each $z_1, z_2 \in \mathbb{D}$, using (13) we have

$$|w(z_1) - w(z_2)| = \left| \int\limits_{[z_1,z_2]} w_z dz + w_{\bar{z}} d\bar{z} \right| \geq \inf_{z\in\mathbb{D}} \lambda_w(z)|z_1 - z_2| \geq \frac{(1-k)d}{4}|z_1 - z_2|.$$

This completes the proof. □

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Jian-Feng Zhu, School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China.

E-mail address: flandy@hqu.edu.cn

Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions

Sotiris K. Ntouyas [a,b], Jessada Tariboon [c]

[a] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr
[b] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[c] Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800 Thailand
e-mail: jessada.t@sci.kmutnb.ac.th

Abstract

In this paper, we study a boundary value problem consisting from a fractional differential inclusion of Riemann-Liouville Langevin type subject to Katugampola fractional integral conditions. Some new existence results for convex as well as non-convex multivalued maps are obtained by using standard fixed point theorems. Enlighten examples illustrating the obtained results are also presented.

Key words and phrases: Fractional differential inclusions; generalized fractional integral; Katugampola fractional integral; nonlocal boundary conditions; fixed point theorems
AMS (MOS) Subject Classifications: 34A60; 26A33; 34A08

1 Introduction

In this manuscript, we investigate the sufficient conditions of existence of solutions for the following fractional Langevin inclusion subject to the generalized nonlocal fractional integral conditions of the form

$$\begin{cases} D^{p_1}(D^{p_2}+\lambda)x(t) \in F(t,x(t)), \qquad 0<t<T, \\ x(0)=0, \\ x(\eta)=\sum_{i=1}^{n}\alpha_i \dfrac{\rho_i^{1-q_i}}{\Gamma(q_i)}\displaystyle\int_0^{\xi_i}\frac{s^{\rho_i-1}x(s)}{(t^{\rho_i}-s^{\rho_i})^{1-q_i}}ds := \sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}x(\xi_i), \end{cases} \tag{1}$$

where D^{p_i} denote the Riemann-Liouville fractional derivative of order p_i, $i=1,2$, $0<p_1,p_2\leq 1$, $1<p_1+p_2\leq 2$, λ is a given constant, ${}^{\rho_i}I^{q_i}$ are the generalized fractional integral of orders $q_i>0$, $\rho_i>0$, η,ξ_i arbitrary, with $\eta,\xi_i\in(0,T)$, $\alpha_i\in\mathbb{R}$ for all $i=1,2,\ldots,n$ and $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The search for the existence of solutions to nonlinear fractional boundary value problems has expanded greatly over the past years. For examples and recent development of the topic, see [1]-[11] and the references cited therein. In fractional calculus, the fractional derivatives are defined via fractional integrals. There are several known forms of the fractional integrals which have been studied extensively for their applications. The most known fractional integrals are the Caputo, Riemann-Liouville and the Hadamard fractional integral.

A new fractional integral, called *generalized Riemann-Liouville fractional integral*, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, was introduced in [12], [13]. See Definition 2.5 below. This integral is now known as *"Katugampola fractional integral"* see for example

[14, pp 15, 123]. The existence and uniqueness results for the Caputo-Katugampola derivative is given in [15]. For some recent work with this new operator and similar operators, for example, see [16]-[18] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [19]. For some new developments on the fractional Langevin equation, see, for example, [20]-[24].

The present paper is motivated by a recent paper [25], where it is considered problem (1) with F single valued. Existence and uniqueness results were proved in [25] by using a variety of fixed point theorems, such as Banach contraction principle, Krasnoselskii fixed point theorem, Leray-Schauder nonlinear alternative and Leray-Schauder degree theory. Here, we cover the multi-valued case. We establish some existence results for the problem (1), when the right hand side is convex as well as non-convex valued. In the first result, we use the nonlinear alternative of Leray-Schauder type while in the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. The third result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. Examples illustrating the obtained results are also presented. The methods used are well known, however their exposition in the framework of problem (1) is new.

2 Preliminaries

2.1 Basic material of fractional calculus

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1 *[2] The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$\mathcal{I}^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} v(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function defined by $\Gamma(p) = \int_0^\infty e^{-s} s^{p-1} ds$.

Definition 2.2 *[2] The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-p-1} v(s) ds, \quad n-1 \leq p < n,$$

where $n = [p]+1$, $[p]$ denotes the integer part of a real number p, provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.3 *[2] Let $p > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation $D^p x(t) = 0$ has a unique solution $x(t) = \sum_{i=1}^n c_i t^{p-i}$, and the following formula holds: $\mathcal{I}^p D^p x(t) = x(t) + \sum_{i=1}^n c_i t^{p-i}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 \leq p < n$.*

Lemma 2.4 *([2], page 71) Let $\alpha > 0$ and $\beta > 0$. Then the following properties hold:*

$$\mathcal{I}^\alpha (x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}.$$

Definition 2.5 *([13]) The Katugampola fractional integral of order $q > 0$ and $\rho > 0$, of a function $f(t)$, for all $0 < t < \infty$, is defined as*

$$ {}^{\rho}I^{q} f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} v(s)}{(t^{\rho} - s^{\rho})^{1-q}} \, ds, $$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.6 *([25]) Let constants $q > 0$ and $p > 0$. Then the following formula holds*

$$ {}^{\rho}I^{q} t^{p} = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)} \frac{t^{p+\rho q}}{\rho^{q}}. \tag{2} $$

Lemma 2.7 *([25]) Let $0 < p_1, p_2 \leq 1$, $1 < p_1 + p_2 \leq 2$, $q_i, \rho_i > 0$, η, $\xi_i \in (0, T)$, $\alpha_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$ and $h \in C([0, T], \mathbb{R})$. Then x is a solution of the problem*

$$ D^{p_1}(D^{p_2} + \lambda) x(t) = h(t), \qquad 0 < t < T, \tag{3} $$

$$ x(0) = 0, \qquad x(\eta) = \sum_{i=1}^{n} \alpha_i \, {}^{\rho_i}I^{q_i} x(\xi_i), \tag{4} $$

if and only if

$$ \begin{aligned} x(t) \;=\; & \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} h(\eta) - \lambda \, \mathcal{I}^{p_2} x(\eta) \\ & - \sum_{i=1}^{n} \alpha_i \, {}^{\rho_i}I^{q_i} \left(\mathcal{I}^{p_1+p_2} h(s) - \lambda \, \mathcal{I}^{p_2} x(s) \right)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} h(t) - \lambda \, \mathcal{I}^{p_2} x(t), \end{aligned} \tag{5} $$

where

$$ \Omega = \sum_{i=1}^{n} \frac{\alpha_i \Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{\Gamma\left(\frac{p_1+p_2+\rho_i-1}{\rho_i}\right)}{\Gamma\left(\frac{p_1+p_2+\rho_i q_i+\rho_i-1}{\rho_i}\right)} \frac{\xi_i^{p_1+p_2+\rho_i q_i+\rho_i-1}}{\rho_i^{q_i}} - \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \eta^{p_1+p_2-1} \neq 0. \tag{6} $$

2.2 Basic material for multivalued maps

Here, we outline some basic concepts of multivalued analysis [26, 27].

Let $C([0, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|, \; t \in [0, T]\}$. Also by $L^1([0, T], \mathbb{R})$, we denote the space of functions $x : [0, T] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^T |x(t)| dt$.

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.

(ii) is *bounded* on bounded sets if $G(Y) = \cup_{x \in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$ (i.e. $\sup_{x \in Y} \{\sup\{|y| : y \in G(x)\}\} < \infty$).

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.

(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) is said to be *measurable* if for every $y \in X$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

3 Existence results

Let $\mathcal{C} = C([0,T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Throughout of this paper, for convenience of proving, we let the notations $\mathcal{I}^z v(s)(y)$ and ${}^{\rho}I^z v(s)(y)$ defined by

$$\mathcal{I}^z v(s)(y) = \frac{1}{\Gamma(z)} \int_0^y (y-s)^{z-1} v(s) ds \quad \text{and} \quad {}^{\rho}I^z v(s)(y) = \frac{\rho^{1-z}}{\Gamma(z)} \int_0^y \frac{s^{\rho-1} v(s)}{(y^\rho - s^\rho)^{1-z}} ds,$$

where $z > 0$ and $y \in [0,T]$.

To simplify the notations, we use the following constants:

$$\Lambda_1 = \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|}, \tag{7}$$

$$\Lambda_2 = \frac{T^{p_2}}{\Gamma(1+p_2)} + \Lambda_1 \left(\frac{\eta^{p_2}}{\Gamma(1+p_2)} + \sum_{i=1}^{n} |\alpha_i| \left[\frac{1}{\Gamma(1+p_2)} \frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}} \frac{\Gamma\left(\frac{p_2+\rho_i}{\rho_i}\right)}{\Gamma\left(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i}\right)} \right] \right). \tag{8}$$

Definition 3.1 *A function $x \in AC^2([0,T], \mathbb{R})$ is a solution of the problem (1) if $x(0) = 0,\ x(\eta) = \sum_{i=1}^{n} \alpha_i\ {}^{\rho_i}I^{q_i} x(\xi_i)$, and there exists a function $v \in L^1([0,T], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,T]$ and*

$$x(t) = \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[\mathcal{I}^{p_1+p_2} v(\eta) - \lambda\ \mathcal{I}^{p_2} x(\eta) - \sum_{i=1}^{n} \alpha_i\ {}^{\rho_i}I^{q_i} \left(\mathcal{I}^{p_1+p_2} v(s) - \lambda\ \mathcal{I}^{p_2} x(s) \right)(\xi_i) \right] + \mathcal{I}^{p_1+p_2} v(t) - \lambda\ \mathcal{I}^{p_2} x(t).$$

3.1 The Carathéodory case

In this subsection, we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Definition 3.2 *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [0,T]$.

For each $y \in \mathcal{C}$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } [0,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 3.3 *([26, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 3.4 *([28]) Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 3.5 *(Nonlinear alternative for Kakutani maps)[29]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ and $\nu \in (0,1)$ with $u \in \nu F(u)$.

Theorem 3.6 *Assume that:*

(H_1) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \quad \text{for each} \quad (t,x) \in [0,T] \times \mathbb{R};$$

(H_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)} > \frac{\Lambda_1}{(1-|\lambda|\Lambda_2)}\left\{\left[\mathcal{I}^{p_1+p_2}p(s)(\eta) + \sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\left(\mathcal{I}^{p_1+p_2}p(s)(\tau)\right)(\xi_i)\right] + \mathcal{I}^{p_1+p_2}p(s)(T)\right\} := \Omega_1, \quad |\lambda|\Lambda_2 < 1,$$

where Λ_1 and Λ_2 are defined by (7) and (8) respectively.

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. Define the operator $\mathcal{F}:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ by

$$\mathcal{F}(x) = \left\{ \begin{array}{l} h\in\mathcal{C}: \\ h(t)=\left\{ \begin{array}{l} \dfrac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\dfrac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ -\displaystyle\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg] \\ +\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t) \end{array}\right. \end{array}\right\} \tag{9}$$

for $v\in S_{F,x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the boundary value problem (1).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder Nonlinear alternative (Lemma 3.5). The proof consists of several steps.

Step 1. $\mathcal{F}(x)$ is convex for each $x\in\mathcal{C}$.

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore, we omit the proof.

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$.

For a positive number ρ, let $B_\rho=\{x\in\mathcal{C}:\|x\|\leq\rho\}$ be a bounded ball in $\mathcal{C}$. Then, for each $h\in\mathcal{F}(x), x\in B_\rho$, there exists $v\in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t). \end{aligned}$$

Then, we have

$$\begin{aligned} |h(x)| &\leq \Bigg|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t)\Bigg| \\ &\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\eta)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\eta) \\ &\quad +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\tau)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg] \\ &\quad +\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(t)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(t) \\ &\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\psi(\|x\|)\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta) \\ &\quad +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\psi(\|x\|)\mathcal{I}^{p_1+p_2}p(s)(T) \\ &\quad +\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}|\lambda|\|x\|\Bigg[\mathcal{I}^{p_2}(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg]+|\lambda|\|x\|\mathcal{I}^{p_2}(T) \\ &\leq \psi(\|x\|)\Bigg\{\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta) \\ &\quad +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&+|\lambda|\|x\|\Bigg\{\frac{T^{p_2}}{\Gamma(1+p_2)}+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{p_2}}{\Gamma(1+p_2)}\\
&+\sum_{i=1}^{n}|\alpha_i|\Bigg[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma\left(\frac{p_2+\rho_i}{\rho_i}\right)}{\Gamma\left(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i}\right)}\Bigg]\Bigg)\Bigg\}\\
=\;&\psi(\|x\|)\Lambda_1\left[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\;{}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\right]+|\lambda|\|x\|\Lambda_2,
\end{aligned}
$$

and consequently,

$$
\|h\| \;\leq\; \psi(\rho)\Lambda_1\left[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\;{}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\right]+|\lambda|\rho\Lambda_2.
$$

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Let $\tau_1,\tau_2\in[0,T]$ with $\tau_1<\tau_2$ and $x\in B_\rho$. For each $h\in\mathcal{F}(x)$, we obtain

$$
\begin{aligned}
|h(\tau_2)-h(\tau_1)| \;\leq\;& \left|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}}{\Omega}\Big[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta)\right.\\
&\left.-\sum_{i=1}^{n}\alpha_i\;{}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Big]\right|\\
&+\big|\mathcal{I}^{p_1+p_2}v(s)(t_2)-\mathcal{I}^{p_1+p_2}v(s)(t_1)\big|+\big|\lambda\,\mathcal{I}^{p_2}x(s)(t_2)-\lambda\,\mathcal{I}^{p_2}x(s)(t_1)\big|\\
\leq\;&\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{|t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}|}{|\Omega|}\Big[\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\eta)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\eta)\\
&+\sum_{i=1}^{n}\alpha_i\;{}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\tau)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Big]\\
&+\big|\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(t_2)-\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(t_1)\big|\\
&+\big|\lambda\,\mathcal{I}^{p_2}x(s)(t_2)-\lambda\,\mathcal{I}^{p_2}x(s)(t_1)\big|\\
\leq\;&\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{|t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}|}{|\Omega|}\Big[\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)(\eta)+|\lambda|\,\rho\mathcal{I}^{p_2}(\eta)\\
&+\sum_{i=1}^{n}\alpha_i\;{}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)(\tau)+|\lambda|\,\rho\mathcal{I}^{p_2}(\tau)\Big)(\xi_i)\Big]\\
&+\big|\left(\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)\right)(t_2)-\left(\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)\right)(t_1)\big|\\
&+|\lambda|\rho\big|\left(\mathcal{I}^{p_2}\right)(t_2)-\left(\mathcal{I}^{p_2}\right)(t_1)\big|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $\tau_2-\tau_1\to 0$. As $\mathcal{F}$ satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{F}:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph (Lemma 3.3). Thus, in our next step, we show that

Step 4. $\mathcal{F}$ has a closed graph.

Let $x_n\to x_*, h_n\in\mathcal{F}(x_n)$ and $h_n\to h_*$. Then, we need to show that $h_*\in\mathcal{F}(x_*)$. Associated with $h_n\in\mathcal{F}(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$
h_n(t) \;=\; \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Big[\mathcal{I}^{p_1+p_2}v_n(s)(\eta)-\lambda\;\mathcal{I}^{p_2}x(s)(\eta)
$$

$$-\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v_n(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v_n(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t).$$

Thus it suffices to show that there exists $v_*\in S_{F,x_*}$ such that for each $t\in[0,T]$,

$$\begin{aligned}h_*(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v_*(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta)\\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v_*(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v_*(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t).\end{aligned}$$

Let us consider the linear operator $\Theta: L^1([0,T],\mathbb{R})\to\mathcal{C}$ given by

$$\begin{aligned}v\mapsto\Theta(v)(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta)\\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t).\end{aligned}$$

Observe that

$$\begin{aligned}\|h_n(t)-h_*(t)\| &= \Bigg\|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}(v_n(s)-v_*(s))(\eta)\\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}(v_n(s)-v_*(s))(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}(v_n(s)-v_*(s))(t)\Bigg\|\to 0,\end{aligned}$$

as $n\to\infty$. Thus, it follows by Lemma 3.4 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$\begin{aligned}h_*(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v_*(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta)\\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v_*(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v_*(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t)\end{aligned}$$

for some $v_*\in S_{F,x_*}$.

Step 5. We show there exists an open set $U\subseteq\mathcal{C}$ with $x\notin\theta\mathcal{F}(x)$ for any $\theta\in(0,1)$ and all $x\in\partial U$.

Let $\theta\in(0,1)$ and $x\in\theta\mathcal{F}(x)$. Then there exists $v\in L^1([0,T],\mathbb{R})$ with $v\in S_{F,x}$ such that, for $t\in[0,T]$, we have

$$\begin{aligned}x(t) &= \theta\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta)\\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\theta\mathcal{I}^{p_1+p_2}v(s)(t)-\theta\lambda\,\mathcal{I}^{p_2}x(s)(t).\end{aligned}$$

Using the computations of the second step above, we have

$$\|x\| \leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\eta)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\eta)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\tau)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg]\\
&+\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(T)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(T)\\
\leq\ & \psi(\|x\|)\Bigg\{\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)\\
&+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\}\\
&+|\lambda|\|x\|\Bigg\{\frac{T^{p_2}}{\Gamma(1+p_2)}+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{p_2}}{\Gamma(1+p_2)}\\
&+\sum_{i=1}^{n}|\alpha_i|\Bigg[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma\left(\frac{p_2+\rho_i}{\rho_i}\right)}{\Gamma\left(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i}\right)}\Bigg]\Bigg)\Bigg\}\\
=\ & \psi(\|x\|)\Bigg\{\Lambda_1\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]\\
&+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\}+|\lambda|\|x\|\Lambda_2,
\end{aligned}
$$

which implies that

$$
\frac{\|x\|}{\psi(\|x\|)}\ \leq\ \frac{\Lambda_1}{(1-|\lambda|\Lambda_2)}\Bigg\{\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\}.
$$

In view of (H_3), there exists M such that $\|x\|\neq M$. Let us set

$$U=\{x\in\mathcal{C}:\|x\|<M\}.$$

Note that the operator $\mathcal{F}:\overline{U}\to\mathcal{P}(\mathcal{C})$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $x\in\partial U$ such that $x\in\theta\mathcal{F}(x)$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.5), we deduce that $\mathcal{F}$ has a fixed point $x\in\overline{U}$ which is a solution of the problem (1). This completes the proof. □

3.2 The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [30] for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G:X\to\mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y\in X:G(y)\cap B\neq\emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v\in\mathcal{A}$ and measurable $\mathcal{J}\subset[0,T]=J$, the function $u\chi_{\mathcal{J}}+v\chi_{J-\mathcal{J}}\in\mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.7 *Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F}:C([0,T]\times\mathbb{R})\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x)=\{w\in L^1([0,T],\mathbb{R}):w(t)\in F(t,x(t))\text{ for a.e. }t\in[0,T]\},$$

which is called the Nemytskii operator associated with F.

Definition 3.8 *Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 3.9 *([31]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,T],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.10 *Assume that (H_2), (H_3) and the following condition holds:*

(H_4) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- *(a) $(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- *(b) $x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,T]$;*

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 3.9, there exists a continuous function $f : AC^2([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,T],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}_{RL}D^q x(t) = f(x(t)), \quad 0 < t < T, \quad 1 < \alpha \leq 2, \\ x(0) = 0, \quad x(\eta) = \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i}I^{q_i} x(\xi_i). \end{cases} \tag{10}$$

Observe that if $x \in AC^2([0,T],\mathbb{R})$ is a solution of (10), then x is a solution to the problem (1). In order to transform the problem (10) into a fixed point problem, we define the operator $\overline{\mathcal{F}}$ as

$$\begin{aligned} \overline{\mathcal{F}}x(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} f(x(s))(\eta) - \lambda\, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i}I^{q_i} \Big(\mathcal{I}^{p_1+p_2} f(x(s))(\tau) - \lambda\, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} f(x(s))(t) - \lambda\, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.6. So, we omit it. This completes the proof. □

3.3 The Lipschitz case

In this subsection, we prove the existence of solutions for the problem (1) with a not necessary nonconvex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [32].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b)\},$$

where $d(A,b) = \inf_{a \in A} d(a;b)$ and $d(a,B) = \inf_{b \in B} d(a;b)$. Then $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space (see [33]).

Definition 3.11 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

(a) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x),N(y)) \leq \gamma d(x,y) \text{ for each } x,y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 3.12 *([32]) Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.13 *Assume that:*

(A_1) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*

(A_2) *$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [0,T]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0,T], \mathbb{R}^+)$ and $d(0, F(t,0)) \leq m(t)$ for almost all $t \in [0,T]$.*

Then the boundary value problem (1) has at least one solution on $[0,T]$ if

$$\Omega_2 := \Lambda_1 \left[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i| \ {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \right] + \mathcal{I}^{p_1+p_2} m(s)(T) < 1.$$

Proof. Consider the operator $\mathcal{F}$ defined by (9). Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathcal{C}$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [34]). Now, we show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 3.12. We show that $\mathcal{F}(x) \in \mathcal{P}_{cl}(\mathcal{C})$ for each $x \in \mathcal{C}$. Let $\{u_n\}_{n\geq 0} \in \mathcal{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{C}$. Then $u \in \mathcal{C}$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v_n(s)(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v_n(s)(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v_n(s)(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,T], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$, we have

$$\begin{aligned} u_n(t) \to v(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v(s)(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v(s)(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v(s)(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

Hence, $u \in \mathcal{F}(x)$.

Next, we show that there exists $\delta < 1$ such that

$$H_d(\mathcal{F}(x), \mathcal{F}(\bar{x})) \leq \delta \|x - \bar{x}\| \quad \text{for each} \quad x, \bar{x} \in AC^2([0,T], \mathbb{R}).$$

Let $x, \bar{x} \in AC^2([0,T], \mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v_1(s)(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v_1(s)(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v_1(s)(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

By (A_2), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0,T].$$

Define $U : [0,T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [34]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,T]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,T]$, let us define

$$\begin{aligned} h_2(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v_2(s)(\eta) - \lambda\, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i\, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v_2(s)(\tau) - \lambda\, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v_2(s)(t) - \lambda\, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \Bigg[\mathcal{I}^{p_1+p_2} |v_1(s) - v_2(s)|(\eta) \\ &\quad + \sum_{i=1}^{n} |\alpha_i|\, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} |v_1(s) - v_2(s)|(\tau) \Big] + \mathcal{I}^{p_1+p_2} |v_1(s) - v_2(s)|(t) \\ &\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \Bigg[\mathcal{I}^{p_1+p_2} m(s) \|x - \bar{x}\|(\eta) \\ &\quad + \sum_{i=1}^{n} |\alpha_i|\, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s) \|x - \bar{x}\|(\tau) \Big) \Bigg] + \mathcal{I}^{p_1+p_2} m(s) \|x - \bar{x}\|(T) \\ &= \Bigg\{ \Lambda_1 \Bigg[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i|\, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \Bigg] \\ &\quad + \mathcal{I}^{p_1+p_2} m(s)(T) \Bigg\} \|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \Bigg\{ \Lambda_1 \Bigg[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i|\, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \Bigg] + \mathcal{I}^{p_1+p_2} m(s)(T) \Bigg\} \|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\mathcal{F}(x), \mathcal{F}(\bar{x})) &\leq \Bigg\{ \Lambda_1 \Bigg[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i|\, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \Bigg] \\ &\quad + \mathcal{I}^{p_1+p_2} m(s)(T) \Bigg\} \|x - \bar{x}\|. \end{aligned}$$

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 3.12 that $\mathcal{F}$ has a fixed point x which is a solution of (1). This completes the proof. □

3.4 Examples

In this section, we will illustrate our main results with the help of some examples.

Example 3.14 *Let us consider the following Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions*

$$\begin{cases} D^{2/3}\left(D^{4/5}+\dfrac{1}{8}\right)x(t)\in F_1(t,x(t)), \quad t\in\left(0,\dfrac{2}{3}\right), \\ x(0)=0, \\ x\left(\dfrac{2}{9}\right)=\dfrac{3}{4}{}^{\frac{\sqrt{2}}{5}}I^{\frac{1}{3}}x\left(\dfrac{1}{9}\right)+\dfrac{1}{\sqrt{7}}{}^{\frac{\sqrt{3}}{8}}I^{\frac{1}{\pi}}x\left(\dfrac{1}{3}\right)+\dfrac{\sqrt{2}}{5}{}^{\frac{4}{e^2}}I^{\frac{2}{3}}x\left(\dfrac{4}{9}\right) \\ \qquad\qquad +\dfrac{11}{15}{}^{\frac{2}{\sqrt{5}}}I^{\frac{6}{13}}x\left(\dfrac{5}{9}\right), \end{cases} \tag{11}$$

where

$$F_1(t,x)=\left[\frac{(t^{1/2}+1)}{20}\left(\frac{|x|\sin|x|}{5(3+2|x|)}+\frac{e^{-2t}}{3}\right),\frac{(t^{1/2}+1)}{15}\left(\frac{x^2}{3(1+|x|)}+\frac{e^{-t}}{2}\right)\right]. \tag{12}$$

Here $p_1=2/3$, $p_2=4/5$, $\lambda=1/8$, $T=2/3$, $\eta=2/9$, $n=4$, $\alpha_1=3/4$, $\rho_1=\sqrt{2}/5$, $q_1=1/3$, $\xi_1=1/9$, $\alpha_2=1/\sqrt{7}$, $\rho_2=\sqrt{3}/8$, $q_2=1/\pi$, $\xi_2=1/3$, $\alpha_3=\sqrt{2}/5$, $\rho_3=4/e^2$, $q_3=2/3$, $\xi_3=4/9$, $\alpha_4=11/15$, $\rho_4=2/\sqrt{5}$, $q_4=6/13$, $\xi_4=5/9$. From these constants, we can find that $\Omega=0.2660602470$, $\Lambda_1=4.756155970$, $\Lambda_2=5.624515148$ and also $|\lambda|\Lambda_2=0.7030643935<1$. It is obvious that the condition (H_1) is satisfied.

For $f\in F_1$, we have

$$\begin{aligned} |f| &\leq \max\left(\frac{(t^{1/2}+1)}{20}\left(\frac{|x|\sin|x|}{5(3+2|x|)}+\frac{e^{-2t}}{3}\right),\frac{(t^{\frac{1}{2}}+1)}{15}\left(\frac{x^2}{3(1+|x|)}+\frac{e^{-t}}{2}\right)\right) \\ &\leq \frac{(t^{1/2}+1)}{15}\left(\frac{1}{3}|x|+\frac{1}{2}\right), \quad t\in(0,2/3),\ x\in\mathbb{R}. \end{aligned}$$

Therefore, we have

$$\|F_1(t,x)\|_{\mathcal{P}}=\sup\{|y|\,:\,y\in F_1(t,x)\}\leq p(t)\psi(|x|), \quad t\in(0,2/3)\ x\in\mathbb{R},$$

where $p(t)=(t^{1/2}+1)/15$, $\psi(|x|)=(1/3)|x|+(1/2)$. This means that the condition (H_2) is fulfilled. By direct computation, we have $\Omega_1=1.123809144$ and also there exists a constant $M>0.8984766718$ satisfying condition (H_3).

Therefore, all the conditions of Theorem 3.6 are satisfied. So, the problem (11) with $F_1(t,x)$ given by (12) has at least one solution on $[0,2/3]$.

Example 3.15 *Let us consider the following Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions*

$$\begin{cases} D^{6/7}\left(D^{8/9}+\dfrac{1}{12}\right)x(t)\in F_2(t,x(t)), \quad t\in\left(0,\dfrac{3}{2}\right), \\ x(0)=0, \\ x\left(\dfrac{2}{3}\right)=\dfrac{2}{\sqrt{11}}{}^{\frac{3}{4}}I^{\frac{5}{8}}x\left(\dfrac{1}{3}\right)+\dfrac{4}{7}{}^{\frac{\sqrt{3}}{5}}I^{\frac{1}{\sqrt{6}}}x\left(\dfrac{1}{2}\right)+\dfrac{\pi}{e^2}{}^{\frac{2}{13}}I^{\frac{3}{\sqrt{8}}}x\left(\dfrac{5}{6}\right) \\ \qquad\qquad +\dfrac{5}{4}{}^{\frac{2}{9}}I^{\frac{3}{\pi^2}}x\left(\dfrac{7}{6}\right)+\dfrac{\sqrt{3}}{9}{}^{\frac{\sqrt{11}}{15}}I^{\frac{13}{17}}x\left(\dfrac{4}{3}\right), \end{cases} \tag{13}$$

where

$$F_2(t,x)=\left[0,\frac{(t^{1/3}+1)}{24}\left(\frac{x^2+2|x|}{(1+|x|)}\right)+\frac{t}{2}\right]. \tag{14}$$

Here $p_1 = 6/7$, $p_2 = 8/9$, $\lambda = 1/12$, $T = 3/2$, $\eta = 2/3$, $n = 5$, $\alpha_1 = 2/\sqrt{11}$, $\rho_1 = 3/4$, $q_1 = 5/8$, $\xi_1 = 1/3$, $\alpha_2 = 4/7$, $\rho_2 = \sqrt{3}/5$, $q_2 = 1/\sqrt{6}$, $\xi_2 = 1/2$, $\alpha_3 = \pi/e^2$, $\rho_3 = 2/13$, $q_3 = 3/\sqrt{8}$, $\xi_3 = 5/6$, $\alpha_4 = 5/4$, $\rho_4 = 2/9$, $q_4 = 3/\pi^2$, $\xi_4 = 7/6$, $\alpha_5 = \sqrt{3}/9$, $\rho_5 = \sqrt{11}/15$, $q_5 = 13/17$, $\xi_5 = 4/3$. From given constants, we can find that $\Omega = 2.105868955$, $\Lambda_1 = 0.7738927855$. In addition, we have

$$\sup\{|x| : x \in F_2(t,x)\} \leq \frac{(t^{1/3}+1)}{24}\left(\frac{x^2+2|x|}{(1+|x|)}\right) + \frac{t}{2},$$

which yields

$$H_d(F_2(t,x), F_2(t,y)) \leq \frac{(t^{1/3}+1)}{12}|x-y|.$$

Choosing $m(t) = (t^{1/3}+1)/12$, we obtain $H_d(F_2(t,x), F_2(t,y)) \leq m(t)|x-y|$ such that $d(0, F_2(t,0)) \leq m(t)$. By the previous setting, we find that $\Omega_2 = 0.3397697571 < 1$.

Thus all assumptions of Theorem 3.13 are fulfilled. Therefore, by the conclusion of Theorem 3.13, we deduce that the problem (13) with $F_2(t,x)$ given by (14) has at least one solution on $[0, 3/2]$.

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On the Riemann-Liouville fractional Hermite-Hadamard-type inequalities for differentiable α-preinvex mappings

Lianzi Chen Tingsong Du* Shasha Zhao and Sheng Zheng

Department of Mathematics, College of Science, China Three Gorges University, Yichang, 443002, P. R. China.
E-mail: tingsongdu@ctgu.edu.cn, chenlianzictgu@gmail.com
zhaoshashactgu@gmail.com, zhengshengctgu@gmail.com
* Corresponding author

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Abstract

In the paper, by discovering a Riemann-Liouville fractional integral identity involving twice differentiable preinvex mappings, the authors establish the right-sided new Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals for α-preinvex functions. The new fractional integral inequalities are then applied to some special means.

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1 Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

referred to as Hermite-Hadamard inequality, is one of the most famous results for convex functions. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements and

new inequalities connected with the Hermite-Hadamard inequality. The reader may refer to [7, 12, 14, 16, 20, 21, 26, 27, 34] and the references therein.

Let us recall some necessary definitions and preliminary results which are used for further discussion.

Definition 1.1 *([3, 32]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$ if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$. The invex set S is also called an η-connected set.*

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. See [3, 33], for example.

Definition 1.2 *([3]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by*

$$P_{xv} = \{z | z = x + t\eta(y, x), t \in [0, 1]\}.$$

A significant generalization of convex mappings is that of preinvex mappings introduced by Weir and Mond in [32].

Definition 1.3 *([32]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0, 1]$ we have*

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y).$$

The function f is said to be preincave if and only if $-f$ is preinvex.

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$. Further, there exist preinvex functions which are not convex.

Moreover, Wang et al. gave the so-called α-preinvex function in [29] as follows.

Definition 1.4 *([29]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be α-preinvex with respect to η for $\alpha \in (0, 1]$, if every $x, y \in S$ and $t \in [0, 1]$,*

$$f\big(x + t\eta(y, x)\big) \leq (1 - t^\alpha)f(x) + t^\alpha f(y).$$

Certainly, α-preinvex mapping means just preinvex mapping when $\alpha = 1$.

For recent results on some new generalizations, refinements of integral inequalities involved with the preinvex functions, one can see [4, 13, 17–19, 23] and the references therein.

We also need the following fractional calculus background.

Definition 1.5 *([25]) Let $f \in L^1[a, b]$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) \mathrm{d}t, \quad a < x$$

and

$$J^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\mathrm{d}t,\quad x<b$$

respectively, where $\Gamma(.)$ *is Gamma function and its definition is* $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}\mathrm{d}u$. *It is to be noted that* $J^0_{a^+}f(x)=J^0_{b^-}f(x)=f(x)$.

In the case $\alpha=1$, the Riemann-Liouville fractional integral reduces to the classical and usual integral.

In [28], Sarikaya et al. established the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.1 *Let* $f:[a,b]\to\mathbb{R}$ *be a positive function with* $0\le a<b$ *and* $f\in L^1[a,b]$. *If* f *is convex function on* $[a,b]$, *then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\le\frac{f(a)+f(b)}{2}\tag{1.2}$$

Observe that for $\alpha=1$, the inequalities (1.2) reduces to the classical Hermite-Hadamard inequality (1.1).

For some recent results associated with the fractional integral inequalities, one can consult [1, 2, 5, 8–10, 15, 28, 30].

In the recently published article [25] by Qaisar et al., they obtained Riemann-Liouville fractional Hadamard-type integral inequalities for mappings whose absolute value of first derivatives are preinvex, and in the paper [11] Dragomir et al. also found some Hadamard-type fractional integral inequalities for differentiable mappings whose absolute value of second derivatives are convex. Motivated and inspired by this idea, in the present paper, by discovering a Riemann-Liouville fractional integral identity involving twice differentiable preinvex mappings, the authors establish the right-sided new Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals for α-preinvex functions. The new fractional integral inequalities are then applied to some special means.

2 Main Results

To derive main results in this section, we prove the following Lemma.

Lemma 2.1 *Let* $A\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta:A\times A\to\mathbb{R}$ *and Let* $a,b\in A$ *with* $a<a+\eta(b,a)$. *Assume that* $f:A\to\mathbb{R}$ *be a twice differentiable mapping. If* f'' *is preinvex on* A *and* f'' *is integrable on the* η*-path* $P_{ac}:c=a+\eta(b,a)$, *then the following identity for Riemann-Liouville fractional integral with* $\alpha>0$ *holds:*

$$\begin{aligned}&\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\\&=\frac{\eta^2(b,a)}{2}\int_0^1\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}f''\big(a+t\eta(b,a)\big)\mathrm{d}t,\end{aligned}\tag{2.1}$$

where $\Gamma(\alpha)=\int_0^\infty e^{-t}u^{\alpha-1}\mathrm{d}u$.

Proof. Set

$$I=\frac{\eta^2(b,a)}{2}\int_0^1\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}f''\big(a+t\eta(b,a)\big)\mathrm{d}t.$$

Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, we have $a+t\eta(b,a)\in A$. Integrating by part yields that

$$\begin{aligned}
I=&\frac{\eta^2(b,a)}{2}\bigg[\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{(\alpha+1)\eta(b,a)}f'\big(a+t\eta(b,a)\big)\Big|_0^1\\
&-\int_0^1\frac{-(\alpha+1)t^\alpha+(\alpha+1)(1-t)^\alpha}{(\alpha+1)\eta(b,a)}f'\big(a+t\eta(b,a)\big)\mathrm{d}t\bigg]\\
=&\frac{\eta^2(b,a)}{2}\bigg[\frac{t^\alpha-(1-t)^\alpha}{\eta^2(b,a)}f\big(a+t\eta(b,a)\big)\Big|_0^1\\
&-\int_0^1\frac{\alpha t^{\alpha-1}+\alpha(1-t)^{\alpha-1}}{\eta^2(b,a)}f\big(a+t\eta(b,a)\big)\mathrm{d}t\bigg]\\
=&\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\alpha}{2}\bigg[\int_0^1\Big(t^{\alpha-1}+(1-t)^{\alpha-1}\Big)f\big(a+t\eta(b,a)\big)\mathrm{d}t\bigg].
\end{aligned}$$

Let $u=a+t\eta(b,a)$, then $\mathrm{d}u=\eta(b,a)\mathrm{d}t$, and using the reduction formula $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)(\alpha>0)$ for Euler gamma function, we have

$$\frac{\alpha}{2}\int_0^1 t^{\alpha-1}f\big(a+t\eta(b,a)\big)\mathrm{d}t=\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}J^\alpha_{(a+\eta(b,a))^-}f(a)$$

and similarly we get

$$\frac{\alpha}{2}\int_0^1(1-t)^{\alpha-1}f\big(a+t\eta(b,a)\big)\mathrm{d}t=\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}J^\alpha_{a^+}f\big(a+\eta(b,a)\big).$$

Thus, we have conclusion (2.1).

Remark 2.1 *Applying Lemma 2.1 for $\eta(b,a)=b-a$, we can obtain the Lemma 2.1 in [31], which may be discovered also in [22]. Furthermore, let $\alpha=1$, we can get lemma 1 in [24].*

With the help of Lemma 2.1, new upper bound for the right-hand side of (1.2) for α-preinvex functions via the Riemann-Liouville fractional integral is presented in the following theorem.

Theorem 2.1 *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. Suppose that $f:A\to\mathbb{R}$ be a twice differentiable mapping and f'' is integrable on the η-path $P_{ac}:c=a+\eta(b,a)$.*

If $|f''|$ is α-preinvex on A then the following inequality for fractional integrals with $0 < \alpha \leq 1$ holds:

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\ \leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\bigg[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|\\ &+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|\bigg].\end{aligned} \tag{2.2}$$

Proof. Since $a+t\eta(b,a)\in A$ for every $t\in[0,1]$, by using the properties of modulus on Lemma 2.1, we can obtain that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\ \leq&\frac{\eta^2(b,a)}{2}\int_0^1\left|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t.\end{aligned}$$

Using the α-preinvexity of $|f''|$, we have

$$\begin{aligned}&\int_0^1\left|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\ \leq&\frac{1}{\alpha+1}\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)\Big((1-t^{\alpha})|f''(a)|+t^{\alpha}|f''(b)|\Big)\mathrm{d}t\\ \leq&\frac{1}{\alpha+1}\bigg[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|\\ &+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|\bigg].\end{aligned}$$

To prove the second inequality, we used the following fact that

$$\begin{aligned}&\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}-t^{\alpha}+t^{2\alpha+1}\big)\mathrm{d}t=\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)},\\ &\int_0^1(t^{\alpha}-t^{2\alpha+1})\mathrm{d}t=\frac{1}{2\alpha+2},\end{aligned}$$

and

$$\int_0^1 t^{\alpha}(1-t)^{\alpha+1}\mathrm{d}t=\beta(\alpha+1,\alpha+2),$$

where the Beta function,

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad\forall x,y>0,$$

which completes the proof.

Another Riemann-Liouville fractional Hermit-Hadamard-type inequality for powers in terms of the second derivatives is obtained below.

Theorem 2.2 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a+\eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ be a twice differentiable mapping and f'' is integrable on the η-path $P_{ac} : c = a + \eta(b,a)$. Assume that $q \in \mathbb{R}$, $q \geq 1$ such that $|f''|^q$ is α-preinvex on A, then the following inequality for fractional integrals with $0 < \alpha \leq 1$ holds:*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\&\leq\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(1-\frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}}.\end{aligned} \tag{2.3}$$

Proof. Since $a+t\eta(b,a) \in A$ for every $t \in [0,1]$, by using the properties of modulus on Lemma 2.1 and using the well-known power-mean integral inequality for $q \geq 1$, we can obtain that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\&\leq\frac{\eta^2(b,a)}{2}\int_0^1\left|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\&\leq\frac{\eta^2(b,a)}{2(\alpha+1)}\bigg(\int_0^1 1\mathrm{d}t\bigg)^{1-\frac{1}{q}}\bigg[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^q\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\\&\leq\frac{\eta^2(b,a)}{2(\alpha+1)}\bigg[\int_0^1\Big(1-\frac{1}{2^{q\alpha}}\Big)|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}.\end{aligned}$$

To prove the third inequality above, we used the following inequality

$$\begin{aligned}\big(1-(1-t)^{\alpha+1}-t^{\alpha+1}\big)^q&\leq 1-[(1-t)^{\alpha+1}+t^{\alpha+1}]^q\\&\leq 1-(2^{-\alpha})^q\\&\leq 1-\frac{1}{2^{q\alpha}}\end{aligned}$$

for any $t \in [0,1]$ with $q \geq 1$, and also using the α-preinvexity of $|f''|^q$, that is

$$\begin{aligned}\int_0^1|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t&\leq\int_0^1\big[(1-t^{\alpha})|f''(a)|^q+t^{\alpha}|f''(b)|^q\big]\mathrm{d}t\\&=\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q.\end{aligned} \tag{2.4}$$

Therefore, we can get the required results (2.3).

Corollary 2.1 *With the same assumptions given in Theorem 2.2, if* $|f''(x)| \leq M$ *on* $[a, a+\eta(b,a)]$, *we can deduce that*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{M\eta^2(b,a)}{2(\alpha+1)}\Big(1-\frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}.$$

Another similar result may be presented in the following theorem.

Theorem 2.3 *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : A\times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $a < a+\eta(b,a)$. *Suppose that* $f : A \to \mathbb{R}$ *be a twice differentiable mapping and* f'' *is integrable on the* η*-path* $P_{ac} : c = a+\eta(b,a)$. *Assume that* $p \in \mathbb{R}$, $p > 1$ *with* $q = \frac{p}{p-1}$ *such that* $|f''|^q$ *is* α*-preinvex on* A, *then the following inequality for fractional integrals with* $0 < \alpha \leq 1$ *holds:*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}}. \tag{2.5}$$

Proof. Since $a+t\eta(b,a) \in A$ for every $t \in [0,1]$, by using the properties of modulus on Lemma 2.1 and making use of the well-known Hölder's integral inequality for $q > 1$, we can obtain that

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{\eta^2(b,a)}{2}\int_0^1\left|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t$$
$$\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\int_0^1|1-t^{\alpha+1}-(1-t)^{\alpha+1}|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big)^{\frac{1}{q}},$$

where we use the following inequality

$$\big(1-(1-t)^{\alpha+1}-t^{\alpha+1}\big)^p \leq 1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)} \tag{2.6}$$

for any $t \in [0,1]$, which follows from

$$(A-B)^p \leq A^p - B^p \tag{2.7}$$

for any $A > B \geq 0$ and $p > 1$.
By applying (2.4) and (2.6), we can get (2.5). Hence the proof is completed.

Corollary 2.2 *With the same assumptions given in Theorem 2.3, if* $|f''(x)| \leq M$ *on* $[a, a+\eta(b,a)]$, *we obtain that*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{M\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}.$$

A different approach leads to the following result.

Theorem 2.4 *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : A\times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $a < a+\eta(b,a)$. *Suppose that* $f : A \to \mathbb{R}$ *be a twice differentiable mapping and* f'' *is integrable on the* η*-path* $P_{ac} : c = a+\eta(b,a)$. *Assume that* $p \in \mathbb{R}$, $p > 1$ *with* $q = \frac{p}{p-1}$ *such that* $|f''|^q$ *is* α*-preinvex on* A, *then the following inequality for fractional integrals with* $0 < \alpha \leq 1$ *holds:*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\ \leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\bigg[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q\\ &+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q\bigg]^{\frac{1}{q}}.\end{aligned} \tag{2.8}$$

Proof. Since $a+t\eta(b,a) \in A$ for every $t \in [0,1]$, by utilizing the properties of modulus on Lemma 2.1 and using the Hölder's integral inequality for $q > 1$, we can obtain that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\ \leq&\frac{\eta^2(b,a)}{2}\int_0^1\Big|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\Big||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\ \leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\bigg[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)\mathrm{d}t\bigg]^{1-\frac{1}{q}}\\ &\times\bigg[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\\ =&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\bigg[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\bigg]^{\frac{1}{q}}.\end{aligned}$$

Using the α-preinvexity of $|f''|^q$, i.e. the inequality (2.4), we have

$$\begin{aligned}
&\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\\
\leq&\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)\Big((1-t^\alpha)|f''(a)|^q+t^\alpha|f''(b)|^q\Big)\mathrm{d}t\\
=&\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q\\
&+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q.
\end{aligned}$$

Thus, we get the desired inequality (2.8).

Corollary 2.3 *With the same assumptions given in Theorem 2.4, if $|f''(x)|\leq M$ on $[a,a+\eta(b,a)]$, we obtain that*

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\big]\right|\\
&\leq\frac{M\alpha\eta^2(b,a)}{2(\alpha+1)(\alpha+2)}.
\end{aligned}$$

Finally we shall prove the following result.

Theorem 2.5 *Suppose that all the assumptions of Theorem 2.4 are satisfied. Then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\big]\right|\\
\leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\Big]^{\frac{q-1}{q}}\\
&\times\left\{\left[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\right]|f''(a)|^q\right.\\
&\left.+\Big[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(b)|^q\right\}.
\end{aligned}\tag{2.9}$$

Proof. Since $a+t\eta(b,a)\in A$ for every $t\in[0,1]$, by using the properties of modulus on Lemma 2.1 and making use of the well-known Hölder's integral inequality for $q>1$, we can obtain that

$$
\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\
\leq&\frac{\eta^2(b,a)}{2}\int_0^1\Big|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\Big||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\
\leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{\frac{q-p}{q-1}}\mathrm{d}t\Big]^{\frac{q-1}{q}}\\
&\times\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\
\leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\Big]^{\frac{q-1}{q}}\\
&\times\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big]^{\frac{1}{q}}
\end{aligned}
$$

Using the α-preinvexity of $|f''|^q$, we have

$$
\begin{aligned}
&\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\\
\leq&\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}\Big((1-t^{\alpha})|f''(a)|^q+t^{\alpha}|f''(b)|^q\Big)\mathrm{d}t\\
\leq&\int_0^1\Big(1-t^{(\alpha+1)p}-(1-t)^{(\alpha+1)p}\Big)\Big((1-t^{\alpha})|f''(a)|^q+t^{\alpha}|f''(b)|^q\Big)\mathrm{d}t\\
\leq&\int_0^1\Big(1-(1-t)^{(\alpha+1)p}-t^{(\alpha+1)p}-t^{\alpha}+t^{\alpha}(1-t)^{(\alpha+1)p}+t^{\alpha+(\alpha+1)p}\Big)|f''(a)|^q\mathrm{d}t\\
&+\int_0^1\Big(t^{\alpha}-t^{\alpha}(1-t)^{(\alpha+1)p}-t^{\alpha+(\alpha+1)p}\Big)|f''(b)|^q\mathrm{d}t\\
=&\Big[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(a)|^q\\
&+\Big[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(b)|^q.
\end{aligned}
$$

Here, we use

$$(1-(1-t)^{\alpha+1}-t^{\alpha+1})^q\leq 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}\tag{2.10}$$

for any $t\in[0,1]$, which follows from

$$(A-B)^q\leq A^q-B^q\tag{2.11}$$

for any $A>B\geq 0$ and $q\geq 1$.
Thus, we get the desired inequality (2.9).

Corollary 2.4 *From Theorems (2.2),(2.3),(2.4) and (2.5), we have*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|$$
$$\leq min\{K_1,K_2,K_3,K_4\},$$

where

$$K_1=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(1-\frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}},$$

$$K_2=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}},$$

$$K_3=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\Big[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q$$
$$+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q\Big]^{\frac{1}{q}},$$

$$K_4=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\Big]^{\frac{q-1}{q}}$$
$$\times\Big\{\Big[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(a)|^q$$
$$+\Big[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(b)|^q\Big\}.$$

3 Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1 *[6] A function $M:\mathbb{R}^2_+\to\mathbb{R}_+$, is called a Mean function if it has the following properties:*
(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry: $M(x,y)=M(y,x)$,
(3) Reflexivity: $M(x,x)=x$,
(4) Monotonicity: If $x\leq x'$ and $y\leq y'$, then $M(x,y)\leq M(x',y')$,
(5) Internality: $min\{x,y\}\leq M(x,y)\leq max\{x,y\}$.

We consider some means for arbitrary positive real numbers $a>0$ and $b>0$, define $A:=A(a,b)=\frac{a+b}{2}$, $G:=G(a,b)=\sqrt{ab}$, $H:=H(a,b)=\frac{2ab}{a+b}$,
$P_r:=P_r(a,b)=\Big(\frac{a^r+b^r}{2}\Big)^{\frac{1}{r}}, r\geq 1$

$$I:=I(a,b)=\begin{cases}\frac{1}{e}\Big(\frac{b^b}{a^a}\Big)^{\frac{1}{b-a}}, & a\neq b,\\ a, & a=b,\end{cases}\qquad L:=L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a}, & a\neq b,\\ a, & a=b,\end{cases}$$

and

$$L_p := L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq 0,-1, \text{ and } a \neq b, \\ L(a,b), & p=-1 \text{ and } a \neq b, \\ I(a,b), & p=0 \text{ and } a \neq b, \\ a, & a=b. \end{cases}$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a+\eta(b,a)] \times [a, a+\eta(b,a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain various inequalities for these means below:

Setting $\eta(b,a) = M(b,a)$ in (2.2), (2.3), (2.5), (2.8) and (2.9), one can derive the following interesting inequalities concerning means:

$$(1)\left|\frac{f(a)+f\big(a+M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+M(b,a)\big) + J^{\alpha}_{(a+M(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\bigg[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)} + \beta(\alpha+1,\alpha+2)\Big)|f''(a)|$$
$$+ \Big(\frac{1}{2\alpha+2} - \beta(\alpha+1,\alpha+2)\Big)|f''(b)|\bigg],$$

$$(2)\left|\frac{f(a)+f\big(a+M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+M(b,a)\big) + J^{\alpha}_{(a+M(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\Big(1-\frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q + \frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}},$$

$$(3)\left|\frac{f(a)+f\big(a+M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+M(b,a)\big) + J^{\alpha}_{(a+M(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\bigg[\frac{\alpha}{\alpha+1}|f''(a)|^q + \frac{1}{\alpha+1}|f''(b)|^q \mathrm{d}t\bigg]^{\frac{1}{q}},$$

$$(4)\left|\frac{f(a)+f\big(a+M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+M(b,a)\big) + J^{\alpha}_{(a+M(b,a))^-}f(a)\big]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\bigg[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)} + \beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q$$
$$+ \Big(\frac{1}{2\alpha+2} - \beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q\bigg]^{\frac{1}{q}},$$

and

$$
\begin{aligned}
(5)&\left|\frac{f(a)+f\big(a+M(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+M(b,a)\big)+J^{\alpha}_{(a+M(b,a))^-}f(a)\right]\right| \\
&\leq\frac{M^2(b,a)}{2(\alpha+1)}\left[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\right]^{\frac{q-1}{q}} \\
&\quad\times\left\{\left[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\right]|f''(a)|^q\right. \\
&\quad\left.+\left[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\right]|f''(b)|^q\right\}.
\end{aligned}
$$

Letting $M = A$, G, H, P_r, I, L, L_p in (1), (2), (3), (4) and (5), we get the inequalities involving means for a particular choice of a twice differentiable α-preinvex function f, and the details are left to the interested reader.

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CONVOLUTION PROPERTIES FOR CERTAIN SUBCLASSES OF MEROMORPHIC BOUNDED FUNCTIONS

HANAN DARWISH, **ABD EL-MONEIM LASHIN**, AND **SULIMAN SOWILEH**

ABSTRACT. By making use of the Hadamard product, we derive necessary and sufficient conditions for certain meromorphic function to be in the class $\mathcal{S}^*(\lambda,\gamma,M)(\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, M \geq 1, \lambda \in \mathbb{C})$ which unifies the classes of bounded starlike and convex functions of complex order. By using Al-Oboudi operator a more general class $\mathcal{S}^*(n,\lambda,\gamma,M)$ related to $\mathcal{S}^*(\lambda,\gamma,M)$ is also considered. Several properties of the class $\mathcal{S}^*(n,\lambda,\gamma,M)$ are also obtained.

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Key Words. Univalent meromorphic functions, bounded starlike functions of complex order, bounded convex functions of complex order, λ-starlike functions, Hadamard product, subordination.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and let Σ denote the class of all meromorphic functions having the form:

$$f(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disc

$$E^* = \{z : z \in \mathbb{C}, \quad 0 < |z| < 1\} =: E\backslash\{0\}.$$

The familiar Hadamard product (or convolution) of two functions $f(z)$ given by (1.1 and $g(z)$ is given by

$$g(z) = z^{-1} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.2}$$

is defined by

$$(f * g)(z) = z^{-1} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

An analytic function f is said to be subordinate to another analytic function g, written symbolically as follows:

$$f(z) \prec g(z) \ (z \in \mathbb{E}),$$

if there exists a function $\omega(z)$, analytic in $\mathbb{E}$ with

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{E}),$$

such that

$$f(z) = g(w(z)) \ (z \in \mathbb{E}).$$

Furthermore, if the function $g(z)$ is univalent in $\mathbb{E}$, then we have the following equivalence, (cf., e.g., [5], [9], [10] *):*

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{E}) \subset g(\mathbb{E}).$$

Making use of the principal of subordination between analytic functions, Aouf [2] defined the subclasses $\mathcal{S}^*(\gamma, M)$ and $\mathcal{C}(\gamma, M)$ of the class Σ as follows:

(1.4)

$$\mathcal{S}^*(\gamma, M) = \left\{ f \in \Sigma : -\frac{zf'(z)}{f(z)} \prec \frac{[\gamma(1+m) - m]\, z + 1}{1 - mz} \ \left(\gamma \in \mathbb{C}^*,\ m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right) \right\}$$

or, equivalently,

$$\left| \frac{\gamma - 1 - \dfrac{zf'(z)}{f(z)}}{\gamma} - M \right| < M \ \left(m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right), \tag{1.5}$$

and

(1.6)

$$\mathcal{C}(\gamma, M) = \left\{ f \in \Sigma : -\frac{zf''(z)}{f'(z)} \prec 2 + \frac{\gamma(1+m)z}{1 - mz}\ ,\ \left(\gamma \in \mathbb{C}^*,\ m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right) \right\}$$

or, equivalently,

$$\left| \frac{\gamma - 2 - \dfrac{zf''(z)}{f'(z)}}{\gamma} - M \right| < M \ \left(m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right). \tag{1.7}$$

From inequalities (1.4) and (1.6), we get

$$f(z) \in \mathcal{C}(\gamma, M) \Leftrightarrow -zf'(z) \in \mathcal{S}^*(\gamma, M). \tag{1.8}$$

First let us define the class $S^*(\lambda, \gamma, M)$ which unifies the classes of bounded meromorphic starlike and convex functions of complex order.

Definition 1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}^*(\lambda, \gamma, M)(\lambda \in \mathbb{C}, \gamma \in \mathbb{C}^*, M \geq 1)$ of bounded meromorphic $\lambda-$starlike functions of complex order, if and only if for fixed M, $\frac{f(z)f'(z)}{z} \neq 0$ and*

$$\left| 1 - \frac{1}{\gamma} \left(1 + \frac{\lambda z \left(zf'(z) \right)' + (1+\lambda) z f'(z)}{\lambda z f'(z) + (1+\lambda) f(z)} \right) - M \right| < M \quad (z \in \mathbb{E}), \tag{1.9}$$

or, equivalently,

(1.10)

$$\mathcal{S}^*(\lambda, \gamma, M) = \left\{ f \in \Sigma : -\frac{\lambda z \left(zf'(z) \right)' + (1+\lambda) z f'(z)}{\lambda z f'(z) + (1+\lambda) f(z)} \prec \frac{[\gamma(1+m) - m]\, z + 1}{1 - mz} \right\}$$

$$\left(\lambda \in \mathbb{C},\ \gamma \in \mathbb{C}^*,\ m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right)$$

One can easily show that $f \in \mathcal{S}^*(\lambda,\gamma,M)$ if and only if there is a function $g \in \mathcal{S}^*(1,M)$ such that

$$\lambda z f^{'}(z) + (1+\lambda) f(z) = \frac{(zg(z))^{\gamma}}{z} \tag{1.11}$$

It was shown in [8] $g \in \mathcal{S}^*(1,M)$ if and only if for $z \in \mathbb{E}$

$$-\frac{zg^{'}(z)}{g(z)} = \frac{1+\omega(z)}{1-m\omega(z)}, \quad \omega(0)=0,\ |\omega(z)|<1 \ \textit{and} \ \ m = 1-\frac{1}{M}. \tag{1.12}$$

Thus from (1.11) and (1.12) follows that $f \in \mathcal{S}^*(\lambda,\gamma,M)$ if and only if for $M \geq 1,\ \lambda \in \mathbb{C}$ and $z \in \mathbb{E}$

$$\frac{\lambda z\left(zf^{'}(z)\right)^{'} + (1+\lambda) z f^{'}(z)}{\lambda z f^{'}(z) + (1+\lambda) f(z)} = -\frac{[\gamma(1+m)-m]\,\omega(z)+1}{1-m\omega(z)}. \tag{1.13}$$

By specializing λ, γ and M, we get the following subclasses studied by earlier authors:

Remark 1.

(i) $\mathcal{S}^*(0,\gamma,\infty) =: \mathcal{S}^*(\gamma)$, with $\gamma \in \mathbb{C}^*$, (see Aouf [2]);
(ii) $\mathcal{S}^*(-1,\gamma,\infty) =: \mathcal{C}(\gamma)$, with $\gamma \in \mathbb{C}^*$, (see Aouf [2]);
(iii) $\mathcal{S}^*(0,1-a,M) =: \mathcal{S}_M^*(a)$, with $0 \leq a < 1$, (see Kaczmarski [8]);
(iv)$\mathcal{S}^*(-1,1-a,M) =: \mathcal{C}_M(a)$, with $0 \leq a < 1$, (see Aouf [2]);
(v) $\mathcal{S}^*(0,1,\infty) =: \mathcal{S}^*(1)$, with $0 \leq a < 1$, (see Clunie [7]);
(vi) $\mathcal{S}^*(-1,1,\infty) =: \mathcal{C}(1)$, with $0 \leq a < 1$, (see Aouf [2]);
(vii) $\mathcal{S}^*(0,1-a,\infty) =: \mathcal{S}^*(1-a)$, with $0 \leq a < 1$, (see Kaczmarski [8] and Pommerenke [11]);
(viii) $\mathcal{S}^*(-1,1-a,\infty) =: \mathcal{C}(1-a)$,with $0 \leq a < 1$, (see Aouf [2]);
(ix) $\mathcal{S}^*(0,(1-a)e^{-i\beta}\cos\beta,M) =: \mathcal{S}_M^*(a,\beta)$, with $0 \leq a < 1,\ |\beta| \leq \frac{\pi}{2})$, (see Kaczmarski [8]);
(x) $\mathcal{S}^*(-1,(1-a)e^{-i\beta}\cos\beta,M) =: \mathcal{C}_M(a,\beta)$,with $0 \leq a < 1,\ |\beta| \leq \frac{\pi}{2})$, (see Aouf [2]);
(xi) $\mathcal{S}^*(0,(1-a)e^{-i\beta}\cos\beta,\infty) =: \mathcal{S}^*(a,\beta)$, with $0 \leq a < 1,\ |\beta| \leq \frac{\pi}{2})$, (see Kaczmarski [8]);
(xii) $\mathcal{S}^*(-1,(1-a)e^{-i\beta}\cos\beta,\infty) =: \mathcal{C}(a,\beta)$,with $0 \leq a < 1,\ |\beta| \leq \frac{\pi}{2})$, (see Aouf [2]).

For $f(z) \in \Sigma$, Al-Oboudi and Al-Zkeri [1] defined the following operator $D^n f (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,3,\ldots\})$ which is called the Al-Oboudi operator:

$$\begin{aligned}
D^0 f(z) &= f(z), \\
D^1 f(z) &= (1-\mu) f(z) + \mu \frac{(z^2 f(z))'}{z}, \quad \mu \geq 0 \\
&= (1+\mu) f(z) + \mu z f'(z) = D_\mu f(z), \\
D^2 f(z) &= D_\mu D^1 f(z). \\
D^n f(z) &= D_\mu \left(D^{n-1} f(z)\right),\ n \in \mathbb{N}
\end{aligned} \tag{1.14}$$

From (1.1) and (1.14) we get

$$D^n f(z) = z^{-1} + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n a_k z^k \quad (z \in \mathbb{E}^*). \tag{1.15}$$

With the aid of Al-Oboudi operator, we introduce the class $\mathcal{S}^*(n,\lambda,\gamma,M)$ as follows:

Definition 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(n,\lambda,\gamma,M)$ if and only if for fixed M,*

$$\left|1-\frac{1}{\gamma}\left(1+\frac{\lambda z\left[z\left(D^nf(z)\right)'\right]'+(1+\lambda)z\left(D^nf(z)\right)'}{\lambda z\left(D^nf(z)\right)'+(1+\lambda)D^nf(z)}\right)-M\right|<M \quad (z\in\mathbb{E})$$

where, $M\geq 1,\gamma\in C^*,\lambda\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

We note that $\mathcal{S}^*(0,\lambda,\gamma,M)\equiv\mathcal{S}^*(\lambda,\gamma,M)$.

In this paper we will investigate some convolution properties of the class $\mathcal{S}^*(\lambda,\gamma,M)$. Using these properties, we find the necessary and sufficient condition, and containment property for the subclass $\mathcal{S}^*(n,\lambda,\gamma,M)$. The results obtained here extend some known results in [3], [4] and [6].

2. Convolution properties

Unless otherwise mentioned, we assume throughout this article that $\gamma\in\mathbb{C}^*, M\geq 1,\lambda\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

Theorem 1. *The function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(\lambda,\gamma,M)$ if and only if*

$$z\left[f(z)*\left\{(1+\lambda)\frac{(C-1)\,z+1}{z(1-z)^2}+\lambda\frac{2(C-1)z^2+3z-1}{z(1-z)^3}\right\}\right]\neq 0\ (z\in\mathbb{E}) \tag{2.1}$$

where $C=C_\theta=\frac{e^{-i\theta}-m}{\gamma(1+m)},\ \theta\in[0,2\pi)$.

Proof. First suppose $f(z)$ defined by (1.1) is in the class $\mathcal{S}^*(\lambda,\gamma,M)$, we have

$$-\frac{\lambda z\left(zf'(z)\right)'+(1+\lambda)zf'(z)}{\lambda zf'(z)+(1+\lambda)f(z)}\prec\frac{[\gamma(1+m)-m]\,z+1}{1-mz}\ (z\in\mathbb{E}), \tag{2.2}$$

since the left-hand side of (2.2) is analytic in $\mathbb{E}$, it follows $\lambda zf'(z)+(1+\lambda)f(z)\neq 0$ for all $z\in\mathbb{E}^*$, i.e. $\lambda z^2f'(z)+(1+\lambda)zf(z)\neq 0,\ z\in\mathbb{E}$, so (2.1) holds for $C=0$. By using the principle of subordination, we can write (2.2) as

$$-\frac{\lambda z\left(zf'(z)\right)'+(1+\lambda)zf'(z)}{\lambda zf'(z)+(1+\lambda)f(z)}=\frac{[\gamma(1+m)-m]\omega(z)+1}{1-m\omega(z)}(z\in\mathbb{E}),$$

which is equivalent to

$$-\frac{z\left[\lambda\left(zf'(z)\right)+(1+\lambda)f(z)\right]'}{\lambda zf'(z)+(1+\lambda)f(z)}\neq\frac{[\gamma(1+m)-m]e^{i\theta}+1}{1-me^{i\theta}},\quad (\ z\in\mathbb{E},\ \theta\in[0,2\pi)).$$

or

$$-z\left[\lambda\left(zf'(z)\right)+(1+\lambda)f(z)\right]'\left(1-me^{i\theta}\right)-\left[\lambda zf'(z)+(1+\lambda)f(z)\right]\left[[\gamma(1+m)-m]\,e^{i\theta}+1\right]\neq 0. \tag{2.3}$$

Since

$$f(z)=f(z)*\frac{1}{z(1-z)}\ \text{ and }\ -zf'(z)=f(z)*\left[\frac{1}{z(1-z)^2}-\frac{2}{(1-z)^2}\right]. \tag{2.4}$$

Applying (2.4) it is not difficult to verify that

$$\lambda z f^{'}(z)+(1+\lambda)f(z)=f(z)*\frac{(\lambda-1)z+1}{z(1-z)^2}. \tag{2.5}$$

Since $z(f*g)^{'}=f*zg^{'}$, we can write

$$z\left[\lambda\left(zf^{'}(z)\right)+(1+\lambda)f(z)\right]^{'}=f(z)*\frac{2(\lambda-1)z^2+3z-1}{z(1-z)^3}. \tag{2.6}$$

Using (2.5) and (2.6) in (2.3), we get

$$z[f(z)*\{-(1+\lambda)(1-z)[\gamma(1+m)e^{i\theta}+\left([\gamma(1+m)-m]e^{i\theta}+1\right)z] \tag{2.7}$$

$$+\lambda(1-z)\gamma(1+m)e^{i\theta}-2\lambda(1-me^{i\theta})z^2+2(1-z)\gamma(1+m)e^{i\theta}z\}/z(1-z)^3]\neq 0.$$

The left hand side of (2.7) may be written as

$$z[f(z)*\{-(1+\lambda)(1-z)[\gamma(1+m)e^{i\theta}+\left(1-me^{i\theta}-\gamma(1+m)e^{i\theta}\right)z]$$
$$+\lambda\gamma(1+m)e^{i\theta}-3\lambda\gamma(1+m)e^{i\theta}z-2\lambda\left(1-me^{i\theta}\right)z^2+2\lambda\gamma(1+m)e^{i\theta}z^2]\}/z(1-z)^3\].$$

Equation (2.7) can be rewritten in the form

$$z\left[f(z)*\left\{(1+\lambda)\frac{\left(\frac{e^{-i\theta}-m}{\gamma(1+m)}-1\right)z+1}{z(1-z)^2}+\lambda\frac{2(\frac{e^{-i\theta}-m}{\gamma(1+m)}-1)z^2+3z-1}{z(1-z)^3}\right\}\right]\neq 0$$

where $z\in\mathbb{E},\ \theta\in[0,2\pi)$. Thus we have the first part of the proof.

(ii) Conversely, since (2.1) holds for $C=0$, then $\lambda z^2 f^{'}(z)+(1+\lambda)zf(z)\neq 0$ for all $z\in\mathbb{E}$, hence the function $\varphi(z)=-\frac{z\left[\lambda z f^{'}(z)+(1+\lambda)f(z)\right]^{'}}{\lambda z f^{'}(z)+(1+\lambda)f(z)}$ is analytic in $\mathbb{E}$ (i.e. it is regular at $z_0=0$, with $\varphi(0)=1$). Since (2.7) is equivalent to (2.1), we have

$$-\frac{z\left[\lambda z f^{'}(z)+(1+\lambda)f(z)\right]^{'}}{\lambda z f^{'}(z)+(1+\lambda)f(z)}\neq\frac{[\gamma(1+m)-m]e^{i\theta}+1}{1-me^{i\theta}}\quad(z\in\mathbb{E},\ \theta\in[0,2\pi)). \tag{2.8}$$

Assume that

$$\varphi(z)=-\frac{z\left[\lambda z f^{'}(z)+(1+\lambda)f(z)\right]^{'}}{\lambda z f^{'}(z)+(1+\lambda)f(z)},\ \psi(z)=\frac{[\gamma(1+m)-m]e^{i\theta}+1}{1-me^{i\theta}}.$$

The relation (2.8) means that $\varphi(\mathbb{E})\cap\psi(\partial\mathbb{E})=\emptyset$. Thus, the simply connected domain $\varphi(\mathbb{E})$ is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{E})$. From this, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, this implies that $f(z)\in\mathcal{S}^*(\lambda,\gamma,M)$. Thus the proof of Theorem 1 is completed. □

Remark 2.

(i) Taking $\lambda=0$ in Theorem 1, we obtain the result obtained by Aouf [4, Theorem 2.1].

(ii) Taking $\lambda=-1$ in Theorem 1, we obtain the result obtained by Aouf [4, Theorem 2.3].

(iii) Taking $\lambda=0$ and $m=1$ in Theorem 1, we obtain the result obtained by Bulboacă et al. [6, Theorem 1, with $A=1$ and $B=-1$] and Aouf et al. [3, Theorem 4, with $\lambda=0,\ A=1$ and $B=-1$].

(iv) Taking $\lambda = 0,\ \gamma = m = 1$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Ponnusamy [12, Theorem 4, with $\lambda = 0,\ A = 1$ and $B = -1$].
(iiv) Taking $\lambda = 0,\ m = 1,\ \gamma = (1-\alpha)e^{-i\mu}\cos\mu\ (\mu \in \mathbb{R},\ |\mu| \leq \frac{\pi}{2},\ 0 \leq \alpha < 1)$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Ravichandran et al. [13, Theorem 1.2 with $p = 1$].

Theorem 2. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to be in the class $\mathcal{S}^*(n,\lambda,\gamma,M)$ is that*

$$1 + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n [\lambda(k+1)+1] a_k z^{k+1} \neq 0 \tag{2.9}$$

and

$$1 + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n \left[\tfrac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right] [(1+\lambda+\lambda k] a_k z^{k+1} \neq 0 \tag{2.10}$$

for all $\theta \in [0, 2\pi)$ and $z \in \mathbb{E}$.

Proof. From Theorem 1, we have $f(z) \in \mathcal{S}^*(n,\lambda,\gamma,M)$ if and only if

$$z\left[D^n f(z) * \left\{(1+\lambda)\frac{(C-1)\,z+1}{z(1-z)^2} + \lambda\frac{2(C-1)z^2+3z-1}{z(1-z)^3}\right\}\right] \neq 0\ (z \in \mathbb{E}) \tag{2.11}$$

for all $C = C_\theta = \frac{e^{-i\theta}-m}{\gamma(1+m)},\ \ (0 \leq \theta < 2\pi)$, and also for $C = 0$. From (1.15) and the equations

$$\frac{1}{z\,(1-z)} = z^{-1} + \sum_{k=0}^{\infty} z^k, \quad \frac{1}{z(1-z)^2} = z^{-1} + \sum_{k=0}^{\infty} (k+2)\,z^k, \tag{2.12}$$

it is not difficult to show that (2.10) holds for $C = 0$ iff (2.9) satisfied. The left hand side of (2.11) may be written as

$$z\left[D^n f(z) * \left\{(1+\lambda)\left[\frac{1-C}{z\,(1-z)} + \frac{C}{z(1-z)^2}\right] + \lambda\left[\frac{2C}{z(1-z)^3} - \frac{4C-1}{z(1-z)^2} + \frac{2(C-1)}{z\,(1-z)}\right]\right\}\right]. \tag{2.13}$$

Using (1.15), (2.12) and the formula

$$\frac{1}{z(1-z)^3} = z^{-1} + \sum_{k=0}^{\infty} \frac{(k+2)\,(k+3)}{2} z^k$$

Equation (2.13) can be written as

$$1 + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n \left[\tfrac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right] [1+\lambda+\lambda k] a_k z^{k+1}.$$

Thus, the proof of Theorem 2 is completed. □

Theorem 3. *If the function $f(z)$ given by (1.1) and satisfy the inequality*

$$\sum_{k=0}^{\infty} (k+1+|\gamma|)[\lambda(k+1)+1][\mu(k+1)+1]^n |a_k| \leq |\gamma| \tag{2.14}$$

then $f(z) \in \mathcal{S}^*(n,\lambda,\gamma,M)$.

Proof. Since

$$\left|\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right| \leq \frac{(k+1+|\gamma|)}{|\gamma|}$$

then

$$\begin{aligned}
&\left|1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^n\left[\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right][\lambda(k+1)+1]a_k z^{k+1}\right| \\
\geq\ & 1-\sum_{k=0}^{\infty}[\mu(k+1)+1]^n\left|\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right|[\lambda(k+1)+1]\,|a_k|\,|z|^{k+1} \\
\geq\ & 1-\sum_{k=0}^{\infty}\frac{(k+1+|\gamma|)}{|\gamma|}[\lambda(k+1)+1]\,[\mu(k+1)+1]^n\,|a_k|>0 \qquad (z\in\mathbb{E})\,.
\end{aligned}$$

Which implies that inequality (2.14). Thus this completes the proof of Theorem 3. □

Theorem 4. *For $\lambda \in \mathbb{C}$, we have $\mathcal{S}^*(n+1,\lambda,\gamma,M) \subset \mathcal{S}^*(n,\lambda,\gamma,M)$.*

Proof. If $f(z) \in \mathcal{S}^*(n+1,\lambda,\gamma,M)$, then Theorem 2 gives

$$1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^{n+1}\,[\lambda(k+1)+1]a_k z^{k+1} \neq 0$$

and

$$1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^{n+1}\left[\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right][\lambda(k+1)+1]\,a_k z^{k+1} \neq 0 \tag{2.15}$$

we can write (2.15) as

$$\left[1+\sum_{k=0}^{\infty}[\mu(k+1)+1]\,z^{k+1}\right]*\left[1+\sum_{k=0}^{\infty}\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\,[\mu(k+1)+1]^n\,[\lambda(k+1)+1]\,a_k z^{k+1}\right]\neq 0. \tag{2.16}$$

But

$$\left[1+\sum_{k=0}^{\infty}[\mu(k+1)+1]\,z^{k+1}\right]*\left[1+\sum_{k=0}^{\infty}\frac{1}{[\mu(k+1)+1]}z^{k+1}\right]=1+\sum_{k=0}^{\infty}z^{k+1}. \tag{2.17}$$

By using the property, if $f\neq 0$ and $g*h\neq 0$, then $f*(g*h)\neq 0$, (2.16) can be written as

$$1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^n\left[\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right][\lambda(k+1)+1]\,a_k z^{k+1}\neq 0. \tag{2.18}$$

In view of Theorem 2, we conclude that $f(z)\in\mathcal{S}^*(n,\lambda,\gamma,M)$. □

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Department of Mathematics Faculty of Science, Mansoura University Mansoura, 35516, EGYPT.
E-mail address: Darwish333@yahoo.com

Department of Mathematics Faculty of Science, Mansoura University Mansoura, 35516, EGYPT.
E-mail address: aylashin@mans.edu.eg

Department of Mathematics Faculty of Science, Mansoura University Mansoura, 35516, EGYPT.
E-mail address: s_soileh@yahoo.com

ON A GENERALIZED DEGENERATE λ-q-DAEHEE NUMBERS AND POLYNOMIALS

JIN-WOO PARK

Abstract. In [4], Daehee numbers and polynomials are introduced by T. Kim et al. In this paper, we consider the generalized λ-q-Daehee polynomials by using the bosonic p-adic q-integral and give some relations between the generalized λ-q-Daehee polynomials and special polynomials.

1. Introduction

Let d be fixed positive integer and let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

We set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z},\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \left\{x \in X | x \equiv a \pmod{dp^N}\right\},$$

where $a \in \mathbb{Z}$ and $0 \le a < dp^n$.

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic bosonic integral on* $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [8, 9, 10])}. \tag{1.1}$$

If we put $f_n(x) = f(x+n)$, then, by (1.1), we can derive the following very useful integral identity;

$$q^n I_q(f_n) - I_q(f) = (q-1) \sum_{j=0}^{n-1} q^j f(j) + \frac{q-1}{\log q} \sum_{j=0}^{n-1} f'(j) q^j, \tag{1.2}$$

where $f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}$.

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As is well known, the *generalized q-Bernoulli numbers $B_{n,\chi,q}$ attached to χ* are defined by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}t}{q^d e^{dt}-1}\sum_{a=0}^{d-1}\chi(a)q^a e^{at}=\sum_{n=0}^{\infty}B_{n,\chi,q}\frac{t^n}{n!},\ \text{(see [3, 7, 11, 12, 17, 18])}.$$

The Stirling numbers of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l\ (x\geq 0),$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}$$

(see [18, 16]). Note that

$$(\log(x+1))^n=n!\sum_{l=n}^{\infty}S_1(l,n)\frac{x^l}{l!},\ (n\geq 0),$$

(see [18, 16]).

Recently, q-Daehee numbers and polynomials are introduced by Kim et. al. in [8], and have been studied by many mathematicians, and possess many interesting properties (see [1, 4-6, 13-15]). In this paper, we consider the generalized λ-q-Daehee polynomials and numbers by using the bosonic p-adic q-integral, and give some relations between the generalized λ-q-Daehee numbers and polynomials and special numbers and polynomials.

2. The generalized degenerate λ-q-Daehee polynomials attached to χ

From now on, we assume that $t\in\mathbb{C}$ with $|t|_p<p^{-\frac{1}{p-1}}$ and $u,\ \lambda\in\mathbb{Z}_p$. Let χ be the Dirichlet character with conductor $d\in\mathbb{N}=\{1,2,\ldots\}$ with $d\equiv 1\ (\mathrm{mod}\ 2)$.

The *generalized degenerate λ-q-Daehee polynomials $D_{n,\chi,\lambda,q}(x)$ attached to χ* are defined by the generating function to be

$$\begin{aligned}&\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j+x}\\&=\sum_{n=0}^{\infty}D_{n,\chi,\lambda,q}(x|u)\frac{t^n}{n!},\end{aligned}\tag{2.1}$$

where $t\in\mathbb{C}_p$ and $|ut|_p<p^{-\frac{1}{p-1}}$. In the special case $x=0$, $D_{n,\chi,\lambda,q}(0|u)=D_{n,\chi,\lambda,q}(u)$ are called *generalized degenerate λ-q-Daehee numbers attached to χ*.

By replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\chi,\lambda,q}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}&=\frac{q-1+\frac{q-1}{\log q}\lambda t}{q^d e^{\lambda dt}-1}\sum_{j=0}^{d-1}\chi(j)q^j e^{\lambda jt}\\&=\sum_{m=0}^{\infty}\lambda^m B_{m,\chi,q}^{(r)}\frac{t^m}{m!},\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\ =&\sum_{n=0}^{\infty}\frac{D_{n,\chi,\lambda,q}(u)}{n!}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{1}{m!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^m\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{k=0}^{m}\sum_{n=0}^{m}D_{k,\chi,\lambda,q}(u)u^{n-l-k}S_2(m,k)S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $m \geq 0$, we have*

$$\lambda^n B_{n,\chi,q} = \sum_{m=0}^{k}\sum_{k=0}^{m}\sum_{n=0}^{m} D_{k,\chi,\lambda,q}(u)u^{n-l-k}S_2(m,k)S_2(n,m).$$

If taking $f(x) = \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^x$ in (1.2), we can have

$$\begin{aligned}&q^d\int_X \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(x+d)}d\mu_q(x)+\int_X\chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x)\\ =&\left(q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)\right)\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j}.\end{aligned} \tag{2.4}$$

By (2.4), we can easily have

$$\begin{aligned}&\int_X\chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x)\\ =&\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j}\\ =&\sum_{n=0}^{\infty}D_{n,\chi,\lambda,q}(u)\frac{t^n}{n!},\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\int_X\chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x)\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_X\chi(x)(\lambda x)_m d\mu_q(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{n,\chi,\lambda,q}(u) = \sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_X\chi(x)(\lambda x)_m d\mu_q(x).$$

By (2.1), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}(x|u)\frac{t^n}{n!}\\
=&\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log\left(1+ut\right)\right)}{q^d\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda j+x}\\
=&\left(\sum_{m=0}^{\infty} D_{m,\chi,\lambda,q}(u)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{x}{l}u^{n-l}l!S_1(n,l)\frac{t^n}{n!}\right)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{x}{l}\binom{n}{m}u^{n-m-l}l!S_1(n-m,l)D_{m,\chi,\lambda,q}(u)\frac{t^n}{n!}.
\end{aligned}\tag{2.7}$$

So, by (2.7), we can have

$$D_{n,\chi,\lambda,q}(x|u)=\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{x}{l}\binom{n}{m}u^{n-m-l}l!S_1(n-m,l)D_{m,\chi,\lambda,q}(u).\tag{2.8}$$

For $r\in\mathbb{N}$, let us consider the *generalized degenerate λ-q-Daehee numbers of order r attached to χ* as follows:

$$\begin{aligned}
&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log\left(1+ut\right)\right)}{q^d\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda j}\right)^r\\
=&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log\left(1+ut\right)\right)}{q^d\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda d}-1}\right)^r\\
&\times\sum_{a_1,\dots,a_r=0}^{d-1}\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda(a_1+\cdots+a_r)}\\
=&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}^{(r)}(u)\frac{t^n}{n!}.
\end{aligned}\tag{2.9}$$

By (2.5), we can see that

$$\begin{aligned}
&\int_X\cdots\int_X\chi(x_1)\cdots\chi(x_r)\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda(x_1+\cdots+x_r)}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
=&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log\left(1+ut\right)\right)}{q^d\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda d}-1}\right)^r\\
&\times\sum_{a_1,\dots,a_r=0}^{d-1}\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}\left(1+\frac{1}{u}\log\left(1+ut\right)\right)^{\lambda(a_1+\cdots+a_r)}.
\end{aligned}\tag{2.10}$$

Thus, by (2.9) and (2.10), we get

$$\begin{aligned}
&D_{n,\chi,\lambda,q}^{(r)}(u)\\
&=\sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(\lambda x_1+\cdots+\lambda x_r)_m d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)\sum_{l=0}^{m} S_1(l,m)\lambda^l (x_1+\cdots+x_r)^l d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}\lambda^l S_1(n,m)S_1(l,m)B_{l,\chi,q}^{(r)},
\end{aligned} \tag{2.11}$$

where $B_{l,\chi,q}^{(r)}$ are the l-th generalized q-Bernoulli numbers of order r attached to χ, given by

$$\left(\sum_{a=0}^{d-1}\frac{1-q+\frac{1-q}{\log q}t}{1-q^d e^{td}}\chi(a)q^a e^{at}\right)^r = \sum_{n=0}^{\infty} B_{n,\chi,q}^{(r)}\frac{t^n}{n!} \text{ (see [11])}.$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, we have*

$$D_{n,\chi,\lambda,q}^{(r)}(u) = \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}\lambda^l S_1(n,m)S_1(l,m)B_{l,\chi,q}^{(r)}.$$

By replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.9), we can get

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}^{(r)}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\
&=\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{q^d e^{\lambda dt}-1}\right)^r \chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}e^{\lambda(a_1+\cdots+a_r)t}\\
&=\sum_{m=0}^{\infty}\lambda^m B_{m,\chi,q}^{(r)}\frac{t^m}{m!},
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}^{(r)}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{D_{n,\chi,\lambda,q}^{(r)}(u)}{n!}u^{-n}n!\sum_{m=n}^{\infty} S_2(m,n)\frac{1}{m!}\left(u(e^t-1)\right)^m\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{s=0}^{m} D_{n,\chi,\lambda,q}^{(r)}(u)u^{m-s}S_2(m,s)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\lambda^m B_{m,\chi,q}^{(r)} = \sum_{m=0}^{n} \sum_{s=0}^{m} D_{n,\chi,\lambda,q}^{(r)}(u) u^{m-s} S_2(m,s) S_2(n,m).$$

From (2.9), we can consider the *generalized λ-q-Daehee polynomials of order r attached to χ* as follows:

$$\begin{aligned}
&\left(\frac{q-1+\frac{q-1}{\log q}\lambda \log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1} \sum_{j=0}^{d-1} \chi(j) q^j \left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j} \right)^r \\
&\quad \times \left(1+\frac{1}{u}\log(1+ut)\right)^x \\
&= \left(\frac{q-1+\frac{q-1}{\log q}\lambda \log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1} \right)^r \\
&\quad \times \sum_{a_1,\ldots,a_r=0}^{d-1} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} \left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(a_1+\cdots+a_r)+x} \\
&= \sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}^{(r)}(x|u) \frac{t^n}{n!}
\end{aligned} \tag{2.14}$$

By (2.14),

$$\begin{aligned}
&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r) \left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x_1+\cdots+\lambda x_r+x} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{q-1+\frac{q-1}{\log q}\lambda \log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1} \right)^r \\
&\quad \times \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} \left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(a_1+\cdots+a_r)+x},
\end{aligned} \tag{2.15}$$

and so, from (2.14) and (2.15)

$$\begin{aligned}
&D_{n,\chi,\lambda,q}^{(r)}(x|u) \\
&= \sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(\lambda x_1+\cdots+\lambda x_r+x)_m d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{m=0}^{n} u^{n-m} S_1(n,m) \\
&\quad \times \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r) \sum_{l=0}^{m} S_1(l,m) \lambda^l (\lambda x_1+\cdots+\lambda x_r+x)^l d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{m=0}^{n} \sum_{l=0}^{m} u^{n-m} \lambda^l S_1(n,m) S_1(l,m) B_{l,\chi,q}^{(r)}\left(\frac{x}{\lambda}\right).
\end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$D_{n,\chi,\lambda,q}^{(r)}(x|u) = \sum_{m=0}^{n} \sum_{l=0}^{m} u^{n-m} \lambda^l S_1(n,m) S_1(l,m) B_{l,\chi,q}^{(r)} \left(\frac{x}{\lambda}\right).$$

In (2.14), by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$, we can get

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}^{(r)}(x|u) \frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!} \\
&= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{q^d e^{\lambda d t}-1}\right)^r \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} e^{\lambda(a_1+\cdots+a_r)t+xt} \qquad (2.17) \\
&= \sum_{m=0}^{\infty} \lambda^m B_{m,\chi,q}^{(r)} \left(\frac{x}{\lambda}\right) \frac{t^m}{m!},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}^{(r)}(x|u) \frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{D_{n,\chi,\lambda,q}^{(r)}(x|u)}{n!} u^{-n} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{1}{m!} \left(u(e^t-1)\right)^m \qquad (2.18) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{s=0}^{m} D_{n,\chi,\lambda,q}^{(r)}(x|u) u^{m-s} S_2(m,s) S_2(n,m) \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$\lambda^m B_{m,\chi,q}^{(r)} \left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n} \sum_{s=0}^{m} D_{n,\chi,\lambda,q}^{(r)}(x|u) u^{m-s} S_2(m,s) S_2(n,m).$$

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Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: a0417001@knu.ac.kr

On the $m-$extension of Fibonacci $p-$functions with period k

Yasin YAZLIK[a,*], Cahit KÖME[a]

[a]*Department of Mathematics, Nevşehir Hacı Bektaş Veli University, 50300, Turkey*

Abstract

Let $f_{p,m}$ be a real valued function on $\mathbb{R}$, p be nonnegative integer, k be a positive integer and m be a nonnegative real number. For all $x \in \mathbb{R}$, $f_{p,m}(x+(p+1)k) = mf_{p,m}(x+pk) + f_{p,m}(x)$, we call this function $m-$extension of Fibonacci $p-$function with period k. In this paper, we present basic properties of $m-$extension of Fibonacci $p-$functions with period k. Specifying p and m, we obtain Fibonacci ($p=1$, $m=1$) and Pell ($p=1$, $m=2$) functions. Furthermore, we define $m-$extension of odd Fibonacci $p-$functions with period k. Moreover, we analyze some properties by using notion of $f-$even and $f-$odd functions with period k. We also demonstrate the products and quotients of these functions and provide new results in the development of Fibonacci functions with period k.

Keywords: $m-$extension of Fibonacci $p-$function with period k, $m-$extension of odd Fibonacci $p-$function with period k, $f-$even function with period k, $f-$odd functions with period k.

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1. Introduction

Fibonacci numbers is one of the most popular and fascinating linear sequences in mathematics and related fields. The classical Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$, for $n \in \mathbb{N}$, with initial conditions $F_0 = 0, \quad F_1 = 1$. Up until now, many authors have studied the sums, representations, properties, relations with another mathematical topics, applications and generalizations of the Fibonacci sequence extensively (see [1–15]). Falcon introduced kth Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ that arises in the study of the recursive application of two geometrical transformations used in the well known four triangle longest edge (4TLE) partition[2]. In [7], Yazlik and Taskara defined generalized $k-$Horadam sequence and proved the properties of this sequence by means of determinant. Stakhov and Rozin presented, one of the important mathematical discoveries of the modern Golden Section and Fibonacci numbers theory, Fibonacci $p-$numbers and some properties of this sequence, $F_p(n) = F_p(n-1) + F_p(n-p-1)$, in [10]. Later on, the authors defined the

*Corresponding Author

Email addresses: yyazlik@nevsehir.edu.tr (Yasin YAZLIK), cahitkome@gmail.com (Cahit KÖME)

$m-$extension of the Fibonacci $p-$numbers as

$$F_{p,m}(n+p+1) = mF_{p,m}(n+p) + F_{p,m}(n) \tag{1}$$

with initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, F_{p,m}(3) = m^2, \ldots, F_{p,m}(p+1) = m^p$, where $p, n \in \mathbb{N}$ and m is positive real number. For different values of p and m in equation (1), it can be reduced into different numerical sequences. For example, if $(p, m) = (1, 1)$, the Fibonacci sequence is obtained as $F_{n+2} = F_{n+1} + F_n$. If $(p, m) = (1, 2)$, the Pell sequence is obtained as $P_{n+2} = 2P_{n+1} + P_n$. If $p = 1$ and $m = k$, the k-Fibonacci sequence is obtained as $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ [9]. Recently, one of the important application of these integer sequences is continuous functions. Han et al.,[16], considered Fibonacci functions on the real numbers $\mathbb{R}$, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\quad f(x+2) = f(x+1)+f(x)$. Also they presented some properties of these functions by using the concept of $f-$even and $f-$odd functions. Moreover, they showed that if f is Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. Afterwards, Sroysang extended Fibonacci functions to Fibonacci functions with period k as $f(x+2k) = f(x+k)+f(x)$ for all $x \in \mathbb{R}$ in [17]. In [18], Rabago defined the second order linear recurrent function with period k, $w(x+2k) = rw(x+k)+sw(x)$, where r, s are nonnegative real numbers, which is generalization of the Fibonacci function with period k.

Up until now, authors investigated some properties of the continuous functions of the second order linear recursive integer sequences. In this paper, we extend these properties to the continuous function in terms of $m-$extension of Fibonacci $p-$numbers which is defined by the $(p+1)th$ order linear recursive relation. We present some properties of the $m-$extension of Fibonacci $p-$functions with period k using the concept of $f-$even and $f-$odd functions with period k. We also define $m-$extension of odd Fibonacci $p-$functions with period k, investigate the product and the limit of $m-$extension of Fibonacci $p-$functions with period k.

2. $m-$extension of Fibonacci $p-$functions with period k

In this section we define $m-$extension of Fibonacci $p-$functions with period k and present some properties of these functions.

Definition 2.1. *Let k be a positive integer, p be nonnegative integer and m be a nonnegative real number. A function $f_{p,m} : \mathbb{R} \to \mathbb{R}$ is called an $m-$extension of Fibonacci $p-$function with period k if it satisfies the equation*

$$f_{p,m}(x + (p+1)k) = mf_{p,m}(x + pk) + f_{p,m}(x), \quad \forall x \in \mathbb{R}. \tag{2}$$

Taking $(p, m) = (1, 1)$ and $(p, m) = (1, 2)$ in (2), we obtain Fibonacci and Pell function with period k, respectively (see [17, 18]).

Example 2.1. Let α be the positive real number that satisfies the equation $\alpha^{p+1} = m\alpha^p + 1$, k be a positive integer, p be a nonnegative integer. Then, $f_{p,m}(x) = \alpha^{\frac{x}{k}}$ is an $m-$extension of Fibonacci $p-$function with period k.

The following are special cases of the previous example:

1. If $(p, m) = (1, 1)$ then the function $f_{1,1}(x) = \phi^{\frac{x}{k}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is known as golden ratio, is an example of $m-$extension of Fibonacci $p-$function with period k in [17].
2. If $(p, m) = (1, 2)$ then the function $f_{1,2}(x) = \sigma^{\frac{x}{k}}$, where $\sigma = 1 + \sqrt{2}$ is known as silver ratio, is an example of $m-$extension of Pell $p-$function with period k in [18].

Proposition 2.1. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of Fibonacci $p-$function with period k. Assume that $f_{p,m}$ is s times differentiable. Then $\{f'_{p,m}, f''_{p,m}, \ldots, f^{(s)}_{p,m}\}$ are also $m-$extension of odd Fibonacci $p-$functions with period k.*

Proposition 2.2. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of Fibonacci $p-$function with period k. Define $g_t(x) = f_{p,m}(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, $g_t(x)$ is also an $m-$extension of Fibonacci $p-$function with period k.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} g_t(x + (p+1)k) &= f_{p,m}(x + (p+1)k + t) \\ &= m f_{p,m}(x + pk + t) + f_{p,m}(x + t) \\ &= m g_t(x + pk) + g_t(x) \end{aligned}$$

is an $m-$extension of Fibonacci $p-$function with period k. □

Example 2.2. Let p be a nonnegative integer, k be positive integer and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by

$$g_t(x) = \alpha^{\frac{x+t}{k}}, \qquad \forall x \in \mathbb{R}, \tag{3}$$

then $g_t(x)$ is an $m-$extension of Fibonacci $p-$function with period k.

As special cases of the previous example, we have

1. If $(p, m) = (1, 1)$, then the function $g_t(x) = f_{1,1}(x + t) = \phi^{\frac{x+t}{k}}$ is an example of $m-$extension of Fibonacci $p-$function with period k in [17].
2. If $(p, m) = (1, 2)$, then the function $g_t(x) = f_{1,2}(x + t) = \sigma^{\frac{x}{k}}$ is an example of $m-$extension of Pell $p-$function with period k in [18].

Theorem 2.1. *Let $f_{p,m}$ be an $m-$extension of Fibonacci $p-$function with period k and $F_{p,m}$ be an $m-$extension of Fibonacci $p-$sequence with the initial conditions $F_{p,m}(0)=0, F_{p,m}(1)=1, F_{p,m}(2)=m, \ldots, F_{p,m}(p)=m^{p-1}$. Then, for $n \geq 2p$ and $\forall x \in \mathbb{R}$,*

$$f_{p,m}(x+nk) = F_{p,m}(n-p+1)f(x+pk) + \sum_{i=0}^{p-1} F_{p,m}(n-p-i)f(x+ik). \tag{4}$$

Proof. We prove the theorem by induction on n. For $n=2p$, we get

$$\begin{aligned} f_{p,m}(x+2pk) &= mf_{p,m}(x+(2p-1)k) + f_{p,m}(x+(p-1)k) \\ &= m\Big[mf_{p,m}(x+(2p-2)k) + f_{p,m}(x+(p-2)k)\Big] \\ &\quad + f_{p,m}(x+(p-1)k) \\ &= m^2 f_{p,m}(x+(2p-2)k) + f_{p,m}(x+(p-1)k) \\ &\quad + mf_{p,m}(x+(p-2)k) \\ &= m^3 f_{p,m}(x+(2p-3)k) + f_{p,m}(x+(p-1)k) \\ &\quad + mf_{p,m}(x+(p-2)k) + m^2 f_{p,m}(x+(p-3)k). \end{aligned}$$

Continuing this process $(p-3)$ times, we have

$$\begin{aligned} f_{p,m}(x+2pk) &= m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\ &\quad + mf_{p,m}(x+(p-2)k) + \cdots + m^{p-1} f_{p,m}(x). \end{aligned}$$

By considering the initial conditions of the $m-$extension of Fibonacci $p-$sequence, we obtain

$$\begin{aligned} f_{p,m}(x+2pk) &= F_{p,m}(p+1)f_{p,m}(x+pk) + F_{p,m}(1)f_{p,m}(x+(p-1)k) \\ &\quad + F_{p,m}(2)f_{p,m}(x+(p-2)k) + \cdots + F_{p,m}(p-1)f_{p,m}(x+k) \\ &\quad + F_{p,m}(p)f_{p,m}(x). \end{aligned}$$

Assume that equation (4) is true for $n \geq 2p+1$. Then we write

$$\begin{aligned}
f_{p,m}(x+(n+1)k) &= mf_{p,m}(x+nk) + f_{p,m}(x+(n-p)k) \\
&= m\Big[F_{p,m}(n-p+1)f_{p,m}(x+pk) \\
&\quad +F_{p,m}(n-2p+1)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+F_{p,m}(n-p)f_{p,m}(x)\Big] \\
&\quad +F_{p,m}(n-2p+1)f_{p,m}(x+pk) \\
&\quad +F_{p,m}(n-3p+1)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+F_{p,m}(n-2p)f_{p,m}(x) \\
&= (mF_{p,m}(n-p+1)+F_{p,m}(n-2p+1))f_{p,m}(x+pk) \\
&\quad +(mF_{p,m}(n-2p+1)+F_{p,m}(n-3p+1))f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+(mF_{p,m}(n-p)+F_{p,m}(n-2p))f_{p,m}(x) \\
&= F_{p,m}(n-p+2)f(x+pk)+\sum_{i=0}^{p-1}F_{p,m}(n+1-p-i)f(x+ik),
\end{aligned}$$

which completes the proof. □

Corollary 2.1. *Let $f_{p,m}$ be an $m-$extension of Fibonacci $p-$function with period k and $F_{p,m}$ be the sequence of $m-$extension of Fibonacci $p-$numbers. Then, for any $x \in \mathbb{R}$ and $n \geq 2p$,*

$$\alpha^n = F_{p,m}(n-p+1)\alpha^p + \sum_{i=0}^{p-1}F_{p,m}(n-p-i)\alpha^i. \tag{5}$$

Proof. From example (2.1), we say that $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, k is a positive integer, is an $m-$extension of Fibonacci $p-$function with period k, so it satisfies the Equation(2), for all $x \in \mathbb{R}$, i.e.

$$\begin{aligned}
\alpha^{\frac{x+nk}{k}} &= f_{p,m}(x+nk) \\
&= F_{p,m}(n-p+1)f(x+pk)+\sum_{i=0}^{p-1}F_{p,m}(n-p-i)f(x+ik) \\
&= \alpha^{\frac{x}{k}+p}F_{p,m}(n-p+1)+\alpha^{\frac{x}{k}}F_{p,m}(n-p)+\alpha^{\frac{x}{k}+1}F_{p,m}(n-p-1) \\
&+ \alpha^{\frac{x}{k}+2}F_{p,m}(n-p-2)+\cdots+\alpha^{\frac{x}{k}+p-1}F_{p,m}(n-2p+1).
\end{aligned}$$

Upon simplifying, we get

$$\alpha^n = F_{p,m}(n-p+1)\alpha^p + \sum_{i=0}^{p-1}F_{p,m}(n-p-i)\alpha^i, \tag{6}$$

which is desired. □

3. $m-$extension of odd Fibonacci $p-$functions with period k

In this section, we present the $m-$extension of odd Fibonacci $p-$function with period k and analyze some properties of these functions.

Definition 3.1. *Let p be a nonnegative integer, m be a nonnegative real number and k be a positive integer. A function $f_{p,m} : \mathbb{R} \to \mathbb{R}$ is said to be $m-$extension of odd Fibonacci $p-$function with period k, if $f_{p,m}$ satisfies*

$$f_{p,m}(x + (p+1)k) = -mf_{p,m}(x + pk) + f_{p,m}(x), \quad \forall x \in \mathbb{R}. \tag{7}$$

Example 3.1. Let α be the positive real number that satisfies the equation $\alpha^{p+1} = m\alpha^p + 1$, k be a positive integer, p be a nonnegative integer. Therefore $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, for all $x \in \mathbb{R}$, is an $m-$extension of odd Fibonacci $p-$function with period k.

Proposition 3.1. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of odd Fibonacci $p-$function with period k. Assume that $f_{p,m}$ is s times differentiable. Then $\{f'_{p,m}, f''_{p,m}, \ldots, f^{(s)}_{p,m}\}$ are also $m-$extension of odd Fibonacci $p-$functions with period k.*

Proposition 3.2. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of odd Fibonacci $p-$function with period k. Define $g_t(x) = f_{p,m}(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, g_t is also an $m-$extension of odd Fibonacci $p-$function with period k.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} g_t(x + (p+1)k) &= f_{p,m}(x + (p+1)k + t) \\ &= -mf_{p,m}(x + pk + t) + f_{p,m}(x + t) \\ &= -mg_t(x + pk) + g_t(x). \end{aligned}$$

Therefore, $g_t(x)$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

4. Products of $m-$extension of Fibonacci $p-$functions with period k

In this section, we present the product of $m-$extension of Fibonacci $p-$functions with period k by using the concept of $f-$even and $f-$odd functions with period k which are defined in [16].

Definition 4.1 ([16])**.** *Let $k \in \mathbb{N}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be such that if $\varphi h \equiv 0$ where $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h \equiv 0$. The map φ is said to be an $f-$even and $f-$odd function with period k if $\varphi(x + k) = \varphi(x)$ and if $\varphi(x + k) = -\varphi(x)$, respectively, for any $x \in \mathbb{R}$.*

Theorem 4.1. *Let k be a positive integer, $\varphi : \mathbb{R} \to \mathbb{R}$ be an $f-$even function with period k and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be a continuous function.Then, $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k if and only if $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k.*

Proof. First, we assume that $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. For any $x \in \mathbb{R}$,

$$\begin{aligned}(\varphi f_{p,m})(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k)\\ &= \varphi(x+(p+1)k)\Big[mf_{p,m}(x+pk)+f_{p,m}(x)\Big]\\ &= m\varphi(x+pk)f_{p,m}(x+pk)+\varphi(x)f_{p,m}(x)\\ &= m(\varphi f_{p,m})(x+pk)+(\varphi f_{p,m})(x).\end{aligned}$$

Therefore, $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k. Next, assume that $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k, then

$$\begin{aligned}\varphi(x+k)f_{p,m}(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k)\\ &= (\varphi f_{p,m})(x+(p+1)k)\\ &= m(\varphi f_{p,m})(x+pk)+(\varphi f_{p,m})(x)\\ &= m\varphi(x+pk)f_{p,m}(x+pk)+\varphi(x)f_{p,m}(x)\\ &= \varphi(x+k)\Big[mf_{p,m}(x+pk)+f_{p,m}(x)\Big].\end{aligned}$$

Thus, $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. This completes the proof. □

Example 4.1. Let k be a positive integer and define $\gamma(x) = x - \lfloor x \rfloor$ which is an example of $f-$even function. Moreover, recall that the function $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, where α is positive real root of the characteristic equation $\alpha^{p+1} - m\alpha^p - 1 = 0$, is an $m-$extension of Fibonacci $p-$function with period k. By using Theorem 4.1, for all $x \in \mathbb{R}$

$$(\gamma f_{p,m})(x) = (x - \lfloor x \rfloor)\alpha^{\frac{x}{k}} \tag{8}$$

is an example of an $m-$extension of Fibonacci $p-$function with period k.

Theorem 4.2. *Let k be a positive integer, $\varphi : \mathbb{R} \to \mathbb{R}$ be an $f-$even function with period k and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k if and only if $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k.*

Proof. First, assume that $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k, for any $x \in \mathbb{R}$

$$\begin{aligned}(\varphi f_{p,m})(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k)\\ &= \varphi(x+(p+1)k)\left[-mf_{p,m}(x+pk)+f_{p,m}(x)\right]\\ &= -m\varphi(x+pk)f_{p,m}(x+pk)+\varphi(x)f_{p,m}(x)\\ &= -m(\varphi f_{p,m})(x+pk)+(\varphi f_{p,m})(x).\end{aligned}$$

Therefore, $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k, for any $x \in \mathbb{R}$, then

$$\begin{aligned}
\varphi(x+k)f_{p,m}(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k) \\
&= (\varphi f_{p,m})(x+(p+1)k) \\
&= -m(\varphi f_{p,m})(x+pk) + (\varphi f_{p,m})(x) \\
&= -m\varphi(x+pk)f_{p,m}(x+pk) + \varphi(x)f_{p,m}(x) \\
&= \varphi(x+k)\left[-mf_{p,m}(x+pk) + f_{p,m}(x)\right].
\end{aligned}$$

Thus, $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k. This completes the proof. □

Example 4.2. Let k be a positive integer and define $\gamma(x) = x - \lfloor x \rfloor$ which is an example of $f-$even function [16]. Moreover, recall that the function $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, where α is positive real root of the characteristic equation $\alpha^{p+1} + m\alpha^p - 1 = 0$, is an $m-$extension of odd Fibonacci $p-$function with period k. By using Theorem (4.2), for all $x \in \mathbb{R}$

$$(\gamma f_{p,m})(x) = (x - \lfloor x \rfloor)\alpha^{\frac{x}{k}} \tag{9}$$

is an example of an $m-$extension of odd Fibonacci $p-$function with period k.

Theorem 4.3. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that the following conditions are satisfied:

(C1) f_{p,m_1} is an $f-$even function,

(C2) f_{p,m_2} is an $f-$odd function,

(C3) if p is odd then $m_1 = m_2$,

(C4) if p is even then $m_1 = -m_2$,

(C5) $\mu = m_1.m_2$.

Then $(f_{p,m_1}f_{p,m_2})(x)$ is also an $m-$extension of Fibonacci $p-$function with period k.

Proof. Assume that f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k and the

conditions *(C1),(C2),(C3),(C4)* and *(C5)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= [m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x)]\\ &\quad [m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x)]\\ &= m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)+m_1 f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad +m_2 f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= \mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x), \quad \forall x\in\mathbb{R}.\end{aligned}$$

Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of Fibonacci $p-$function with period k. □

Theorem 4.4. *Let k be a positive integer, f_{p,m_1} be an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} be an $m-$extension of odd Fibonacci $p-$function with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x), \quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = -m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x), \quad \forall x\in\mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that (C6), (C9) and one the following conditions (C7) and (C8) are satisfied:

(C6) if p is odd or even then $m_1 = m_2$,

(C7) f_{p,m_1} and f_{p,m_2} are both $f-$even functions,

(C8) f_{p,m_1} and f_{p,m_2} are both $f-$odd functions,

(C9) $\mu = m_1m_2$

Then, $(f_{p,m_1}f_{p,m_2})$ is also an $m-$extension of odd Fibonacci $p-$function with period k.

Proof. First assume that f_{p,m_1} is an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} is an $m-$extension of odd Fibonacci $p-$function with period k and the conditions *(C6)*, *(C9)* and *(C7)* are

satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= -m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)+m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= -m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\end{aligned}$$

$\forall x\in\mathbb{R}$. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} is an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} is an $m-$extension of odd Fibonacci $p-$function with period k and the conditions *(C6)*, *(C9)* and *(C8)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= -m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad +m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= -m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\end{aligned}$$

$\forall x\in\mathbb{R}$. Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. This proves the theorem. □

Theorem 4.5. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k)=m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x),\quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k)=m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x),\quad \forall x\in\mathbb{R},$$

where m_1,m_2 are nonnegative real numbers. Suppose that (C10),(C11) and one the conditions (C7) and (C8) are satisfied:

(C10) if p is odd or even then $m_1=-m_2$,

(C11) $\mu = -m_1.m_2$.

Then, $(f_{p,m_1} f_{p,m_2})$ *is an* $m-$*extension of odd Fibonacci* $p-$*function with period* k.

Proof. First assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C7)*, *(C10)* and *(C11)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1} f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k) f_{p,m_2}(x+(p+1)k) \\
&= \Big[m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x)\Big]\Big[m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x)\Big] \\
&= m_1 m_2 f_{p,m_1}(x+pk) f_{p,m_2}(x+pk) \\
&\quad + f_{p,m_1}(x) f_{p,m_2}(x) + m_1 f_{p,m_1}(x+pk) f_{p,m_2}(x) \\
&\quad + m_2 f_{p,m_2}(x+pk) f_{p,m_1}(x) \\
&= m_1 m_2 f_{p,m_1}(x+pk) f_{p,m_2}(x+pk) \\
&\quad + f_{p,m_1}(x) f_{p,m_2}(x) \\
&= -\mu (f_{p,m_1} f_{p,m_2})(x+pk) + (f_{p,m_1} f_{p,m_2})(x),
\end{aligned}$$

$\forall x \in \mathbb{R}$. Therefore, $(f_{p,m_1} f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C8)*, *(C10)* and *(C11)* are satisfied. Then the same result can be obtained. Therefore, $(f_{p,m_1} f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

Theorem 4.6. *Let* k *be a positive integer,* f_{p,m_1} *and* f_{p,m_2} *be two* $m-$*extension of odd Fibonacci* $p-$*functions with period* k *satisfying*

$$\begin{aligned}
f_{p,m_1}(x+(p+1)k) &= -m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x), \quad \forall x \in \mathbb{R} \\
f_{p,m_2}(x+(p+1)k) &= -m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x), \quad \forall x \in \mathbb{R},
\end{aligned}$$

where m_1, m_2 *are nonnegative real numbers. Suppose that the conditions (C1),(C2),(C3),(C4) and (C5) are satisfied. Then* $(f_{p,m_1} f_{p,m_2})(x)$ *is an* $m-$*extension of Fibonacci* $p-$*function with period* k.

Proof. Assume that f_{p,m_1} and f_{p,m_2} be two $m-$extension of odd Fibonacci $p-$functions with period k and

the conditions *(C1),(C2),(C3),(C4)* and *(C5)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= \mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\quad \forall x\in\mathbb{R}.\end{aligned}$$

Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of Fibonacci $p-$function with period k. □

Theorem 4.7. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of odd Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k)=-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x),\quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k)=-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x),\quad \forall x\in\mathbb{R},$$

where m_1,m_2 are nonnegative real numbers. Suppose that (C10),(C11) and one the conditions (C7) and (C8) are satisfied. Then, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k.

Proof. First assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C7)*, *(C10)* and *(C11)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\end{aligned}$$

$\forall x\in\mathbb{R}$. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C8)*, *(C10)* and *(C11)* are satisfied. Then the same result can be obtained. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

5. Quotients of $m-$extension of Fibonacci $p-$functions with period k

In this section, we discuss the limit of quotients of $m-$extension of Fibonacci $p-$functions with period k.

Theorem 5.1. *If $f_{p,m}$ is an $m-$extension of Fibonacci $p-$functions with period k, then the limit of the quotient $\frac{f_{p,m}(x+k)}{f_{p,m}(x)}$ exists.*

Proof. Let $k \in \mathbb{N}$, $m \in \mathbb{R}^+$, p be a nonnegative integer and $n \geq 2p$. Consider the quotient $Q(x) = \frac{f_{p,m}(x+k)}{f_{p,m}(x)}$, where $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. We have two possibilities such that either $Q(x) < 0$ or $Q(x) > 0$. First, suppose that $Q(x) < 0$ then without loss of generality, $f_{p,m}(x) > 0$ and $f_{p,m}(x+k) < 0$. Therefore,

$$\begin{aligned}
f_{p,m}(x+2pk) &= mf_{p,m}(x+(2p-1)k) + f_{p,m}(x+(p-1)k) \\
&= m^2 f_{p,m}(x+(2p-2)k) + f_{p,m}(x+(p-1)k) \\
&\quad + mf_{p,m}(x+(p-2)k) \\
&= m^3 f_{p,m}(x+(2p-3)k) + f_{p,m}(x+(p-1)k) \\
&\quad + mf_{p,m}(x+(p-2)k) + m^2 f_{p,m}(x+(p-3)k) \\
&\vdots \\
&= m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-2} f_{p,m}(x+k) + m^{p-1} f_{p,m}(x) \\
&= F_{p,m}(p+1) f_{p,m}(x+pk) + F_{p,m}(1) f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - F_{p,m}(p-1) f_{p,m}(x+k) + F_{p,m}(p) f_{p,m}(x),
\end{aligned}$$

$$\begin{aligned}
f_{p,m}(x+(2p+1)k) &= mf_{p,m}(x+2pk) + f_{p,m}(x+pk) \\
&= m\Big[m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-2} f_{p,m}(x+k) + m^{p-1} f_{p,m}(x)\Big] + f(x+pk) \\
&= (m^{p+1}+1) f_{p,m}(x+pk) + mf_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-1} f_{p,m}(x+k) + m^p f_{p,m}(x) \\
&= F_{p,m}(p+2) f_{p,m}(x+pk) + F_{p,m}(2) f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - F_{p,m}(p) f_{p,m}(x+k) + F_{p,m}(p+1) f_{p,m}(x).
\end{aligned}$$

$$\begin{aligned} f_{p,m}(x+(2p+2)k) &= mf_{p,m}(x+(2p+1)k)+f_{p,m}(x+(p+1)k) \\ &= m\Big[(m^{p+1}+1)f_{p,m}(x+pk)+mf_{p,m}(x+(p-1)k) \\ &\quad +\cdots-m^{p-1}f_{p,m}(x+k)+m^{p}f_{p,m}(x)\Big] \\ &\quad +mf(x+pk)+f(x) \\ &= (m^{p+2}+2m)f_{p,m}(x+pk)+m^{2}f_{p,m}(x+(p-1)k) \\ &\quad +\cdots-m^{p}f_{p,m}(x+k)+(m^{p+1}+1)f_{p,m}(x) \\ &= F_{p,m}(p+3)f_{p,m}(x+pk)+F_{p,m}(3)f_{p,m}(x+(p-1)k) \\ &\quad +\cdots-F_{p,m}(p+1)f_{p,m}(x+k)+F_{p,m}(p+2)f_{p,m}(x). \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} f_{p,m}(x+nk) &= F_{p,m}(n-p+1)f_{p,m}(x+pk) \\ &= +F_{p,m}(n-2p+1)f_{p,m}(x+(p-1)k) \\ &= +\cdots-F_{p,m}(n-p-1)f_{p,m}(x+k)+F_{p,m}(n-p)f_{p,m}(x) \end{aligned}$$

and

$$\begin{aligned} f_{p,m}(x+(n+1)k) &= F_{p,m}(n-p+2)f_{p,m}(x+pk) \\ &= +F_{p,m}(n-2p+2)f_{p,m}(x+(p-1)k) \\ &= +\cdots-F_{p,m}(n-p)f_{p,m}(x+k)+F_{p,m}(n-p+1)f_{p,m}(x), \end{aligned}$$

where $F_{p,m}$ is an $m-$extension of Fibonacci $p-$sequence with the initial conditions, $F_{p,m}(0)=0$, $F_{p,m}(1)=1$, $F_{p,m}(2)=m$, ..., $F_{p,m}(p)=m^{p-1}$. Given $x'\in\mathbb{R}$, there exists $x\in\mathbb{R}$ such that $x'=x+nk$. Therefore,

$$\begin{aligned} \frac{f_{p,m}(x'+k)}{f_{p,m}(x')} &= \frac{f_{p,m}(x+(n+1)k)}{f_{p,m}(x+nk)} \\ &= \left(\begin{array}{l} F_{p,m}(n-p+2)f_{p,m}(x+pk)+\cdots-F_{p,m}(n-p)f_{p,m}(x+k)+ \\ F_{p,m}(n-p+1)f_{p,m}(x) \\ \hline F_{p,m}(n-p+1)f_{p,m}(x+pk)+\cdots-F_{p,m}(n-p-1)f_{p,m}(x+k)+ \\ F_{p,m}(n-p)f_{p,m}(x) \end{array}\right) \\ &= \left(\begin{array}{l} F_{p,m}(n-p+2)\Big[f_{p,m}(x+pk)+\cdots-\dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+ \\ \dfrac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)\Big] \\ \hline F_{p,m}(n-p+1)\Big[f_{p,m}(x+pk)+\cdots-\dfrac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+ \\ \dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)\Big] \end{array}\right). \end{aligned}$$

$$\lim_{n\to\infty}\left(\frac{f_{p,m}(x'+k)}{f_{p,m}(x')}\right) = \lim_{n\to\infty}\left(\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\begin{array}{l} f_{p,m}(x+pk)+\cdots-\dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+ \\ \dfrac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x) \\ \hline f_{p,m}(x+pk)+\cdots-\dfrac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+ \\ \dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x) \end{array}\right)$$

$$= \left(\lim_{n\to\infty}\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\begin{array}{l} f_{p,m}(x+pk)+\cdots-\lim\limits_{n\to\infty}\dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+ \\ \lim\limits_{n\to\infty}\dfrac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x) \\ \hline f_{p,m}(x+pk)+\cdots-\dfrac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+ \\ \dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x) \end{array}\right).$$

Let $N=n+1$. If $n\to\infty$ then $N\to\infty$. So, we can write the above expression as

$$\left(\lim_{n\to\infty}\frac{f_{p,m}(x'+k)}{f_{p,m}(x')}\right) = \lim_{n\to\infty}\left(\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\begin{array}{l} f_{p,m}(x+pk)+\cdots-\lim\limits_{N\to\infty}\dfrac{F_{p,m}(N-p-1)}{F_{p,m}(N-p+1)}f_{p,m}(x+k)+ \\ \lim\limits_{N\to\infty}\dfrac{F_{p,m}(N-p)}{F_{p,m}(N-p+1)}f_{p,m}(x) \\ \hline f_{p,m}(x+pk)+\cdots-\lim\limits_{n\to\infty}\dfrac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+ \\ \lim\limits_{n\to\infty}\dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x) \end{array}\right)$$

$$= \alpha_m\left(\begin{array}{l} f_{p,m}(x+pk)+\cdots-\lim\limits_{N\to\infty}\dfrac{F_{p,m}(N-p-1)}{F_{p,m}(N-p+1)}f_{p,m}(x+k)+ \\ \lim\limits_{N\to\infty}\dfrac{F_{p,m}(N-p)}{F_{p,m}(N-p+1)}f_{p,m}(x) \\ \hline f_{p,m}(x+pk)+\cdots-\lim\limits_{n\to\infty}\dfrac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+ \\ \lim\limits_{n\to\infty}\dfrac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x) \end{array}\right) = \alpha_m.$$

Here α_m is the unique positive real root of the characteristic equation of $m-$extension of Fibonacci $p-$sequence. Next, suppose that $Q(x)>0$, without loss of generality we assume $f_{p,m}(x)>0$, $f_{p,m}(x+k)>0$. Identically, we can easily obtain that $\lim_{n\to\infty}\left(\frac{f_{p,m}(x+(n+1)k)}{f_{p,m}(x+nk)}\right)=\alpha_m$. Hence we omit the proof. □

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FOURIER SPECTRAL METHODS FOR STOCHASTIC SPACE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY SPECIAL ADDITIVE NOISES

FANG LIU, MONZORUL KHAN AND YUBIN YAN *

Abstract. Fourier spectral methods for solving stochastic space fractional partial differential equations driven by special additive noises in one-dimensional case are introduced and analyzed. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by some appropriate approximations in the space direction. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods. For the linear problem, we obtain the precise error estimates in the L_2 norm and find the relations between the error bounds and the fractional powers. For the nonlinear problem, we introduce the numerical algorithms and MATLAB codes based on the FFT transforms. Our numerical algorithms can be adapted easily to solve other stochastic space fractional partial differential equations with multiplicative noises. Numerical examples for the semilinear stochastic space fractional partial differential equations are given.

Key words. Space fractional partial differential equations, stochastic partial differential equations, Fourier spectral method, error estimates

AMS subject classifications. 65M12; 65M06; Secondary 65M70;35S10

1. Introduction. Fourier spectral methods for solving the following stochastic space fractional partial differential equation are considered in this work, with $1/2 < \alpha \leq 1$,

$$\frac{du(t)}{dt} + A^{\alpha}u(t) = f(u(t)) + \frac{dW(t)}{dt}, \quad 0 < t < T, \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

Here A is an unbounded positive self-adjoint operator, u_0 is an initial value and $f(u)$ is a nonlinear term. The space-time white noise $W(t)$ will be defined below.

Let H be a separable Hilbert space and $\|\cdot\|$, $(\cdot,\cdot)$ denote the norm and inner product in H. Let $A : \mathcal{D}(A) \subset H \to H$ be a positive definite self-adjoint operator such that A^{-1} is compact on H. From this we infer the existence of a complete orthonormal basis $\{e_k\}_{k\geq 0}$ for H of eigenfunctions of A such that the associated sequence of eigenvalues $\{\lambda_k\}$ form an increasing unbounded sequence.

Using the basis $\{e_k\}$ we may also define the fractional powers of A. Given $1/2 < \alpha \leq 1$ define

$$H^{2\alpha} := \mathcal{D}(A^{\alpha}) = \{v \in H : \sum_k \lambda_k^{2\alpha}|(v, e_k)|^2 < \infty\},$$

and

$$A^{\alpha}v := \sum_k \lambda_k^{\alpha}(v, e_k)e_k, \ v \in \mathcal{D}(A^{\alpha}), \tag{1.3}$$

*Fang Liu: Department of Mathematics, Lvliang University, Lishi, P.R. China, 033000, (13935870281@163.com), Monzorul Khan: Department of Mathematics, University of Chester, Thornton Science Park, Pool Lane, Ince, CH2 4NU, UK, (sohel_ban@yahoo.com), Yubin Yan: Department of Mathematics, University of Chester, Thornton Science Park, Pool Lane, Ince, CH2 4NU, UK, (y.yan@chester.ac.uk). Dr. Yubin Yan is the corresponding author.

with the associated Hilbert norm defined by

$$\|A^{\alpha}v\|^2 = \sum_k \lambda_k^{2\alpha}|(v,e_k)|^2.$$

The special space-time noise considered in this work is

$$\frac{dW(t)}{dt} = \sum_{k=1}^{\infty} \sigma_k(t)\dot{\beta}_k(t)e_k, \tag{1.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k = 1, 2, \ldots$ is the derivative of the standard Brownian motions $\beta_k(t)$, $k = 1, 2, \ldots$ and $\sigma_k(t), k = 1, 2, \ldots$ are some appropriate functions of t. In particular, when $\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0$, the noise (1.4) reduces to

$$\frac{dW(t)}{dt} = \sum_{k=1}^{\infty} \bar{\gamma}_k^{1/2}\dot{\beta}_k(t)e_k,$$

which is a so-called H-valued Wiener process with the covariance operator Q and the linear operator $Q : H \to H$ is a trace class operator, that is $\mathrm{Tr}(Q) = \sum_{k=1}^{\infty} \bar{\gamma}_k < \infty$ where $Qe_k = \bar{\gamma}_k e_k, k = 1, 2, \ldots$.

Let us here give two possible operators in (1.1)-(1.2). One is $A = -\Delta$ with the homogeneous Dirichlet boundary condition, $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$, where $\Delta = \partial^2/\partial x^2$ denotes the Laplacian. In this case, A has the eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $e_k = \sqrt{2}\sin k\pi x, k = 1, 2, \ldots$. Our error estimates in this work are based on these eigenvalues and eigenfunctions. Another one is $A = I - \Delta$ with periodic boundary conditions, $\mathcal{D}(A) = H_{per}^2(-\pi, \pi)$. Here $H_{per}^2(-\pi, \pi)$ denotes the completion with respect to the $H^2(-\pi, \pi)$ norm of the set of $u \in C^{\infty}([-\pi, \pi])$ such that the pth derivative $u^{(p)}(-\pi) = u^{(p)}(\pi)$ for $p = 0, 1, \ldots$. It is a Hilbert space with the $H^2(-\pi, \pi)$ inner product, see [24, Definition 1.47]. In this case, A has the eigenvalues $\lambda_1 = 1, \lambda_{2k} = 1 + k^2, \lambda_{2k+1} = 1 + k^2$ and eigenfunctions $e_1(x) = \frac{1}{\sqrt{2\pi}}, e_{2k}(x) = \frac{1}{\sqrt{\pi}}\sin kx, e_{2k+1}(x) = \frac{1}{\sqrt{\pi}}\cos kx, k = 1, 2, \ldots$, see [24, Example 1.84].

We obtain the detailed error estimates, i.e., Theorems 2.1, 3.1, 3.3 below for the linear stochastic space fractional partial differential equation subject to the Dirichlet boundary conditions. More precisely, we shall consider the error estimates for the following linear problem, with $1/2 < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^{\alpha}u(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad 0 < t < T,\ 0 < x < 1, \tag{1.5}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{1.6}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1. \tag{1.7}$$

Here the space-time noise $\frac{\partial^2 W(t,x)}{\partial t \partial x} = \frac{dW(t)}{dt}$ is define by (1.4).

For the linear stochastic space fractional partial differential equation subject to the periodic boundary conditions, we may obtain the similar error estimates as in Theorems 2.1, 3.1, 3.3. For the length of the paper, we will not give the detailed proofs for the error estimates in this case. However, in the numerical examples in Section 4, we shall consider the spectral method for the semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions to illustrate the experimentally determined convergence orders.

The stochastic partial differential equations driven by the white noise (the covariance operator $Q = I$) often have poor regularity estimates. In the physical world, to take into account the short and long range correlations of the stochastic effects, both white noise and colored noises may be considered. There are many situations where colored noises model the reality more closely, and there are also instances where the important stochastic effects are the noises acting on a few selected frequencies. For example one may choose $\sigma_k(t) = \frac{\cos t}{k^3}$. [12]

Space-fractional partial differential equations are widely used to model complex phenomena, for example, quasi-geostrophic flows, fast rotating fluids, dynamic of the frontogenesis in meteorology, diffusion in fractal or disordered medium, pollution problems, mathematical finance and the transport problems, see, e.g., [3], [7], [21], [36], [2].

Let us here consider two examples which apply the fractional Laplacian in the physical models. The first example is about the surface quasi-geostrophic (SQG) equation,

$$\partial_t\theta + \vec{u}\cdot\nabla\theta + \kappa(-\Delta)^\alpha\theta = 0,$$

where $\kappa \geq 0$ and $\alpha > 0$, $\theta = \theta(x_1, x_2, t)$ denotes the potential temperature, $\vec{u} = (u_1, u_2)$ is the velocity field determined by θ. When $\kappa > 0$, the SQG equation takes into account the dissipation generated by a fractional Laplacian. The SQG equation with $\kappa > 0$ and $\alpha = 1/2$ arises in geophysical studies of strongly rotating fluids. For the dissipative SQG equation, $\alpha = 1/2$ appears to be a critical index. In the subcritical case when $\alpha > 1/2$, the dissipation is sufficient to control the nonlinearity and the global regularity is a consequence of global a priori bound. In the critical case when $\alpha = 1/2$, the global regularity issue is more delicate. The mystery in the supercritical case $\alpha < 1/2$ is only partially uncovered at the moment. [9]

The second example is about the wave propagation in complex solids, especially viscoelastic materials (for example Polymers).[4]. In this case, the relaxation function has the form $k(t) = ct^{-\nu}, 0 < \nu < 1,\ c \in \mathbb{R}$, instead of the exponential form known in the standard models. This polynomial relaxation is due to the non uniformity of the material. The far field is then described by a Burgers equation with the leading operator $(-\Delta)^{\frac{1+\nu}{2}}$ instead of the Laplacian

$$\partial_t u = -(-\Delta)^{\frac{1+\nu}{2}}u + \partial_x(u^2).$$

This equation also describes the far-field evolution of acoustic waves propagating in a gas-filled tube with a boundary layer.

Frequently, the initial value or the coefficients of the equation are random, therefore it is natural to consider the stochastic space-fractional partial differential equations. The existence, uniqueness and regularities of the solutions of stochastic space-fractional partial differential equations have been extensively studied, see, e.g., [3], [7], [10], [28]. In this work, we will focus on the case $1/2 < \alpha \leq 1$ since the existence and uniqueness and regularity of the solution in this case is well understood in literature, see [11, Theorem 1.3]. However the numerical methods for solving space-fractional stochastic partial differential equations are quite restricted even for the case $1/2 < \alpha \leq 1$. Debbi and Dozzi [11] introduced a discretization of the fractional Laplacian and used it to elaborate an approximation scheme for fractional heat equation perturbed by a multiplicative cylindrical white noise. As far as we know [11] is the only existing paper in the literature of dealing with the numerical approach of this

kind of problems. In this work, we will use the ideas developed in [1] and [12], to consider the numerical methods for solving stochastic space fractional partial differential equations, see also [19], [8], [20]. We first approximate the noise by using piecewise constant functions and then obtain the approximate solution $\hat{u}(t)$ of the exact solution $u(t)$. Finally we provide error estimates in L^2-norm for $u(t) - \hat{u}(t)$.

For the deterministic space fractional partial differential equations, many numerical methods are available in literature. There are two approaches to define the fractional Laplacian. One approach is by using the eigenvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the boundary conditions as in (1.3). Another approach is by using the left-handed and right-handed Riemann-Liouville fractional derivatives. For the deterministic space fractional partial differential equations defined by the Riemann-Liouville fractional derivatives, many numerical methods are available, e.g., finite difference methods [14]-[15], [26], [31]-[32], finite element methods [13], [18] and the spectral methods [22]-[23]. For the deterministic space fractional partial differential equations defined by (1.3), some numerical methods are also available, see, e.g., matric transfer technique (MTT) [14], [15], [6], Fourier spectral method [5]. In this work, we will use Fourier spectral method to solve the stochastic space fractional partial differential equations. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem.

Let $N_t \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_t} = T$ be the time partition of $[0, T]$ and Δt the time step size. To find the approximate solution of (1.5)-(1.7), we approximate the noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ by the piecewise constant functions in the time direction defined by, with $l = 1, 2, ..., N_t$, [12]

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \sigma_k^M(t) e_k(x) \Big(\sum_{l=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,l} \chi_l(t) \Big), \tag{1.8}$$

where

$$\eta_{k,l} = \frac{1}{\sqrt{\Delta t}} \int_{t_{l-1}}^{t_l} d\,\beta_k(t) = \frac{1}{\sqrt{\Delta t}} \Big(\beta_k(t_l) - \beta_k(t_{l-1}) \Big) \in \mathcal{N}(0,1), \tag{1.9}$$

and

$$\chi_l(t) = \begin{cases} 1, & t \in [t_{l-1}, t_l],\ l = 1, 2, \ldots, N_t, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\sigma_k^M(t)$ is the approximation of $\sigma_k(t)$ in the space direction. For example, we can choose, with some positive integer $M > 0$,

$$\sigma_k(t) = \frac{\cos t}{k^3}, \qquad \sigma_k^M(t) = \begin{cases} \sigma_k(t), & k \le M, \\ 0, & k > M. \end{cases}$$

More precisely, replacing $\sigma_k(t)$ by $\sigma_k^M(t)$, we get the noise approximation in space, and replacing $\dot{\beta}_k(t)$ by $\sum_{j=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,j} \chi_j(t)$, we get the noise approximation in time.

Substituting $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ with $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ in (1.5)-(1.7), we get

$$\frac{\partial \hat{u}(t,x)}{\partial t} + (-\Delta)^{\alpha}\hat{u}(t,x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}, \quad 0 < t < T,\ 0 < x < 1, \tag{1.10}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \quad 0 < t < T, \tag{1.11}$$

$$\hat{u}(0,x) = u_0(x), \quad 0 < x < 1. \tag{1.12}$$

Note that $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ now is a function in $L^2((0,T)\times(0,1))$ and therefore we can solve (1.10)-(1.12) by using any numerical methods for deterministic space fractional partial differential equations. Assume that $\{\sigma_k(t)\}$ and its derivative are uniformly bounded, [12]

$$|\sigma_k(t)| \le \beta_k, \quad |\sigma_k'(t)| \le \gamma_k, \quad \forall\, t \in [0,T], \tag{1.13}$$

and the coefficients $\{\sigma_k^M\}$ are constructed such that

$$|\sigma_k(t) - \sigma_k^M(t)| \le \alpha_k^M, \quad |\sigma_k^M(t)| \le \beta_k^M, \quad |(\sigma_k^M)'(t)| \le \gamma_k^M, \quad \forall\, t \in [0,T], \tag{1.14}$$

with positive sequences $\{\alpha_k^M\}$ being arbitrarily chosen, $\{\beta_k^M\}$ and $\{\gamma_k^M\}$ being related to $\{\beta_k\}$ and $\{\gamma_k\}$. Further we assume that

$$\beta_k^M \le k^{-\tilde{\alpha}}, \quad \text{for some } 0 \le \tilde{\alpha} < 1/2. \tag{1.15}$$

Let $\mathbb{E}$ denote the expectation, in Theorem 2.1, we prove that, with $1/2 < \alpha \le 1$ and $0 \le \tilde{\alpha} < 1/2$,

$$\begin{aligned} &\mathbb{E}\int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2\, dxdt \\ &\le C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^{\alpha}} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^{\alpha}\beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big). \end{aligned} \tag{1.16}$$

Let $J \in \mathbb{N}$, we denote

$$S_J = \text{span}\{e_1, e_2, \ldots, e_J\},$$

and define by $P_J : H \to S_J$ the projection from H to S_J,

$$P_J v = \sum_{j=1}^{J} (v, e_j) e_j. \tag{1.17}$$

The Fourier spectral method of (1.10)-(1.12) is to find $\hat{u}_J(t) \in S_J$ such that, with $\hat{g}(t,x) := \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$.

$$\frac{\partial \hat{u}_J(t,x)}{\partial t} + (-\Delta)^{\alpha}\hat{u}_J(t,x) = P_J\hat{g}(t,x), \quad 0 < t < T,\ 0 < x < 1, \tag{1.18}$$

$$\hat{u}_J(t,0) = \hat{u}_J(t,1) = 0, \quad 0 < t < T, \tag{1.19}$$

$$\hat{u}_J(0,x) = P_J u_0(x), \quad 0 < x < 1, \tag{1.20}$$

In Theorem 3.1, we prove that, with $1/2 < \alpha \le 1$ and $0 \le \tilde{\alpha} < 1/2$,

$$\|\hat{u}(t) - \hat{u}_J(t)\|^2 \le C\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}} \int_0^t \|\hat{g}(s)\|^2\, ds. \tag{1.21}$$

Combining Theorem 2.1 with Theorem 3.1, we have, with $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \leq 1$,

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2\, dxdt \\
&\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big) \\
&+ C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^{\infty} \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^{\infty} (\beta_k^M)^2\Big).
\end{aligned}$$

The paper is organized as follows. In Section 2, we consider the approximation of noise. In Section 3, we introduce the Fourier spectral methods for solving the approximated space fractional partial differential equations and the error estimates for the linear stochastic space fractional partial differential equations are proved. In Section 4, we consider the numerical examples for solving the semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions. From now on we denote by C a generic constant, which may not be the same at different occurrences.

2. Approximate the noise and regularity. It is well known that the mild solution of (1.5)-(1.7) has the following form

$$u(t,x) = \int_0^1 G_\alpha(t,x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, dW(s,y), \tag{2.1}$$

where

$$G_\alpha(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j^\alpha t} e_j(x) e_j(y),$$

and the stochastic integral $\int_0^t \int_0^1 G_\alpha(t-s,x,y)\, dW(s,y)$ is well-defined. The existence and uniqueness of the solutions of (1.5)-(1.7) are discussed in, e.g., [10], [11], [28] and the references cited therein.

Similarly the mild solution of (1.10)-(1.12) has the form of, see, e.g., [12]

$$\hat{u}(t,x) = \int_0^1 G_\alpha(t,x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, d\widehat{W}(s,y), \tag{2.2}$$

THEOREM 2.1. *Let u and $\hat{u}$ be the solutions of* (1.5)-(1.7) *and* (1.10)-(1.12)*, respectively. Assume that the assumptions* (1.13)-(1.15) *hold. Then we have*

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2\, dxdt \\
&\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big).
\end{aligned} \tag{2.3}$$

Proof. See the Appendix. □

REMARK 2.2. *When $\alpha = 1$, Theorem 2.1 should reduce to the Theorem 3.3 in [12]. However one term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$ of the bounds in (3.20) in Theorem 3.3 [12] is missing. The term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$ comes from the estimates II_1 and*

II_3 of the estimate for $II = \mathbb{E}\int_0^T \int_0^1 F_2(t,x)\,dxdt$ in (4.12). The authors in [12] only considered the estimate II_2 and neglected the terms II_1 and II_3 which would produce the term $\Delta t^{1/2+\tilde{\alpha}}, 0 \le \tilde{\alpha} < 1/2$. (See the estimates for the term II in [12, p.1441]). In Theorem 2.1, we include the terms $O(\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}})$.

THEOREM 2.3. *Let $\hat{u}$ be the solution of (1.10)-(1.12). Assume that the assumptions (1.13)-(1.15) hold. Further assume that $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \le 1$ and $\mathbb{E}\|A^\alpha u_0\|^2 < \infty$. Then*

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big|\frac{\partial \hat{u}(t,x)}{\partial t}\Big|^2 dxdt \le C\Big(\Delta t\mathbb{E}\|A^\alpha u_0\|^2 + \Delta t\sum_{k=1}^{\infty}\lambda_k^\alpha(\beta_k^M)^2 + \sum_{k=1}^{\infty}(\beta_k^M)^2\Big), \tag{2.4}$$

and

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \big|A^\alpha \hat{u}(t,x)\big|^2 dxdt \le C\Big(\Delta t\mathbb{E}\|A^\alpha u_0\|^2 + \Delta t\sum_{k=1}^{\infty}\lambda_k^\alpha(\beta_k^M)^2\Big). \tag{2.5}$$

Proof. Assume that, with $0 < t \le t_{j+1}$,

$$\hat{u}(t,x) = \sum_{k=1}^{\infty}\hat{u}_k(t)e_k(x), \tag{2.6}$$

and, with $\hat{u}_k(0) = (u_0, e_k)$, $k = 1, 2, \ldots,$

$$\hat{u}(0,x) = u_0(x) = \sum_{k=1}^{\infty}\hat{u}_k(0)e_k(x).$$

Substituting (2.6) into (1.10), we get, with $0 < t \le t_{j+1}$,

$$\frac{d\hat{u}_k(t)}{dt} + \lambda_k^\alpha \hat{u}_k(t) = \sigma_k^M(t)\Big(\sum_{l=1}^{j+1}\frac{1}{\sqrt{\Delta t}}\eta_{k,l}\chi_l(t)\Big), \tag{2.7}$$

which implies that, with $0 < t \le t_{j+1}$,

$$\hat{u}_k(t) = e^{-\lambda_k^\alpha t}\hat{u}_k(0) + \int_0^t e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)\Big(\sum_{l=1}^{j+1}\frac{1}{\sqrt{\Delta t}}\eta_{k,l}\chi_l(s)\Big)\,ds. \tag{2.8}$$

Let us first show (2.4). Note that $\{e_k\}$ is an orthonormal basis in $H = L^2(0,1)$, we have, by (2.7),

$$\begin{aligned}
&\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big|\frac{\partial \hat{u}(t,x)}{\partial t}\Big|^2 dxdt = \mathbb{E}\sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}\Big|\frac{d\hat{u}_k(t)}{dt}\Big|^2 dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\Big(\int_{t_j}^{t_{j+1}}|\lambda_k^\alpha\hat{u}_k(t)|^2\,dt + \int_{t_j}^{t_{j+1}}\Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\sum_{l=1}^{j+1}\eta_{k,l}\chi_l(t)\Big|^2 dt\Big)\\
&= 2\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}|\hat{u}_k(t)|^2\,dt + 2\mathbb{E}\sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}\Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\eta_{k,j+1}\chi_{j+1}(t)\Big|^2 dt\\
&= 2(I + II).
\end{aligned}$$

For I, we have, by (2.8), with $t_l^* = t_l, 1 \le l \le j$ and $t_l^* = t,\ l = j+1$,

(2.9)

$$
\begin{aligned}
I &\le 2\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\left|e^{-\lambda_k^\alpha t}\hat{u}_k(0)\right|^2 dt + 2\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\Big|\sum_{l=1}^{j+1}\frac{\eta_{k,l}}{\sqrt{\Delta t}}\int_{t_{l-1}}^{t_l^*}e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)\,ds\Big|^2 dt\\
&= 2\mathbb{E}\sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}e^{-2\lambda_k^\alpha t}\big(A^\alpha u_0,e_k\big)^2 dt + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j+1}\frac{1}{\Delta t}\Big(\int_{t_{l-1}}^{t_l^*}e^{-\lambda_k^\alpha(t-s)}\sigma_k^M(s)\,ds\Big)^2 dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\big(A^\alpha u_0,e_k\big)^2\Delta t + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j+1}\frac{1}{\Delta t}\Big(\int_{t_{l-1}}^{t_l^*}e^{-2\lambda_k^\alpha(t-s)}\big(\sigma_k^M(s)\big)^2 ds\Big)\Big(\int_{t_{l-1}}^{t_l^*}1^2\,ds\Big)\,dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\big(A^\alpha u_0,e_k\big)^2\Delta t + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}\Big(\int_0^t e^{-2\lambda_k^\alpha(t-s)}\big(\sigma_k^M(s)\big)^2 ds\Big)\,dt\\
&\le 2\mathbb{E}\sum_{k=1}^{\infty}\big(A^\alpha u_0,e_k\big)^2\Delta t + 2\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\big(\beta_k^M\big)^2\int_{t_j}^{t_{j+1}}\frac{1-e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha}\,dt\\
&\le 2\mathbb{E}\|A^\alpha u_0\|^2\Delta t + \Delta t\sum_{k=1}^{\infty}\lambda_k^\alpha\big(\beta_k^M\big)^2,
\end{aligned}
$$

where in the last inequality, we use the fact $1 - e^{-2\lambda_k^\alpha t} \le 1$.

For II, we have

$$
II = \mathbb{E}\sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}\Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\eta_{k,j+1}\chi_{j+1}(t)\Big|^2 dt = \sum_{k=1}^{\infty}\int_{t_j}^{t_{j+1}}\Big(\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\Big)^2 dt \le \sum_{k=1}^{\infty}\big(\beta_k^M\big)^2.
$$

Combining I with II we get (2.4). Similarly we have,

$$
\begin{aligned}
&\mathbb{E}\int_{t_j}^{t_{j+1}}\int_0^1|A^\alpha\hat{u}(t)|^2\,dxdt = \mathbb{E}\int_{t_j}^{t_{j+1}}\|A^\alpha\hat{u}(t,x)\|^2\,dt\\
&= \mathbb{E}\int_{t_j}^{t_{j+1}}\Big(\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\hat{u}_k^2(t)\Big)\,dt = \mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\int_{t_j}^{t_{j+1}}|\hat{u}_k(t)|^2\,dt = I,
\end{aligned}
$$

which implies (2.5) also holds. Together these estimates complete the proof of Theorem 2.3.

□

3. Fourier spectral method. Denote $E_\alpha(t) = e^{-tA^\alpha}, 1/2 < \alpha \le 1$, where A^α is defined by (1.3). The mild solution of (1.10)-(1.12) has the form of, with $\hat{g}(t) = \frac{\partial^2\widehat{W}(t,x)}{\partial t\partial x}$,

$$
\hat{u}(t) = E_\alpha(t)\hat{u}_0 + \int_0^t E_\alpha(t-s)\hat{g}(s)\,ds, \quad \hat{u}(0) = u_0. \tag{3.1}
$$

Similarly the solution of (1.18)-(1.20) has the form of

$$
\hat{u}_J(t) = E_\alpha(t)P_J\hat{u}_0 + \int_0^t E_\alpha(t-s)P_J\hat{g}(s)\,ds, \quad \hat{u}(0) = P_J u_0. \tag{3.2}
$$

THEOREM 3.1. *Assume that $\hat{u}$ and $\hat{u}_J$ are the solutions of* (1.10)-(1.12) *and* (1.18)-(1.20)*, respectively. Let $0 \le r < 1/2$ and let $u_0 \in H$. Then we have*

$$\big\|A^{r/2}\big(\hat{u}(t)-\hat{u}_J(t)\big)\big\|^2 \le C\big\|u_0-P_Ju_0\big\|^2 + C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}\int_0^t \|\hat{g}(s)\|^2\,ds. \tag{3.3}$$

In particular, with $r=0$,

$$\|\hat{u}(t)-\hat{u}_J(t)\|^2 \le C\|u_0-P_Ju_0\|^2 + C\frac{1}{(J+1)^{2\alpha}}\int_0^t \|\hat{g}(s)\|^2\,ds. \tag{3.4}$$

To prove Theorem 3.1, we need the following smoothing property for the solution operator $E_\alpha(t)$.

LEMMA 3.2.

1. *Let $s>0$. We have, with $1/2<\alpha\le 1$,*

$$\|A^sE_\alpha(t)\| \le Ct^{-\frac{s}{\alpha}}e^{-\delta t},\ t>0,$$

for some constants $C=C(s,\alpha)>0$ and $\delta=\delta(\alpha)>0$.

2. *Let $P_J: H\to S_J$ be defined by* (1.17)*. We have*

$$\|E_\alpha(t)(I-P_J)v\| \le e^{-t\lambda_{J+1}^\alpha}\|v\|,\ t>0.$$

Proof. Recall that A is positive definite and A has the eigenvalues $0<\lambda_1<\lambda_2<\lambda_3<\dots$. For any function $h(\cdot)$, we have

$$\|h(A)\| = \sup_{\lambda\in\sigma(A)}|h(\lambda)|,$$

where $\sigma(A)$ denotes the set of eigenvalues of A. Thus, with $\delta=\frac{1}{2}\lambda_1^\alpha$,

$$\begin{aligned}
&\|A^sE_\alpha(t)\| = \|A^sE_\alpha(t/2)E_\alpha(t/2)\| \le \|A^sE_\alpha(t/2)\|\|E_\alpha(t/2)\|\\
&= \sup_{\lambda\in\sigma(A)}\big(\lambda^se^{-\frac{t}{2}\lambda^\alpha}\big)\cdot\sup_{\lambda\in\sigma(A)}\big(e^{-\frac{t}{2}\lambda^\alpha}\big) = \sup_{\lambda\in\sigma(A)}\Big(\frac{\big(\frac{t}{2}\lambda^\alpha\big)^{s/\alpha}}{e^{\frac{t}{2}\lambda^\alpha}}\Big(\frac{t}{2}\Big)^{-s/\alpha}\Big)e^{-\frac{t}{2}\lambda_1^\alpha}\\
&\le C(t/2)^{-s/\alpha}e^{-\delta t} \le Ct^{-s/\alpha}e^{-\delta t},
\end{aligned}$$

which shows (1). Further (2) follows from

$$\|E_\alpha(t)(I-P_J)v\| = \Big(\sum_{j=J+1}^\infty e^{-2t\lambda_j^\alpha}(v,e_j)^2\Big)^{1/2} \le e^{-t\lambda_{J+1}^\alpha}\|v\|.$$

Together these estimates complete the proof of Lemma 3.2.
□

Proof. [Proof of Theorem 3.1] Subtracting (3.2) from (3.1), we get

$$\hat{u}(t)-\hat{u}_J(t) = E_\alpha(t)(u_0-P_Ju_0) + \int_0^t E_\alpha(t-s)\big(\hat{g}(s)-P_J\hat{g}(s)\big)\,ds = I+II. \tag{3.5}$$

For I, we have, with $0\le r<1/2$,

$$\begin{aligned}
\|A^{r/2}I\| &= \|A^{\frac{r}{2}}E_\alpha(t)\big(u_0-P_Ju_0\big)\|\\
&= \Big(\sum_{j=J+1}^\infty e^{-2t\lambda_j^\alpha}\lambda_j^r(u_0,e_j)^2\Big)^{1/2} \le e^{-t\lambda_{J+1}^\alpha}\|A^{r/2}(u_0-P_Ju_0)\|.
\end{aligned}$$

For II, we have, by Lemma 3.2, for some $\gamma \in (0,1)$,

$$\begin{aligned}\|A^{r/2}II\| &= \Big\| \int_0^t A^{r/2}E_\alpha(t-s)\big(I-P_J\big)\hat{g}(s)\,ds\Big\| \\ &= \Big\| \int_0^t \Big[A^{r/2}E_\alpha\big((1-\gamma)(t-s)\big)\Big]\Big[E_\alpha\big(\gamma(t-s)\big)(I-P_J)\Big]\hat{g}(s)\,ds\Big\| \\ &\le C\int_0^t (t-s)^{-\frac{r}{2\alpha}}e^{-\kappa_\alpha(t-s)}\|\hat{g}(s)\|\,ds,\end{aligned}$$

where $\kappa_\alpha = \delta(1-\gamma)+\lambda_{J+1}^\alpha\gamma$.

By Cauchy- Schwarz inequality, we have

$$\|A^{r/2}II\| \le C\Big(\int_0^t \big((t-s)^{-\frac{r}{2\alpha}}e^{-\kappa_\alpha(t-s)}\big)^2\,ds\Big)^{1/2}\Big(\int_0^t \|\hat{g}(s)\|^2\,ds\Big)^{1/2}.$$

Note that $r<\alpha$, we have, with $\lambda_{J+1}=(J+1)^2\pi^2$,

$$\begin{aligned}\int_0^t \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}}\,ds &\le \int_0^\infty \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}}\,ds \le \frac{\int_0^\infty s^{-r/\alpha}e^{-2s}\,ds}{\kappa_\alpha^{1-r/\alpha}} \le C\frac{1}{\kappa_\alpha^{1-r/\alpha}} \\ &\le C\frac{1}{(\lambda_{J+1}^\alpha)^{1-r/\alpha}} \le C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}.\end{aligned}$$

Thus

$$\|A^{r/2}II\| \le C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}\Big(\int_0^t \|\hat{g}(s)\|^2\,ds\Big)^{1/2}.$$

Together these estimates complete the proof of Theorem 3.1. □

Combining Theorem 2.1 with Theorem 3.1, we have

THEOREM 3.3. *Let u and $\hat{u}_J$ be the solutions of* (1.5)-(1.7) *and* (1.18)-(1.20)*, respectively. Assume that the assumptions* (1.13)-(1.15) *hold. Further assume that $u_0 \in \mathcal{D}(A^\alpha), 1/2<\alpha\le 1$ and $\mathbb{E}\|A^\alpha u_0\|^2<\infty$. Then we have*

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^1 \big(u(t,x)-\hat{u}_J(t,x)\big)^2\,dxdt \\ &\le C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha}+\Delta t^2\sum_{k=1}^\infty\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2+\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big) \\ &+C\mathbb{E}\|u_0-P_Ju_0\|^2+C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t\mathbb{E}\|A^\alpha u_0\|^2+\Delta t\sum_{k=1}^\infty\lambda_k^\alpha(\beta_k^M)^2+\sum_{k=1}^\infty(\beta_k^M)^2\Big).\end{aligned}$$

Proof. Note that

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^1 \big(u(t,x)-\hat{u}_J(t,x)\big)^2\,dxdt \\ &\le 2\mathbb{E}\int_0^T\int_0^1 \big(u(t,x)-\hat{u}(t,x)\big)^2\,dxdt+2\mathbb{E}\int_0^T\int_0^1 \big(\hat{u}(t,x)-\hat{u}_J(t,x)\big)^2\,dxdt \\ &= 2I+2II.\end{aligned}$$

For I, we have, by Theorem 2.1,

$$I \le C\Big(\sum_{k=1}^{\infty}\frac{(\alpha_k^M)^2}{2\lambda_k^\alpha}+\Delta t^2\sum_{k=1}^{\infty}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2+\Delta t^{1+\frac{\bar\alpha}{\alpha}-\frac{1}{2\alpha}}\Big).$$

For II, we have

$$II=\mathbb{E}\int_0^T\|\hat u(t)-\hat u_J(t)\|^2\,dt\le C\mathbb{E}\|u_0-P_Ju_0\|^2+C\frac{1}{(J+1)^{2\alpha}}\mathbb{E}\int_0^T\int_0^t\|\hat g(s)\|^2\,dsdt.$$

Note that $\hat g(s)=\frac{d\hat u(s)}{ds}+(-\Delta)^\alpha\hat u(s)$, we have, by Theorem 2.3,

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^t\|\hat g(s)\|^2\,dsdt\le\mathbb{E}\int_0^T\int_0^t\Big\|\frac{d\hat u(s)}{ds}+(-\Delta)^\alpha\hat u(s)\Big\|^2\,dsdt\\&\le C\mathbb{E}\int_0^T\int_0^T\int_0^1\Big(\Big|\frac{\partial\hat u(s,x)}{\partial s}\Big|^2+|(-\Delta)^\alpha\hat u(s,x)|^2\Big)\,dxdsdt\\&\le C\Big(\Delta t\mathbb{E}\|A^\alpha u_0\|^2+\Delta t\sum_{k=1}^{\infty}\lambda_k^\alpha(\beta_k^M)^2+\sum_{k=1}^{\infty}(\beta_k^M)^2\Big).\end{aligned}$$

Together these estimates complete the proof of Theorem 3.3.
□

4. Numerical simulations. In this section, we will consider the numerical simulation of the Fourier spectral methods for solving the following semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions, with $1/2<\alpha\le 1$, $0<x<1$, $0<t\le T$,

$$\frac{\partial u(t,x)}{\partial t}+\epsilon(-\Delta)^\alpha u(t,x)=f(u(t,x))+\frac{\partial^2 W(t,x)}{\partial t\partial x},\tag{4.1}$$

$$u(t,0)=u(t,1),\quad u_x'(t,0)=u_x'(t,1),\tag{4.2}$$

$$u(0,x)=u_0(x),\tag{4.3}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian defined by using the eigenvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the periodic boundary conditions. Here $f:\mathbb{R}\to\mathbb{R}$ is a smooth function and $\epsilon>0$ denotes the diffusion coefficient. Here we consider the problems with the periodic boundary conditions because we want to compare our numerical results with the results in [24, Example 10.39] where the algorithms of the spectral methods for stochastic semilinear parabolic equation subject to the periodic boundary conditions are given and discussed. One may also consider the algorithms and MATLAB codes for stochastic space fractional partial differential equations with the homogeneous boundary conditions following the approaches in, e.g., [16], [17]. Although the Laplacian is singular in (4.1)-(4.2) due to the periodic boundary conditions, we expect the errors to behave as in Theorem 3.3, see the comments in [24, Corollary 10.38].

Denote $A=-\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A)=H^2_{per}(0,1)$, where $\mathcal{D}(A)=H^2_{per}(0,1)$ is defined in the Introduction section. Then the eigenvalues and eigenfunctions of A can also be expressed by

$$\lambda_k=(2\pi k)^2,\quad e_k=e^{i2\pi kx},\ k\in\mathbb{Z}.$$

The noise has the form of

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k\in\mathbb{Z}} \sigma_k(t)\dot{\beta}_k(t)e_k(x), \tag{4.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k \in \mathbb{Z}$ are the derivatives of the standard Brownian motions $\beta_k(t)$, $k \in \mathbb{Z}$ and $\sigma_k(t), k \in \mathbb{Z}$ are some appropriate functions of t. Here $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. When $\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0, k \in \mathbb{Z}$, the noise (4.4) reduces to

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k\in\mathbb{Z}} \bar{\gamma}_k^{1/2}\dot{\beta}_k(t)e_k(x). \tag{4.5}$$

The approximate noise $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is, with some positive integer $M > 0$,

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k\in\mathbb{Z},|k|\le M} \bar{\gamma}_k^{1/2} e_k(x) \sum_{l=1}^{N_t} \frac{\eta_{k,l}}{\Delta t}\chi_l(t). \tag{4.6}$$

In our numerical example below, we assume that, [24, Example 10.8],

$$\bar{\gamma}_0 = 0, \quad \bar{\gamma}_k = |k|^{-(2r_1+1+\tilde{\epsilon})},\ k \in \mathbb{Z},\ k \neq 0. \tag{4.7}$$

where $\tilde{\epsilon} > 0$ is a very small positive number. When $r_1 = -1/2$, we obtain so-called space-time white noise. When $r_1 = 1$, we obtain the smooth noise.

Let $S_J := \text{span}\{e_0, e_1, \ldots, e_{J/2}, e_{-J/2+1} \ldots, e_{-1}\}$. We assume $J \le M$ where M is determined in (4.5). Here the ordering $0, 1, 2, \ldots, J/2, -J/2+1, \ldots, -1$ is consistent with the ordering in the MATLAB functions **fft** and **ifft** [33]. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_t} = T$, $N_t \in \mathbb{N}$ be the time partition of $[0, T]$ and Δt the time step size with $T = N_t\Delta t$. We use the semi-implicit Euler method to consider the time discretization.

We will consider the convergence rate against the different time steps. Choose $J = 64$. The reference solution is obtained by using the time step size $\Delta tref = T/Nref$ with $Nref = 10^4$. Let $\mathbf{kappa} = [5, 10, 20, 50, 100, 200, 500]$, we will consider the approximate solutions with the different time step sizes $\Delta t_i = \Delta tref * \mathbf{kappa}(i), i = 1, 2, \ldots, 7$. By Theorem 2.1, we have

$$\begin{aligned} &\mathbb{E}\int_0^T \int_0^1 \Big(u(t,x) - \hat{u}(t,x)\Big)^2 dxdt \\ &\le C\Big(\sum_{k\in\mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k\in\mathbb{Z}} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\bar{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big). \end{aligned} \tag{4.8}$$

We remark that here we choose $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. In our numerical example, we will choose, with $\bar{\gamma}_k$ given by (4.7),

$$\begin{aligned} &\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0, k \in \mathbb{Z}, \\ &\sigma_k^M(t) = \begin{cases} \sigma_k(t) = \bar{\gamma}_k^{1/2},\ |k| \le M, \\ 0, \quad |k| > M, \end{cases} \end{aligned}$$

which implies that

$$|\sigma_k^M(t)| \le \beta_k^M, \text{ where } \beta_k^M = \bar{\gamma}_k^{1/2},\ |k| \le M,$$

and

$$|\sigma_k(t) - \sigma_k^M(t)| \le \alpha_k^M, \text{ where } \alpha_k^M = \bar{\gamma}_k^{1/2}, |k| > M.$$

We first observe that for sufficiently large M the convergence order of the L^2 norm of the error in (4.8) is dominated by $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big)$. In fact, we will choose $M = J$ where J is sufficiently large. Then the first term of the right side of (4.8) satisfies, with $\lambda_k = (2\pi k)^2, k \in \mathbb{Z}$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} &= \sum_{|k|>M} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} \le C\Big(\frac{1}{\lambda_{M+1}^\alpha} + \frac{1}{\lambda_{M+2}^\alpha} + \dots\Big)\\ &\le C\Big(\frac{1}{(M+1)^{2\alpha}} + \frac{1}{(M+2)^{2\alpha}} + \dots\Big)\\ &= C\Big(\frac{1}{(J+1)^{2\alpha}} + \frac{1}{(J+2)^{2\alpha}} + \dots\Big).\end{aligned}$$

The second term of the right side of the error in (4.8) is $O(\Delta t^2)$. Hence for sufficiently large J, the convergence order of the L^2 norm of the error in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big)$.

We now consider two cases $r_1 = -1/2$ and $r_1 = 1$ in (4.7). For $r_1 = -1/2$, we may choose $\tilde{\alpha} = 0$ which implies that the convergence order of the L^2 norm in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big) = O\big(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}\big)$. Indeed, $\tilde{\alpha} = 0$ satisfies (1.15), that is,

$$\beta_k^M = \bar{\gamma}_k^{1/2} = |k|^{-\frac{2r_1+1+\tilde{\epsilon}}{2}} = |k|^{-\tilde{\epsilon}/2} \le |k|^{-\tilde{\alpha}}.$$

For $r_1 = 1$, we may choose $\tilde{\alpha} = 1/2 - \bar{\epsilon}$ (since $0 \le \tilde{\alpha} < 1/2$) with arbitrarily small positive number $\bar{\epsilon}$ which implies that the convergence order of the L^2 norm in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big) = O\big(\Delta t^{\frac{1}{2}(1-\frac{\bar{\epsilon}}{\alpha})}\big) \approx O(\Delta t^{1/2})$. Indeed, in this case, $\tilde{\alpha} = 1/2 - \bar{\epsilon}$ satisfies (1.15), that is,

$$\beta_k^M = \bar{\gamma}_k^{1/2} = |k|^{-\frac{2r_1+1+\tilde{\epsilon}}{2}} = |k|^{-\frac{3+\tilde{\epsilon}}{2}} \le |k|^{-\tilde{\alpha}}.$$

Thus we have, by Theorem 2.1, the following error estimates, with $1/2 < \alpha \le 1$ and $r_1 = -1/2$,

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \le C(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}), \tag{4.9}$$

and, with $1/2 < \alpha \le 1$ and $r_1 = 1$

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \le C(\Delta t^{1/2}), \tag{4.10}$$

where the norm is measured in L^2 both for time and space. In particular, when $\alpha = 1, r_1 = -1/2$, we have

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \le C(\Delta t^{1/4}),$$

which is consistent with the standard time discretization error for the stochastic heat equation driven by space-time white noise, see, e.g., [35].

In our numerical experiment below, we choose $f(u) = u - u^3$, $u_0(x) = \sin(2\pi x)$, and $\epsilon = 1$. See the simulation of this problem for $\alpha = 1$ in [30]. We will consider the error estimates $\|\hat{u}(t_n) - u(t_n)\|_{L^2(\Omega,H)}$ at time t_n. We hope to observe the same convergence order as in (4.9) and (4.10).

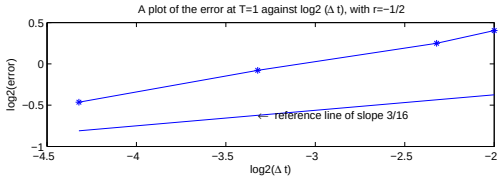


FIG. 1. *A plot of the error at* $T = 1$ *against* $log2(\Delta t)$ *with* $\alpha = 0.8, r_1 = -1/2$

To do this, we consider $\overline{M} = 100$ simulations. For each simulation $\omega_m, m = 1, 2, \ldots, \overline{M}$, we generate J independent Brownian motions $\beta_l, l = 0, 1, \ldots, J/2, -J/2+1, \ldots, -1$ and compute $\hat{u}_J(t_n) \approx \hat{u}(t_n)$ at time $t_n = 1$ by using the different time step sizes. We then compute the following L^2 norm of the error at $t_n = 1$ for the simulation $\omega_m, m = 1, 2, \ldots, \overline{M}$,

$$\epsilon(\Delta t_i, \omega_m) = \epsilon(\Delta t_i, \omega_m, t_n) = \|\hat{u}_J(t_n, \omega_m) - \text{uref}(t_n, \omega_m)\|^2,$$

where the reference ("true ") solution $\text{uref}(t_n, \omega_m)$ is approximated by using the time step $\Delta tref = T/Nref$ and $Jref = J$. We then average $\epsilon(\Delta t_i, \omega_m)$ with respect to ω_m to obtain the following approximation of $\|\hat{u}_J(t_n) - \text{uref}(t_n)\|_{L^2(\Omega,H)}$ for the different time step size Δt_i,

$$S(\Delta t_i) = \Big(\frac{1}{\overline{M}} \sum_{m=1}^{\overline{M}} \epsilon(\Delta t_i, \omega_m)\Big)^{1/2} = \Big(\frac{1}{\overline{M}} \sum_{m=1}^{\overline{M}} \|\hat{u}_J(t_n, \omega_m) - \text{uref}(t_n, \omega_m)\|^2\Big)^{1/2}.$$

For example, in the case $\alpha = 0.8, r_1 = -1/2$, the convergence rate against the time step size is $O(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}) = O(\Delta t^{3/16})$, i.e., with some positive constant C,

$$S(\Delta t_i) \approx C \Delta t_i^{3/16},$$

which implies that

$$\log(S(\Delta t_i)) \approx \log(C) + \frac{3}{16} \log(\Delta t_i), i = 1, 2, \ldots, 7.$$

In Figure 1, we consider the case $\alpha = 0.8, r_1 = -1/2$ and plot the points $\big(\log(\Delta t_i), \log(S(\Delta t_i))\big), i = 1, 2, \ldots, 7$ and we observe that the experimentally determined convergence order is higher than the theoretical order in this case. Here the reference line has the slope $\frac{3}{16}$.

In Figure 2, we consider the case $\alpha = 0.8, r_1 = 1$ and in this case the theoretical convergence order with respect to the time step size is $O(\Delta t^{1/2})$. We plot the points $\big(\log(\Delta t_i), \log(S(\Delta t_i))\big), i = 1, 2, \ldots, 7$ and we observe that the experimentally determined convergence order is also higher than the theoretical order in this case. Here the reference line has the slope $1/2$.

In Figure 3, we consider the convergence rate against the different J. Choose fixed time step $\Delta t = T/N_t$ with $N_t = 10^4$. We then consider the different $J = Jref * (\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^8})$ where $Jref = 2^{10}$.

We will first generate the reference Brownian motions

$$\beta_j(t), j = 0, 1, 2, \ldots, Jref/2, -Jref/2 + 1, \cdots - 1 \tag{4.11}$$

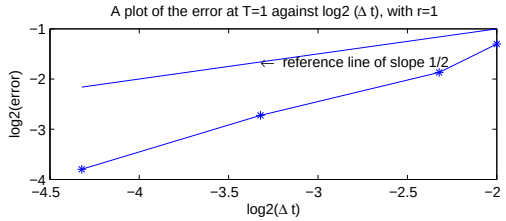


FIG. 2. *A plot of the error at $T = 1$ against $log2(\Delta t)$ with $\alpha = 0.8, r_1 = 1$*

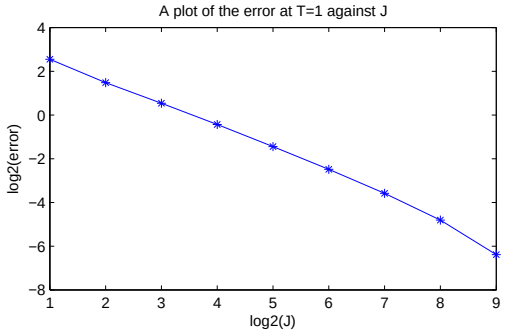


FIG. 3. *A plot of the error at $T = 1$ against the J with $\alpha = 0.8, r_1 = 1$*

for computing the reference ("true") solution uref. When we consider the approximate solution u with J truncated terms, we will use the Brownian motions $\beta_j(t), j = 0, 1, 2, \ldots, J/2, -J/2+1, \cdots -1$ from (4.11).

In Figure 3, we consider the case $\alpha = 0.8, r_1 = 1$ and plot the L^2 norm error against the different J where the L^2 norm error are approximated by using $\overline{M} = 100$ simulations. We indeed observe the spectral convergence with respect to the different J.

Appendix In the Appendix, we shall provide the proof of Theorem 2.1. To do this, we need the following lemma.

LEMMA 4.1. *Let $1/2 < \alpha \leq 1$ and $0 \leq \tilde{\alpha} < 1/2$. We have*

$$\int_0^\infty x^{-2(\tilde{\alpha}+\alpha)}\big(1 - e^{-x^{2\alpha}\Delta t}\big)\, dx \leq C\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}.$$

Proof. With the variable change $y = x^{2\alpha}\Delta t$, we have

$$\int_0^\infty x^{-2(\tilde{\alpha}+\alpha)}\big(1 - e^{-x^{2\alpha}\Delta t}\big)\, dx = C\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big(\int_0^1 + \int_1^\infty\Big)\frac{1-e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy$$

It is easy to see that, with $\alpha \in (1/2, 1]$,

$$\int_1^\infty \frac{1-e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \leq C.$$

Further, we have

$$\Big|\int_0^1 \frac{1-e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy\Big| \leq C\int_0^1 \frac{y}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \leq C\int_0^1 \frac{1}{y^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy < \infty.$$

if $1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}<1$, i.e., $0\le\tilde{\alpha}<1/2$.

Together these estimates complete the proof of Lemma 4.1.

□

Proof. [Proof of Theorem 2.1]

Subtracting (2.2) from (2.1), we have

$$\begin{aligned}
&u(t,x)-\hat{u}(t,x)\\
&=\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\\
&=\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]\\
&+\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\Big]\\
&=F_1(t,x)+F_2(t,x),
\end{aligned}$$

where, with $\eta_{k,l}$ and $\chi_l(t)$ defined as in (1.9),

$$\begin{aligned}
d\,W(s,y)&=\frac{\partial^2 W(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^{\infty}\sigma_k(s)e_k(y)\Big]\,d\beta_k(s)dy,\\
d\,\overline{W}(s,y)&=\frac{\partial^2 \overline{W}(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^{\infty}\sigma_k^M(s)e_k(y)\Big]\,d\beta_k(s)dy,\\
d\,\widehat{W}(s,y)&=\frac{\partial^2 \widehat{W}(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^{\infty}\sigma_k^M(s)\Big(\sum_{l=1}^{N_t}\frac{\eta_{k,l}}{\sqrt{\Delta t}}\chi_l(s)\Big)e_k(y)\Big]\,dsdy.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\int_0^T\int_0^1|u(t,x)-\hat{u}(t,x)|^2\,dxdt&\le C\mathbb{E}\int_0^T\int_0^1 F_1^2(t,x)\,dxdt\\
&+C\mathbb{E}\int_0^T\int_0^1 F_2^2(t,x)\,dxdt=C(I+II).
\end{aligned}$$

For I, we have, by using isometry property and (1.14), with $G_\alpha(t-s,x,y)=\sum_{j=1}^{\infty}e^{-(t-s)\lambda_j^\alpha}e_j(x)e_j(y)$,

$$\begin{aligned}
I&=\mathbb{E}\int_0^T\int_0^1\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2\,dxdt\\
&=\int_0^T\int_0^1\int_0^t\Big[\int_0^1 G_\alpha(t-s,x,y)\Big(\sum_{k=1}^{\infty}\big(\sigma_k(s)-\sigma_k^M(s)\big)e_k(y)\Big)\,dy\Big]^2\,dsdxdt.\\
&=\int_0^T\int_0^t\sum_{k=1}^{\infty}e^{-2(t-s)\lambda_k^\alpha}\big(\alpha_k^M\big)^2\,dsdt=\int_0^T\sum_{k=1}^{\infty}\frac{1-e^{-2t\lambda_k^\alpha}}{2\lambda_k^\alpha}\big(\alpha_k^M\big)^2\,dt\le C\sum_{k=1}^{\infty}\frac{1}{2\lambda_k^\alpha}\big(\alpha_k^M\big)^2.
\end{aligned}$$

For II, we have

$$
\begin{aligned}
II &= \mathbb{E}\int_0^T\int_0^1 \Big\{\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y) - \int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\Big]^2\Big\}dxdt\\
&\le 3\mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1 \Big\{\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y) - \int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\overline{W}(s,y)\Big]^2\\
&+ \Big[\int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\overline{W}(s,y) - \int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\widehat{W}(s,y)\Big]^2\\
&+ \Big[\int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\widehat{W}(s,y) - \int_0^t\int_0^1 G_\alpha(t_j-s,x,y)\,d\widehat{W}(s,y)\Big]^2\Big\}\,dxdt
\end{aligned}
\tag{4.12}
$$

$$\le 3\big(II_1+II_2+II_3\big).$$

For II_2, we have, by isometry property,

$$
\begin{aligned}
II_2 &= \mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\int_0^1 G_\alpha(t_j-s,x,y)\Big(\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)\,dy\Big)\,d\beta_k(s)\\
&- \sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\int_0^1 G_\alpha(t_j-\tilde{s},x,y)\sum_{k=1}^\infty \sigma_k^M(\tilde{s})e_k(y)\,dyd\tilde{s}\Big(\frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\,d\beta_k(s)\Big)\Big]^2\,dxdt\\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\Big\{\int_0^1 G_\alpha(t_j-s,x,y)\Big(\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)\Big)\,dy\\
&- \frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\int_0^1 G_\alpha(t_j-\tilde{s},x,y)\Big(\sum_{k=1}^\infty \sigma_k^M(\tilde{s})e_k(y)\Big)\,dyd\tilde{s}\Big\}^2\,dsdxdt\\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\Big\{\frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\Big[\int_0^1 G_\alpha(t_j-s,x,y)\Big(\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)\Big)\,dy\\
&- \int_0^1 G_\alpha(t_j-\tilde{s},x,y)\Big(\sum_{k=1}^\infty \sigma_k^M(\tilde{s})e_k(y)\Big)\,dy\Big]\,d\tilde{s}\Big\}^2\,dsdxdt\\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^\infty\Big\{\frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\Big[e^{-\lambda_k^\alpha(t_j-s)}\sigma_k^M(s) - e^{-\lambda_k^\alpha(t_j-\tilde{s})}\sigma_k^M(\tilde{s})\Big]\,d\tilde{s}\Big\}^2\,dsdt\\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^\infty\frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2}\Big\{\int_{t_l}^{t_{l+1}}\Big[e^{\lambda_k^\alpha s}\sigma_k^M(s) - e^{\lambda_k^\alpha \tilde{s}}\sigma_k^M(\tilde{s})\Big]\,d\tilde{s}\Big\}^2\,dsdt.
\end{aligned}
$$

By (1.14), we have, with some ξ_l^1,ξ_l^2 which lie between s and $\tilde{s}$,

$$
\begin{aligned}
&\Big|e^{\lambda_k^\alpha s}\sigma_k^M(s) - e^{\lambda_k^\alpha \tilde{s}}\sigma_k^M(\tilde{s})\Big| = \Big|\Big(e^{\lambda_k^\alpha s}-e^{\lambda_k^\alpha\tilde{s}}\Big)\sigma_k^M(s) + e^{\lambda_k^\alpha\tilde{s}}(\sigma_k^M(s)-\sigma_k^M(\tilde{s})\Big|\\
&\le \Big|\Big(\lambda_k^\alpha e^{\lambda_k^\alpha\xi_l^1}\Delta t\Big)\sigma_k^M(s) + e^{\lambda_k^\alpha\tilde{s}}\Big((\sigma_k^M)'(\xi_l^2)\Big)\Delta t\Big|\\
&\le \Big|\lambda_k^\alpha e^{\lambda_k^\alpha t_{l+1}}\beta_k^M\Delta t + e^{\lambda_k^\alpha t_{l+1}}\gamma_k^M\Delta t\Big| \le e^{\lambda_k^\alpha t_{l+1}}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)\Delta t.
\end{aligned}
$$

Hence

$$II_2 \le \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^{\infty}\frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2}\Big[e^{2\lambda_k^\alpha t_{l+1}}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2\Delta t^4\Big]\,dsdt$$
$$\le \Delta t^2\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{k=1}^{\infty}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2\,dsdt \le C\Delta t^2\sum_{k=1}^{\infty}\Big(\lambda_k^\alpha\beta_k^M+\gamma_k^M\Big)^2,$$

where we use the inequality $e^{-2\lambda_k^\alpha(t_j-t_{l+1})}\le 1$ for $l=0,1,2,\ldots,j-1$.

For II_1, we have

$$II_1 = \mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)-\int_0^{t_j}\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2dxdt$$
$$\le 2\mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_0^{t_j}\int_0^1\Big(G_\alpha(t-s,x,y)-G_\alpha(t_j-s,x,y)\Big)\,d\overline{W}(s,y)\Big]^2dxdt$$
$$+2\mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_{t_j}^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2dxdt = 2(II_1^1+II_1^2).$$

For II_1^1, we have, by the isometry property and (1.14),

$$II_1^1 = \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^{t_j}\sum_{k=1}^{\infty}\Big(e^{-\lambda_k^\alpha(t-s)}-e^{-\lambda_k^\alpha(t_j-s)}\Big)^2(\sigma_k^M(s))^2\,dsdt$$
$$\le \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{k=1}^{\infty}(\beta_k^M)^2\int_0^{t_j}\Big(e^{-\lambda_k^\alpha(t-s)}-e^{-\lambda_k^\alpha(t_j-s)}\Big)^2dsdt$$

Note that

$$\int_0^{t_j}\Big(e^{-\lambda_k^\alpha(t-s)}-e^{-\lambda_k^\alpha(t_j-s)}\Big)^2ds = \int_0^{t_j}e^{-2\lambda_k^\alpha(t-s)}\Big(1-e^{-\lambda_k^\alpha(t_j-t)}\Big)^2ds$$
$$= \Big(1-e^{-\lambda_k^\alpha(t_j-t)}\Big)^2\frac{e^{-2\lambda_k^\alpha(t-t_j)}-e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha} \le \frac{\Big(1-e^{-\lambda_k^\alpha(t-t_j)}\Big)^2}{2\lambda_k^\alpha}.$$

Hence, we have

$$II_1^1 \le \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\Big(\sum_{k=1}^{\infty}(\beta_k^M)^2\Big)\frac{\Big(1-e^{-\lambda_k^\alpha(t-t_j)}\Big)^2}{2\lambda_k^\alpha}\,dt \le C\sum_{k=1}^{\infty}(\beta_k^M)^2\frac{\Big(1-e^{-\lambda_k^\alpha\Delta t}\Big)^2}{2\lambda_k^\alpha}.$$

By (1.15) and Lemma 4.1, we have

$$II_1^1 \le C\sum_{k=1}^{\infty}k^{-2\tilde\alpha}\frac{\Big(1-e^{-\lambda_k^\alpha\Delta t}\Big)^2}{2\lambda_k^\alpha} \le C\int_1^{\infty}x^{-2(\tilde\alpha+\alpha)}\big(1-e^{-x^{2\alpha}\Delta t}\big)\,dx \le C\Delta t^{1+\frac{\tilde\alpha}{\alpha}-\frac{1}{2\alpha}}.$$

For II_1^2, we have, by isometry property and (1.14) and (1.15),

$$
\begin{aligned}
II_1^2 &= \mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\Big[\int_{t_j}^{t}\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2\,dxdt\\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t}\sum_{k=1}^{\infty} e^{-2\lambda_k^\alpha(t-s)}\big(\sigma_k^M(s)\big)^2\,dsdt \le \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t}\sum_{k=1}^{\infty}\Big(k^{-2\tilde\alpha}e^{-2\lambda_k^\alpha(t-s)}\Big)\,dsdt\\
&\le C\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{k=1}^{\infty}\Big[k^{-2\tilde\alpha}\Big(\frac{1-e^{-2\lambda_k^\alpha\Delta t}}{\lambda_k^\alpha}\Big)\Big]\,dt = C\sum_{k=1}^{\infty}\Big[k^{-2\tilde\alpha}\Big(\frac{1-e^{-2\lambda_k^\alpha\Delta t}}{\lambda_k^\alpha}\Big)\Big]\\
&\le C\int_0^\infty \frac{1-e^{-2x^{2\alpha}\Delta t}}{x^{2\alpha+2\tilde\alpha}}\,dx \le C\int_0^\infty x^{-2(\tilde\alpha+\alpha)}\big(1-e^{-x^{2\alpha}\Delta t}\big)\,dx.
\end{aligned}
$$

By Lemma 4.1, we have

$$II_1^2 \le C\Delta t^{1+\frac{\tilde\alpha}{\alpha}-\frac{1}{2\alpha}}. \tag{4.13}$$

Similarly we may show, with $0 \le \tilde\alpha < 1/2$,

$$II_3 \le C\Delta t^{1+\frac{\tilde\alpha}{\alpha}-\frac{1}{2\alpha}}.$$

Together these estimates complete the proof of Theorem 2.1.

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□

On anti-periodic type boundary value problems of sequential fractional differential equations of order $q \in (2, 3]$

Ahmed Alsaedi, Mohammed H. Aqlan, Bashir Ahmad

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: aalsaedi@hotmail.com, aqlan76@gmail.com, bashirahmad_qau@yahoo.com

Abstract

We investigate a new kind of anti-periodic type boundary value problems of sequential fractional differential equation of order $q \in (2, 3]$. We make use of Banach's contraction mapping principle to obtain the uniqueness result while the existence of solutions is established via Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative. The paper concludes with some illustrative examples.

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1 Introduction

Boundary value problems constitute an important field of research and arise in several disciplines such as applied mathematics, control theory, mechanical structures and physics. The literature on the topic ranges from theoretical aspects of existence and uniqueness of solutions to analytic and numerical methods for finding solutions of the problems. Linear and nonlinear, singular and nonsingular, well-posed and ill-posed, local and nonlocal, free and fixed problems are well known types of boundary value problems. In relation to the boundary conditions, considerable attention has been given to two-point, multi-point, periodic/anti-periodic and integral boundary value problems. In particular, anti-periodic boundary conditions are found to be quite significant and important in the mathematical modeling of certain physical processes and phenomena, for example, wavelets, physics, trigonometric polynomials in the study of interpolation problems, etc., for example, see [1] and the references cited therein.

Differential and integral operators of fractional-order appear in the mathematical modelling of several phenomena occurring in engineering and scientific disciplines such as biological sciences, ecology, control theory, aerodynamics, fluid dynamics, polymer rheology, regular variation in thermodynamics, etc. For more details and explanation, for instance, see [2, 3, 4]. The interest in the study of fractional-order operators is mainly due to nonlocal nature of such operators which takes into account memory and hereditary properties of some important and useful materials and processes.

In recent years, fractional-order boundary value problems involving a variety of boundary conditions have been studied by several researchers. For details and examples, we refer the reader to a series of papers ([5]-[10]). For some works on sequential fractional differential equations, for example, see ([11]-[15]).

Anti-periodic boundary value problems of fractional-order have also been investigated in the literature ([16]-[19]). However, the study of sequential fractional differential equations equipped with anti-periodic boundary conditions has not been investigated yet.

In this paper, we consider a nonlinear anti-periodic boundary value problem of sequential fractional

differential equations given by

$$\begin{cases} ({}^cD^q + k\ {}^cD^{q-1})u(t) = f(t,u(t)),\ 2 < q \le 3,\ 0 < t < T, \\ \alpha_1 u(0) + \gamma_1 u(T) = a, \quad \alpha_2 u'(0) + \gamma_2 u'(T) = b, \quad \alpha_3 u''(0) + \gamma_3 u''(T) = c, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $\alpha_i, \gamma_i, (i = 1,2,3), a, b, c \in \mathbb{R}$, $k \in \mathbb{R}^+$ and f is a given continuous function.

2 Preliminaries and an auxiliary lemma

First of all, let us recall some basic definitions [2, 3].

Definition 2.1 *The fractional integral of order r with the lower limit zero for a function f is defined as*

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, \quad r > 0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$.

Definition 2.2 *The Riemann-Liouville fractional derivative of order $r > 0$, $n-1 < r < n$, $n \in N$, is defined as*

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 *The Caputo derivative of order r for a function $f : [0,\infty) \to R$ can be written as*

$${}^cD^r f(t) = D_{0+}^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < r < n.$$

Remark 2.4 *If $f(t) \in C^n[0,\infty)$, then*

$${}^cD^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

The following lemma plays a pivotal role in defining the solution for problem (1).

Lemma 2.5 *Let $h \in AC([0,T]), \mathbb{R})$. Then the following linear boundary value problem*

$$\begin{cases} ({}^cD^q + k\ {}^cD^{q-1})u(t) = h(t),\ 2 < q \le 3,\ 0 < t < T, \\ \alpha_1 u(0) + \gamma_1 u(T) = a, \quad \alpha_2 u'(0) + \gamma_2 u'(T) = b, \quad \alpha_3 u''(0) + \gamma_3 u''(T) = c \end{cases} \tag{2}$$

is equivalent to the fractional integral equation

$$\begin{aligned} u(t) \quad &= \nu_1(t) + \int_0^t e^{-k(t-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)dx \Big) ds + \nu_2(t) \int_0^T e^{-k(T-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x)dx \Big) ds \\ &+ \nu_3(t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} h(s)ds + \nu_4(t) \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} h(s)ds, \end{aligned} \tag{3}$$

where

$$\begin{aligned}
\nu_1(t) &= \frac{a}{\lambda_1} - \frac{\gamma_1 Tb}{\lambda_1\lambda_2} - \frac{c}{k^2\lambda_1\lambda_2\delta_3}\Big(\lambda_2\delta_1 + k\delta_2\gamma_1 T\Big) + \frac{ce^{-kt}}{k^2\delta_3} + \frac{t}{\lambda_2}\Big(b + \frac{c\delta_2}{k\delta_3}\Big),\\
\nu_2(t) &= \frac{\gamma_3}{\delta_3\lambda_1}\Big(\delta_1 - \frac{\delta_3\gamma_1}{\gamma_3}\Big) - \frac{k\gamma_1 T}{\lambda_1\lambda_2}\Big(\gamma_2 - \frac{\delta_2\gamma_3}{\delta_3}\Big) - \frac{\gamma_3 e^{-kt}}{\delta_3} + \frac{kt}{\lambda_2}\Big(\gamma_2 - \frac{\delta_2\gamma_3}{\delta_3}\Big),\\
\nu_3(t) &= \frac{-\delta_1\gamma_3}{k\lambda_1\delta_3} - \frac{\gamma_1 T}{\lambda_1\lambda_2}\Big(\frac{\delta_2\gamma_3}{\delta_3} - \gamma_2\Big) + \frac{\gamma_3 e^{-kt}}{k\delta_3} + \frac{t}{\lambda_2}\Big(\frac{\delta_2\gamma_3}{\delta_3} - \gamma_2\Big)\\
\nu_4(t) &= \frac{\gamma_3}{k^2\delta_3\lambda_1\lambda_2}\Big(\lambda_2\delta_1 + \delta_2 k\gamma_1 T\Big) - \frac{\gamma_3 e^{-kt}}{k^2\delta_3} - \frac{\gamma_3\delta_2 t}{k\delta_3\lambda_2},
\end{aligned} \tag{4}$$

$$\delta_i = \alpha_i + \gamma_i e^{-kT},\ i = 1,2,3,\ \delta_3 \neq 0,\ \lambda_1 = \alpha_1 + \gamma_1 \neq 0,\ \lambda_2 = \alpha_2 + \gamma_2 \neq 0. \tag{5}$$

Proof. Rewrite the equation $({}^cD^q + k\ {}^cD^{q-1})u(t) = h(t)$ as

$${}^cD^{q-1}(D + k)u(t) = h(t). \tag{6}$$

Applying the operator I^{q-1} on both sides of (6), and solving the resulting equation, we get

$$u(t) = A_0 e^{-kt} + A_1 + A_2 t + \int_0^t e^{-k(t-s)} I^{q-1}h(s)ds, \tag{7}$$

where A_0, A_1 and A_2 are arbitrary constants and

$$I^{q-1}h(t) = \int_0^t \frac{(t-x)^{q-2}}{\Gamma(q-1)}h(x)dx.$$

Differentiating (7) with respect to t, we obtain

$$u'(t) = -kA_0 e^{-kt} + A_2 - k\int_0^t e^{-k(t-s)} I^{q-1}h(s)ds + I^{q-1}h(t), \tag{8}$$

$$u''(t) = k^2 A_0 e^{-kt} + k^2\int_0^t e^{-k(t-s)} I^{q-1}h(s)ds - kI^{q-1}h(t) + I^{q-2}h(t). \tag{9}$$

Using the boundary conditions of (2) in (7)-(9), we get

$$\delta_1 A_0 + \lambda_1 A_1 + A_2\gamma_1 T + \gamma_1\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds = a, \tag{10}$$

$$-k\delta_2 A_0 + \lambda_2 A_2 + \gamma_2\Big(-k\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds + I^{q-1}h(T)\Big) = b, \tag{11}$$

$$A_0 k^2\delta_3 + \gamma_3\Big(k^2\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds - kI^{q-1}h(T) + I^{q-2}h(T)\Big) = c. \tag{12}$$

Solving the system (10)-(12) for A_0, A_1 and A_2, we find that

$$A_0 = \frac{1}{k^2\delta_3}\Big\{c - \gamma_3\Big(k^2\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds - kI^{q-1}h(T) + I^{q-2}h(T)\Big)\Big\},$$

$$A_1 = \frac{a}{\lambda_1} - \Big(\frac{\gamma_1 Tb}{\lambda_1\lambda_2}\Big) - \Big(\frac{\delta_1}{k^2\delta_3\lambda_1} + \frac{\delta_2\gamma_1 T)}{k\delta_3\lambda_1\lambda_2}\Big)c$$

$$+\Big(\frac{\delta_1\gamma_3}{\delta_3\lambda_1} - \frac{\gamma_1 T}{\lambda_1}\Big(\frac{\gamma_2 k}{\lambda_2} - \frac{k\delta_2\gamma_3}{\delta_3\lambda_2}\Big) - \frac{\gamma_1}{\lambda_1}\Big)\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds$$

$$-\Big(\frac{\delta_1\gamma_3}{k\delta_3\lambda_1} + \frac{\gamma_1 T}{\lambda_1}\Big(\frac{\delta_2\gamma_3}{\delta_3\lambda_2} - \frac{\gamma_2}{\lambda_2}\Big)\Big)I^{q-1}h(T) + \Big(\frac{\delta_1\gamma_3}{k^2\delta_3\lambda_1} + \frac{\delta_2\gamma_3\gamma_1 T}{k\delta_3\lambda_1\lambda_2}\Big)I^{q-2}h(T),$$

$$A_2 = \frac{b}{\lambda_2} + \frac{\delta_2 c}{k\delta_3\lambda_2} + \Big(\frac{\gamma_2 k}{\lambda_2} - \frac{k\delta_2\gamma_3}{\delta_3\lambda_2}\Big)\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds$$

$$+\Big(\frac{\delta_2\gamma_3}{\delta_3\lambda_2} - \frac{\gamma_2}{\lambda_2}\Big) I^{q-1}h(T) - \frac{\delta_2\gamma_3}{k\delta_3\lambda_2} I^{q-2}h(T),$$

where we have used (5). Substituting the values of A_0, A_1 and A_2 in (7) and using the notations (4), we obtain the solution (3). By direct computation, it is easy to show that (3) satisfies the problem (2). This completes the proof. □

3 Main Result

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|, t \in [0,T]\}$.
Via Lemma 2.5, we transform the problem (1) to an equivalent fixed point problem as

$$u = \mathcal{H}u, \tag{13}$$

where $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{H}u)(t) &= \nu_1(t) + \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f(x,u(x))dx\Big)ds \\ &+\nu_2(t)\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f(x,u(x))dx\Big)ds \\ &+\nu_3(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,u(s))ds + \nu_4(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,u(s))ds.\end{aligned} \tag{14}$$

Notice that the problem (1) has solutions if the operator equation (13) has fixed points.
For computational convenience, we set

$$Q = \sup_{t\in[0,T]}\Big\{\frac{t^{q-1}(1-e^{-kt})}{k\Gamma(q)} + \frac{|\nu_2(t)|T^{q-1}(1-e^{-kT})}{k\Gamma(q)} + \frac{|\nu_3(t)|T^{q-1}}{\Gamma(q)} + \frac{|\nu_4(t)|T^{q-2}}{\Gamma(q-1)}\Big\}. \tag{15}$$

Now we are in a position to present our first result which deals with the existence of a unique solution of the problem (1) and is based on Banach's contraction mapping principle.

Theorem 3.1 *Assume that $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous functions satisfying the Lipschitz condition:*

(A_1) there exists a positive number ℓ such that $|f(t,u) - f(t,v)| \le \ell|u-v|$, $\forall t \in [0,T]$, $u, v \in \mathbb{R}$.

Then the problem (1) has a unique solution on $[0,T]$ if $\ell < 1/Q$, where Q is given by (15).

Proof. Let us fix $r \ge \dfrac{QM + \|\nu_1\|}{1-\ell Q}$, where $\sup_{t\in[0,T]}|f(t,0)| = M$ and Q is given by (15), and define a set $B_r = \{u \in \mathcal{C} : \|u\| \le r\}$. In the first step, we show that $\mathcal{H}B_r \subset B_r$, where the operator $\mathcal{H}$ is defined by (14). For any $u \in B_r, t \in [0,T]$, we have

$$\begin{aligned}|f(t,u(t))| &= |f(t,u(t)) - f(t,0) + f(t,0)| \le |f(t,u(t)) - f(t,0)| + |f(t,0)| \\ &\le \ell\|u\| + M \le \ell r + M.\end{aligned}$$

Then, for $u \in B_r$, we obtain

$$\|(\mathcal{H}u)\| \le \sup_{t\in[0,T]}\Big\{|\nu_1(t)| + \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}|f(x,u(x))|dx\Big)ds$$

$$
\begin{aligned}
&+ \quad |\nu_2(t)| \int_0^T e^{-k(T-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} |f(x,u(x))| dx \Big) ds \\
&+ \quad |\nu_3(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))| ds + |\nu_4(t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s,u(s))| ds. \Big\} \\
&\leq \quad (\ell r + M) \sup_{t\in[0,T]} \Big\{ \int_0^t e^{-k(t-s)} \Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} dx \Big) ds \\
&+ \quad |\nu_2(t)| \int_0^T e^{-k(T-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} dx \Big) ds \\
&+ \quad |\nu_3(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + |\nu_4(t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} ds. \Big\} + \|\nu_1\| \\
&\leq \quad (\ell r + M) Q + \|\nu_1\| \leq r.
\end{aligned}
$$

This shows that $\mathcal{H}B_r \subset B_r$. Next we show that the operator $\mathcal{H}$ is a contraction. Let $u, v \in \mathcal{C}$. Then

$$
\begin{aligned}
\|\mathcal{H}u - \mathcal{H}v\| \quad &\leq \quad \sup_{t\in[0,T]} \Big\{ \int_0^t e^{-k(t-s)} \Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} |f(x,u(x)) - f(x,v(x))| dx \Big) ds \\
&+ \quad |\nu_2(t)| \int_0^T e^{-k(T-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \Big| f(x,u(x)) - f(x,v(x)) \Big| dx \Big) ds \\
&+ \quad |\nu_3(t)| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \Big| f(s,u(s)) - f(s,v(s)) \Big| ds \\
&+ \quad |\nu_4(t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \Big| f(s,u(s)) - f(s,v(s)) \Big| ds \Big\} \\
&\leq \quad \ell Q \parallel u - v \parallel,
\end{aligned}
$$

where we have used (15). By the given assumption: $\ell < 1/Q$, it follows that the operator $\mathcal{H}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{H}$ has a fixed point, which equivalently means that the problem (1) has a unique solution on $[0,T]$. □

Now we show the existence of solutions for the problem (1) by means of Krasnoselskiis fixed point theorem, which is stated below for the reader's convenience.

Lemma 3.2 *(Krasnoselskii's fixed point theorem [20]) Let $\mathcal{Y}$ be a closed bounded, convex and nonempty subset of a Banach space $\mathcal{X}$. Let φ_1, φ_2 be the operators such that (i) $\varphi_1 y_1 + \varphi_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) φ_1 is compact and continuous and (iii) φ_2 is a contraction mapping. Then there exists $y \in \mathcal{Y}$ such that $y = \varphi_1 y + \varphi_2 y$.*

Theorem 3.3 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(t,x)| \leq g(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, where $g \in C([0,T], \mathbb{R}^+)$, with $\sup_{t\in[0,T]} |g(t)| = \|g\|$. In addition, it is assumed that $\ell Q_1 < 1$, where*

$$
Q_1 \quad = \sup_{t\in[0,T]} \Big\{ \frac{|\nu_2(t)| T^{q-1}(1-e^{-kT})}{k\Gamma(q)} + \frac{|\nu_3(t)| T^{q-1}}{\Gamma(q)} + \frac{|\nu_4(t)| T^{q-2}}{\Gamma(q-1)} \Big\}. \tag{16}
$$

Then the problem (1) has at least one solution on $[0,T]$.

Proof. With $\bar{r} \geq Q\|g\| + \|\nu_1\|$ (Q is given by (15)), we define operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on $B_{\bar{r}} = \{u \in \mathcal{C} : \|u\| \leq \bar{r}\}$ as follows

$$
(\mathcal{H}_1 u)(t) = \int_0^t e^{-k(t-s)} \Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f(x,u(x)) dx \Big) ds,
$$

$$\begin{aligned}(\mathcal{H}_2u)(t) &= \nu_1(t)+\nu_2(t)\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}f(x,u(x))dx\Big)ds\\ &+\nu_3(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))ds+\nu_4(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}f(s,u(s))ds.\end{aligned}$$

For $u,v\in B_{\bar{r}}$, it is easy to verify that $\|\mathcal{H}_1u+\mathcal{H}_2v\|\le Q\|g\|+\|\nu_1\|$. Thus, $\mathcal{H}_1u+\mathcal{H}_2v\in B_{\bar{r}}$. Using the assumption (A_1) and (3.3),one can get $\|\mathcal{H}_2u-\mathcal{H}_2v\|\le \ell Q_1\parallel u-v\parallel$, which implies that the operator $\mathcal{H}_2$ is a contraction in view of the given condition: $\ell Q_1<1$.
Continuity of f implies that the operator $\mathcal{H}_1$ is continuous. Also, $\mathcal{H}_1$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{H}_1u\|\le\frac{(1-e^{-kT})T^{q-1}\|g\|}{k\Gamma(q)}.$$

Finally, we establish that the operator $\mathcal{H}_1$ is compact. Letting $\sup_{(t,u)\in[0,T]\times B_{\bar{r}}}|f(t,u)|=f_{\bar{r}}$, for $t_1,t_2\in[0,T]$, we have

$$\begin{aligned}&\|(\mathcal{H}_1u)(t_2)-(\mathcal{H}_1u)(t_1)\|\\ \le\ & f_{\bar{r}}\left(|e^{-kt_2}-e^{-kt_1}|\int_0^{t_1}e^{ks}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}dx\Big)ds+\int_{t_1}^{t_2}e^{-k(t_2-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}dx\Big)ds\right)\\ &\to 0 \text{ as } t_2-t_1\to 0,\end{aligned}$$

independent of u. Thus the operator $\mathcal{H}_1$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, the operator $\mathcal{H}_1$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 3.2 are satisfied. In consequence, by the conclusion of Lemma 3.2, the problem (1) has at least one solution on $[0,T]$. □

In our last result, we prove the existence of solutions the problem(1) by applying Leray-Schauder nonlinear alternative.

Lemma 3.4 *(Nonlinear alternative for single valued maps [20]). Let $\mathcal{S}$ be a closed, convex subset of a Banach space $\mathcal{E}$, and $\mathcal{V}$ be an open subset of $\mathcal{S}$ with $0\in\mathcal{V}$. Suppose that $\mathcal{A}:\overline{\mathcal{V}}\to\mathcal{S}$ is continuous and compact (that is, $\mathcal{A}(\overline{\mathcal{V}})$ is a relatively compact subset of $\mathcal{S}$) map. Then either*

(i) $\mathcal{A}$ has a fixed point in $\overline{\mathcal{V}}$, or

(ii) there is a $v\in\partial\mathcal{V}$ (the boundary of $\mathcal{V}$ in $\mathcal{S}$) and $\lambda\in(0,1)$ with $v=\lambda\mathcal{A}(v)$.

Theorem 3.5 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(A_3) *there exist a function $p\in C([0,T],\mathbb{R}^+)$, and a nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)|\le p(t)\psi(\|u\|)$, $\forall(t,u)\in[0,T]\times\mathbb{R}$;*

(A_4) *there exists a constant $\overline{M}>0$ such that $\overline{M}/\overline{Q}>1$, where*

$$\overline{Q}=\|\nu_1\|+\|p\|\psi(\|\overline{M}\|)Q. \tag{17}$$

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. We complete the proof in several steps. Firstly we show that the operator $\mathcal{H}:\mathcal{C}\to\mathcal{C}$ defined by (14) maps bounded sets into bounded sets in $\mathcal{C}$. For the positive number r, let $B_r=\{u\in\mathcal{C}:\|u\|\le r\}$ be a bounded set in $\mathcal{C}$. Then, for $u\in B_r$, we have

$$\begin{aligned}|(\mathcal{H}u)(t)| &\le |\nu_1(t)|+\int_0^t e^{-k(t-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds\\ &+ |\nu_2(t)|\int_0^T e^{-k(T-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds\\ &+ |\nu_3(t)|\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds+|\nu_4(t)|\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds.\\ &\le |\nu_1(t)|+\|p\|\psi(\|u\|)\Big\{\frac{t^{q-1}(1-e^{-kt})}{k\Gamma(q)}+|\nu_2(t)|\frac{T^{q-1}(1-e^{-kT})}{k\Gamma(q)}+|\nu_3(t)|\frac{T^{q-1}}{k\Gamma(q)}+|\nu_4(t)|\frac{T^{q-2}}{k\Gamma(q-1)}\Big\},\end{aligned}$$

which implies that $\|(\mathcal{H}u)\| \leq \|\nu_1(t)\| + \|p\|\psi(r)Q$, where Q is given by (15).
Next we show that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in B_r$, where B_r is a bounded set in $\mathcal{C}$. Then we obtain

$$
\begin{aligned}
&|(\mathcal{H}u)(t_2) - (\mathcal{H}u)(t_1)| \leq |\nu_1(t_2) - \nu_1(t_1)| \\
&+\Big|\int_0^{t_2} e^{-k(t_2-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f(x,u(x))dx\Big)ds - \int_0^{t_1} e^{-k(t_1-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f(x,u(x))dx\Big)ds\Big| \\
&+|\nu_2(t_2) - \nu_2(t_1)| \int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+|\nu_3(t_2) - \nu_3(t_1))| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |\nu_4(t_2) - \nu_4(t_1)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds \\
&\leq \Big|(\frac{b}{\lambda_2} + \frac{\delta_2 c}{k\lambda_2\delta_3})(t_2-t_1) + \frac{c}{k^2\delta_2}(e^{-kt_2} - e^{-kt_1})\Big| + \|p\|\psi(r)\Big[\frac{(1-e^{-k(t_2-t_1)})}{k\Gamma(q)}\Big(t_1^{q-1}(1-e^{-kt_1}) + t_2^{q-1}\Big) \\
&+\frac{T^{q-2}e^{-kt_1}}{\Gamma(q)}\Big|\frac{\gamma_3}{\delta_3}\Big|\Big\{\frac{T}{k}\Big(2-e^{-kT}\Big) + \frac{q-1}{k^2}\Big\}\Big(1-e^{-k(t_2-t_1)}\Big) \\
&+\frac{T^{q-2}}{\Gamma(q)}\Big\{\Big|\frac{(\gamma_2\delta_3 - \gamma_3\delta_2)T}{\lambda_2\delta_3}\Big|\Big(2-e^{-kT}\Big) + \frac{q-1}{k}\Big|\frac{\delta_2\gamma_3}{\delta_3\gamma_2}\Big|\Big\}(t_2-t_1)\Big].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. As $\mathcal{H}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is completely continuous. The conclusion will follow form the Leray-Schauder nonlinear alternative (Lemma 3.4) once we have proved the boundedness of the set of all solutions to equations $u = \lambda\mathcal{H}u$ for $\lambda \in [0,1]$. Let u be a solution. Then, for $t \in [0,T]$, and using the computations in proving that $\mathcal{H}$ is bounded, we have

$$
\begin{aligned}
|u(t)| &= |\lambda(\mathcal{H}u)(t)| \leq |\nu_1(t)| + \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+ |\nu_2(t)| \int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+ |\nu_3(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |\nu_4(t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds \\
&\leq |\nu_1(t)| + \|p\|\psi(\|u\|)\Big\{\frac{t^{q-1}(1-e^{-kt})}{k\Gamma(q)} + |\nu_2(t)|\frac{T^{q-1}(1-e^{-kT})}{k\Gamma(q)} + |\nu_3(t)|\frac{T^{q-1}}{k\Gamma(q)} + |\nu_4(t)|\frac{T^{q-2}}{k\Gamma(q-1)}\Big\}.
\end{aligned}
$$

In consequence, we get

$$\|u\|/\Big[\|\nu_1\| + \|p\|\psi(\|u\|)Q\Big] \leq 1.$$

In view of (A_4), there exists $\overline{M}$ such that $\|u\| \neq \overline{M}$. Let us set $U = \{u \in \mathcal{C} : \|u\| < \overline{M}\}$. Note that the operator $\mathcal{H} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda\mathcal{H}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathcal{H}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1). □

Example 3.6 *Consider the following anti-periodic fractional boundary value problem:*

$$
\begin{cases}
({}^cD^{5/2} + 2^cD^{3/2})u(t) = \dfrac{\sin u}{25} + e^{-t}\cos t,\ t \in [0,2], \\
u(0) + u(2) = 1,\ \ u'(0) - (1/2)u'(2) = 2,\ \ u''(0) + (1/4)u''(2) = 1,
\end{cases} \tag{18}
$$

where $f(t,u(t)) = \frac{\sin u}{25} + e^{-t}\cos t$, $T = 2$, $k = 2$, $\alpha_1 = 1$, $\gamma_1 = 1$, $\alpha_2 = 1$, $\gamma_2 = -1/2$, $\alpha_3 = 1$, $\gamma_3 = 1/4$, $a = 1$, $b = 2$, $c = 1$. *With the given data, we find that the values of* Q *and* Q_1 *respectively given by (15) and* (16) *are* $Q \simeq 7.557935$, $Q_1 \simeq 6.513574$.

(a) *For the applicability of Theorem 3.1, we have that* $\ell and = 1/25$ *as* $|f(t,u) - f(t,v)| \leq \frac{1}{25}|u - v|$ *and* $\ell Q \simeq 0.302317 < 1$. *Thus all the conditions of Theorem 3.1 are satisfied. Hence the conclusion of Theorem 3.1 implies that there exists a unique solution for problem (18)on* $[0, 2]$.

(b) *Observe that* $|f(t,u)| \leq g(t) = \frac{1}{25} + e^{-t}\cos t$ *with* $\|g\| = \frac{26}{25}$ *and* $\ell Q_1 \simeq 0.260543 < 1$. *Thus all the conditions of Theorem (3.3) are satisfied. Hence, by the conclusion of Theorem (3.3), the problem (18) has at least one solution on* $[0, 2]$.

(c) *Obviously* $|f(t,u)| \leq 1/25 + e^{-t}\cos t$. *Taking* $\psi(\|u\|) = 1$, $p(t) = 1/25 + e^{-t}\cos t$, *we have* $\overline{Q} = \|\nu_1\| + \|p\|\psi(\|\overline{M}\|)Q \simeq 13.224428$ *(*$\overline{Q}$ *is given by (17)) so that* $\overline{M} > 13.224428$. *Thus all the conditions of Theorem 3.5 are satisfied. Hence it follows by the conclusion of Theorem 3.5 that the problem (18) has at least one solution on* $[0, 2]$.

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

JUNG RYE LEE[1], CHOONKIL PARK[2*], DONG YUN SHIN[3*], AND SUNGSIK YUN[4]

Abstract. Let

$$M_1 f(x,y) := \frac{3}{4} f(x+y) - \frac{1}{4} f(-x-y) + \frac{1}{4} f(x-y) + \frac{1}{4} f(y-x) - f(x) - f(y),$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities

$$N(M_1 f(x,y),t) \geq N\left(\rho M_2 f(x,y),t\right) \tag{0.1}$$

where ρ is a fixed real number with $|\rho| < 1$, and

$$N\left(M_2 f(x,y),t\right) \geq N\left(\rho M_1 f(x,y),t\right) \tag{0.2}$$

where ρ is a fixed real number with $|\rho| < \frac{1}{2}$.

1. Introduction and preliminaries

Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [15, 21, 48]. In particular, Bag and Samanta [3], following Cheng and Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 25, 26] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [3, 25, 26, 27] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [25, 28].

Definition 1.2. [3, 28, 26, 27] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

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*Corresponding authors.

Definition 1.3. [3, 28, 26, 27] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

The stability problem of functional equations originated from a question of Ulam [47] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [46] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 9, 10, 14, 22, 24, 29, 34, 35, 36, 40, 41, 42, 43, 44, 45, 49, 50]).

We recall a fundamental result in fixed point theory.

Theorem 1.4. [6, 13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [7, 8, 30, 31, 38]).

Park [32, 33] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

2. Additive-quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$.

We need the following lemma to prove the main results.

Lemma 2.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$N(M_1 f(x,y),t) \geq N(\rho M_2 f(x,y),t) \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$, then f is the Cauchy additive mapping.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (2.1), then f is the quadratic mapping.*

Proof. (i) Letting $y = x$ in (2.1), we get $N(f(2x) - 2f(x), t) = 1$ for all $t > 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &= N(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t) \\ &= N(\rho(f(x+y) - f(x) - f(y)), t) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$ by (N_5).

(ii) Letting $y = x$ in (2.1), we get $N\left(\frac{1}{2} f(2x) - 2f(x), t\right) = 1$ for all $t > 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &N\left(\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y), t\right) \\ &= N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N\left(\rho\left(\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y)\right), t\right) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$ by (N_5). □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4} \varphi(2x, 2y) \leq \frac{L}{2} \varphi(2x, 2y) \tag{2.4}$$

for all $x, y \in X$.
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N(M_1 f(x,y),t) \geq \min\left\{N(\rho M_2 f(x,y),t), \frac{t}{t + \varphi(x,y)}\right\} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. (i) Letting $y = x$ in (2.5), we get

$$N\left(f(2x) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x)} \tag{2.8}$$

and so

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \tag{2.9}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [23, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.9) that $N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right)=\frac{1}{2}A(x) \tag{2.10}$$

for all $x\in X$. Since $f:X\to Y$ is odd, $A:X\to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M=\{g\in S: d(f,g)<\infty\}.$$

This implies that A is a unique mapping satisfying (2.10) such that there exists a $\mu\in(0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$;

(2) $d(J^nf,A)\to 0$ as $n\to\infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty}2^nf\left(\frac{x}{2^n}\right)=A(x)$$

for all $x\in X$;

(3) $d(f,A)\leq\frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,A)\leq\frac{L}{2-2L}.$$

This implies that the inequality (2.6) holds.

By (2.5),

$$N\left(2^nM_1f\left(\frac{x}{2^n},\frac{y}{2^n}\right),2^nt\right)\geq\min\left\{N\left(2^nM_2f\left(\frac{x}{2^n},\frac{y}{2^n}\right),2^nt\right),\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\right\}$$

and so

$$N\left(2^nM_1f\left(\frac{x}{2^n},\frac{y}{2^n}\right),t\right)\geq\min\left\{N\left(2^nM_2f\left(\frac{x}{2^n},\frac{y}{2^n}\right),t\right),\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}\right\}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}=1$ for all $x,y\in X$ and all $t>0$,

$$N\left(M_1A(x,y),t\right)\geq N\left(\rho M_2A(x,y),t\right)$$

for all $x,y\in X$ and all $t>0$. By Lemma 2.1, the mapping $A:X\to Y$ is Cauchy additive.

(ii) Letting $y=x$ in (2.5), we get

$$N\left(\frac{1}{2}f\left(2x\right)-2f(x),t\right)\geq\frac{t}{t+\varphi(x,x)} \tag{2.11}$$

and so

$$N\left(f\left(x\right)-4f\left(\frac{x}{2}\right),t\right)\geq\frac{\frac{t}{2}}{\frac{t}{2}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)}=\frac{t}{t+2\varphi\left(\frac{x}{2},\frac{x}{2}\right)} \tag{2.12}$$

for all $x\in X$.

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=4g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.12) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.13}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.7) holds.

By (2.5),

$$N\left(4^n M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 4^n t\right) \geq \min\left\{N\left(4^n M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\}$$

and so

$$N\left(4^n M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \min\left\{N\left(4^n M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(M_1 Q(x,y), t\right) \geq N\left(\rho M_2 Q(x,y), t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N(M_1 f(x,y),t) \geq \min\left\{N\left(\rho M_2 f(x,y),t\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\} \tag{2.14}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right) \geq \frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.14). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x)-Q(x),t\right) \geq \frac{(2^p-4)t}{(2^p-4)t+4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p+\|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{2.15}$$

for all $x, y \in X$..
(i) *Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x)-A(x),t\right) \geq \frac{(2-2L)t}{(2-2L)t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x)-Q(x),t\right) \geq \frac{(2-2L)t}{(2-2L)t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.
(i) It follows from (2.8) that

$$N\left(f(x)-\frac{1}{2}f(2x),\frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
(ii) It follows from (2.11) that

$$N\left(f(x)-\frac{1}{4}f(2x),\frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying* (2.14). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.14). *Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-2^p)t}{(4-2^p)t + 4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

3. Additive-quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < \frac{1}{2}$.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$N(M_2 f(x, y), t) \geq N(\rho M_1 f(x, y), t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$, then f is the Cauchy additive mapping.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and* (3.1), *then f is the quadratic mapping.*

Proof. (i) Letting $y = 0$ in (3.1), we get $N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) = 1$ for all $t > 0$. So

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &\geq N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ &= N(\rho(f(x+y) - f(x) - f(y)), t) \end{aligned}$$

for all $t > 0$ and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$ by (N_5).

(ii) Letting $y = 0$ in (3.1), we get $N\left(4f\left(\frac{x}{2}\right) - f(x), t\right)$ for all $t > 0$. So

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{3.3}$$

for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned} &N\left(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y), t\right) \\ &\geq N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ &= N\left(\rho\left(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\right), t\right) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all $x, y \in X$ by (N_5). □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying* (2.4).
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y),t\right) \geq \min\left\{N\left(\rho M_1 f(x,y),t\right), \frac{t}{t+\varphi(x,y)}\right\} \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right) \geq \frac{(1-L)t}{(1-L)t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (3.4). *Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x)-Q(x),t\right) \geq \frac{(1-L)t}{(1-L)t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. (i) Letting $y = 0$ in (3.4), we get

$$N\left(f(x)-2f\left(\frac{x}{2}\right),t\right) = N\left(2f\left(\frac{x}{2}\right)-f(x),t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [23, Lemma 2.1]).

The rest of the proof is similar to the proof of Theorem 2.2 (i).

(ii) Letting $y = 0$ in (3.4), we get

$$N\left(f(x)-4f\left(\frac{x}{2}\right),t\right) = N\left(4f\left(\frac{x}{2}\right)-f(x),t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.6}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2 (ii). □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y)-\rho M_1 f(x,y),t\right) \geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right) \geq \frac{(2^p-2)t}{(2^p-2)t+2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (2.15).*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.4). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.
(i) It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{2Lt}{2Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.
(ii) It follows from (3.6) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{t}{4}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{4}f(2x), Lt\right) \geq \frac{4Lt}{4Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.
The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.7). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-2^p)t}{(4-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

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[1]Department of Mathematics, Daejin University, Kyunggi 11159, Korea
E-mail address: jrlee@hdaejin.ac.kr

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

[3]Department of Mathematics, t University of Seoul, Seoul 02504, Korea
E-mail address: dyshin@uos.ac.kr

[4]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
E-mail address: ssyun@hs.ac.kr

Hyers-Ulam stability of set-valued functional equations: a fixed point approach

Sungsik Yun, Choonkil Park* and Hassan Azadi Kenary*

Abstract. In [36], Park proved the Hyers-Ulam stability of set-valued functional equations by using the direct method.

In this paper, we prove the Hyers-Ulam stability of set-valued functional equations by using the fixed point method.

1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering paper by Aumann [5] and Debreu [14] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [3], McKenzie [29], the momographs by Hindenbrand [20], Aubin and Frankowska [4], Castaing and Valadier [8], Klein and Thompson [26] and the survey by Hess [19].

The stability problem of functional equations originated from a question of Ulam [50] concerning the stability of group homomorphisms. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [49] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [13] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 17, 18, 22, 23], [41]–[48]).

In [25], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [28], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

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*Corresponding authors.

Hyers-Ulam stability of set-valued functional equations

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [9, 15] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 11, 31, 39]).

Let Y be a Banach space. We define the following:

2^Y : the set of all subsets of Y;

$C_b(Y)$: the set of all closed bounded subsets of Y;

$C_c(Y)$: the set of all closed convex subsets of Y;

$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.

On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$\begin{aligned} d(x, C) &= \inf\{\|x - y\| : y \in C\}, \qquad x \in Y, \\ s(x^*, C) &= \sup\{\langle x^*, x\rangle : x \in C\}, \qquad x^* \in Y^*. \end{aligned}$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \qquad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.2. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*

(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;

(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [8]). Debreu [14] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

Lemma 1.3. [14] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*

(a) $j(A \oplus B) = j(A) + j(B)$;

(b) $j(\lambda A) = \lambda j(A)$;

(c) $h(A, B) = \|j(A) - j(B)\|_u$;

(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$

for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [7]). In this case, the Debreu integral of f in Ω is the unique element $(D)\int_\Omega f d\nu \in C_{cb}(Y)$ such tha $j((D)\int_\Omega f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6], [32]–[35], [37, 38]).

Using the fixed point method, we prove the additive set-valued functional equation, the quadratic set-valued functional equation, the cubic set-valued functional equation and the quartic set-valued functional equation.

Throughout this paper, let X be a real vector space and Y a Banach space.

2. Stability of the additive set-valued functional equation

Using the fixed point method, we prove the Hyers-Ulam stability of the additive set-valued functional equation.

Definition 2.1. [27] Let $f : X \to C_{cb}(Y)$. The additive set-valued functional equation is defined by

$$f(x + y) = f(x) \oplus f(y)$$

for all $x, y \in X$. Every solution of the additive set-valued functional equation is called an *additive set-valued mapping*.

Note that there are some examples in [27].

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying

$$h(f(x + y), f(x) \oplus f(y)) \leq \varphi(x, y) \tag{2.1}$$

for all $x, y \in X$. Then there exists a unique additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), A(x)) \leq \frac{L}{2-2L}\varphi(x,x) \tag{2.2}$$

for all $x \in X$.

Proof. Let $y = x$ in (2.1). Since $f(x)$ is convex, we get

$$h(f(2x), 2f(x)) \leq \varphi(x,x) \tag{2.3}$$

and so

$$h\left(f(x), 2f\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi\left(x,x\right) \tag{2.4}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \leq \mu\varphi(x,x), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g,f) = \varepsilon$. Then

$$h(g(x), f(x)) \leq \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$h(Jg(x), Jf(x)) \quad = \quad h\left(2g\left(\frac{x}{2}\right), 2f\left(\frac{x}{2}\right)\right) = 2h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \leq L\varphi(x,x)$$

for all $x \in X$. So $d(g,f) = \varepsilon$ implies that $d(Jg, Jf) \leq L\varepsilon$. This means that

$$d(Jg, Jf) \leq Ld(g,f)$$

for all $g, f \in S$.

It follows from (2.4) that $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$h(f(x), A(x)) \leq \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$h\left(2^n f\left(\frac{x+y}{2^n}\right), 2^n f\left(\frac{x}{2^n}\right) \oplus 2^n f\left(\frac{y}{2^n}\right)\right) \leq 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq L^n \varphi(x, y),$$

which tends to zero as $n \to \infty$ for all $x, y \in X$. Thus $A(x+y) = A(x) \oplus A(y)$, as desired. □

Corollary 2.3. *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h(f(x+y), f(x) \oplus f(y)) \leq \theta(||x||^p + ||y||^p) \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique additive set-valued mapping $A : X \to Y$ satisfying

$$h(f(x), A(x)) \leq \frac{2\theta}{2^p - 2}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (2.1). Then there exists a unique additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), A(x)) \leq \frac{1}{2-2L}\varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$h\left(f(x), \frac{1}{2} f(2x)\right) \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $1 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (2.6). Then there exists a unique additive set-valued mapping $A : X \to Y$ satisfying*

$$h(f(x), A(x)) \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

3. Stability of the quadratic set-valued functional equation

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic set-valued functional equation.

Definition 3.1. [27] Let $f : X \to C_{cb}(Y)$. The quadratic set-valued functional equation is defined by

$$2f(x+y) \oplus 2f(x-y) = f(2x) \oplus f(2y)$$

for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is called a *quadratic set-valued mapping*.

Note that there are some examples in [27].

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x,2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \leq \frac{L}{4-4L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.1). Since $f(x)$ is convex, we get

$$h(f(2x), 4f(x)) \leq \varphi(x,0) \tag{3.2}$$

and

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{4}\varphi(x,0) \tag{3.3}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \leq \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that

$$d(Jg, Jf) \leq Ld(g,f)$$

for all $g, f \in S$.

It follows from (3.3) that $d(f, Jf) \leq \frac{L}{4}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $p > 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \theta(||x||^p + ||y||^p) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to Y$ satisfying

$$h(f(x), Q(x)) \leq \frac{\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.1). Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \leq \frac{1}{4 - 4L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.2) that

$$h\left(f(x), \frac{1}{4}f\left(2x\right)\right) \leq \frac{1}{4}\varphi\left(x, 0\right)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Corollary 3.5. *Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.4). Then there exists a unique quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h(f(x), Q(x)) \leq \frac{\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

4. Stability of the cubic set-valued functional equation

Using the fixed point method, we define a cubic set-valued functional equation and prove the Hyers-Ulam stability of the cubic set-valued functional equation.

Definition 4.1. [36] Let $f : X \to C_{cb}(Y)$. The cubic set-valued functional equation is defined by

$$f(2x+y) \oplus f(2x-y) = 2f(x+y) \oplus 2f(x-y) \oplus 12f(x)$$

for all $x, y \in X$. Every solution of the cubic set-valued functional equation is called a *cubic set-valued mapping*.

Theorem 4.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{8}\varphi(2x,2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(f(2x+y) \oplus f(2x-y), 2f(x+y) \oplus 2f(x-y) \oplus 12f(x)) \leq \varphi(x,y) \tag{4.1}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), C(x)) \leq \frac{L}{16-16L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (4.1). Since $f(x)$ is convex, we get

$$h(2f(2x), 16f(x)) \leq \varphi(x,0) \tag{4.2}$$

and

$$h\left(f(x), 8f\left(\frac{x}{2}\right)\right) \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{16}\varphi(x,0) \tag{4.3}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \leq \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that

$$d(Jg, Jf) \leq Ld(g,f)$$

for all $g, f \in S$.

It follows from (4.3) that $d(f, Jf) \leq \frac{L}{16}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 4.3. *Let $p > 3$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h(f(2x+y) \oplus f(2x-y), 2f(x+y) \oplus 2f(x-y) \oplus 12f(x)) \leq \theta(||x||^p + ||y||^p) \tag{4.4}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping $C : X \to Y$ satisfying

$$h(f(x), C(x)) \leq \frac{\theta}{2(2^p - 8)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 4.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (4.1). Then there exists a unique cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), C(x)) \leq \frac{1}{16 - 16L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (4.2) that

$$h\left(f(x), \frac{1}{8}f\left(2x\right)\right) \leq \frac{1}{16}\varphi\left(x, 0\right)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 4.2. □

Corollary 4.5. *Let $3 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (4.4). Then there exists a unique cubic set-valued mapping $C : X \to Y$ satisfying*

$$h(f(x), C(x)) \leq \frac{\theta}{2(8 - 2^p)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

5. Stability of the quartic set-valued functional equation

Using the fixed point method, we define a quartic set-valued functional equation and prove the Hyers-Ulam stability of the quartic set-valued functional equation.

Definition 5.1. [36] Let $f : X \to C_{cb}(Y)$. The quartic set-valued functional equation is defined by

$$f(2x+y) \oplus f(2x-y) \oplus 6f(y) = 4f(x+y) \oplus 4f(x-y) \oplus 24f(x)$$

for all $x, y \in X$. Every solution of the quartic set-valued functional equation is called a *quartic set-valued mapping*.

Theorem 5.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{16}\varphi(2x,2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(f(2x+y) \oplus f(2x-y) \oplus 6f(y), 4f(x+y) \oplus 4f(x-y) \oplus 24f(x)) \leq \varphi(x,y) \tag{5.1}$$

for all $x, y \in X$. Then there exists a unique quartic set-valued mapping $T : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{L}{32 - 32L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (5.1). Since $f(x)$ is convex, we get

$$h(2f(2x), 32f(x)) \leq \varphi(x,0) \tag{5.2}$$

and

$$h\left(f(x), 16f\left(\frac{x}{2}\right)\right) \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{32}\varphi(x,0) \tag{5.3}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \leq \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that

$$d(Jg, Jf) \leq Ld(g,f)$$

for all $g, f \in S$.

It follows from (5.3) that $d(f, Jf) \leq \frac{L}{32}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 5.3. *Let $p > 4$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h(f(2x+y) \oplus f(2x-y) \oplus 6f(y), 4f(x+y) \oplus 4f(x-y) \oplus 24f(x)) \leq \theta(||x||^p + ||y||^p) \tag{5.4}$$

for all $x, y \in X$. Then there exists a unique quartic set-valued mapping $T : X \to Y$ satisfying

$$h(f(x), T(x)) \leq \frac{\theta}{2(2^p - 16)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.2 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{4-p}$ and we get the desired result. □

Theorem 5.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 16L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (5.1). Then there exists a unique quartic set-valued mapping $T : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{32-32L}\varphi(x,0)$$

for all $x \in X$.

Proof. It follows from (5.2) that

$$h\left(f(x), \frac{1}{16}f(2x)\right) \leq \frac{1}{32}\varphi(x,0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 5.2. □

Corollary 5.5. *Let $4 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (5.4). Then there exists a unique quartic set-valued mapping $T : X \to Y$ satisfying*

$$h(f(x), T(x)) \leq \frac{\theta}{2(2^p-16)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.4 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. □

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Sungsik Yun
Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
E-mail address: ssyun@hs.ac.kr

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Hassan Azadi Kenary
Department of Mathematics, College of Science, Yasouj University, Yasouj 75914-353, Iran
E-mail address: azadi@mail.yu.ac.ir

On precompactness of the Hausdorff fuzzy metric on closed sets

Chang-qing Li[a, *], Yan-lan Zhang[b]
[a]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, China
[b]College of Computer, Minnan Normal University, Zhangzhou, Fujian 363000, China
Email: helen_smile0320@163.com, zyl_1983_2004@163.com

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In the paper, we construct a Hausdorff fuzzy metric on the family of nonempty closed subsets of a stationary and F-bounded fuzzy metric space. Using the construction of the Hausdorff fuzzy metric, we prove three equivalent characterizations for the given fuzzy metric space to be precompact. Furthermore, several examples are given.

Keywords: Fuzzy metric, Continuous t-norm, The Hausdorff fuzzy metric, Closed subset, Precompact.

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1 Introduction

Fuzzy metric is an important notion in Fuzzy Topology. Many authors have introduced the concept of fuzzy metric from different points of view [2, 3, 12, 13]. In particular, George and Veeramani [3] obtained the concept of fuzzy metric with the help of continuous t-norms in 1994. Later, it was proved that the topological space induced by the fuzzy metric space is metrizable in [8]. This version of fuzzy metric determines the class of spaces that are tightly connected with the class of metrizable topological spaces. Hence it is interesting to study the version of fuzzy metric. Some contributions to the study of fuzzy metric spaces can be found in [4, 5, 6, 14, 15, 16, 19, 20].

In order to study the hyperspaces in a fuzzy metric space, Rodríguez-López and Romaguera [17] gave a definition of Hausdorff fuzzy metric on the family

*Corresponding author.
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of nonempty compact sets. Unfortunately, the Hausdorff fuzzy metric defined by the authors does not provide a fuzzy metric when one consider the family of nonempty closed and F-bounded subsets of a given fuzzy metric space. In [17], Rodríguez-López and Romaguera illustrated the result above with the help of a example. It is a nature problem to explore under what condition the Hausdorff fuzzy metric defined by Rodríguez-López and Romaguera on the family of nonempty closed and F-bounded subsets of a given fuzzy metric space can provide a fuzzy metric. This is done in the present paper.

We construct a Hausdorff fuzzy metric on the family of nonempty closed subsets of a stationary and F-bounded fuzzy metric space. Also, we prove three necessary and sufficient conditions for the given fuzzy metric space to be precompact. Moreover, we give some illustrative examples.

2 Preliminaries

Throughout the paper the letter **N** shall denote the set of all nature numbers. Our basic reference for general topology is [1].

Definition 2.1 [3] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-norms: $a * b = \min\{a, b\}; a * b = a \cdot b; a * b = \max\{a + b - 1, 0\}$.

Definition 2.2 [3] A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a fuzzy metric on X.

Definition 2.3 [3] Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1), t > 0$ and $x \in X$. The set

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the *open ball with center x and radius r with respect to t*.

Obviously, $\{B_M(x, r, t) | x \in X, t > 0, r \in (0, 1)\}$ forms a base of a topology in X. The topology is denoted by τ_M and is known to be metrizable (see [8]).

Lemma 2.4 [3] *Let $(X, M, *)$ be a fuzzy metric space. Then, for each $x \in X$, $\{B_M(x, \frac{1}{n}, \frac{1}{n}) | n \in \mathbf{N}\}$ is a neighborhood base at x for the topology τ_M.*

Definition 2.5 [3] Let (X, d) be a metric space. Define $a * b = ab$ for all $a, b \in [0, 1]$, and let M_d be the function on $X \times X \times (0, \infty)$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called *the standard fuzzy metric induced by d.*

Definition 2.6 [3] Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be *F-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

We call $(X, M, *)$ a *F-bounded fuzzy metric space* provided that X is F-bounded. Clearly, a subset of an F-bounded fuzzy metric space is F-bounded.

Definition 2.7 [10] A fuzzy metric space $(X, M, *)$ is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M(x, y, \cdot)$ is constant. In the case we write $M(x, y)$ and $B_M(x, r)$ instead of $M(x, y, t)$ and $B_M(x, r, t)$, respectively.

Lemma 2.8 [17] *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.*

3 The Hausdorff fuzzy metric on Cld(X)

Given a fuzzy metric space $(X, M, *)$, we will denote by $\mathcal{P}(X)$, Cld(X) and Fin(X), the set of nonempty subsets, the set of nonempty closed subsets and the set of nonempty finite subsets of (X, τ_M), respectively. For every $C \in \mathcal{P}(X)$, $a \in X$ and $t > 0$, let $M(a, C, t) := \sup_{c \in C} M(a, c, t)$, $M(C, a, t) := \sup_{c \in C} M(c, a, t)$ (see Definition 2.4 of [20]). It is clear that $M(a, C, t) = M(C, a, t)$.

Lemma 3.1 *Let $(X, M, *)$ be a fuzzy metric space, $a, c \in X$, $D \in \mathcal{P}(X)$ and $t, s \in (0, \infty)$. Then $M(a, D, t + s) \geq M(a, c, t) * M(c, D, s)$.*

Proof Note that, for each $d \in D$,

$$M(a, D, t + s) \geq M(a, d, t + s) \geq M(a, c, t) * M(c, d, s).$$

It follows from continuity of $*$ that

$$M(a, D, t + s) \geq M(a, c, t) * M(c, D, s).$$

Let $(X, M, *)$ be a fuzzy metric space, $A, C \in$ Cld(X) and $t > 0$, define H_M: Cld$(X)\times$ Cld$(X) \times (0, \infty) \to [0, 1]$ by

$$H_M(A, C, t) = \min\{\inf_{a \in A} M(a, C, t), \inf_{c \in C} M(A, c, t)\}.$$

If $(X, M, *)$ is a stationary fuzzy metric space, then we write $H_M(A, C)$, $M(a, C)$ and $M(A, c)$ instead of $H_M(A, C, t)$, $M(a, C, t)$ and $M(A, c, t)$, respectively.

Theorem 3.2 *Let $(X, M, *)$ be a stationary and F-bounded fuzzy metric space. Then (Cld(X),$H_M, *$) is a fuzzy metric space.*

Proof Let $A, C, D \in \mathrm{Cld}(X)$.

Obviously, (i), (ii), (iii) and (v) in Definition 2.2 hold.

Now, we are going to prove that (iv) in Definition 2.2 is satisfied, i.e., $H_M(A, D) \geq H_M(A, C) * H_M(C, D)$. Let $a \in A$. Then we can choose a sequence $\{c_n^a\}_{n\in\mathbf{N}}$ in C such that

$$\lim_{n\to+\infty} M(a, c_n^a) = M(a, C).$$

Since $\{M(c_n^a, D)\}_{n\in\mathbf{N}}$ is a sequence in [0,1], there is a subsequence $\{c_{n_l}^a\}_{l\in\mathbf{N}}$ of $\{c_n^a\}_{n\in\mathbf{N}}$ such that the sequence $\{M(c_{n_l}^a, D)\}_{l\in\mathbf{N}}$ converges to some point of [0,1]. It follows from Lemma 3.1 that

$$M(a, D) \geq M(a, c_{n_k}^a) * M(c_{n_k}^a, D)$$

for every $c_{n_k}^a \in \{c_{n_l}^a | l \in \mathbf{N}\}$. Therefore, by continuity of $*$, we have

$$\begin{aligned} M(a, D) &\geq \lim_{k\to+\infty} M(a, c_{n_k}^a) * \lim_{k\to+\infty} M(c_{n_k}^a, D) \\ &= M(a, C) * \lim_{k\to+\infty} M(c_{n_k}^a, D). \end{aligned}$$

According to continuity of $*$, we deduce that

$$\begin{aligned} \inf_{a\in A} M(a, D) &\geq \inf_{a\in A} M(a, C) * \inf_{a\in A} \lim_{k\to+\infty} M(c_{n_k}^a, D) \\ &\geq \inf_{a\in A} M(a, C) * \inf_{a\in A} \{M(c_{n_k}^a, D) \mid k \in \mathbf{N}\} \\ &\geq \inf_{a\in A} M(a, C) * \inf_{c\in C} M(c, D) \\ &\geq H_M(A, C) * H_M(C, D). \end{aligned}$$

Analogously, we get

$$\inf_{d\in D} M(A, d) \geq H_M(A, C) * H_M(C, D).$$

Hence

$$H_M(A, D) \geq H_M(A, C) * H_M(C, D).$$

$(H_M, *)$ will be called *the Hausdorff fuzzy metric on Cld(X).*

Next we will give two examples.

Example 3.3 Let $X = (1, 10]$. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define M by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a stationary and F-bounded fuzzy metric space (see [7]). So, by Theorem 3.2, (Cld(X),$H_M, *$) is a fuzzy metric space.

Example 3.4 Let $X = [3, 5]$. Denote $a*b = \max\{a+b-1, 0\}$ for all $a, b \in [0, 1]$ and let M be a fuzzy set on $X \times X \times (0, \infty)$ defined as follows:

$$M(x, y, t) = \begin{cases} 1 & x = y, \\ \frac{1}{x} + \frac{1}{y} & x \neq y, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a stationary and F-bounded fuzzy metric space (see [18]). Thus, according to Theorem 3.2, (Cld(X),$H_M, *$) is a fuzzy metric space.

More examples of stationary and F-bounded fuzzy metric spaces may be found in [7, 11, 18].

4 Precompactness of the Hausdorff fuzzy metric

We start this section by recalling the concept of precompact.

Definition 4.1 [8] A fuzzy metric space $(X, M, *)$ is called *precompact* if for each $r \in (0, 1)$ and $t > 0$, there is a finite subset A of X such that $X = \bigcup_{a \in A} B_M(a, r, t)$.

Theorem 4.2 *Let Y be a dense subspace of a stationary and F-bounded fuzzy metric space $(X, M, *)$. Then Fin(Y) is dense in (Cld(X), $H_M, *$) if and only if $(X, M, *)$ is precompact.*

Proof Assume that Fin(Y) is dense in (Cld(X), $H_M, *$). Let $r \in (0, 1)$. Since Fin(Y) $\subseteq$ Fin(X), we get

$$\text{Fin}(X) \cap B_{H_M}(X, r) \neq \emptyset.$$

Take $A \in \text{Fin}(X) \cap B_{H_M}(X, r)$. Then $A \in \text{Fin}(X)$ and $H_M(X, A) > 1 - r$. Hence

$$\inf_{x \in X} M(x, A) > 1 - r.$$

Let $x \in X$. Then $M(x, A) > 1 - r$. Since $A \in \text{Fin}(X)$, there exists an $a \in A$ such that

$$M(x, a) = M(x, A) > 1 - r.$$

We have

$$x \in B_M(a, r).$$

So

$$X \subset \bigcup_{a \in A} B_M(a, r).$$

It follows that $(X, M, *)$ is precompact.

Conversely, assume that $(X, M, *)$ is precompact. Let $D \in \text{Cld}(X)$ and $\varepsilon \in (0, 1)$. Then, by the continuity of $*$, there exists a $\delta \in (0, \varepsilon)$ such that

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

We only need to verify that $\mathrm{Fin}(Y) \cap B_{H_M}(D, \varepsilon) \neq \emptyset$. Since $(X, M, *)$ is precompact, there exists a $C = \{c_1, c_2, \cdots, c_n\} \in \mathrm{Fin}(X)$ such that

$$X = \bigcup_{i=1}^{n} B_M(c_i, \delta).$$

Now, we can find $C' = \{c_{n_1}, c_{n_2}, \cdots, c_{n_k}\} \subset C$ such that

$$D \subset \bigcup_{i=1}^{k} B_M(c_{n_i}, \delta)$$

and

$$D \cap B_M(c_{n_i}, \delta) \neq \emptyset (i = 1, 2, \cdots, k).$$

Take $a_{n_i} \in D \cap B_M(c_{n_i}, \delta)$ $(i = 1, 2, \cdots, k)$. Denote $A = \{a_{n_1}, a_{n_2}, \cdots, a_{n_k}\}$. Since Y is a dense subspace of X, we can find an $e_{n_i} \in B_M(c_{n_i}, \delta) \cap Y$ for every $i \in \{1, 2, \cdots, k\}$. So $E = \{e_{n_1}, e_{n_2}, \cdots, e_{n_k}\} \in \mathrm{Fin}(Y)$. For $e_{n_i} \in E$ $(i = 1, 2, \cdots, k)$, we have

$$M(D, e_{n_i}) \geq M(a_{n_i}, e_{n_i}) \geq M(a_{n_i}, c_{n_i}) * M(c_{n_i}, e_{n_i}) \geq (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Hence

$$\inf_{e \in E} M(D, e) = \min\{M(D, e_{n_i}) | i = 1, 2, \cdots, k\} > 1 - \varepsilon.$$

On the other hand, let $d \in D$. Then there exists a $c_{n_i} \in C'$ such that

$$d \in B_M(c_{n_i}, \delta).$$

Hence

$$M(d, E) \geq M(d, e_{n_i}) \geq M(d, c_{n_i}) * M(c_{n_i}, e_{n_i}) \geq (1 - \delta) * (1 - \delta).$$

So

$$\inf_{d \in D} M(d, E) \geq (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Hence

$$H_M(D, E) = \min\{\inf_{d \in D} M(d, E), \inf_{e \in E} M(D, e)\} > 1 - \varepsilon,$$

that is, $E \in B_{H_M}(D, \varepsilon)$. Consequently,

$$E \in \mathrm{Fin}(Y) \cap B_{H_M}(D, \varepsilon).$$

We complete the proof.

Let $(X, M, *)$ be a fuzzy metric space and $A \subset X$. We will denote by $M|_{A \times A \times (0, \infty)}$ the restriction of M on $A \times A \times (0, \infty)$. It is easy to see that $M|_{A \times A \times (0, \infty)}$ is a fuzzy metric on A. We will simply write $M|_A$ instead of $M|_{A \times A \times (0, \infty)}$ when confusion is not possible.

Lemma 4.3 *Let $(X, M, *)$ be a precompact fuzzy metric space and $A \subset X$. Then $(A, M|_A, *)$ is precompact.*

Proof Let $\varepsilon \in (0,1)$ and $t > 0$. Then there exists a $\delta \in (0, \varepsilon)$ such that $(1-\delta) * (1-\delta) > 1-\varepsilon$. Because of precompactness of $(X, M, *)$, we can find finite points $x_1, x_2, \cdots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} B_M(x_i, \delta, \frac{t}{2}).$$

Since $A \subset X$, there exists $\{x_{n_1}, x_{n_2}, \cdots, x_{n_k}\} \subset \{x_1, x_2, \cdots, x_n\}$ such that

$$A \subset \bigcup_{j=1}^{k} B_M(x_{n_j}, \delta, \frac{t}{2})$$

and

$$A \cap B_M(x_{n_j}, \delta, \frac{t}{2}) \neq \emptyset (j = 1, 2, \cdots, k).$$

Choose $y_{n_j} \in A \cap B_M(x_{n_j}, \delta, \frac{t}{2})$ $(j = 1, 2, \cdots, k)$. Then, for each $z \in B_M(x_{n_j}, \delta, \frac{t}{2})$, we have

$$M(z, y_{n_j}, t) \geq M(z, x_{n_j}, \frac{t}{2}) * M(x_{n_j}, y_{n_j}, \frac{t}{2}) \geq (1-\delta) * (1-\delta) > 1-\varepsilon.$$

So

$$B_M(x_{n_j}, \delta, \frac{t}{2}) \subset B_M(y_{n_j}, \varepsilon, t).$$

Hence

$$A \subset \bigcup_{j=1}^{k} B_M(x_{n_j}, \delta, \frac{t}{2}) \subset \bigcup_{j=1}^{k} B_M(y_{n_j}, \varepsilon, t).$$

Whence

$$A = (\bigcup_{j=1}^{k} B_M(y_{n_j}, \varepsilon, t)) \cap A = \bigcup_{j=1}^{k} (B_M(y_{n_j}, \varepsilon, t) \cap A) = \bigcup_{j=1}^{k} B_{M|_A}(y_{n_j}, \varepsilon, t).$$

We are done.

Theorem 4.4 *Let $(X, M, *)$ be a stationary and F-bounded fuzzy metric space. Then $(Cld(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact.*

Proof Suppose that $(\mathrm{Cld}(X), H_M, *)$ is precompact. For each $x, y \in X$, we have $H_M(\{x\}, \{y\}) = M(x, y)$. So we can regard X as a subset of $\mathrm{Cld}(X)$ and M as $H_M|_{\{\{x\}|x \in X\}}$. It follows from Lemma 4.3 that $(X, M, *)$ is precompact.

Conversely, suppose that $(X, M, *)$ is precompact. Let $\varepsilon \in (0,1)$ and $D \in \mathrm{Cld}(X)$. Then, by precompactness of $(X, M, *)$, we can find an $F \in \mathrm{Fin}(X)$ such that

$$X = \bigcup_{x \in F} B_M(x, \frac{\varepsilon}{2}).$$

Therefore, there exists an $F_D \subset F$ such that

$$D \subset \bigcup_{x \in F_D} B_M(x, \frac{\varepsilon}{2}).$$

Also, we can find $x_D \in B_M(x, \frac{\varepsilon}{2}) \cap D$ for every $x \in F_D$. Note that there exists an $x_y \in F_D$ such that

$$y \in B_M(x_y, \frac{\varepsilon}{2})$$

for every $y \in D$. It follows that

$$M(y, F_D) \geq M(y, x_y) > 1 - \frac{\varepsilon}{2}.$$

Hence

$$\inf_{y \in D} M(y, F_D) \geq 1 - \tfrac{\varepsilon}{2} > 1 - \varepsilon.$$

On the other hand, for each $x \in F_D$, we get

$$M(D, x) \geq M(x_D, x) > 1 - \frac{\varepsilon}{2}.$$

Hence

$$\inf_{x \in F_D} M(D, x) \geq 1 - \frac{\varepsilon}{2} > 1 - \varepsilon.$$

So

$$H_M(D, F_D) > 1 - \varepsilon,$$

i.e., $D \in B_{H_M}(F_D, \varepsilon)$. Since F is a finite set, we see that $\mathcal{F} = \{F_D | D \in \mathrm{Cld}(X)\}$ is a finite family. Observe that

$$\mathrm{Cld}(X) = \bigcup_{F_D \in \mathcal{F}} B_{H_M}(F_D, \varepsilon).$$

It follows that $(\mathrm{Cld}(X), H_M, *)$ is precompact.

Definition 4.5 Let $(X, M, *)$ be a fuzzy metric space, $Y \subset X$, $r \in (0, 1)$ and $t > 0$. Y is said to be *fuzzy r discrete with respect to t* if $M(x, y, t) < 1 - r$ whenever $x, y \in Y$ and $x \neq y$.

Definition 4.6 Let $(X, M, *)$ be a fuzzy metric space and $Y \subset X$. Y is called a *fuzzy uniformly discrete set* provided that there exist $r \in (0, 1)$ and $t > 0$ such that Y is fuzzy r discrete with respect to t.

According to Zorn's lemma, it is straightforward to show that, by the inclusion relationship of the sets, X has a maximal subset which is fuzzy r discrete with respect to t for all $r \in (0, 1)$ and $t > 0$.

Lemma 4.7 *Let $(X, M, *)$ be a fuzzy metric space and Y be a fuzzy uniformly discrete subset of X. Then Y is a closed set in (X, τ_M).*

Proof By assumption, we can find some $r_0 \in (0,1)$ and $t_0 > 0$ such that Y is fuzzy r_0 discrete with respect to t_0. According to the continuity of $*$, there exists a $\varepsilon \in (0, r_0)$ such that

$$(1-\varepsilon) * (1-\varepsilon) > 1 - r_0.$$

Let $x \notin Y$. To complete our proof, it suffices to prove that there exists a open set U of x in X such that $U \cap Y = \emptyset$. For every $y, z \in B_M(x, \varepsilon, \frac{t_0}{2})$, we have

$$M(y, z, t_0) \geq M(x, y, \frac{t_0}{2}) * M(x, z, \frac{t_0}{2}) \geq (1-\varepsilon) * (1-\varepsilon) > 1 - r_0.$$

So $B_M(x, \varepsilon, \frac{t_0}{2})$ contains at most one point of Y. If $B_M(x, \varepsilon, \frac{t_0}{2}) \cap Y = \emptyset$, then $B_M(x, \varepsilon, \frac{t_0}{2})$ is the required open set. If $B_M(x, \varepsilon, \frac{t_0}{2}) \cap Y = \{a\}$, then, by the Hausdorffness of $(X, M, *)$, we can choose an $n \in \mathbf{N}$ such that

$$a \notin B_M(x, \frac{1}{n}, \frac{1}{n}).$$

So we get

$$x \in B_M(x, \varepsilon, \frac{t_0}{2}) \cap B_M(x, \frac{1}{n}, \frac{1}{n}),$$

with

$$B_M(x, \varepsilon, \frac{t_0}{2}) \cap B_M(x, \frac{1}{n}, \frac{1}{n}) \cap Y = \emptyset,$$

which implies that $B_M(x, \varepsilon, \frac{t_0}{2}) \cap B_M(x, \frac{1}{n}, \frac{1}{n})$ is the required open set. We are done.

Lemma 4.8 *Let $(X, M, *)$ be a fuzzy metric space and Y be an uncountable fuzzy uniformly discrete subset of X. Then X is not separable.*

Proof Suppose that X is separable. Then we take a countable and dense subset A in X. According to assumption, we can find some $r_0 \in (0,1)$ and $t_0 > 0$ such that $M(x, y, t_0) < 1 - r_0$ for all $x, y \in Y$ and $x \neq y$. According to continuity of $*$, there exists a $\varepsilon \in (0, r_0)$ such that

$$(1-\varepsilon) * (1-\varepsilon) > 1 - r_0.$$

Pick $a, b \in Y$, with $a \neq b$. We conclude that

$$B_M(a, \varepsilon, \frac{t_0}{2}) \cap B_M(b, \varepsilon, \frac{t_0}{2}) = \emptyset.$$

Indeed, otherwise, we can take a $c \in B_M(a, \varepsilon, \frac{t_0}{2}) \cap B_M(b, \varepsilon, \frac{t_0}{2})$. Then

$$M(a, b, t_0) \geq M(a, c, \frac{t_0}{2}) * M(c, b, \frac{t_0}{2}) \geq (1-\varepsilon) * (1-\varepsilon) > 1 - r_0,$$

which contradicts $M(a, b, t_0) < 1 - r_0$. So $\{B_M(y, \varepsilon, \frac{t_0}{2}) | y \in Y\}$ is an uncountable and pair-wise disjoint open family. Since A is dense in X, we see that

$B_M(y, \varepsilon, \frac{t_0}{2}) \cap A \neq \emptyset$ for all $y \in Y$. This shows that A is uncountable, which is a contradiction. We complete the proof.

Lemma 4.9 [**8**] *Let $(X, M, *)$ be a precompact fuzzy metric space. Then it is separable.*

Theorem 4.10 *Let $(X, M, *)$ be a stationary and F-bounded fuzzy metric space. Then $(X, M, *)$ is precompact if and only if (Cld(X),$H_M, *)$ is separable.*

Proof Suppose that $(X, M, *)$ is precompact. then, according to Theorem 4.4, we deduce that (Cld(X),$H_M, *)$ is precompact. So, by Lemma 4.9, we see that (Cld(X),$H_M, *)$ is separable.

Conversely, Let (Cld(X),$H_M, *)$ be separable. Suppose that $(X, M, *)$ fails to be precompact. Then there exists an infinite fuzzy uniformly discrete subset Y of X. Observe that $\mathcal{P}(Y)$ is the set of nonempty subsets of Y. Then $\mathcal{P}(Y)$ is uncountable. Take $A, C \in \mathcal{P}(Y)$. Then, by Lemma 4.7, we see that $A, C \in$ Cld(X). Now, for each $a \in A$ and $c \in C$, we can find $r_0 \in (0, 1)$ such that

$$M(a, c) < 1 - r_0.$$

So

$$M(a, C) \leq 1 - r_0.$$

It follows that

$$\inf_{a \in A} M(a, C) \leq 1 - r_0.$$

Similarly, we obtain

$$\inf_{c \in C} M(A, c) \leq 1 - r_0.$$

Hence

$$H_M(A, C) \leq 1 - r_0 < 1 - \frac{r_0}{2},$$

which implies that $\mathcal{P}(Y)$ is a fuzzy uniformly discrete subset of Cld(X). Thus, by Lemma 4.8, Cld(X) fails to be separable, which is a contradiction. Consequently, $(X, M, *)$ is precompact.

From Theorem 4.2, Theorem 4.4 and Theorem 4.10 we immediately deduce the next corollary.

Corollary 4.11 *Let Y be a dense subspace of a stationary and F-bounded fuzzy metric space $(X, M, *)$. Then the following are equivalent.*

(i) *$(X, M, *)$ is precompact.*

(ii) *Fin(Y) is dense in (Cld(X), $H_M, *)$.*

(iii) *(Cld(X), $H_M, *)$ is precompact.*

(iv) *(Cld(X),$H_M, *)$ is separable.*

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Semilocal convergence of a modified Chebyshev-like's method for solving nonlinear equations under generalized weak condition *

Lin Zheng † 1, 2

1. *School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu 233030, China;*
2. *Department of Mathematics, Shanghai University, Shanghai 200444, China*

Abstract: In this paper, the semilocal convergence of the modified Chebyshev-like's method is established by using recurrence relations under generalized weak condition. We prove an existence-uniqueness theorem and give a priori error bound which demonstrates the R-order convergence of the method. Moreover, the dynamical behavior of this method is also studied. Finally, numerical examples are presented to demonstrate our approach.

Keywords: Nonlinear equations; Chebyshev-like's method; Recurrence relations; Semilocal convergence; A priori error bounds; Dynamics.

1 Introduction

A number of problems arisen from scientific and engineering areas often needs to find the solution of nonlinear equations in Banach spaces

$$F(x) = 0, \tag{1.1}$$

where F is a third-order Fréchet-differentiable operator defined on a convex subset Ω of a Banach space X with values in a Banach space Y.

Generally, iterative methods are often used to solve this problem [1]. Newton's method being a second-order method is one of best known of these methods. The convergence of Newton's method in Banach spaces was established by Kantorovich in [2]. The convergence of the sequence obtained by the iterative expression is derived from the convergence of majorizing sequences. This technique has been used by many authors in order to establish the order of convergence of the variants of Newton's methods [3-9]. An alternative approach is developed to establish this convergence by using recurrence relations. The approach is also a very popular technique to

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†Corresponding author.
E-mail addresses: hbzhenglin@126.com.

establish the convergence of iterative methods. For example, it has been successfully applied to the convergence analysis of Newton's method and some high-order methods [10-23].

In [9], we introduce a modified Chebyshev-like's method given by

$$x_{n+1} = x_n - \Big[I + \frac{1}{2}K_F(x_n) + \frac{1}{2}K_F^2(x_n)\Big]\Gamma_n F(x_n), \tag{1.2}$$

where $\Gamma_n = [F'(x_n)]^{-1}$, $K_F(x_n) = \Gamma_n F''(u_n)\Gamma_n F(x_n)$ and $u_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$.

By assuming that

(A1) Γ_0 exist and $\|\Gamma_0 F(x_0)\| \leq \eta$,

(A2) $\|\Gamma_0\| \leq \beta$,

(A3) $\|F^{''}(x)\| \leq M, x \in \Omega$,

(A4) $\|F'''(x)\| \leq N, x \in \Omega$,

(A5) $\|F^{'''}(x) - F^{'''}(y)\| \leq L\|x - y\|, \ \ \forall\ x, y \in \Omega$,

we have analyzed the semilocal convergence of the method given by (1.2) by majorizing sequences and proved the R-order is improved to four, the computation efficiency and error estimate were also given. Numerical applications shows this method can solve some equations successfully.

But under assumptions (A1)-(A5), we can not study the solution of some equations. Such as the nonlinear integral equation of mixed Hammerstein type, which is given by

$$x(s) + \sum_{i=1}^{m} \int_a^b G_i(s,t)H_i(x(t))dt = u(s), \quad s \in [a,b], \tag{1.3}$$

where $-\infty < a < b < \infty$, u, H_i and G_i, for $i = 1, 2, \cdots, m$, are known functions and x is a solution to be determined. The problem is from the dynamic model of a chemical reactor [24]. On the condition that $H_i'''(x(t))$ is (L_i, q_i)-Hölder continuous in Ω, $i = 1, 2, \ldots, m$, then corresponding operator $F : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[F(x)](s) = x(s) + \sum_{i=1}^{m} \int_a^b G_i(s,t)H_i(x(t))dt - u(s), \quad s \in [a,b], \tag{1.4}$$

is such that its third Fréchet derivative is neither Lipschitz continuous nor Hölder continuous in Ω while, for an example, we consider the max-norm. For this case,

$$\|F'''(x) - F'''(y)\| \leq \sum_{i=1}^{m} L_i\|x - y\|^{q_i}, \ L_i \geq 0, q_i \in [0,1], \quad x, y \in \Omega. \tag{1.5}$$

Because of the importance of nonlinear integral equation of mixed Hammerstein type, several authors [18-20] have considered a mild condition

$$\|F'''(x) - F'''\| \leq \omega(\|x - y\|), \ x, \ y \in \Omega, \tag{1.6}$$

where $\omega(z)$ is nondecreasing continuous real valued function for $z > 0$, such that $\omega(0) \geq 0$, on F''' to study the semilocal convergence of some iterative methods.

In the paper, the semilocal convergence of the modified Chebyshev-like's method is established by using recurrence relations under the assumption that F''' satisfies the ω-continuity condition (1.6), An existence-uniqueness theorem is also established for the solution along with its a priori error bounds. Moreover, the dynamical behavior of modified Chebyshev-like's method is also studied. Finally numerical examples are presented to demonstrate our approach.

2 Recurrence relations

Let $x_0 \in \Omega$ and the nonlinear operator $F : \Omega \subset X \to Y$ be continuously third-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. We assume that

(C1) Γ_0 exist and $\|\Gamma_0 F(x_0)\| \leq \eta$,

(C2) $\|\Gamma_0\| \leq \beta$,

(C3) $\|F''(x)\| \leq M, x \in \Omega$,

(C4) $\|F'''(x)\| \leq N, x \in \Omega$,

(C5) $\|F'''(x) - F'''(y)\| \leq \omega(\|x - y\|), \ \forall\ x, y \in \Omega$, where $\omega(z)$ is non-decreasing continuous real function for $z > 0$ and satisfy $\omega(0) \geq 0$,

(C6) there exists a non-negative real function $\varphi(t) \leq 1$, such that $\omega(tz) \leq \varphi(t)\omega(z)$, for $t \in [0, 1]$, $z \in (0, +\infty)$.

Notice that condition (C6) is not restrictive, since we can always consider $\varphi(t) = 1$, as a consequence of ω is non-decreasing function, but its interest is to sharp the priori error bounds.

Firstly we give the following lemma to show an approximation of operator F, which will be used in the latter developments.

Lemma 1 Assume that the nonlinear operator $F : \Omega \subset X \to Y$ is continuously third-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. Then we have

$$\begin{aligned} F(x_{n+1}) = &-\frac{1}{2}\big[F''(u_n) - F''(x_n)\big](y_n - x_n)K_F(x_n)(y_n - x_n) \\ &+ \frac{1}{2}F''(x_n)(y_n - x_n)K_F^2(x_n)(y_n - x_n) \\ &+ \int_0^1 F''\big(y_n + t(x_{n+1} - y_n)\big)(1 - t)dt(x_{n+1} - y_n)^2 \\ &- \frac{1}{6}\int_0^1 \Big[F'''\Big(x_n + \frac{1}{3}t(y_n - x_n)\Big) - F'''(x_n)\Big]dt(y_n - x_n)^3 \\ &+ \frac{1}{2}\int_0^1 \Big[F'''\big(x_n + t(y_n - x_n)\big) - F'''(x_n)\Big](1 - t)^2 dt(y_n - x_n)^3 \\ &+ \int_0^1 F'''\big(x_n + t(y_n - x_n)\big)(1 - t)dt(y_n - x_n)^2(x_{n+1} - y_n). \end{aligned} \tag{2.1}$$

Now we consider the following scalar functions

$$g(t) = 1 + \frac{1}{2}t + \frac{1}{2}t^2, \tag{2.2}$$

$$h(t) = \frac{1}{1 - tg(t)}, \tag{2.3}$$

$$\ell(t, u, v) = \frac{1}{8}t^3(t^2 + 2t + 5) + \frac{1}{12}(3t + 5)tu + \frac{I_1 + 3I_2}{6}v, \tag{2.4}$$

where $I_1 = \int_0^1 \varphi(\frac{1}{3}t)dt$ and $I_2 = \int_0^1 \varphi(t)(1 - t)^2 dt$.

Let $\Phi(t) = g(t)t - 1$. Since $\Phi(0) = -1$ and $\Phi(1) = 1 > 0$, then $\Phi(t)$ has at least a zero in $(0, 1)$. Let s is the smallest positive zero of the scalar function $g(t)t - 1$.

Denote $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, $c_0 = w(\eta)\beta\eta^2$ and $d_0 = h(a_0)\ell(a_0, b_0, c_0)$. Next, some properties of the functions g, h, ℓ defined in (2.2)-(2.4) are given in the following lemma.

Lemma 2 Let the real functions g, h and ℓ be given in (2.2)-(2.4). Then
(i) $g(t)$ and $h(t)$ are increasing and $g(t) > 1, h(t) > 1$ for $t \in (0, s)$,
(ii) $\ell(t, u, v)$ is increasing for $t \in (0, s), u > 0, v > 0$.

For $n = 0$, the existence of Γ_0 implies the existence of y_0, u_0. This gives us

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta,$$

and

$$\|u_0 - x_0\| = \frac{1}{3}\Big\|\Gamma_0 F(x_0)\Big\| \leq \frac{1}{3}\eta.$$

This means that $y_0,\ u_0 \in B(x_0, R\eta)$, where $R = g(a_0)/\big(1 - d_0\big)$. Furthermore, we have

$$\begin{aligned}\|K_F(x_0)\| &= \|\Gamma_0 F''(u_0)\Gamma_0 F(x_0)\| \\ &\leq \|\Gamma_0\|\|F''(u_0)\|\|\Gamma_0 F(x_0)\| \\ &\leq M\beta\eta = a_0.\end{aligned}$$

We can obtain

$$\begin{aligned}\|x_1 - x_0\| &= \Big\|I + \frac{1}{2}K_F(x_0) + \frac{1}{2}K_F^2(x_0)\Big\|\|\Gamma_0 F(x_0)\| \\ &\leq \Big[1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2\Big]\|y_0 - x_0\| = g(a_0)\|y_0 - x_0\|. \qquad (2.5)\end{aligned}$$

From the assumption $d_0 < 1/h(a_0) < 1$, it follows that $x_1 \in B(x_0, R\eta)$. We also have

$$\begin{aligned}\|x_1 - y_0\| &\leq \Big\|\frac{1}{2}K_F(x_0) + \frac{1}{2}K_F^2(x_0)\Big\|\|\Gamma_0 F(x_0)\| \\ &\leq \frac{a_0(1 + a_0)}{2}\|y_0 - x_0\|. \qquad (2.6)\end{aligned}$$

By $a_0 < s$ and $g(a_0) < g(s)$, we have

$$\begin{aligned}\|I - \Gamma_0 F'(x_1)\| &\leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \\ &\leq M\beta\|x_1 - x_0\| \leq a_0 g(a_0) < 1.\end{aligned}$$

It follows by the Banach lemma that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\|\Gamma_1\| \leq \frac{1}{1 - a_0 g(a_0)}\|\Gamma_0\| = h(a_0)\|\Gamma_0\|. \qquad (2.7)$$

By Lemma 1, we can get

$$\begin{aligned}\|F(x_1)\| &\leq \frac{a_0^2}{2}M\eta^2 + \frac{a_0}{6}N\eta^3 + \frac{1}{2}M\|x_1 - y_0\|^2 \\ &\quad + \frac{1}{6}I_1 w(\eta)\eta^3 + \frac{1}{2}I_2 w(\eta)\eta^3 + \frac{1}{2}N\eta^2\|x_1 - y_0\|. \qquad (2.8)\end{aligned}$$

Then from (2.7) and (2.8), we have

$$\begin{aligned}\|y_1 - x_1\| &= \|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\|\|F(x_1)\| \\ &\leq d_0\|y_0 - x_0\|. \qquad (2.9)\end{aligned}$$

Because of $g(a_0) > 1$, we obtain

$$\begin{aligned} \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\leq \Big(g(a_0) + d_0\Big)\eta \\ &< g(a_0)\Big(1 + d_0\Big)\eta < R\eta, \end{aligned} \tag{2.10}$$

which shows $y_1 \in B(x_0, R\eta)$.

In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leq Mh(a_0)\|\Gamma_0\|d_0\|y_0 - x_0\| = a_0 h(a_0)d_0, \tag{2.11}$$

$$N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leq Nh(a_0)\|\Gamma_0\|d_0^2\|y_0 - x_0\|^2 = b_0 h(a_0)d_0^2, \tag{2.12}$$

$$\begin{aligned} \omega(\|y_1 - x_1\|)\|\Gamma_1\|\|y_1 - x_1\|^2 &\leq h(a_0)\|\Gamma_0\|\omega(d_0\|y_0 - x_0\|)d_0^2\|y_0 - x_0\|^2 \\ &\leq c_0 h(a_0)d_0^2\varphi(d_0), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq g(a_1)\|y_1 - x_1\| + g(a_0)\|y_0 - x_0\| \\ &\leq (1 + d_0)g(a_0)\|y_0 - x_0\| < R\eta. \end{aligned} \tag{2.14}$$

Since

$$\begin{aligned} &\|I - \Gamma_1 F'(x_2)\| \leq \|\Gamma_1\|\|F'(x_1) - F'(x_2)\| \\ &\leq M\|\Gamma_1\|\|x_2 - x_1\| \leq a_0 h(a_0)d_0 g(a_0 h(a_0)d_0) < 1, \end{aligned}$$

and by the Banach lemma, $\Gamma_2 = [F'(x_2)]^{-1}$ exists and

$$\|\Gamma_2\| \leq h(a_0 h(a_0)d_0)\|\Gamma_1\|. \tag{2.15}$$

Hence x_2 is well defined.

We now write $a_0 h(a_0)d_0 = a_1$, $b_0 h(a_0)d_0^2 = b_1$, $c_0 h(a_0)d_0^2\varphi(d_0) = c_1$ and define for $n \geq 0$

$$\begin{aligned} a_{n+1} &= a_n h(a_n)d_n, \\ b_{n+1} &= b_n h(a_n)d_n^2, \\ c_{n+1} &= c_n h(a_n)d_n^2\varphi(d_n), \\ d_{n+1} &= h(a_{n+1})\ell(a_{n+1}, b_{n+1}, c_{n+1}). \end{aligned} \tag{2.16}$$

Later developments will require the following lemma, where some properties of the previous scalar sequences are given.

Lemma 3 Let the real functions g, h and ℓ be given in (2.2)-(2.4). If

$$0 < a_0 < s \quad \text{and} \quad h(a_0)d_0 < 1, \tag{2.17}$$

then we have

(i) $h(a_n) > 1$ and $h(a_n)d_n < 1$ for $n \geq 0$,

(ii) the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are decreasing,

(iii) $g(a_n)a_n < 1$ and $h(a_n)d_n < 1$ for $n \geq 0$.

Our next goal is to guarantee that (1.2) is well defined. To do this, the system of recurrence relations given in the next lemma must be satisfied. The proof follows by using a similar way that the above-mentioned and invoking the induction hypothesis.

Lemma 4 Let the assumptions of Lemma 3 and the conditions (C1)-(C6) hold. Then the following items are true for all $n \geq 1$:

(i) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq h(a_{n-1})\|\Gamma_{n-1}\|$,

(ii) $\|y_n - x_n\| = \|\Gamma_n F(x_n)\| \leq d_{n-1}\|y_{n-1} - x_{n-1}\| \leq d_0^n\|y_0 - x_0\| < \eta$,

(iii) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leq a_n$,

(iv) $N\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \leq b_n$,

(v) $\omega(\|y_n - x_n\|)\|\Gamma_n\|\|y_n - x_n\|^2 \leq c_n$,

(vi) $\|x_{n+1} - x_n\| \leq g(a_n)\|y_n - x_n\|$,

(vii) $\|y_n - x_0\| \leq R\eta$ and $\|x_{n+1} - x_0\| \leq g(a_0)\frac{1-d_0^{n+1}}{1-d_0}\|y_0 - x_0\| < R\eta$, where $R = \frac{g(a_0)}{1-d_0}$.

3 Semilocal convergence

We are now interested in proving that sequence (1.2) is convergent. To do this, we will see that (1.2) is a Cauchy sequence. We will give some properties of the scalar sequence $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ in the following lemma.

Lemma 5 Let the real functions g, h and ℓ be given in (2.2)-(2.4). Let $\tau \in (0,1)$, then $g(\tau t) < g(t), h(\tau t) < h(t)$ and $\ell(\tau t, \tau^2 u, \tau^2 v) < \tau^2 \ell(t,u,v)$ for $t \in (0,s)$.

Lemma 6 Under the assumptions of Lemma 3. Let $\gamma = h(a_0)d_0$, $\delta = 1/h(a_0)$. For $n \geq 0$, we have

$$a_{n+1} \leq \gamma^{3^n} a_n \leq \gamma^{\frac{3^{n+1}-1}{2}} a_0, \tag{3.1}$$

$$b_{n+1} < \left(\gamma^{3^n}\right)^2 b_n < \gamma^{3^{n+1}-1} b_0, \tag{3.2}$$

$$c_{n+1} < \left(\gamma^{3^n}\right)^2 c_n < \gamma^{3^{n+1}-1} c_0, \tag{3.3}$$

$$d_{n+1} = h(a_{n+1})\ell(a_{n+1}, b_{n+1}, c_{n+1}) \leq \delta\gamma^{3^{n+1}}. \tag{3.4}$$

Proof The Lemma will be proved by induction. Since $a_1 = \gamma a_0$, by the above-mentioned assumption, we get $\gamma < 1$. We also get

$$b_1 = b_0 h(a_0) d_0^2 < \gamma^2 b_0,$$

and

$$c_1 = c_0 h(a_0) d_0^2 \varphi(d_0) < \gamma^2 c_0,$$

as $\varphi(d_0) \leq 1$.

Suppose (3.1)-(3.3) hold for $n = k$, then

$$\begin{aligned} a_{k+1} &= a_k h(a_k) d_k = a_k h^2(a_k)\ell(a_k, b_k, c_k) \\ &\leq \gamma^{3^{k-1}} a_{k-1} h^2\left(\gamma^{3^{k-1}} a_{k-1}\right)\ell\left(\gamma^{3^{k-1}} a_{k-1}, \left(\gamma^{3^{k-1}} b_{k-1}\right)^2, \left(\gamma^{3^{k-1}} c_{k-1}\right)^2\right) \\ &\leq \gamma^{3^{k-1}} a_{k-1} h^2(a_{k-1})\left(\gamma^{3^{k-1}}\right)^2 \ell(a_{k-1}, b_{k-1}, c_{k-1}) = \gamma^{3^k} a_k. \end{aligned}$$

We also have

$$b_{k+1} = b_k h(a_k) d_k^2 < \left(\frac{a_{k+1}}{a_k}\right)^2 b_k < \left(\gamma^{3^k}\right)^2 b_k,$$

and

$$c_{k+1} = c_k h(a_k) d_k^2 \varphi(d_k) < \left(\frac{a_{k+1}}{a_k}\right)^2 c_k \varphi(d_k) \leq \left(\gamma^{3^k}\right)^2 c_k,$$

as $\varphi(d_k) \leq 1$. Hence

$$\begin{aligned}
a_{k+1} &\leq \gamma^{3^k} a_k \leq \gamma^{3^k}\gamma^{3^{k-1}} \cdots \gamma^{3^0} a_0 = \gamma^{\frac{3^{k+1}-1}{2}} a_0, \\
b_{k+1} &< \left(\gamma^{3^k}\right)^2 b_k < \left(\gamma^{3^k}\right)^2 \left(\gamma^{3^{k-1}}\right)^2 \cdots \left(\gamma^{3^0}\right)^2 b_0 = \gamma^{3^{k+1}-1} b_0, \\
c_{k+1} &< \left(\gamma^{3^k}\right)^2 c_k < \left(\gamma^{3^k}\right)^2 \left(\gamma^{3^{k-1}}\right)^2 \cdots \left(\gamma^{3^0}\right)^2 c_0 = \gamma^{3^{k+1}-1} c_0.
\end{aligned}$$

Thus (3.1)-(3.3) hold by induction. Furthermore

$$\begin{aligned}
d_{n+1} &= h(a_{n+1})\ell(a_{n+1}, b_{n+1}, c_{n+1}) \leq h\Big(\gamma^{\frac{3^{n+1}-1}{2}} a_0\Big)\ell\Big(\gamma^{\frac{3^{n+1}-1}{2}} a_0, \gamma^{3^{n+1}-1} b_0, \gamma^{3^{n+1}-1} c_0\Big) \\
&= \gamma^{3^{n+1}} \frac{h(a_0)\ell(a_0, b_0, c_0)}{\gamma} = \delta\gamma^{3^{n+1}}.
\end{aligned}$$

The proof is completed.

3.1 Convergence theorem

Now we give a theorem to show the existence and uniqueness of the solution and the domain in which it is located, along with a priori error bounds, which lead to the R-order of convergence of iteration (1.2).

Theorem 1 Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ be a third-order Fréchet differentiable on a non-empty open convex subset Ω. g, h, ℓ are defined by (2.2)-(2.4). Let $a_0 = M\beta\eta, b_0 = N\beta\eta^2$ and $c_0 = w(\eta)\beta\eta^2$ satisfy $0 < a_0 < s$ and $h(a_0)d_0 < 1$, $\overline{B(x_0, R\eta)} \in \Omega$ where $R = g(a_0)/\big(1-d_0\big)$. Assume that $x_0 \in \Omega$ and all conditions (C1)-(C6) hold. Then starting from x_0, the sequence $\{x_n\}$ generated by the modified Chebyshev-like's method (1.2) converges to a solution x^* of $F(x) = 0$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. Moreover, a priori error estimate is given by

$$\|x_n - x^*\| \leq g(a_0)\eta\delta^n\gamma^{\frac{3^n-1}{2}} \frac{1}{1-\delta\gamma^{3^n}}, \tag{3.5}$$

where $\gamma = h(a_0)d_0$ and $\delta = 1/h(a_0)$.

Proof It is sufficient to show that $\{x_n\}$ is a Cauchy sequence in order to establish the convergence of $\{x_n\}$.

From Lemma 4 and Lemma 6, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq g(a_n)\|y_n - x_n\| \leq g(a_n)d_{n-1}\|y_{n-1} - x_{n-1}\| \\
&\leq \cdots \leq g(a_n)\|y_0 - x_0\| \prod_{j=0}^{n-1} d_j \\
&\leq g(a_n)\eta \prod_{j=0}^{n-1} (\delta\gamma^{3^j}) = g(a_n)\eta\delta^n\gamma^{\frac{3^n-1}{2}},
\end{aligned}$$

where $\gamma = h(a_0)d_0 < 1$ and $\delta = 1/h(a_0) < 1$. Hence,

$$\begin{aligned}
\|x_{m+n} - x_n\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq g(a_{m+n-1})\|y_{m+n-1} - x_{m+n-1}\| + \cdots + g(a_n)\|y_n - x_n\| \\
&\leq g(a_{m+n-1})\gamma^{\frac{3^{m+n-1}-1}{2}}\delta^{m+n-1}\eta + \cdots + g(a_n)\gamma^{\frac{3^n-1}{2}}\delta^n\eta \\
&\leq g(a_n)\delta^n\big[\gamma^{\frac{3^{m+n-1}-1}{2}}\delta^{m-1} + \cdots + \gamma^{\frac{3^n-1}{2}}\big]\eta \\
&= g(a_n)\delta^n\gamma^{\frac{3^n-1}{2}}\Big[\gamma^{\frac{3^n[3^{m-1}-1]}{2}}\delta^{m-1} + \cdots + \gamma^{\frac{3^n[3-1]}{2}}\delta + 1\Big]\eta.
\end{aligned}$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \geq 0$, we have $(1+x)^k - 1 \geq kx$. Thus,

$$\|x_{m+n} - x_n\| \leq g(a_0)\delta^n\gamma^{\frac{3^n-1}{2}}\ \frac{1-\gamma^{m\cdot 3^n}\delta^m}{1-\gamma^{3^n}\delta}\eta. \tag{3.6}$$

It follows that $\{x_n\}$ is a Cauchy sequence. So there exists a x^* such that $\lim\limits_{n\to\infty} x_n = x^*$.

For $m \geq 1$ and $n = 0$, we get

$$\|x_m - x_0\| \leq g(a_0)\frac{1-\gamma^m\delta^m}{1-\gamma\delta}\eta < R\eta.$$

Hence, $x_m \in B(x_0, R\eta)$, for all $m \geq 0$. By letting $n = 0$, $m \to \infty$ in (3.6), we obtain

$$\|x^* - x_0\| \leq R\eta. \tag{3.7}$$

This shows $x^* \in \overline{B(x_0, R\eta)}$.

Now we prove that x^* is a solution of $F(x) = 0$. Since

$$\begin{aligned}
\|F'(x_n)\| &\leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \\
&\leq \|F'(x_0)\| + M\|x_n - x_0\| \\
&\leq \|F'(x_0)\| + MR\eta,
\end{aligned}$$

we can obtain

$$\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\| \leq (\|F'(x_0)\| + MR\eta)\|\Gamma_n F(x_n)\|. \tag{3.8}$$

Since

$$\|\Gamma_n F(x_n)\| \leq d_{n-1}\|y_{n-1} - x_{n-1}\| = \cdots = \eta\Big(\prod_{i=0}^{n-1} d_i\Big) \leq \eta\delta^n\gamma^{\frac{3^n-1}{2}},$$

by letting $n \to \infty$, we obtain $\|\Gamma_n F(x_n)\| \to 0$, and $\|F(x_n)\| \to 0$ in (3.8). Hence, by the continuity of F in Ω, we obtain $F(x^*) = 0$.

Now we prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. Firstly we can obtain $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. Since $R = g(a_0)/\big(1 - d_0\big) < 1/a_0$, then we have

$$\frac{2}{M\beta} - R\eta = \Big(\frac{2}{a_0} - R\Big)\eta > \frac{1}{a_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subseteq B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$.

Let x^{**} be another zero of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. By Taylor theorem, we have

$$0 = F(x^{**}) - F(x^*) = \int_0^1 F'\big((1-t)x^* + tx^{**}\big)dt(x^{**} - x^*). \tag{3.9}$$

Since

$$\begin{aligned}
&\|\Gamma_0\|\left\| \int_0^1 \Big[F'\big((1-t)x^* + tx^{**}\big) - F'(x_0)\Big]dt\right\| \\
&\leq M\beta \int_0^1 \big[(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|\big]dt \\
&< \frac{M\beta}{2}\Big[R\eta + \frac{2}{M\beta} - R\eta\Big] = 1,
\end{aligned}$$

it follows by the Banach lemma that $\int_0^1 F'\big((1-t)x^* + tx^{**}\big)dt$ is invertible and hence $x^{**} = x^*$.

By letting $m \to \infty$ in (3.6), we obtain (3.5) and furthermore

$$\|x_n - x^*\| \leq \frac{g(a_0)\eta}{\gamma^{1/3}(1-d_0)}\big(\gamma^{1/3}\big)^{3^n}. \tag{3.10}$$

This means that the modified Chebyshev-like's method given by (1.2) is of R-order of convergence at least three.

3.2 R-order of convergence

Now we consider the following mixed condition

$$\|F'''(x) - F'''(y)\| \leq \sum_{i=1}^{m} L_i\|x-y\|^{q_i}, L_i \geq 0, q_i \in [0,1], x, y \in \Omega.$$

By choosing $w(\mu) = \sum_{i=1}^m (L_i\mu^{q_i})$, we have $w(t\mu) = \sum_{i=1}^m (L_i t^{q_i}\mu^{q_i})$, since $t \in [0,1]$, $q_i \in [0,1]$, then $\varphi(t) = t^p$, where $p = min\{q_1, q_2, \cdots, q_m\}$.

Now we consider that $\varphi(t) = t^p$, where $p \in (0,1]$. In this situation

$$I_1 = \int_0^1 \varphi\big(\frac{1}{3}t\big)dt = \frac{1}{3^p} \cdot \frac{1}{1+p}, \tag{3.11}$$

and

$$I_2 = \int_0^1 \varphi(t)(1-t)^2 dt = \frac{1}{(1+p)(2+p)(3+p)}. \tag{3.12}$$

The sequence $\{c_n\}$ is reduced to

$$c_{n+1} = c_n h(a_n) d_n^{2+p}, \quad n \geq 1.$$

Moveover

$$\ell(\tau t, \tau^2 u, \tau^{2+p} v) < \tau^{3+p}\ell(t,u,v), \quad \text{for } \tau \in (0,1),\ p \in [0,1].$$

Then

$$\begin{aligned}
a_{n+1} &\leq \gamma^{(3+p)^n} a_n \leq \gamma^{\frac{(3+p)^{n+1}-1}{2+p}} a_0, \quad n \geq 0, \\
b_{n+1} &< \gamma^{2(3+p)^n} b_n < \gamma^{\frac{2[(3+p)^{n+1}-1]}{2+p}} b_0, \quad n \geq 0, \\
c_{n+1} &< \gamma^{(2+p)(3+p)^n} c_n < \gamma^{(3+p)^{n+1}-1} c_0, \quad n \geq 0.
\end{aligned}$$

Consequently, the new a prior error estimates for iteration (1.2) are:

$$\|x^* - x_n\| \leq g(a_0)\eta\gamma^{\frac{(3+p)^n-1}{2+p}}\frac{\delta^n}{1-\gamma^{(3+p)^n}\delta}, \quad n \geq 0,$$

so that

$$\|x^* - x_n\| \leq \frac{g(a_0)\eta}{\gamma^{\frac{1}{2+p}}(1-\gamma\delta)}\Big(\gamma^{\frac{1}{2+p}}\Big)^{(3+p)^n}, \quad n \geq 0,$$

and the R-order of convergence is then at least $3+p$.

Remark Notice that $w(z) = Lz,\ L \geq 0$, if F''' is Lipschitz continuous in Ω and the R-order of convergence of iteration (1.2) is now at least four. And if F''' is (L,p)-Hölder continuous in Ω, then $w(z) = Lz^p,\ L \geq 0$, so that (1.2) is of R-order of convergence at least $3+p$.

4 Application

We illustrate the previous study with an application to the following nonlinear integral equations.

Example 1. Consider the mixed Hammerstein type integral equation [24]:

$$x(s) = 1 + \frac{1}{3}\int_0^1 G(s,t)\Big(x(t)^{10/3} + x(t)^4\Big)dt, \qquad s,t \in [0,1], \tag{4.1}$$

where $x \in X$. Here $X = C[0,1]$ is the space of continuous functions on $[0,1]$ with the max-norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|.$$

And the kernel G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

The analysis and computation of these types of equations are justified by the dynamic model of a chemical reactor.

Solving (4.1) is equivalent to solve $F(x) = 0$, where $F : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[F(x)](s) = x(s) - 1 - \frac{1}{3}\int_0^1 G(s,t)\Big(x(t)^{10/3} + x(t)^4\Big)dt, \quad s \in [0,1], \tag{4.2}$$

and Ω is a suitable non-empty open convex domain. Note that the first, second and third Fréchet derivatives of the operator F are given by

$$[F'(x)y](s) = y(s) - \frac{1}{3}\int_0^1 G(s,t)\Big(\frac{10}{3}x(t)^{7/3} + 4x(t)^3\Big)y(t)dt, \tag{4.3}$$

$$[F''(x)yz](s) = -\frac{1}{3}\int_0^1 G(s,t)\Big(\frac{70}{9}x(t)^{4/3} + 12x(t)^2\Big)z(t)y(t)dt, \tag{4.4}$$

$$[F'''(x)yzu](s) = -\frac{1}{3}\int_0^1 G(s,t)\Big(\frac{280}{27}x(t)^{1/3} + 24x(t)\Big)u(t)z(t)y(t)dt. \tag{4.5}$$

Observe that F''' is neither Lipschitz continuous nor Hölder continuous in Ω, but the operator F satisfies the assumptions of Theorem 1, so that a solution of (4.1) can be approximated by (1.2).

Now we consider $\Omega = B(0, 3/2) \subseteq X$ as an open convex nonempty domain and choose $x_0(s) = 1$ as an initial approximation solution. One can easily obtain

$$\|\Gamma_0\| \leq 36/25 = \beta, \quad \|\Gamma_0 F(x_0)\| \leq 3/25 = \eta,$$

$$\|F''(x)\| \leq 1.6815 = M, \quad \|F'''(x)\| \leq 1.9946 = N,$$

$$\omega(z) = \frac{35}{81}\sqrt[3]{z} + z, \ \varphi(t) = \sqrt[3]{t}, \ I_1 = 0.5200, \quad I_2 = 0.0964.$$

Hence, $a_0 = M\beta\eta = 0.2906$, $\quad b_0 = N\beta\eta^2 = 0.0414$ and $c_0 = \omega(\eta)\beta\eta^2 = 0.0069$, so that

$$\Phi(a_0) = a_0 g(a_0) - 1 \simeq -0.6550 < 0, \quad \text{and} \quad h(a_0)d_0 \simeq 0.0564 < 1.$$

Besides, the solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(1, 0.1480\cdots)} \subseteq \Omega$ and it is unique in $B(1, 0.6780\cdots) \cap \Omega$.

Finally, we discretize (4.1) to transform it into a finite dimensional problem and we apply (1.2) to approximate a solution. This procedure consists of approximating the integral appearing in (4.1) by a numerical quadrature formula. We consider the following Gauss-Legendre formula

$$\int_0^1 v(t)dt \simeq \sum_{i=1}^{m} w_i v(t_i),$$

where the nodes t_i and the weights w_i can be easily computed.

If we denote $x_j = x(t_j)$, $(j = 1, 2, \cdots, m)$, (4.1) becomes the following nonlinear system of equations

$$x_j = 1 + \frac{1}{3}\sum_{k=1}^{m} \alpha_{jk}(x_k^{10/3} + x_k^4), \qquad j = 1, 2, \cdots, m, \tag{4.6}$$

where

$$\alpha_{jk} = \begin{cases} w_k t_k(1 - t_j) & if\ k \leq j, \\ w_k t_j(1 - t_k) & if\ k < j. \end{cases}$$

Now we apply the method given by (1.2) to compute (4.6) and compare it with the Chebyshev-like's method in [30]. Taking into account that we have previously considered the starting function $x_0(s) = 1$, we now choose the vector $x_0 = (1, 1, \cdots, 1)^T$ as the initial iterate. All computations are carried out with double arithmetic precision. In the tests, we take $m = 10, 20$ in (4.6) respectively. Displayed in Tables 1 and 2 is the max-norm of vector functions at each iterative step. The stopping criterion that we consider is $\|F(x_n)\|_\infty \leq 10^{-15}$.

From the numerical results, we can see that the performance of the method (1.2) is better. This means that our method can be of practical interest.

Table 1: Results of system (4.6) with $m = 10$

n	Chebyshev-like method	the method (1.2)
1	3.098738e-004	6.359533e-005
2	1.645750e-011	6.661338e-016
3	6.661338e-016	

Table 2: Results of system (4.6) with $m = 20$

n	Chebyshev-like method	the method (1.2)
1	2.956476e-004	5.838933e-005
2	1.313716e-011	9.992007e-016
3	1.443290e-015	
4	4.440892e-016	

5 Dynamics

The dynamical analysis of a method is becoming a trend in recent publications [25-28] on iterative methods because it allows us to classify the various iterative formulas, not only from the point of view of its order of convergence, but also analyzing how these formulas behave as function of the initial estimate that is taken. Let us first recall some dynamical concepts. Consider a Fréchet differential function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The orbit of $x \in \mathbb{R}^n$ is defined as:

$$x, G(x), G^2(x), \cdots, G^p(x), \cdots .$$

A point x_f is a fixed point of G if $G(x_f) = x_f$. The basin of attraction of x_f is the set of points whose orbits tend to this fixed point

$$\mathcal{A}(x_f) = \{x \in \mathbb{R}^n : G^p(x) \longrightarrow x_f \ \text{ for } \ p \to \infty\}.$$

In this section we study the dynamics of the method (1.2) when applied to the solution of a 2×2 nonlinear system and compare it with the dynamics of Chebyshev-like's method in [30]. We show that the method is generally convergent and depict the attraction basins.

Example 2. Consider the following system [29]

$$\begin{cases} f_1(x) = \dfrac{1}{2} sin(x_1 x_2) - \dfrac{1}{4\pi} x_2 - \dfrac{1}{2} x_1 = 0 \\ f_2(x) = \big(1 - \dfrac{1}{4\pi}\big)\big(e^{2x_1} - e\big) + \dfrac{e}{\pi} x_2 - 2e x_1 = 0 \end{cases}$$

where $0.25 \leq x_1 \leq 1$ and $1.5 \leq x_2 \leq 2\pi$. The exact solutions are $x^* = (\frac{1}{2}, \pi)^T$ and $x^* = (0.29945, 2.83693)^T$.

For the comparisons, we have run the methods with tolerance 10^{-5}, performing a maximum of 20 iterations. The starting points form a uniform grid of size 600×600 in a rectangle of the real plane. The attraction basins have been colored according to the corresponding fixed point.

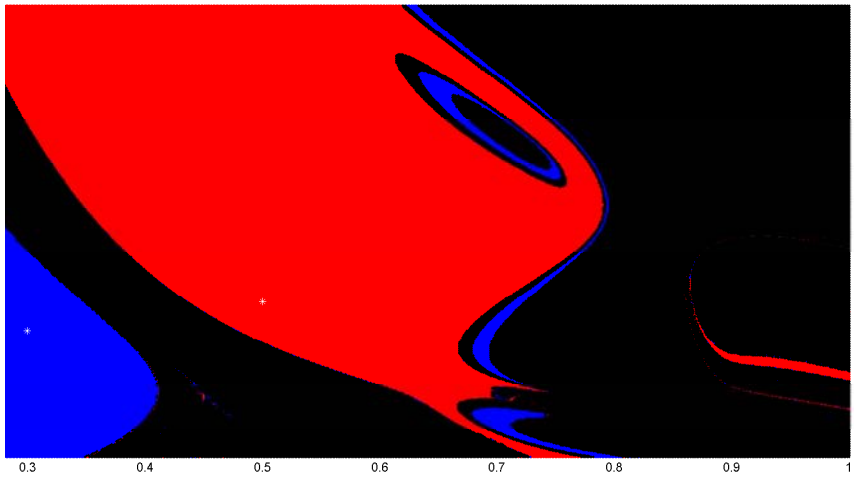


Figure 1: Attraction basins for Chebyshev-like's method

Figures 1 and 2 show the attraction basins of the method (1.2) and Chebyshev-like's method in [30], respectively. According to the figure, for any starting point that arise from the red or blue regions, the methods are converge to the solution in that region, while starting points from other region failed to convergence. The presented basin of attraction show the good performance of the method (1.2) as compared to Chebyshev-like's method in [30].

6 Conclusion

In this paper, the semilocal convergence of the modified Chebyshev-like's method for solving nonlinear equations in Banach spaces is established by using recurrence relations under the assumption that F''' satisfies ω-continuity condition. An existence-uniqueness theorem is given to show the R-order convergence of the method. Also a priori error bounds is given. From the numerical results, we can observe that the performance of our method in this paper is better.

The dynamical behavior of the method (1.2) has been compared with that of Chebyshev-like's method in [30]. The presented basin of attraction have also confirmed better performance of the method (1.2).

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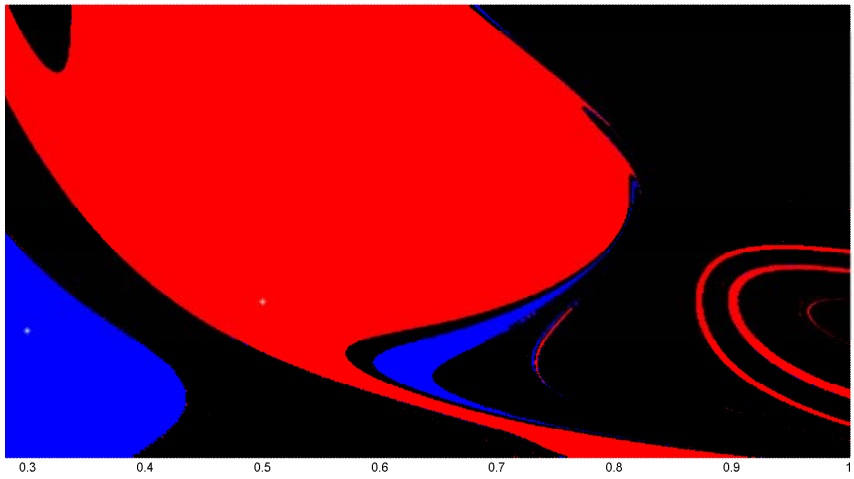


Figure 2: Attraction basins for Modified Chebyshev-like's method

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Generalized Hyers–Ulam stability of sextic functional equation in random normed spaces

Shaymaa Alshybani[a], S. Mansour Vaezpour[a], Reza Saadati[b,*]

[a]*Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran*
[b]*Department of Mathematics, Iran University of Science and Technology, Tehran, Iran*

Abstract

By using the direct and fixed point methods, we establish the general solution and generalized Hyers–Ulam stability of the following sextic functional equation

$$f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)=(n^4+n^2)[f(x+y)+f(x-y)] \\ +2(n^6-n^4-n^2+1)[f(x)+f(y)]$$

in random normed spaces. Also, we present an illustrative example with the Lukasiewicz t-norm that can be a suitable approximation using this sextic function.

1. Introduction

If the values of norms are probability distribution functions, then we have a generalized notion of normed space named random normed space, that was introduced by Sherstnev in [31] and extended by Alsina, Schweizer and Sklar in [1]. The theory of random normed spaces have significant applications in quantum particle physics (see [20]). Also, it has very useful applications in many fields like population dynamics, chaos control, computer programming, nonlinear dynamical system, nonlinear operators, statistical convergence, etc.

On the other hand, one of the most important issues in the theory of functional equations, concerning the famous Ulam stability problem is: when a mapping satisfying a functional equation approximately, must be close to an exact solution of a given functional equation?

Ulam [35] in 1940 raised the first stability problem concerning group homomorphisms. Hyers [12] was the first mathematician to present an affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [28] for linear mappings by considering an unbounded Cauchy difference. Gavruta [10] obtained a generalization of Rassias' theorem, which allows the Cauchy difference to be controlled by a general unbounded function.

*Corresponding author
Email addresses: shmhader@gmail.com (Shaymaa Alshybani), vaez@aut.ac.ir (S. Mansour Vaezpour), rsaadati@iust.ac.ir (Reza Saadati)

The stability problems of a wide class of functional equations have been investigated by a number of authors, and there are many interesting results concerning those problem (see, e.g., [3, 11, 13, 15, 25, 28, 32–34, 36]). Also by using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see, e.g., [5–7, 19, 26]).

The generalized Hyers–Ulam stability of different functional equations in random normed spaces, paranormed spaces, quasi-normed spaces and quasi-β-normed spaces has been studied by many authors (see , e.g., [8, 14, 18, 22–24]). Park and Lee [23] proved the Hyers–Ulam stability of the following additive-quadratic-cubic-quatric functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1.1}$$

in paranormed spaces. The general solution of quintic and sextic functional equations

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120y \tag{1.2}$$

and

$$f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)-6f(x-2y)+f(x-3y)=720f(y) \tag{1.3}$$

was introduced and investigated on the generalized Hyers–Ulam stability in quasi-β-normed spaces via fixed point method by Xu et al. [37].

The general solution and the generalized Hyers–Ulam stability of the sextic functional equation

$$\begin{aligned} f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)&=(n^4+n^2)[f(x+y)+f(x-y)] \\ &\quad +2(n^6-n^4-n^2+1)[f(x)+f(y)] \end{aligned} \tag{1.4}$$

in paranormed spaces was discussed by Ravi and Sabarinathan [29].

In this paper, we present the general solution and generalized Hyers–Ulam stability of the sextic functional equation (1.4) under arbitrary t-norms by direct method and under Min t-norm by fixed point method in random normed spaces and provide an example for random normed spaces with the Lukasiewicz t-norm, by direct method.

2. Preliminaries

Before giving the main result, we present some basic facts related to random normed spaces and some preliminary results. We say that $f:\mathbb{R}\longrightarrow[0,1]$ is a distribution function if and only if it is a monotone, nondecreasing, left continuous, $\inf f(x)=0$ and $\sup f(x)=1$. By $\triangle^+$ we denote a collection of all distribution functions and by D^+ the set of all distribution functions such that $f(x)=0$. If $a\in\mathbb{R}_0$, then $H_a\in D^+$ where

$$H_a(t):=\begin{cases}0 & \text{if } t\le a,\\ 1 & \text{if } t>a.\end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2.1 ([8, 30]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly a t-norm) if T satisfies the following conditions:

1. T is commutative and associative;
2. T is continuous;
3. $T(a,1) = a$ for all $a \in [0,1]$;
4. $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$.

Typical examples of continuous t-norms are $T_p(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1, 0)$ (the Lukasiewicz t-norm).

Recall (see [8, 11]) that if T is a t-norm and x_n is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.

It is known [11] that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

Definition 2.2 ([31]). A random normed space (briefly RN-space) is a triple (X, μ, T) where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

1. $\mu_x(t) = H_0(t)$ for all $t > 0$ iff $x = 0$;
2. $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$ and $\alpha \neq 0$;
3. $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.3 ([18]). Let (X, μ, T) be an RN-space. Then

1. A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$, whenever $n \geq N$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$, whenever $n \geq m \geq N$.
3. An RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.4 ([30]). *If (X, μ, T) is an RN-space and x_n is a sequence such that $x_n \longrightarrow x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Definition 2.5 ([16]). Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if it satisfies

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 2.6 ([4, 9]). *Let (X,d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n, or there exists a positive integer n_0 such that

1. *$d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;*
2. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;*
4. *$d(y, y^*) \leqslant \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

3. Hyers–Ulam stability of the sextic functional equation (1.4) by direct method

In this section, using the direct method, we prove the generalized Hyers–Ulam stability of the sextic functional equation (1.4) in complete RN-spaces. Also, we present an illustrative example with the Lukasiewicz t-norm that can be suitable approximation using this sextic function.

Theorem 3.1. *Let X be a real liner space, (Y, μ, T) a complete RN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{3.1}$$

where

$$\begin{aligned} D_s f(x,y) := {} & f(nx+y) + f(nx-y) + f(x+ny) + f(x-ny) \\ & - (n^4+n^2)[f(x+y) + f(x-y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)] \end{aligned}$$

for all $x, y \in X$ and $t > 0$. If

$$\lim_{m\to\infty} T_{i=1}^{\infty}(\phi_{n^{i+m-1}x,0}(n^{6m+5i}t)) = 1, \tag{3.2}$$

and

$$\lim_{m\to\infty} \phi_{n^m x, n^m y}(n^{6m}t) = 1 \tag{3.3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and the inequality

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{n^{i-1}x,0}(n^{5i}t) \tag{3.4}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (3.1), we get

$$\mu_{f(nx)-n^6 f(x)}(t) \geq \phi_{x,0}(2t) \geq \phi_{x,0}(t) \tag{3.5}$$

for all $x \in X$. Then we get

$$\mu_{\frac{f(nx)}{n^6}-f(x)}(t) \geq \phi_{x,0}(n^6 t), \tag{3.6}$$

therefore,

$$\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}}-\frac{f(n^k x)}{n^{6k}}}(t) \geq \phi_{n^k x,0}(n^{6k+6}t), \tag{3.7}$$

that is,

$$\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}}-\frac{f(n^k x)}{n^6 k}}(\frac{t}{n^{k+1}}) \geq \phi_{n^k x,0}(n^{5(k+1)}t) \tag{3.8}$$

for every $k \in N$, $t > 0$, n positive integer, $n > 1$. As

$$1 > \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + ... + \frac{1}{n^k},$$

by the triangle inequality it follows:

$$\begin{aligned} \mu_{\frac{f(n^m x)}{n^{6m}}-f(x)}(t) &\geq \mu_{\frac{f(n^m x)}{n^{6m}}-f(x)}(\sum_{k=0}^{m-1} \frac{1}{n^{k+1}}t) \\ &\geq T_{k=0}^{m-1}\left(\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}}-\frac{f(n^k x)}{n^{6k}}}(\frac{1}{n^{k+1}}t)\right) \\ &\geqslant T_{k=0}^{m-1}(\phi_{n^k x,0}(n^{5k+5}t) \\ &= T_{i=1}^{m}\left(\phi_{n^{i-1}x,0}(n^{5i}t)\right), \end{aligned} \tag{3.9}$$

$x \in X$, $t > 0$, and $n > 1$. In order to prove the convergence of the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$, we replace x by $n^j x$, and multiplying the left-hand side of (3.9) by $\frac{n^{6j}}{n^{6j}}$, we get

$$\mu_{\frac{f(n^{m+j}x)}{n^{6m+6j}} - \frac{f(n^j x)}{n^{6j}}}(t) \geq T_{i=1}^{m} \left(\phi_{n^{j+i-1}x,0}(n^{6j+5i}t)\right). \tag{3.10}$$

Since the right-hand side of the inequality (3.10) tends to 1 as m and j tend to infinity, the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(n^j x)}{n^{6j}}$$

for all $x \in X$.

Replacing x, y by $n^m x$ and $n^m y$, respectively, in (3.1), then multiplying the right hand-side by $\frac{n^{6m}}{n^{6m}}$, it follows that

$$\mu_{\frac{1}{n^{6m}} D_s f(n^m x, n^m y)}(t) \geq \phi_{n^m x, n^m y}(n^{6m}t)$$

for all $x, y \in X$, and positive integer n, $n > 1$. Taking the limit as $m \to \infty$ we find that S satisfies (1.4), that is, S is a sextic map. To prove (3.4) take the limit as $m \to \infty$ in (3.9).

Finally, to prove the uniqueness of the sextic function S, let us assume that there exists a sextic function r which satisfies (3.4) and equation (1.4). Therefore

$$\begin{aligned} \mu_{r(x)-s(x)}(t) &= \mu_{r(x)-\frac{f(n^j x)}{n^{6j}}+\frac{f(n^j)}{n^{6j}}-s(x)}(t) \\ &\geq T(\mu_{r(x)-\frac{f(n^j x)}{n^{6j}}}(\frac{t}{2}), \mu_{\frac{f(n^j x)}{n^{6j}}-s(x)}(\frac{t}{2})). \end{aligned}$$

Taking the limit as $j \to \infty$, we find $\mu_{r(x)-s(x)}(t) = 1$. Therefore $r = s$. □

Corollary 3.2. *Let X be a real liner space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|} \tag{3.11}$$

for all $x \in X$, $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(1 - \frac{\|x\|}{n^{4i+1}t + \|x\|})$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Corollary 3.3. *Let X be a real liner space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{n^{5i}t}{n^{5i}t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Corollary 3.4. *Let X be a real linear space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and let $L \geq 0$ and p be a real number with $0 < p < 5$ and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\frac{t}{t + Ln^{i(p-5)-p} \| x \|^p})$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + L(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Example 3.5. Let $(X, \|.\|)$ be a Banach algebra and

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Let

$$\varphi_{x,y}(t) = \begin{cases} \max\{1 - \frac{(8n^6)(\|x\|+\|y\|)}{t}, 0\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We note that $\varphi_{x,y}(t)$ is a distribution function and $\lim_{j\to\infty} \varphi_{n^j x, n^j y}(n^{6j}t) = 1$ for all $x, y \in X$ and $t > 0$.

It is easy to show that (X, μ, T_L) is an RN-space (this was essentially proved by Mushtari in [21], see also [27]). Indeed, $\mu_x(t) = 1$, $\forall t > 0$ implies $\frac{\|x\|}{t} = 0$ and hence $x = 0$ for all $x \in X$ and $t > 0$. Obviously, $\mu_{\lambda x}(t) = \mu_x(\frac{t}{\lambda})$ for all $x \in X$ and $t > 0$. Next, for all $x, y \in X$ and $t, s > 0$, we have

$$\begin{aligned} \mu_{x+y}(t + s) &= \max\{1 - \frac{\|x + y\|}{t + s}, 0\} \\ &= \max\{1 - \|\frac{x + y}{t + s}\|, 0\} \\ &= \max\{1 - \|\frac{x}{t + s} + \frac{y}{t + s}\|, 0\} \\ &\geq \max\{1 - \|\frac{x}{t}\| - \|\frac{y}{s}\|, 0\} \\ &= T_L(\mu_x(t), \mu_y(s)). \end{aligned}$$

It is easy to see that (X, μ, T_L) is complete, for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x - y\|}{t}, \quad \forall x, y \in X,$$

and $t > 0$ and $(X, \|.\|)$ is complete. Define a mapping $f : X \longrightarrow X$ by $f(x) = x^6 + \|x\|x_0$ for all $x \in X$, where x_0 is a unit vector in X. A simple computation shows that

$$\begin{aligned}
&\|f(nx+y) + f(nx-y) + f(x+ny) + f(x-ny) \\
&\quad - (n^4+n^2)[f(x+y) + f(x-y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)]\| \\
&= |\, \| \, nx + y \, \| + \| \, nx - y \, \| + \| \, x + ny \, \| + \| \, x - ny \, \| \\
&\quad - (n^2+n^4)[\| \, x + y \, \| + \| \, x - y \, \|] \\
&\quad - 2(n^6 - n^4 - n^2 + 1)[\| \, x \, \| + \| \, y \, \|]| \\
&\leq 2(n^6 + n + 2)(\| \, x \, \| + \| \, y \, \|) \leq 8n^6(\| \, x \, \| + \| \, y \, \|)
\end{aligned}$$

for all $x, y \in X$. Hence $\mu_{D_s f(x,y)}(t) \geq \phi_{x,y}(t)$ for all $x, y \in X$ and $t > 0$. Fix $x \in X$ and $t > 0$. Then it follows that,

$$\begin{aligned}
(T_L)_{i=1}^{\infty}\left(\phi_{n^{i+j-1}x,0}(n^{6j+5i})t\right) &= \max\left\{\sum_{i=1}^{\infty}\left(\phi_{n^{i+j-1}x,0}(n^{6j+5i})t) - 1\right) + 1, 0\right\} \\
&= \max\left\{1 - \frac{8n^5\|x\|}{n^{5j}(n^4-1)t}, 0\right\}
\end{aligned}$$

for all $x \in X$, $n \in \mathbb{N}$ and $t > 0$. Hence

$$\lim_{j\to\infty}(T_L)_{i=1}^{\infty}\left(\varphi_{n^{i+j-1}x,0}(n^{6j+5i})t\right) = 1$$

for all $x \in X$ and $t > 0$. Thus, all the conditions of Theorem 3.1 hold. Since

$$(T_L)_{i=1}^{\infty}\left(\phi_{n^{i-1}x,0}(n^{5i}t)\right) = \max\{1 - \frac{8n^5\|x\|}{(n^4-1)t}, 0\}$$

for all $x \in X$ and $t > 0$, we can deduce that $S(x) = x^6$ is the unique sextic mapping $S : X \longrightarrow X$ such that

$$\mu_{f(x)-s(x)}(t) \geq \max\{1 - \frac{8n^5\|x\|}{(n^4-1)t}, 0\}$$

for all $x \in X$ and $t > 0$.

4. Hyers–Ulam stability of the sextic functional equation (1.4) by fixed point method

In this section, using the fixed point method, we prove the generalized Hyers–Ulam stability of the sextic functional equation (1.4) in complete RN-spaces.

Theorem 4.1. *Let X be a real liner space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{nx,ny}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < n^6,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t) \tag{4.1}$$

for all $x, y \in X$, and $t > 0$, where

$$D_s f(x, y) := f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny)$$

$$- (n^4 + n^2)[f(x+y) + f(x-y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)]$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $g : X \longrightarrow Y$ such that

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(n^6 - \alpha)t) \tag{4.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. Let $y = 0$ in (4.1); we get

$$\mu_{2f(nx)-2n^6 f(x)}(t) \geq \phi_{x,0}(t) \tag{4.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\mu_{\frac{f(nx)}{n^6}-f(x)}(t) \geq \phi_{x,0}(2n^6 t). \tag{4.4}$$

Consider the set

$$E := \{g : X \to Y : \ g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g, h) = \texttt{inf}\{\epsilon > 0 : \mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(2n^6 t)\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [17, Lemma 2.1]). Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(nx)}{n^6}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \frac{\alpha}{n^6}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g, h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{n^6}) &= \mu_{\frac{g(nx)}{n^6}-\frac{h(nx)}{n^6}}(\frac{\epsilon\alpha t}{n^6}) \\ &= \mu_{g(nx)-h(nx)}(\alpha\varepsilon t) \\ &\geq \phi_{nx,0}(2\alpha n^6 t) \end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$\phi_{nx,ny}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < n^6,$$

we have

$$\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{n^6}) \geq \phi_{x,0}(2n^6 t),$$

that is,

$$d_G(g, h) < \epsilon \Longrightarrow d_G(Jg, Jh) < \frac{\alpha}{n^6}\epsilon.$$

This means that

$$d_G(Jg, Jh) < \frac{\alpha}{n^6} d_G(g, h),$$

for all $g, h \in E$. Next, from

$$\mu_{\frac{f(nx)}{n^6}-f(x)}(t) \geq \phi_{x,0}(2n^6 t),$$

it follows that $d_G(f, Jf) \leq 1$. Using Theorem 2.6, we show the existence of a fixed point of J, that is, the existence of a mapping $g : X \longrightarrow Y$ such that $g(nx) = n^6 g(x)$ for all $x \in X$. Since, for all $x \in X$ and $t > 0$,

$$d_G(u, v) < \epsilon \Longrightarrow \mu_{u(x)-v(x)}(t) \geq \phi_{x,0}(\frac{2n^6 t}{\epsilon}),$$

it follows from $d_G(J^n f, g) \longrightarrow 0$ that $lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}} = g(x)$ for all $x \in X$. Also from

$$d_G(f, g) \leq \frac{1}{1-L} d(f, Jf)$$

for all $g, h \in E$, we have $d_G(f, g) \leq \frac{1}{1-\frac{\alpha}{n^6}}$, and it immediately follows that

$$\mu_{g(x)-f(x)}(\frac{n^6}{n^6-\alpha}t) \geqslant \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$. This means that

$$\mu_{g(x)-f(x)}(t) \geqslant \phi_{x,0}(2(n^6-\alpha)t)$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0, \infty)$ satisfying

$$\mu_{g(x)-f(x)}(Ct) \geqslant \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$. This completes the proof. □

Corollary 4.2. *Let X be a real liner space, (Y, μ, T_M) a complete RN-space, and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|} \tag{4.5}$$

for all $x \in X$, $t > 0$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(n^6-\alpha)t + \|x\|}$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|}$$

for all $x \in X$ and $t > 0$ in Theorem 4.1. Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

Corollary 4.3. *Let X be a real liner space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(n^6-\alpha)t}{2(n^6-\alpha)t + \varepsilon\|x_0\|}$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|}$$

for all $x \in X$, and $t > 0$ in Theorem 4.1. Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

Corollary 4.4. *Let X be a real liner space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $0 < p < 6$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(n^6 - \alpha)t}{2(n^6 - \alpha)t + \theta \| x \|^p}$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \theta(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$ in Theorem 4.1. Then we can choose $n^p < \alpha < n^6$ and so we get the desired result. □

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On convergence theorem of a finite family of nonlinear mappings in uniformly convex metric spaces

Atid Kangtunyakarn*
Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

In this paper, we introduce the $S-$mapping generated by a finite family of nonexpansive mapping and real numbers in convex metric space by using concept of the $S-$mapping defined by Kangtunyakarn and Suantai [1]. Then, we prove convergence of Ishikawa iteration generated by the $S-$mapping to a common fixed point of a finite family of nonexpansive mappings in uniformly convex metric space.
Keywords: Convex metric space; Nonexpansive mapping; $S-$mapping.

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1 Introduction

Throughout this paper, we assume that (X, d) is a complete metric space and C is a nonempty closed convex subset of (X, d). A point x is called a fixed point of T if $Tx = x$. We use $F(T)$ to denote the set of fixed point of T. Recalled the following definitions;

Definition 1.1. *The mapping $T : C \to C$ is said to be nonexpansive if*

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

In 1970, Takahashi [9] introduce the following definition as follows:

Definition 1.2. *Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and for all $u \in X$,*

$$d\big(u, W(x, y, \lambda)\big) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

*E-mail: beawrock@hotmail.com

We observe that $W(x,y,\lambda)=\lambda x+(1-\lambda)y$ is a convex structure on a normed linear space. A metric space (X,d) together with a convex structure W is called a *convex metric space* denoted by (X,d,W). A nonempty subset C of X is said to be *convex* if $W(x,y,\lambda)\in C$ for all $x,y\in C$ and $\lambda\in[0,1]$.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [10] and is defined as follows:

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)Tx_n,\quad \forall n\geq 1,$$

where $x_1\in C$, $\{\alpha_n\}\subseteq[0,1]$.

The second iteration process is referred to as Ishikawa's iteration process [5] which is defined recursively by

$$\begin{cases} x_{n+1}=\alpha_n x_n+(1-\alpha_n)Ty_n \\ y_n=\beta_n x_n+(1-\beta_n)Tx_n,\quad \forall n\geq 1, \end{cases} \tag{1.1}$$

where $x_1\in C$, $\{\alpha_n\},\{\beta_n\}\subseteq[0,1]$.

In 2009, Kangtunyakarn and Suantai [1] introduced the mapping generated by a finite family of nonexpansive mapping and family of real numbers as follows:

Definition 1.3. *Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself. For each $j=1,2,...,N$, let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I$ where $I\in[0,1]$ and $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$. They define the mapping $S:C\to C$ as follows:*

$$\begin{aligned} U_0 &= I, \\ U_1 &= \alpha_1^1T_1U_0+\alpha_2^1U_0+\alpha_3^1I, \\ U_2 &= \alpha_1^2T_2U_1+\alpha_2^2U_1+\alpha_3^2I, \\ U_3 &= \alpha_1^3T_3U_2+\alpha_2^3U_2+\alpha_3^3I, \\ &\cdot \\ &\cdot \\ &\cdot \\ U_{N-1} &= \alpha_1^{N-1}T_{N-1}U_{N-2}+\alpha_2^{N-1}U_{N-2}+\alpha_3^{N-1}I, \\ S &= U_N=\alpha_1^NT_NU_{N-1}+\alpha_2^NU_{N-1}+\alpha_3^NI. \end{aligned} \tag{1.2}$$

This mapping is called *S-mapping* generated by $T_1,....,T_N$ and $\alpha_1,\alpha_2,...,\alpha_N$.

In this paper, by using the concept of the $S-$mapping in Definition 1.3, we define the $S-$mapping generated by a finite family of nonexpansive mappings and real numbers in convex metric space. Then, we prove convergence of Ishikawa iteration generated by the $S-$mapping to a common fixed point of a finite family of nonexpansive mappings in uniformly convex metric space.

2 Preliminaries

In this section, we recall some lemmas and definitions to prove our main result as follows:

Definition 2.1. *(See [7]) A convex metric space (X, d, W) is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) < r, d(z, y) < r$ and $d(x, y) \geq r\epsilon$,*

$$d\big(z, W(x, y, \frac{1}{2})\big) \leq (1 - \delta)r.$$

Lemma 2.1. *(See [11], [3]) Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following.*

(i) $W(x, x, \lambda) = x$, $W(x, y, 0) = y$ *and* $W(x, y, 1) = x$.

(ii) $d\big(x, W(x, y, \lambda)\big) = (1 - \lambda)d(x, y)$ *and* $d\big(y, W(x, y, \lambda)\big) = \lambda d(x, y)$.

(iii) $d(x, y) = d\big(x, W(x, y, \lambda)\big) + d\big(W(x, y, \lambda), y\big)$.

(iv) $|\lambda_1 - \lambda_2|d(x, y) \leq d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big)$.

We say that a convex metric space (X, d, W) has the property:

(C) if $W(x, y, \lambda) = W(y, x, 1 - \lambda)$ for all $x, y \in X$ and $\lambda \in [0, 1]$,

(I) if $d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big) \leq |\lambda_1 - \lambda_2|d(x, y)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$,

(H) if $d\big(W(x, y, \lambda), W(x, z, \lambda)\big) \leq (1-\lambda)d(y, z)$ for all $x, y, z \in X$ and $\lambda \in [0, 1]$,

(S) if $d\big(W(x, y, \lambda), W(z, w, \lambda)\big) \leq \lambda d(x, z) + (1 - \lambda)d(y, w)$ for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$.

Remark 2.2. It is easy to see that the property (C) and (H) imply continuity of a convex structure $W : X \times X \times [0, 1] \to X$ and the property (S) implies the property (H). In 2005, Aoyama et al. [3] proved that a convex metric space with property (C) and (H) has the property (S).

In 2011, Phuengrattana and Suantai [8] proved the following lemma as follows;

Lemma 2.3. *(See [8]) Property (C) holds in uniformly convex metric space.*

Remark 2.4. (See [8]) From Lemma 2.3, a uniformly convex metric space (X, d, W) with the property (H) has the property S and the convex structure W is also continuous.

Lemma 2.5. *(See [6]) Let (X, d, W) be a uniformly convex metric space with continuous convex structure. Then for arbitrary positive number ϵ, there exists $\eta = \eta(\epsilon) > 0$ such that*

$$d\big(z, W(x, y, \lambda)\big) \leq (1 - 2\min\{\lambda, 1 - \lambda\}\eta)r$$

for all $r > 0$ and $x, y, z \in X, d(z, x) \leq r, d(z, y) \leq r, d(x, y) \geq r\varepsilon$ and $\lambda \in [0, 1]$.

Lemma 2.6. *(See [2],[4]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ n \geq 1.$$

If

$$\sum_{n=1}^{\infty} \delta_n < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty,$$

then $\lim_{n\to\infty} a_n$ exists.

We introduce the following definition to use in the next section.

Definition 2.2. *Let (X,d,W) be a complete convex metric space and C be a nonempty closed convex subset of (X,d,W). Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into C. For each $j = 1,2,\cdots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $x \in C$, we define the mapping $S : C \times C \times [0,1] \to C$ as follows:*

$$\begin{aligned}
U_0x &= x, \\
U_1x &= W\big(T_1U_0x, W(U_0x, x, \frac{\alpha_2^1}{1-\alpha_1^1}), \alpha_1^1\big), \\
U_2x &= W\big(T_2U_1x, W(U_1x, x, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big), \\
&\vdots \\
U_{N-1}x &= W\big(T_{N-1}U_{N-2}x, W(U_{N-2}x, x, \frac{\alpha_2^{N-1}}{1-\alpha_1^{N-1}}), \alpha_1^{N-1}\big), \\
Sx &= U_Nx = W\big(T_NU_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big).
\end{aligned}$$

This mapping is called $S-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.7. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X,d,W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,3,...,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1,2,...,N-1$, $\alpha_1^N \in (0,1]$ α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1,2,...,N$. Let S be the mapping generated by $T_1,....,T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Proof. From Lemma 2.1 and definition of $S-$mapping, it is easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(S)$. Next, we show that $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. To show this let $x_0 \in F(S)$ and $q \in \bigcap_{i=1}^N F(T_i)$, we have

$$\begin{aligned}
d(q, Sx_0) &= d\bigg(q, W\big(T_NU_{N-1}x_0, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\bigg) \\
&\leq \alpha_1^N d(q, T_NU_{N-1}x_0) + (1-\alpha_1^N) d\bigg(q, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N})\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_1^N d(q, T_N U_{N-1} x_0) + (1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N} d(q, U_{N-1}x_0) \\
&+ (1-\frac{\alpha_2^N}{1-\alpha_1^N}) d(q, x_0)\Big) \\
&= \alpha_1^N d(q, T_N U_{N-1} x_0) + \alpha_2^N d(q, U_{N-1}x_0) + \alpha_3^N d(q, x_0) \\
&\leq (1-\alpha_3^N) d(q, U_{N-1}x_0) + \alpha_3^N d(q, x_0) \\
&\leq (1-\alpha_3^N)\Big((1-\alpha_3^{N-1}) d(q, U_{N-2}x_0) + \alpha_3^{N-1} d(q, x_0)\Big) \\
&\quad +\alpha_3^N d(q, x_0) \\
&= (1-\alpha_3^N)(1-\alpha_3^{N-1}) d(q, U_{N-2}x_0) + \alpha_3^{N-1}(1-\alpha_3^N) d(q, x_0) \\
&\quad +\alpha_3^N d(q, x_0) \\
&= \Pi_{j=N-1}^N (1-\alpha_3^j) d(q, U_{N-2}x_0) + \big(1-\Pi_{j=N-1}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \\
&\vdots \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j) d(q, U_2 x_0) + \big(1-\Pi_{j=3}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&= \Pi_{j=3}^N (1-\alpha_3^j) d\big(q, W\big(T_2U_1x_0, W(U_1x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big)\big) \\
&\quad +\big(1-\Pi_{j=3}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + (1-\alpha_1^2) d\big(q, W(U_1x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2})\big)\Big) \\
&\quad +\big(1-\Pi_{j=3}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + (1-\alpha_1^2)\big(\frac{\alpha_2^2}{1-\alpha_1^2} d(q, U_1x_0) \\
&\quad +(1-\frac{\alpha_2^2}{1-\alpha_1^2}) d(q, x_0)\big)\Big) + \big(1-\Pi_{j=3}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&= \Pi_{j=3}^N (1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + \alpha_2^2 d(q, U_1x_0) + \alpha_3^2 d(q, x_0)\Big) \\
&\quad +\big(1-\Pi_{j=3}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j)\Big((1-\alpha_3^2) d(q, U_1x_0) + \alpha_3^2 d(q, x_0)\big)\Big) \\
&\quad +\big(1-\Pi_{j=3}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&= \Pi_{j=2}^N (1-\alpha_3^j) d(q, U_1x_0) + \big(1-\Pi_{j=2}^N (1-\alpha_3^j)\big) d(q, x_0) \\
&= \Pi_{j=2}^N (1-\alpha_3^j) d\big(q, W\big(T_1U_0x_0, W(U_0x_0, x_0, \frac{\alpha_2^1}{1-\alpha_1^1}), \alpha_1^1\big)\big) \\
&\quad +\big(1-\Pi_{j=2}^N (1-\alpha_3^j)\big) d(q, x_0)
\end{aligned}
$$

$$\begin{aligned} &= \Pi_{j=2}^{N}(1-\alpha_3^j)d\big(q,W\big(T_1x_0,x_0,\alpha_1^1\big)\big)+\big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\ &\leq \Pi_{j=2}^{N}(1-\alpha_3^j)\big(\alpha_1^1d(q,T_1x_0)+(1-\alpha_1^1)d(q,x_0)\big)+\big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\ &\leq \Pi_{j=2}^{N}(1-\alpha_3^j)d(q,x_0)+\big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\ &= d(q,x_0). \end{aligned} \tag{2.1}$$

From (2.1), we have

$$d(q,U_1x_0)=d\big(q,W(T_1x_0,x_0,\alpha_1^1)\big)=d(q,x_0) \text{ and } d(q,T_1x_0)=d(q,x_0).$$

Suppose $x_0\neq T_1x_0$, then we have $d(x_0,T_1x_0)>0$. Choose $r=d(q,x_0)>0$ and $\epsilon=\dfrac{d(x_0,T_1x_0)}{r}$, we have $d(q,T_1x_0)\leq d(q,x_0)=r$, $d(q,x_0)\leq r$ and $d(x_0,T_1x_0)\geq r\epsilon$. From Lemma 2.5, we have

$$d\big(q,W(T_1x_0,x_0,\alpha_1^1)\big)<d(q,x_0) \text{ for } \alpha_1^1\in(0,1).$$

This is a contradiction, we have $x_0\in T_1x_0$, that is, $x_0\in F(T_1)$. Since $x_0=T_1x_0$ definition of U_1 and Lemma 2.1, we have $U_1x_0=x_0$, that is, $x_0\in F(U_1)$. From (2.1) and $x_0=U_1x_0$, we have

$$d(q,U_2x_0)=d\big(q,W\big(T_2x_0,x_0,\alpha_1^2\big)\big)=d(q,x_0) \text{ and } d(q,T_2x_0)=d(q,x_0).$$

Suppose $x_0\neq T_2x_0$, then we have $d(x_0,T_2x_0)>0$. Choose $r_1=d(q,x_0)>0$ and $\epsilon=\dfrac{d(x_0,T_2x_0)}{r_1}$, we have $d(q,T_2x_0)\leq d(q,x_0)=r_1$, $d(q,x_0)\leq r_1$ and $d(x_0,T_2x_0)\geq r_1\epsilon$. From Lemma 2.5, we have

$$d\big(q,W(T_2x_0,x_0,\alpha_1^2)\big)<d(q,x_0) \text{ for } \alpha_1^2\in(0,1).$$

This is a contradiction, we have $x_0=T_2x_0$, that is, $x_0\in F(T_2)$. Since $x_0=T_2x_0$ definition of U_2 and Lemma 2.1, we have $U_2x_0=x_0$, that is, $x_0\in F(U_2)$.
By continuing on this way we can conclude that $x_0\in F(T_i)$ and $x_0\in F(U_i)$ for all $i=1,2,\ldots,N-1$.
Finally, we show that $x_0\in F(T_N)$. From definition of S and Lemma 2.1, we have

$$Sx_0=W\big(T_NU_{N-1}x_0,W(U_{N-1}x_0,x_0,\frac{\alpha_2^N}{1-\alpha_1^N}),\alpha_1^N\big)=W(T_Nx_0,x_0,\alpha_1^N).$$

Since

$$0=d(x_0,Sx_0)=d(x_0,W(T_Nx_0,x_0,\alpha_1^N))=\alpha_1^Nd(T_Nx_0,x_0),$$

we have $x_0=T_Nx_0$, that is, $x_0\in F(T_N)$. Hence $F(S)\subseteq\bigcap_{i=1}^{N}F(T_i)$. □

Remark 2.8. From Theorem 2.7, we have the mapping S is nonexpansive. To show this, let $x, y \in C$. By remark 2.4, we have

$$\begin{aligned}
d(Sx,Sy) &= d\Big(W\big(T_NU_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big), \\
&\quad W\big(T_NU_{N-1}y, W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big) \\
&\leq \alpha_1^N d(T_NU_{N-1}x, T_NU_{N-1}y) \\
&\quad +(1-\alpha_1^N)d\big(W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N})\big) \\
&\leq \alpha_1^N d(T_NU_{N-1}x, T_NU_{N-1}y) \\
&\quad +(1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N}d(U_{N-1}x, U_{N-1}y) + \big(1-\frac{\alpha_2^N}{1-\alpha_1^N}\big)d(x,y)\Big) \\
&\leq \alpha_1^N d(U_{N-1}x, U_{N-1}y) + \alpha_2^N d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y) \\
&= (1-\alpha_3^N)d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y) \\
&\leq (1-\alpha_3^N)\big((1-\alpha_3^{N-1})d(U_{N-2}x, U_{N-2}y) + \alpha_3^{N-1}d(x,y)\big) + \alpha_3^N d(x,y) \\
&= \Pi_{j=n-1}^N(1-\alpha_3^j)d(U_{N-2}x, U_{N-2}y) + \big(1-\Pi_{j=N-1}^N(1-\alpha_3^j)\big)d(x,y) \\
&\leq \\
&\vdots \\
&= \Pi_{j=1}^N(1-\alpha_3^j)d(U_0x, U_0y) + \big(1-\Pi_{j=1}^N(1-\alpha_3^j)\big)d(x,y) \\
&= d(x,y).
\end{aligned}$$

3 Main results

Theorem 3.1. *Let C be a nonempty compact closed convex subset of a complete uniformly convex metric space (X,d,W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,3,...,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1,2,...,N-1$, $\alpha_1^N \in (0,1]$ α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1,2,...,N$. Let S be the mapping generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $x_1 \in C$ and let $\{x_n\}, \{y_n\}$ be sequences generated by*

$$\begin{cases} x_{n+1} = W(x_n, Sy_n, \gamma_n), \\ y_n = W(x_n, Sx_n, \beta_n) \end{cases} \tag{3.1}$$

for all $n \geq 1$ where $\{\gamma_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying $0 < a \leq \gamma_n \leq b < 1$ and $\sum_{n=1}^\infty \beta_n(1-\beta_n) = \infty$. Then the sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F(T_i)$.

Proof. First, we show that $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) = 0$. Assume that $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) =$

$r' > 0$. Let $p \in \bigcap_{i=1}^{N} F(T_i)$, by nonexpansiveness of $S-$mapping, we have

$$\begin{aligned} d(p, x_{n+1}) &= d\big(p, W(x_n, Sy_n, \gamma_n)\big) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d(p, Sy_n) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d(p, y_n) \\ &= \gamma_n d(p, x_n) + (1-\gamma_n) \Big(d\big(p, W(x_n, Sx_n, \beta_n)\big) \Big) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) \Big(\beta_n d(p, x_n) + (1-\beta_n) d(p, Sx_n) \Big) \\ &\leq d(p, x_n). \end{aligned}$$

It implies by Lemma 2.6 that $\lim_{n\to\infty} d(p, x_n)$ exists. Then, we have $\lim_{n\to\infty} d(p, x_n) = r'' > 0$. By nonexpansiveness of S, we have $d(p, Sx_n) \leq d(p, x_n)$. Since $\{d(p, x_n)\}$ is a nonincreasing and $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) = r' > 0$, we have

$$\begin{aligned} d(x_n, Sx_n) &\geq r' \\ &\geq \frac{r'}{d(p, x_n)} d(p, x_n) \\ &\geq \frac{r'}{d(p, x_1)} d(p, x_n) \\ &> 0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

By Lemma 2.5, there exists $\eta = \eta\big(\frac{r'}{d(p,x_1)}\big) > 0$ such that

$$\begin{aligned} d(p, x_{n+1}) &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d\big(p, W(x_n, Sx_n, \beta_n) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) \Big(\big(1 - 2\min\{\beta_n, 1-\beta_n\}\eta\big) d(p, x_n) \Big) \\ &= \gamma_n d(p, x_n) + (1-\gamma_n) d(p, x_n) - 2(1-\gamma_n) \min\{\beta_n, 1-\beta_n\} \eta d(p, x_n) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d(p, x_n) - 2(1-\gamma_n) \beta_n (1-\beta_n) \eta d(p, x_n) \\ &= d(p, x_n) - 2(1-\gamma_n) \beta_n (1-\beta_n) \eta d(p, x_n), \end{aligned}$$

which follows that

$$2(1-b)\beta_n(1-\beta_n)\eta r'' \leq 2(1-\gamma_n)\beta_n(1-\beta_n)\eta d(p, x_n) \leq d(p, x_n) - d(p, x_{n+1}). \quad (3.2)$$

From (3.2), it implies that

$$2(1-b)\eta \sum_{n=1}^{k} \beta_n(1-\beta_n) r'' \leq d(p, x_1) - d(p, x_{k+1}) \quad (3.3)$$

for all $k \geq 1$. Letting $k \to \infty$ in (3.3) and , we have $\infty \leq d(p, x_1) - r'' < \infty$. This is a contradiction, then we have $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) = 0$. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} d(x_{n_j}, Sx_{n_j}) = 0$. Since C is compact,

then there exists a subsequence $\{x_{n_{j_l}}\}$ of $\{x_{n_j}\}$ and p such that $\lim_{l\to\infty} x_{n_{j_l}} = p$. From nonexpansiveness of S, we have

$$\begin{aligned} d(p, Sp) &\leq d(p, x_{n_{j_l}}) + d(x_{n_{j_l}}, Sx_{n_{j_l}}) + d(Sx_{n_{j_l}}, Sp) \\ &\leq 2d(p, x_{n_{j_l}}) + d(x_{n_{j_l}}, Sx_{n_{j_l}}). \end{aligned}$$

Taking $l \to \infty$, it implies that $p \in F(S)$. From Lemma 2.7, we have $p \in \bigcap_{i=1}^{N} F(T_i)$. Since $\lim_{n\to\infty} d(p, x_n)$ exists, we can conclude that $\{x_n\}$ converges to $p \in \bigcap_{i=1}^{N} F(T_i)$.

□

We can prove the following results by using Theorem 3.1.

Corollary 3.2. *Let C be a nonempty compact closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $T : C \to C$ be a nonexpansive mappings with $F(T) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}, \{y_n\}$ be sequences generated by*

$$\begin{cases} x_{n+1} = W(x_n, Ty_n, \gamma_n), \\ y_n = W(x_n, Tx_n, \beta_n) \end{cases} \tag{3.4}$$

for all $n \geq 1$ where $\{\gamma_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ satisfying $0 < a \leq \gamma_n \leq b < 1$ and $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$. Then the sequence $\{x_n\}$ converges to $z \in F(T)$.

Proof. Put $N = 1$ in Theorem 3.1, we obtain the desired result. □

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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Dipartimento di Matematica
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Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
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Ram Verma
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1200 Dallas Drive #824 Denton,
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Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Shin Min Kang[1,2], Arif Rafiq[3], Young Chel Kwun[4,*] and Faisal Ali[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[3]Department of Mathematics and Statistics, Virtual University of Pakistan, Lahore 54000, Pakistan
e-mail: aarafiq@gmail.com

[4]Department of Mathematics, Dong-A University, Busan 49315, Korea
e-mail: yckwun@dau.ac.kr

[5]Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan
e-mail: faisalali@bzu.edu.pk

Abstract

We study the convergence of implicit midpoint type Picard sequence for strongly accretive and strongly pseudocontractive mappings. We have also improved the results of some authors.

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1 Introduction and Preliminaries

Let E be a real Banach space with dual E^*. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if and only if for all $x, y \in D(T)$, the following inequality is satisfied:

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.1}$$

for $r > 0$ and some $t > 1$. If $t = 1$ in inequality (1.1), then T is called *pseudocontractive*.

*Corresponding author

For E, we will denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. As a consequence of a result of Kato [15], it follows from inequality (1.1) that T is strongly pseudocontractive if and only if

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq k\|x-y\|^2 \tag{1.2}$$

holds for all $x, y \in D(T)$ and for some $j(x-y) \in J(x-y)$, where $k = \frac{t-1}{t} \in (0,1)$.

Consequently, it follows easily (again from Kato [15] and inequality (1.2) that T is strongly pseudocontractive if and only if the following inequality holds:

$$\|x-y\| \leq \|x-y+s[(I-T-kI)x - (I-T-kI)y\| \tag{1.3}$$

for all $x, y \in D(T)$ and $s > 0$.

Closely related to the class of pseudocontractive mappings is the class of *accretive operators*. A mapping A with domain $D(A)$ and range $R(A)$ in E is called *accretive* if the following inequality holds:

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\|$$

for all $x, y \in D(A)$ and $s > 0$. Also, as a consequence of Kato [15], this accretive condition can be expressed in terms of the duality map as follows: For each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0. \tag{1.4}$$

Consequently, inequality (1.1) with $t = 1$ yields that A is accretive if and only if $T := (I-A)$ is pseudocontractive. Furthermore, setting $A := (I-T)$, it follows from inequality (1.3) that T is *strongly pseudocontractive* if and only if $(A-kI)$ is *accretive*, and using (1.4), this implies that $T(= I-A)$ is *strongly pseudocontractive* if and only if the following inequality holds

$$\langle Ax - Ay, j(x-y) \rangle \geq k\|x-y\|^2 \tag{1.5}$$

for all $x, y \in D(A)$ and some $k \in (0,1)$. Operators A satisfying inequality (1.5) for all $x, y \in D(A)$ and some $k \in (0,1)$ are called *strongly accretive*. It is then clear that A is strongly accretive if and only if $T := (I-A)$ is strongly pseudocontractive. Thus, the mapping theory for strongly accretive operators is closely related to the fixed point theory of strongly pseudocontractive maps. We shall exploit this connection in the sequel.

The notion of accretive operators was introduced independently in 1967 by Browder [2] and Kato [15]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0 \tag{1.6}$$

is solvable if A is locally Lipschitzian and accretive on E. If u is independent of t, then $Au = 0$ and the solution of this equation corresponds to the equilibrium points of the

system (1.6). Consequently, considerable research efforts have been devoted, especially within the past 15 years or so, to developing constructive techniques for the determination of the kernels of accretive operators in Banach spaces (see [3, 4, 8–12, 14, 16, 17, 19, 20, 22]). Two well known iterative schemes, the *Mann iterative method* (see [18]) and the *Ishikawa iteration scheme* (see [13]) have successfully been employed.

In [16], Liu obtained a fixed point of the strictly pseudocontractive mapping as the limit of an iteratively constructed sequence with error estimation in general Banach spaces.

Theorem 1.1. *Let E be a Banach space, and let K be a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be a Lipschitzian strictly pseudocontractive mapping. If $Fix(T) \neq \emptyset$, where $Fix(T)$ is the fixed point set of T, then $\{x_n\}$ is a sequence in K generated by $x_1 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a seqnence in $(0, 1]$ satisfying

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0 \quad n \to \infty$$

strongly converges to $q \in Fix(T)$ and $Fix(T)$ is a single set.

In [21], Sastry and Babu showed that any fixed point of a Lipschitzian, strictly pseudocontractive mapping T on a closed convex subset K of a Banach space E is necessarily unique, and may be norm approximated by an iterative procedure. They also provided a convergence rate estimate and removes the boundedness assumption on K, generalizing Theorems of Liu.

Theorem 1.2. *Let $(E, \|\cdot\|), K, T, L$ and k be as described above. Let $q \in K$ be a fixed point of T. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1]$ such that for some $\eta \in (0, k)$, for all $n \in \mathbb{N}$,*

$$\alpha_n \leq \frac{k - \eta}{(L + 1)(L + 2 - k)}, \quad \text{while} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Fix $x_1 \in K$. Define for all $n \in \mathbb{N}$,

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T x_n.$$

Then there exists a sequence $\{\beta_n\}$ in $(0, 1)$ with each $\beta_n \geq \frac{\eta}{1+k}\alpha_n$ such that for all $n \in \mathbb{N}$,

$$\|x_{n+1} - q\| \leq \prod_{j=1}^{n} (1 - \beta_j)\|x_1 - q\|.$$

In particular, $\{x_n\}$ converges strongly to q, and q is the unique fixed point of T.

The Mann and Ishikawa iteration schemes are global and their rate of convergence is generally of the order $O(n^{-\frac{1}{2}})$. It is clear that if, for an operator U, *the classical iteration sequence* of the form, $x_{n+1} = Ux_n$, $x_0 \in D(U)$ (the so-called *Picard sequence*) converges, then it is certainly superior and preferred to either the Mann or the Ishikawa sequence since it requires less computations and moreover, its rate of convergence is always at least as fast as that of a geometric progression.

In [5, 6], Chidume proved the following results.

Theorem 1.3. *Let E be an arbitrary real Banach space and $A : E \to E$ be a Lipschitz (with constant $L > 0$) and strongly accretive mapping with strong accretivity constant $k \in (0,1)$. Let x^* denote a solution of the equation $Ax = 0$. Set $\epsilon := \frac{1}{2}\big(\frac{k}{1+L(3+L-k)}\big)$ and define $A_\epsilon : E \to E$ by $A_\epsilon x := x - \epsilon Ax$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}$ in E by*

$$x_{n+1} = A_\epsilon x_n, \quad n \geq 0. \tag{1.7}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \delta^n \|x_0 - x^*\|,$$

where $\delta = 1 - \frac{1}{2}k\epsilon \in (0,1)$ is the Lipschitz constant of the operator A. Moreover, x^* is unique.

Corollary 1.4. *Let E be an arbitrary real Banach space and K be nonempty convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.3) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{1}{2}\big(\frac{k}{1+L(3+L-k)}\big)$ and define $T_{\epsilon_0} : K \to K$ by $T_{\epsilon_0}x = (1-\epsilon_0)x + \epsilon_0 Tx$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in K by*

$$x_{n+1} = T_{\epsilon_0} x_n, \quad n \geq 0. \tag{1.8}$$

Then $\{x_n\}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \delta^n \|x_0 - x^*\|,$$

where $\delta := 1 - \frac{1}{2}k\epsilon_0 \in (0,1)$. Moreover, x^* is unique.

Recently Ćirić et al. [7] presented the following results.

Theorem 1.5. *Let E be an arbitrary real Banach space, $A : E \to E$ be a Lipschitz (with constant $L > 0$) and strongly accretive mapping with strong accretivity constant $k \in (0,1)$. Let x^* denote a solution of the equation $Ax = 0$. Set $\epsilon := \frac{k-\eta}{L(2+L)}$, $\eta \in (0,k)$ and define $A_\epsilon : E \to E$ by $A_\epsilon x := x - \epsilon Ax$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}$ in E by*

$$x_{n+1} = A_\epsilon x_n, \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \theta^n \|x_0 - x^*\|,$$

where $\theta = 1 - \frac{k-\eta}{k-\eta+L(2+L)}\eta \in (0,1)$. Thus the choice $\eta = \frac{k}{2}$ yields $\theta = 1 - \frac{k^2}{2[k+2L(2+L)]}$. Moreover, x^* is unique.

Corollary 1.6. *Let E be an arbitrary real Banach space, K be a nonempty convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.3) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{k-\eta}{L(2+L)}$, $\eta \in (0,k)$ and define $T_{\epsilon_0} : K \to K$ by $T_{\epsilon_0}x = (1-\epsilon_0)x + \epsilon_0 Tx$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ in K by*

$$x_{n+1} = T_{\epsilon_0} x_n, n \geq 0.$$

Then $\{x_n\}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \leq \theta^n \|x_0 - x^*\|,$$

where $\theta := 1 - \frac{k-\eta}{k-\eta+L(2+L)}\eta \in (0,1)$. *Moreover,* x^* *is unique.*

However Kang et al. [14] established the following results.

Theorem 1.7. *Let* E *be an arbitrary real Banach space,* $A : E \to E$ *be a Lipschitz (with constant* $L > 1$*) and strongly accretive mapping with strong accretivity constant* $k \in (0,1)$. *Let* x^* *denote a solution of the equation* $Ax = 0$. *Set* $\epsilon := \frac{k-\eta}{L+(1+L)(k-\eta)}$, $\eta \in (0,k)$ *and define* $A_\epsilon : E \to E$ *by* $A_\epsilon x_n := (1-\epsilon)\, x_{n-1} + \epsilon x_n - \epsilon A x_n$ *for each* $x_n \in E$. *For arbitrary* $x_0 \in E$, *define the sequence* $\{x_n\}$ *in* E *by*

$$x_n = A_\epsilon x_n, \quad n \geq 1.$$

Then $\{x_n\}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \leq \lambda^n \|x_0 - x^*\|,$$

where $\lambda = 1 - \frac{k-\eta}{L+(k-\eta)(1+L+k)}\eta \in (0,1)$. *Thus the choice* $\eta = \frac{k}{2}$ *yields* $\lambda = 1 - \frac{k^2}{2[2L+k(1+L+k)]}$. *Moreover,* x^* *is unique.*

Corollary 1.8. *Let* E *be an arbitrary real Banach space ans* K *be a nonempty closed convex subset of* E. *Let* $T : K \to K$ *be Lipschitz (with constant* $L > 0$*) and strongly pseudocontractive (i.e.,* T *satisfies inequality* (1.3) *for all* $x, y \in K$*). Assume that* T *has a fixed point* $x^* \in K$. *Set* $\epsilon_0 := \frac{k-\eta}{L+(1+L)(k-\eta)}$, $\eta \in (0,k)$ *and define* $A_{\epsilon_0} : K \to K$ *by* $A_{\epsilon_0} x_n := (1-\epsilon_0)\, x_{n-1} + \epsilon_0 x_n - \epsilon_0 A x_n$ *for each* $x_n \in K$. *For arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *in* K *by*

$$x_n = A_{\epsilon_0} x_n, \quad n \geq 1.$$

Then $\{x_n\}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \leq \lambda_0^n \|x_0 - x^*\|,$$

where $\lambda_0 = 1 - \frac{k-\eta}{L+(k-\eta)(1+L+k)}\eta \in (0,1)$. *Thus the choice* $\eta = \frac{k}{2}$ *yields* $\lambda_0 = 1 - \frac{k^2}{2[2L+k(1+L+k)]}$. *Moreover,* x^* *is unique.*

Let H be the Hilbert space. Recently Alghamdi et al. [1] defined the following algorithm.

Algorithm 1.9. *Initialize* $x_n \in H$ *arbitrarily and iterate*

$$x_{n+1} = (1-t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0,$$

where $t_n \in (0,1)$ *for all* n.

For the approximation of fixed points of nonexpansive mappings under the setting of Hilbert spaces, they provide the following results.

Lemma 1.10. *Let $\{x_n\}$ be the sequence generated by Algorithm 1.9. Then*
(i) $\|x_{n+1}-p\| \leq \|x_n-p\|$ *for all* $n \geq 0$ *and* $p \in Fix(T)$,
(ii) $\sum_{n=1}^{\infty} t_n \|x_n - x_{n+1}\|^2 < \infty$,
(iii) $\sum_{n=1}^{\infty} t_n(1-t_n)\|x_n - T(\frac{x_n+x_{n+1}}{2})\|^2 < \infty$.

Lemma 1.11. *Let $\{x_n\}$ be the sequence generated by Algorithm 1.9. Suppose that $t_{n+1}^2 \leq at_n$ for all $n \geq 0$ and $a > 0$. Then*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Lemma 1.12. *Assume that*
(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,
(ii) $\limsup_{n\to\infty} t_n > 0$.
Then the sequence $\{x_n\}$ generated by Algorithm 1.9 satisfies the property

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Theorem 1.13. *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that $\{x_n\}$ is generated by Algorithm 1.9, where the sequence $\{t_n\}$ of parameters satisfies the conditions:*
(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,
(ii) $\limsup_{n\to\infty} t_n > 0$.
Then $\{x_n\}$ converges weakly to a fixed point of T.

In this paper, we study the convergence of implicit Picard sequence for strongly accretive and strongly pseudocontractive mappings. We have also improved the results of [5–7, 14, 16, 19–21].

2 Main results

In the following theorems, $L > 1$ will denote the Lipschitz constant of the operator A and $k > 0$ will denote the strong accretivity constant of A (as in inequality (1.5)). Furthermore, $\epsilon > 0$ is defined by

$$\epsilon := \frac{k-\eta}{L + \frac{1}{2}(1+L)(k-\eta)}, \quad \eta \in (0,k).$$

With these notations, we prove the following theorem.

Theorem 2.1. *Let E be an arbitrary real Banach space, $A : E \to E$ be a Lipschitz and strongly accretive mapping with strong accretivity constant $k \in (0,1)$. Let x^* denote a solution of the equation $Ax = 0$. Define $A_\epsilon : E \to E$ by $A_\epsilon x_n := (1-\epsilon)\, x_{n-1} + \epsilon \frac{x_{n-1}+x_n}{2} - \epsilon A\frac{x_{n-1}+x_n}{2}$ for each $x_n \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in E by*

$$x_n = A_\epsilon x_n, \quad n \geq 1.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \rho^n \|x_0 - x^*\|,$$

where $\rho = 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta \in (0,1)$. Thus the choice $\eta = \frac{k}{2}$ yields $\rho = 1 - \frac{k^2}{4L+k(2+L+k)}$. Moreover, x^* is unique.

Proof. Existence of x^* follows from Theorem 13.1 of [8]. Define $T := (I - A)$ where I denotes the identity mapping on E. Observe that $Ax^* = 0$ if and only if x^* is a fixed point of T. Moreover, T is strongly pseudocontractive (satisfies inequality (1.2) since A satisfies (1.5), and so T also satisfies inequality (1.3) for all $x, y \in E$ and all $s > 0$. Furthermore, the recursion formula $x_n = A_\epsilon x_n$ becomes

$$x_n = (1-\epsilon)x_{n-1} + \epsilon T\left(\frac{x_{n-1}+x_n}{2}\right), \quad n \geq 1. \tag{2.1}$$

Observe that

$$x^* = (1+\epsilon)x^* + \epsilon(I - T - kI)x^* - (1-k)\epsilon x^*,$$

and from the recursion formula (2.1) that

$$\begin{aligned} x_{n-1} = (1+\epsilon)x_n + \epsilon(I-T-kI)\frac{x_{n-1}+x_n}{2} - (1-k)\epsilon\frac{x_{n-1}+x_n}{2} \\ + \epsilon^2\left(x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right), \end{aligned} \tag{2.2}$$

so that

$$\begin{aligned} x_{n-1} - x^* = (1+\epsilon)(x_n - x^*) + \epsilon\left[(I-T-kI)\frac{x_{n-1}+x_n}{2} - (I-T-kI)x^*\right] \\ - (1-k)\epsilon\left(\frac{x_{n-1}+x_n}{2} - x^*\right) + \epsilon^2\left(x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right). \end{aligned}$$

Assume that $x_n \simeq x_{n-1}$, which yields that $\frac{x_{n-1}+x_n}{2} \simeq x_n$. Replace $\frac{x_{n-1}+x_n}{2}$ by x_n in the second term of right hand side, we get

$$\begin{aligned} x_{n-1} - x^* = (1+\epsilon)(x_n - x^*) + \epsilon\left[(I-T-kI)x_n - (I-T-kI)x^*\right] \\ - (1-k)\epsilon\left(\frac{x_{n-1}+x_n}{2} - x^*\right) + \epsilon^2\left(x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right). \end{aligned}$$

This implies, using inequality (1.3) with $s = \frac{\epsilon}{1+\epsilon}$ and $y = x^*$ that

$$\begin{aligned} &\|x_{n-1} - x^*\| \\ &\geq (1+\epsilon)\left[\left\|(x_n - x^*) + \frac{\epsilon}{1+\epsilon}\left[(I-T-kI)x_n - (I-T-kI)x^*\right]\right\|\right] \\ &\quad - (1-k)\epsilon\left\|\frac{x_{n-1}+x_n}{2} - x^*\right\| - \epsilon^2\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\ &\geq (1+\epsilon)\|x_n - x^*\| - (1-k)\frac{\epsilon}{2}\|x_{n-1} - x^*\| \\ &\quad - (1-k)\frac{\epsilon}{2}\|x_n - x^*\| - \epsilon^2\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\ &= -(1-k)\frac{\epsilon}{2}\|x_{n-1} - x^*\| + \left(1 + (1+k)\frac{\epsilon}{2}\right)\|x_n - x^*\| \\ &\quad - \epsilon^2\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|. \end{aligned} \tag{2.3}$$

Observe that

$$\begin{aligned}
&\left\|x_{n-1}-T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\leq \|x_{n-1}+Tx_{n-1}\|+\left\|Tx_{n-1}+T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\leq \|Ax_{n-1}\|+\frac{1}{2}\|x_{n-1}-x_n\|+\left\|Ax_{n-1}-A\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\leq L\|x_{n-1}-x^*\|+\frac{1}{2}\|x_{n-1}-x_n\|+L\left\|x_{n-1}-\frac{x_{n-1}+x_n}{2}\right\| \\
&= L\|x_{n-1}-x^*\|+\frac{1}{2}(1+L)\|x_{n-1}-x_n\| \\
&= L\|x_{n-1}-x^*\|+\frac{1}{2}(1+L)\,\epsilon\left\|x_{n-1}-T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|,
\end{aligned}$$

and so

$$\|x_{n-1}-T\left(\frac{x_{n-1}+x_n}{2}\right)\| \leq \frac{L}{1-\frac{1}{2}(1+L)\,\epsilon}\|x_{n-1}-x^*\|, \tag{2.4}$$

so that from (2.3) we obtain

$$\begin{aligned}
&\left(1+(1-k)\frac{\epsilon}{2}\right)\|x_{n-1}-x^*\| \\
&\geq \left(1+(1+k)\frac{\epsilon}{2}\right)\|x_n-x^*\|-\frac{L\epsilon^2}{1-\frac{1}{2}(1+L)\,\epsilon}\|x_{n-1}-x^*\|.
\end{aligned}$$

Therefore

$$\|x_n-x^*\| \leq \frac{1+(1-k)\frac{\epsilon}{2}+\frac{L\epsilon^2}{1-\frac{1}{2}(1+L)\epsilon}}{1+(1+k)\frac{\epsilon}{2}}\|x_{n-1}-x^*\|, \tag{2.5}$$

and consider

$$\begin{aligned}
\rho &= \frac{1+(1-k)\frac{\epsilon}{2}+\frac{L\epsilon^2}{1-\frac{1}{2}(1+L)\epsilon}}{1+(1+k)\frac{\epsilon}{2}} \\
&= 1-\frac{\epsilon}{1+(1+k)\frac{\epsilon}{2}}\left[k-\frac{L\epsilon}{1-\frac{1}{2}(1+L)\,\epsilon}\right] \\
&= 1-\frac{\epsilon}{1+(1+k)\frac{\epsilon}{2}}\eta \\
&= 1-\frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get

$$\begin{aligned}
\|x_n-x^*\| &\leq \rho\|x_{n-1}-x^*\| \\
&\leq \cdots \leq \rho^n\|x_0-x^*\| \\
&\to 0
\end{aligned}$$

as $n\to\infty$. Hence $x_n\to x^*$ as $n\to\infty$. Uniqueness follows from the strong accretivity property of A. This completes the proof. □

The following is an immediate corollary of Theorem 2.1.

Corollary 2.2. *Let E be an arbitrary real Banach space and K be anonempty closed convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 1$) and strongly pseudocontractive (i.e., T satisfies inequality (1.3) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{k-\eta}{L+\frac{1}{2}(1+L)(k-\eta)}$; $\eta \in (0,k)$ and Define $A_{\epsilon_0} : K \to K$ by $A_{\epsilon_0}x_n := (1-\epsilon_0)\, x_{n-1} + \epsilon_0 \frac{x_{n-1}+x_n}{2} - \epsilon_0 A\frac{x_{n-1}+x_n}{2}$ for each $x_n \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ in K by*

$$x_n = A_{\epsilon_0}x_n, \quad n \geq 1. \tag{2.7}$$

Then $\{x_n\}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \rho_0^n \|x_0 - x^*\|,$$

where $\rho_0 = 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta \in (0,1)$. Thus the choice $\eta = \frac{k}{2}$ yields $\rho_0 = 1 - \frac{k^2}{4L+k(2+L+k)}$. Moreover, x^* is unique.

Proof. Observe that x^* is a fixed point of T if and only if it is a fixed point of T_{ϵ_0}. Furthermore, the recursion formula (2.7) simplifies to the formula

$$x_n = (1-\epsilon_0)x_{n-1} + \epsilon_0 T x_n,$$

which is similar to (2.1). Following the method of computations as in the proof of the Theorem 2.1, we obtain

$$\begin{aligned} \|x_n - x^*\| &\leq \frac{1 + (1-k)\,\frac{\epsilon_0}{2} + \frac{L\epsilon_0^2}{1-\frac{1}{2}(1+L)\epsilon_0}}{1 + (1+k)\,\frac{\epsilon_0}{2}} \|x_{n-1} - x^*\| \\ &= \left(1 - \frac{2\,(k-\eta)}{2L + (k-\eta)\,(2+L+k)}\eta\right) \|x_{n-1} - x^*\|. \end{aligned} \tag{2.8}$$

Set $\rho_0 = 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta$. Then from (2.8) we obtain

$$\begin{aligned} \|x_n - x^*\| &\leq \rho_0 \|x_{n-1} - x^*\| \\ &\leq \cdots \leq \rho_0^n \|x_0 - x^*\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. This completes the proof. □

Remark 2.3. Since $L > 1$, consider

$$\begin{aligned} \rho &= 1 - \frac{2\,(k-\eta)}{2L + (k-\eta)\,(2+L+k)}\eta \\ &= 1 - \frac{k-\eta}{L + (k-\eta)\,(1+L+k)} \\ &\quad - \left(\frac{2}{2L + (k-\eta)\,(2+L+k)} - \frac{1}{L + (k-\eta)\,(1+L+k)}\right)(k-\eta)\,\eta \\ &< 1 - \frac{k-\eta}{L + (k-\eta)\,(1+L+k)} \\ &\quad - \frac{(k-\eta)^2\,(L+k)\,\eta}{(2L + (k-\eta)\,(2+L+k))\,(L + (k-\eta)\,(1+L+k))}. \end{aligned}$$

Thus the relation between Kang et al. [14] and our parameter of convergence, that is, λ and ρ, respectively, is the following:

$$\rho < \lambda.$$

Our convergence parameter ρ shows the overall improvement for λ, and consequently the results of Chidume [5, 6], Ćirić et al. [7] and Kang et al. [14] are improved.

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JORDAN HOMOMORPHISMS IN C^*-TERNARY ALGEBRAS AND JB^*-TRIPLES

MOHAMMAD RAGHEBI MOGHADAM, THEMISTOCLES M. RASSIAS, VAHID KESHAVARZ, CHOONKIL PARK* AND YOUNG SUN PARK

Abstract. In this paper, we investigate Jordan homomorphisms between C^*-ternary algebras and Jordan derivations on C^*-ternary algebras, and Jordan homomorphisms between JB^*-triples and Jordan derivations on JB^*-triples, associated with the following Apollonius type additive functional equation

$$f(z-x)+f(z-y)=-\frac{1}{2}f(x+y)+2f\Big(z-\frac{x+y}{4}\Big).$$

1. Introduction and preliminaries

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it.

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A.

Cayley [3] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [17]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Y. Nambu [19] in 1973, is based on such structures.
There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 40]).
The comments on physical applications of ternary structures can be found in [2, 5, 7, 18].

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\leq\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see [41]).

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $x\circ y:=[x,e,y]$ and $x^*:=[e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then

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*Corresponding author.
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$[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.
A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H\Big([x, y, z]\Big) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$.
A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta\Big([x, y, z]\Big) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [22]).
A $\mathbb{C}$-linear mapping $H : A \to B$ is called a ternary Jordan homomorphism if

$$H\Big([x, x, x]\Big) = [H(x), H(x), H(x)]$$

for all $x \in A$.
A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a ternary Jordan derivation if

$$\delta\Big([xxx]\Big) = [\delta(x)xx] + [x\delta(x)x] + [xx\delta(x)]$$

for all $x \in A$. Suppose that $\mathcal{J}$ is a complex vector space endowed with a real trilinear composition $\mathcal{J} \times \mathcal{J} \times \mathcal{J} \ni (x, y, z) \mapsto \{xy^*z\} \in \mathcal{J}$ which is complex bilinear in (x, z) and conjugate linear in y. Then $\mathcal{J}$ is called a *Jordan triple system* if $\{xy^*z\} = \{zy^*x\}$ and

$$\{\{xy^*z\}u^*v\} + \{\{xy^*v\}u^*z\} - \{xy^*\{zu^*v\}\} = \{z\{yx^*u\}^*v\}$$

hold.

We are interested in Jordan triple systems having a Banach space structure. A complex Jordan triple system $\mathcal{J}$ with a Banach space norm $\|\cdot\|$ is called a *J^*-triple* if, for every $x \in \mathcal{J}$, the operator $x\Box x^*$ is hermitian in the sense of Banach algebra theory. Here the operator $x\Box x^*$ on $\mathcal{J}$ is defined by $(x\Box x^*)y := \{xx^*y\}$. This implies that $x\Box x^*$ has real spectrum $\sigma(x\Box x^*) \subset \mathbb{R}$.
A J^*-triple $\mathcal{J}$ is called a JB^*-triple if every $x \in \mathcal{J}$ satisfies $\sigma(x\Box x^*) \geq 0$ and $\|x\Box x^*\| = \|x\|^2$.
A $\mathbb{C}$-linear mapping $H : \mathcal{J} \to \mathcal{L}$ is called a *JB^*-triple homomorphism* if

$$H\Big(\{xyz\}\Big) = \{H(x)H(y)H(z)\}$$

for all $x, y, z \in \mathcal{J}$.
A $\mathbb{C}$-linear mapping $\delta : \mathcal{J} \to \mathcal{J}$ is called a *JB^*-triple derivation* if

$$\delta\Big(\{xyz\}\Big) = \{\delta(x)yz\} + \{x\delta(y)z\} + \{xy\delta(z)\}$$

for all $x, y, z \in \mathcal{J}$ (see [20]).
A $\mathbb{C}$-linear mapping $H : \mathcal{J} \to \mathcal{L}$ is called a *JB^*-triple Jordan homomorphism* if

$$H\Big(\{xxx\}\Big) = \{H(x)H(x)H(x)\}$$

for all $x \in \mathcal{J}$.
A $\mathbb{C}$-linear mapping $\delta : \mathcal{J} \to \mathcal{J}$ is called a *JB^*-triple Jordan derivation* if

$$\delta\Big(\{xxx\}\Big) = \{\delta(x)xx\} + \{x\delta(x)x\} + \{xx\delta(x)\}$$

for all $x \in \mathcal{J}$

The study of stability problems originated from a famous talk given by Ulam [39] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, Hyers [11] answered affirmatively the question of Ulam for additive mappings between Banach spaces. Th.M. Rassias [25] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Th.M. Rassias [26] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [9] following the same approach as in Th.M. Rassias [25], gave an affirmative solution to this question for $p > 1$. For further research developments in stability of functional equations the readers are referred to the works of Găvruta [10], Jung [16], Park [23], Th.M. Rassias [27]–[30], Th.M. Rassias and Šemrl [31], F. Skof [38] and the references cited therein. See also [32, 33, 34, 35, 36, 37] for functional equations.

In an inner product space, the equality

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\Big\|z - \frac{x+y}{2}\Big\|^2$$

holds, and is called the *Apollonius' identity.* The following functional equation, which was motivated by this equation,

$$Q(z-x) + Q(z-y) = \frac{1}{2}Q(x-y) + 2Q\Big(z - \frac{x+y}{2}\Big), \tag{1.1}$$

is quadratic. For this reason, the function equation (1.1) is called a *quadratic functional equation of Apollonius type*, and each solution of the functional equation (1.1) is said to be a *quadratic mapping of Apollonius type.* Jun and Kim [15] investigated the quadratic functional equation of Apollonius type.

In this paper, employing the above equality (1.1), we introduce a new functional equation, which is called the *Apollonius type additive functional equation* and whose solution of the functional equation is said to be the *Apollonius type additive mapping*:

$$L(z-x) + L(z-y) = -\frac{1}{2}L(x+y) + 2L\Big(z - \frac{x+y}{4}\Big).$$

In this paper, we investigate Jordan homomorphisms and Jordan derivations in C^*-ternary algebras, and Jordan homomorphisms and Jordan derivations in JB^*-triples.

2. Jordan homomorphisms between C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$ and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

In this section, we investigate Jordan homomorphisms between C^*-ternary algebras. The following lemma was proved in [24].

Lemma 2.1. *Let $f : A \to B$ be a mapping such that*

$$\Big\|f(z-x)+f(z-y)+\frac{1}{2}f(x+y)\Big\|_B \leq \Big\|2f(z-\frac{x+y}{4})\Big\|_B$$

for all $x, y, z \in A$. Then f is additive.

The following lemma was proved in [8].

Lemma 2.2. *Let $f : A \to B$ be an additive mapping. Then the following assertions are equivalent*

$$f\Big([x,x,x]\Big) = [f(x), f(x), f(x)]$$

for all $x \in A$, and

$$f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) = [f(x),f(y),f(z)]+[f(y),f(z),f(x)]+[f(z),f(x),f(y)]$$

for all $x, y, z \in A$.

The following lemma was proved in [6].

Lemma 2.3. *Let $f : A \to A$ be an additive mapping. Then the following assertions are equivalent.*

$$f\Big([x,x,x]\Big) = [f(x),x,x]+[x,f(x),x]+[x,x,f(x)]$$

for all $x \in A$, and

$$\begin{aligned} &f\Big([xyz]+[yzx]+[zxy]\Big)\\ &=[f(x),b,c]+[x,f(y),z]+[x,y,f(z)]+[f(y),z,x]+[y,f(z),x]\\ &+[y,z,f(x)]+[f(z),x,y]+[z,f(x),y]+[z,x,f(y)], \end{aligned}$$

for all $x, y, z \in A$.

Theorem 2.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\Big\|f(z-\mu x)+\mu f(z-y)+\frac{1}{2}f(x+y)\Big\|_B \leq \Big\|2f(z-\frac{x+y}{4})\Big\|_B, \tag{2.1}$$

$$\begin{aligned} &\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big)-[f(x),f(y),f(z)]-[f(y),f(z),f(x)]-[f(z),f(x),f(y)]\Big\|_B\\ &\leq \theta\Big(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\Big) \end{aligned} \tag{2.2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ *and all* $x, y, z \in A$. *Then the mapping* $f : A \to B$ *is a* C^**-ternary algebra Jordan homomorphism.*

Proof. Assume $r > 1$.

Let $\mu = 1$ in (2.1). By Lemma 2.1, the mapping $f : A \to B$ is additive.

Letting $y = -x$ and $z = 0$, we get

$$\|f(-\mu x) + \mu f(x)\|_B \leq \|2f(0)\|_B = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So

$$-f(\mu x) + \mu f(x) = f(-\mu x) + \mu f(x) = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [21, Theorem 2.1], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\begin{aligned}
&\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) - [f(x),f(y),f(z)] - [f(y),f(z),f(x)] - [f(z),f(x),f(y)]\Big\|_B \\
&\quad = \lim_{n\to\infty} 8^n \Big\|f\Big(\frac{[x,y,z]}{8^n} + \frac{[y,z,x]}{8^n} + \frac{[z,x,y]}{8^n}\Big) \\
&-\Big[f(\frac{x}{2^n}), f(\frac{y}{2^n}), f(\frac{z}{2^n})\Big] - \Big[f(\frac{y}{2^n}), f(\frac{z}{2^n}), f(\frac{x}{2^n})\Big] - \Big[f(\frac{z}{2^n}), f(\frac{x}{2^n}), f(\frac{y}{2^n})\Big]\Big\|_B \\
&\quad \leq \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}\Big(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}\Big) = 0
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) = [f(x),f(y),f(z)] + [f(y),f(z),f(x)] + [f(z),f(x),f(y)]$$

for all $x, y, z \in A$. Hence the mapping $f : A \to B$ is a C^*-ternary algebra Jordan homomorphism.

Similarly, one obtains the result for the case $r < 1$. □

3. Jordan derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

In this section, we investigate Jordan derivations on C^*-ternary algebras.

Theorem 3.1. *Let* $r \neq 1$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be a mapping satisfying* (2.1) *such that*

$$\begin{aligned}
&\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) - [f(x),b,c] - [x,f(y),z] - [x,y,f(z)] - [f(y),z,x] \\
&- [y,f(z),x] - [y,z,f(x)] - [f(z),x,y] - [z,f(x),y] - [z,x,f(y)]\Big\|_A \qquad (3.1)\\
&\leq \theta\Big(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}\Big)
\end{aligned}$$

for all $x, y, z \in A$. *Then the mapping* $f : A \to A$ *is a* C^**-ternary Jordan derivation.*

Proof. Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.4, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.1) that

$$\begin{aligned}
&\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big)-[f(x),y,z]-[x,f(y),z]-[x,y,f(z)]-[f(y),z,x]\\
&-[y,f(z),x]-[y,z,f(x)]-[f(z),x,y]-[z,f(x),y]-[z,x,f(y)]\|_A\\
&=\lim_{n\to\infty}8^n\Big\|f\Big([\frac{x,y,z}{8^n}]+[\frac{y,z,x}{8^n}]+[\frac{z,x,y}{8^n}]\Big)-\Big[f(\frac{x}{2^n}),\frac{y}{2^n},\frac{z}{2^n}\Big]\\
&-\Big[\frac{x}{2^n},f(\frac{y}{2^n}),\frac{z}{2^n}\Big]-\Big[\frac{x}{2^n},\frac{y}{2^n},f(\frac{z}{2^n})\Big]-\Big[f(\frac{y}{2^n}),\frac{z}{2^n},\frac{x}{2^n}\Big]-\Big[\frac{y}{2^n},f(\frac{z}{2^n}),\frac{x}{2^n}\Big]\\
&-\Big[\frac{y}{2^n},\frac{z}{2^n},f(\frac{x}{2^n})\Big]-\Big[f(\frac{z}{2^n}),\frac{x}{2^n},\frac{y}{2^n}\Big]-\Big[\frac{z}{2^n},f(\frac{x}{2^n}),\frac{y}{2^n}\Big]-\Big[\frac{z}{2^n},\frac{x}{2^n},f(\frac{y}{2^n})\Big]\Big\|_A\\
&\le\lim_{n\to\infty}\frac{8^n\theta}{8^{nr}}\Big(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\Big)=0
\end{aligned}$$

for all $x, y, z \in A$. So

$$\begin{aligned}
f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big)&=[f(x),y,z]+[x,f(y),z]+[x,y,f(z)]+[f(y),z,x]\\
&+[y,f(z),x]+[y,z,f(x)]+[f(z),x,y]+[z,f(x),y]+[z,x,f(y)]
\end{aligned}$$

for all $x, y, z \in A$.

Thus the mapping $f : A \to A$ is a C^*-ternary Jordan derivation.

Similarly, one obtains the result for the case $r < 1$. □

4. Jordan homomorphisms between JB^*-triples

Throughout this paper, assume that $\mathcal{J}$ is a JB^*-triple with norm $\|\cdot\|_{\mathcal{J}}$ and that $\mathcal{L}$ is a JB^*-triple with norm $\|\cdot\|_{\mathcal{L}}$.

In this section, we investigate Jordan homomorphisms between JB^*-triples.

Theorem 4.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : \mathcal{J} \to \mathcal{L}$ be a mapping such that*

$$\|f(z-\mu x)+\mu f(z-y)+\frac{1}{2}f(x+y)\|_{\mathcal{L}} \quad\le\quad \|2f(z-\frac{x+y}{4})\|_{\mathcal{L}}, \tag{4.1}$$

$$\begin{aligned}
&\Big\|f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)-\{f(x)f(y)f(z)\}-\{f(y)f(z)f(x)\}-\{f(z)f(x)f(y)\}\Big\|_{\mathcal{L}}\\
&\le\theta\Big(\|x\|_{\mathcal{J}}^{3r}+\|y\|_{\mathcal{J}}^{3r}+\|z\|_{\mathcal{J}}^{3r}\Big)
\end{aligned} \tag{4.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{L}$ is a JB^*-triple Jordan homomorphism.

Proof. Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.4, the mapping $f : \mathcal{J} \to \mathcal{L}$ is $\mathbb{C}$-linear.

It follows from (4.2) that

$$\begin{aligned}&\Big\|f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)-\{f(x)f(y)f(z)\}-\{f(y)f(z)f(x)\}-\{f(z)f(x)f(y)\}\Big\|_{\mathcal{L}}\\&\quad=\lim_{n\to\infty}8^n\Big\|f\Big(\frac{\{xyz\}}{2^n.2^n.2^n}+\frac{\{yzx\}}{2^n.2^n.2^n}+\frac{\{zxy\}}{2^n.2^n.2^n}\Big)-\Big\{f(\frac{x}{2^n})f(\frac{y}{2^n})f(\frac{z}{2^n})\Big\}\\&-\Big\{f(\frac{y}{2^n})f(\frac{z}{2^n})f(\frac{x}{2^n})\Big\}-\Big\{f(\frac{z}{2^n})f(\frac{x}{2^n})f(\frac{y}{2^n})\Big\}\Big\|_{\mathcal{L}}\\&\quad\le\lim_{n\to\infty}\frac{8^n\theta}{8^{nr}}\Big(\|x\|_{\mathcal{J}}^{3r}+\|y\|_{\mathcal{J}}^{3r}+\|z\|_{\mathcal{J}}^{3r}\Big)=0\end{aligned}$$

for all $x,y,z\in\mathcal{J}$. Thus

$$f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)=\{f(x)f(y)f(z)\}+\{f(y)f(z)f(x)\}+\{f(z)f(x)f(y)\}$$

for all $x,y,z\in\mathcal{J}$. Hence the mapping $f:\mathcal{J}\to\mathcal{L}$ is a JB^*-triple Jordan homomorphism.

Similarly, one obtains the result for the case $r<1$. □

5. Jordan derivations on JB^*-triples

Throughout this paper, assume that $\mathcal{J}$ is a JB^*-triple with norm $\|\cdot\|_{\mathcal{J}}$.

In this section, we investigate Jordan derivations on JB^*-triples.

Theorem 5.1. *Let $r\neq1$ and θ be nonnegative real numbers, and let $f:\mathcal{J}\to\mathcal{J}$ be a mapping satisfying* (4.1) *such that*

$$\begin{aligned}&\Big\|f(\{xyz\}+\{yzx\}+\{zxy\})-\{f(x)yz\}-\{xf(y)z\}-\{xyf(z)\}-\{f(y)zx\}\\&-\{yf(z)x\}-\{yzf(x)\}-\{f(z)xy\}-\{zf(x)y\}-\{zxf(y)\}\Big\|_A\\&\le\theta\Big(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\Big)\end{aligned}\tag{5.1}$$

for all $x,y,z\in\mathcal{J}$. Then the mapping $f:\mathcal{J}\to\mathcal{J}$ is a JB^*-triple Jordan derivation.

Proof. Assume $r>1$.

By the same reasoning as in the proof of Theorem 2.4, the mapping $f:\mathcal{J}\to\mathcal{J}$ is $\mathbb{C}$-linear.

It follows from (5.1) that

$$\begin{aligned}&\Big\|f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)-\{f(x)yz\}-\{xf(y)z\}-\{xyf(z)\}-\{f(y)zx\}\\&-\{yf(z)x\}-\{yzf(x)\}-\{f(z)xy\}-\{zf(x)y\}-\{zxf(y)\}\Big\|_{\mathcal{J}}\\&=\lim_{n\to\infty}8^n\Big\|f\Big(\frac{\{xyz\}}{2^n.2^n.2^n}+\frac{\{yzx\}}{2^n.2^n.2^n}+\frac{\{zxy\}}{2^n.2^n.2^n}\Big)-\Big\{f(\frac{x}{2^n})\frac{y}{2^n}\frac{z}{2^n}\Big\}\\&-\Big\{\frac{x}{2^n}f(\frac{y}{2^n})\frac{z}{2^n}\Big\}-\Big\{\frac{x}{2^n}\frac{y}{2^n}f(\frac{z}{2^n})\Big\}-\Big\{f(\frac{y}{2^n})\frac{z}{2^n}\frac{x}{2^n}\Big\}-\Big\{\frac{y}{2^n}f(\frac{z}{2^n})\frac{x}{2^n}\Big\}\\&-\Big\{\frac{y}{2^n}\frac{z}{2^n}f(\frac{x}{2^n})\Big\}-\Big\{f(\frac{z}{2^n})\frac{x}{2^n}\frac{y}{2^n}\Big\}-\Big\{\frac{z}{2^n}f(\frac{x}{2^n})\frac{y}{2^n}\Big\}-\Big\{\frac{z}{2^n}\frac{x}{2^n}f(\frac{y}{2^n})\Big\}\Big\|_{\mathcal{J}}\\&\le\lim_{n\to\infty}\frac{8^n\theta}{8^{nr}}\Big(\|x\|_{\mathcal{J}}^{3r}+\|y\|_{\mathcal{J}}^{3r}+\|z\|_{\mathcal{J}}^{3r}\Big)=0\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So $f(\{xyz\}+\{yzx\}+\{zxy\}) = \{f(x)yz\}+\{xf(y)z\}+\{xyf(z)\}+\{f(y)zx\}+\{yf(z)x\}+\{yzf(x)\}+\{f(z)xy\}+\{zf(x)y\}+\{zxf(y)\}$
for all $x, y, z \in \mathcal{J}$.

Thus the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^*-triple Jordan derivation.

Similarly, one obtains the result for the case $r < 1$. □

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Mohammad Raghebi Moghadam, Department of Mathematics, Tabiat Modares University, P. O. Box 14115-134, Tehran, Iran
E-mail address: mohammadraghebi@gmail.com

Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece
E-mail address: trassias@math.ntua.gr

Vahid Keshavaz, Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran
E-mail address: v.keshavarz68@yahoo.com

Choonkil Park, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

Young Sun Park, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: pppppys@hanyang.ac.kr

THE GENERIC STABILITY OF KKM POINTS IN PMT SPACES

M. TATARI, S. M. VAEZPOUR, AND REZA SAADATI*

Abstract. In this paper, we consider the generic stability of generalized KKM points and present a result concerning the generic continuity of set-valued mappings in PMT spaces. Then we prove that almost all of generalized KKM points of probabilistic upper semicontinuous set-valued mappings defined on compact subsets of such spaces are stable in the sense of Baire category theory. Also, we discuss on existence of the essential component of generalized KKM points.

Keywords: KKM point, PMT space, Hausdorff distance, Generic stability, Essential component, Generic continuity

1. Introduction

In 2003, Yu et.al. [1], introduced the concept of KKM points of a KKM mapping $G : X \to K(X)$, from a bounded complete convex subset X of a normed linear space E into nonempty compact subsets of X. By Fort theorem, they prove that if M be the collection of all KKM mappings G, then there exists a dense residual subset Q of M such that for each $G \in Q$, G is essential. They also proved there exists at least one essential component of KKM points for each $G \in M$; (see also [2, 3]). In this paper, we present a result concerning generic continuity of set-valued mappings based upon extensions of Fort's theorems in probabilistic metric type spaces.

2. Preliminaries

First, let us give the background and auxiliary results which will be needed. For more details see [4, 5, 6, 7, 8].

Definition 2.1. ([9, 10]) mapping $F : (-\infty, \infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$. If in addition $F(0) = 0$, then F is called a distance distribution function. The set of all distance distribution functions $(d.d.f)$is denoted by Δ^+. The maximal element for Δ^+ in this order is the $d.d.f$, ϵ_0, given by

$$\epsilon_0(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ 1 & \text{if } t > 0\,. \end{cases}$$

Definition 2.2. ([9]) A triangular norm (shorter t-norm) is a binary operation T on $[0, 1]$, i.e., a function $T : [0, 1]^2 \to [0, 1]$ which satisfies the following conditions:

(1) T is associative and commutative;
(2) $T(a, 1) = a$ for all $a \in [0, 1]$;
(3) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

In particular, a $t-$norm T is said to be continuous if it is a continuous function in $[0, 1]^2$. A $t-$norm is called sup-continuous if $\sup_{\lambda\in\Lambda} T(a_\lambda, b) = T(\sup_{\lambda\in\Lambda} a_\lambda, b)$ for any family $\{a_\lambda : \lambda \in \Lambda\} \subset [0, 1]$ and $b \in [0, 1]$. The operatins $T_L(a, b) = \max(a + b - 1, 0)$, $T_M(a, b) = \min\{a, b\}$ and $T_p(a, b) = ab$ on $[0, 1]$ are T norms.

Lemma 2.3. ([11]) *Let T be a $t-$norm.*

(1) *If T is left-continuous, then T satisfies $\sup_{0<a<1} T(a, b) = b$ for all $b \in [0, 1]$;*
(2) *If $\sup_{0<a<1} T(a, b) = b$ for all $b \in [0, 1]$, then T satisfies $\sup_{0<a<1} T(a, a) = 1$.*

*The corresponding author.

Definition 2.4. A probabilistic metric type space (PMT space) is a triple (X, F, T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X$ into Δ^+ such that, if $F_{x,y}$ denote the value of F at the pair (x, y), the following conditions hold:

(PMT1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PMT2) $F_{x,y}(t) = F_{y,x}(t)$;
(PMT3) $F_{x,y}(K(s+t)) \geq T(F_{x,z}(s), F_{z,y}(t))$ for any $x, y, z \in M$, $t, s \geq 0$ for some constant $K \geq 1$;

Observe that if $K = 1$, then the PMT space is a probabilistic metric space, however it does not hold true when $K > 1$.

Example 2.5. ([12]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^p dx < \infty$, where $p > 0$ is a real number. Define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ \frac{t}{t+\int_0^1 |f(x)-g(x)|^p dx)^{\frac{1}{p}}} & \text{if } t > 0\,. \end{cases}$$

Then (X, F, T_p) is a PMT space with $K = 2^{\frac{1}{p}}$.

In [12], the authors proved that every PMT space (X, F, T), generated a topology τ on X which has as a base the family of sets of the form $\{B_x(r, t) : x \in X\,, 0 < r < 1\,, t > 0\}$, where $B_x(r, t) = \{y \in X : F_{x,y}(t) > 1 - r\}$ for all $r \in (0, 1)$ and $t > 0$, and (X, F, T) is a Hausdorff topological space. In virtue of this topology τ, a sequence $\{x_n\}$ in (X, F, T) is said to be convergent to x (we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a Cauchy sequence in (X, F, T) if for any given $t > 0$ and $r \in (0, 1)$, there exists $N = N(\varepsilon, \lambda) \in Z^+$ such that $F_{x_n,x_m}(t) > 1 - r$, whenever $n, m \geq N$. Let $t > 0$ and $r \in (0, 1]$, A is said to have a finite (r, t)-net if there exists a finite set $S \subset A$ such that $A \subset \bigcup_{x\in S} B_x(r, t)$, i.e. , for each $y \in A$ there is $x \in S$ such that $F_{x,y}(t) > 1 - r$. A is said to be totally bounded if for each $t > 0$ and $r \in (0, 1]$, A has a finite (r, t)-net. A is said to be probabilistically bounded($P-$bounded) if $\sup_{t>0} \inf_{x,y\in A} F_{x,y}(t) = 1$. Let $P(X)$ denote the class of all nonempty subsets of X. We use the notions:

(1) $P_{cl}(X) = \{Y \in P(X) :\ Y \textit{ is closed}\}$;
(2) $P_{bd}(X) = \{Y \in P(X) :\ Y \textit{ is probabilistic bounded}\}$;
(3) $P_{cp}(X) = \{Y \in P(X) :\ Y \textit{ is compact}\}$;
(4) $P_{cl,bd}(X) = P_{cl}(X) \bigcap P_{bd}(X)$.

Let $\Psi : X \to X$ be a mapping. Ψ is said to be closed if $\Psi A \in P_{cl}(X)$ for each $A \in P_{cl}(X)$. It is said to be bounded if $\Psi A \in P_{bd}(X)$ for each $A \in P_{bd}(X)$.

Lemma 2.6. ([12]) *Let (X, F, T) be a PMT space. Let $A \subset X$.*

(1) *A is compact if and only if A is sequentially compact;*
(2) *If A is compact, then A is closed and totally bounded;*
(3) *If A is totally bounded, then $A \in P_{bd}(X)$ and $\bar{A}$ is also totally bounded.*

3. Probabilistic Hausdorff distance type

Given $x \in X$, $B \in P(X)$, the "probabilistic distance type" from x to B is defined as

$$F_{x,B}(t) = F_{B,x}(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ \sup_{s<t} \sup_{y\in B} F_{x,y}(s) & \text{if } t \in (0, \infty)\,, \end{cases}$$

with the convention $F_{x,\emptyset} = 1 - \epsilon_0$.

Given $A, B \in P(X)$, the "probabilistic type distance" from A to B is defined as

$$F_{A,B}(t) = F_{B,A}(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ \sup_{s<t} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s) & \text{if } t \in (0, \infty)\,. \end{cases}$$

For convenience, we write $\tilde{F}_{A,B}(s) = \inf_{x\in A} \sup_{y\in B} F_{x,y}(s)$. Then $F_{A,B}(t) = \sup_{s<t} \tilde{F}_{A,B}(s)$. Based on the above formulas, we can obtain the following definition.

Definition 3.1. ([13, 14]) Let (X, F, T) be a PMT space and let $A, B \in P(X)$. The probabilistic Hausdorff distance type between A and B is a mapping $H_{A,B} : [0, \infty) \to [0, 1]$ defined by

$$\begin{aligned} H_{A,B}(t) &= \sup_{s<t} \min\{\tilde{F}_{A,B}(s), \tilde{F}_{B,A}(s)\} \\ &= \sup_{s<t} \min\{\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)\}, \quad t \in (0, \infty), \end{aligned}$$

where $H_{A,B}(0) = 0$.

According to the above definition, some results related to the probabilistic Hausdorff distance type can be obtained.

Lemma 3.2. ([13]) *Let (X, F, T) be a PMT space, Let $A, B, C \in P(X), x, y \in X$ and $s, t \in \Re^+$. Then*

(1) $F_{x,B}(K(s + t)) \geq T(F_{x,y}(s), F_{y,B}(t))$;
(2) $H_{x,B}(K(s + t)) \geq T(F_{x,A}(s), H_{A,B}(t))$;
(3) $F_{A,B}(K(s + t)) \geq T(F_{A,C}(s), F_{C,B}(t))$.

Theorem 3.3. *Let (X, F, T) be a PMT space. Then $(P_{cl,bd}(X), H, T)$ is also a PMT space.*

Proof. The conditions (PMT1) and (PMT2) are obvious. Now, we have to show that the condition (PMT3) is satisfied.

Let $A, B, C \in P_{cl,bd}(X)$. If at least one of these three sets is empty, by Definition (3.1) it can easily be verified that the inequality is true. Moreover, if $s = 0$ or $t = 0$, the inequality is also obvious. Thus, we assume that these three sets are non-empty and $t > 0$, $s > 0$.

Set $u < s$, $v < t$. For every $x \in A$ we may assume that $\sup_{y\in B} F_{x,y}(u) > 0$. Then for each $\epsilon \in (0, \sup_{y\in B} F_{x,y}(u))$ there exists $y_x \in B$ such that

$$\sup_{y\in B} F_{x,y}(u) - \epsilon \leq F_{x,y_x}(u) \ .$$

Moreover, since F is a probabilistic metric type, it follows that

$$T(F_{x,y}(u), \sup_{z\in C} F_{x,z}(v)) \leq \sup_{z\in C} F_{x,z}(K(u + v)) \ ,$$

for every $y \in B$ and constant $K > 0$. Thus we can obtain

$$T((\sup_{y\in B} F_{x,y}(u) - \epsilon), \inf_{y\in B} \sup_{z\in C} F_{y,z}(v)) \leq T(F_{x,y_x}(u), \sup_{z\in C} F_{y_x,z}(v)) \leq \sup_{z\in C} F_{x,z}(K(u + v))$$

By the arbitrariness of ϵ and the continuity of T we have

$$T(\sup_{y\in B} F_{x,y}(u), \inf_{y\in B} \sup_{z\in C} F_{y,z}(v)) \leq \sup_{z\in C} F_{x,z}(K(u + v)). \tag{3.1}$$

Then we have that

$$T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(u), \inf_{y\in B} \sup_{z\in C} F_{y,z}(v)) \leq \inf_{x\in A} \sup_{z\in C} F_{x,z}(K(u + v)) \ ,$$

which implies that

$$T(\tilde{F}_{A,B}(u), \tilde{F}_{B,C}(v)) \leq \tilde{F}_{A,C}(K(u + v)). \tag{3.2}$$

In addition, if $\sup_{y\in B} F_{y,y}(u) = 0$ for every $x \in A$, then the inequality (1) still holds. So the inequality (2) is also true in this case. Analogously, we also obtain that

$$T(\tilde{F}_{B,A}(u), \tilde{F}_{C,B}(v)) \leq \tilde{F}_{C,A}(K(u + v)).$$

Therefore, we have

$$\begin{aligned} T \quad (\quad & \min\{\tilde{F}_{A,B}(u), \tilde{F}_{B,A}(u)\}, \min\{\tilde{F}_{B,C}(v), \tilde{F}_{C,B}(v)\} \) \\ \leq \quad & \min\{T(\tilde{F}_{A,B}(u), \tilde{F}_{B,C}(v)), T(\tilde{F}_{B,A}(u), \tilde{F}_{C,B}(v))\} \\ \leq \quad & \min\{\tilde{F}_{A,C}(K(u + v)), \tilde{F}_{C,A}(K(u + v))\}. \end{aligned}$$

Furthermore, we can get that

$$\begin{aligned} T \quad (\quad & \sup_{u<s} \min\{\tilde{F}_{A,B}(u), \tilde{F}_{B,A}(u)\}, \sup_{v<t} \min\{\tilde{F}_{B,C}(v), \tilde{F}_{C,B}(v)\}) \\ \leq \quad & \sup_{u<s,\, v<t} \min\{\tilde{F}_{A,C}(K(u+v)), \tilde{F}_{C,A}(K(u+v))\} \\ \leq \quad & \sup_{u+v<s+t} \min\{\tilde{F}_{A,C}(K(u+v)), \tilde{F}_{C,A}(K(u+v))\}. \end{aligned}$$

Then, it follows from the above inequality that

$$T(H_{A,B}(s), H_{B,C}(t)) \leq H_{A,C}(K(s+t)) \ .$$

Hence, we conclude that $(P_{cl,bd}(X), H, T)$ is a PMT space. This completes the proof. □

As a consequence of Theorem (3.3) and Lemma (2.6) we have the following result.

Corollary 3.4. *Let (X, F, T) be a PMT space. Then (P_{cp}, H, T) is also a PMT space.*

Theorem 3.5. *Let (X, F, T) be a complete PMT space. Then $(P_{cl,bd}(X), H, T)$ is a complete PMT space.*

Proof. By Theorem (3.3), $(P_{cl,bd}(X), H, T)$ is a PMT space. Now let $\{A_n\}_{n=1}^{\infty} \subset P_{cl,bd}(X)$, $A_n \to A$ with respect to H. We shall prove that $A \in P_{cl,bd}(X)$. Take an arbitrary number $\lambda \in (0, 1]$. By continuity of T, we have $\sup_{0<a<1} T(a, a) = 1$, applying the Lemma (2.3), we have $\mu \in (0, \lambda]$ and $\nu \in (0, \lambda]$ such that

$$T(1-\mu, 1-\mu) > 1-\lambda \quad and \quad T(1-\nu, 1-\nu) > 1-\mu. \tag{3.3}$$

The convergence of $\{A_n\}$ implies that there exists $N \in Z^+$ such that

$$H_{A,A_n}(1) > 1-\nu \quad for\, all\, n \geq N. \tag{3.4}$$

Since A_N is probabilistically bounded, we have $\sup_{t>0} inf_{x,y\in A_N} F_{x,y}(t) = 1$. Thus, there exists $M = M(\nu) > 0$ such that $F_{x,y}(M/K) > 1-\nu$ for all $x, y \in A_N$. Suppose that $u, w \in A$ are two arbitrary points. From (4) it follows that there exists $x, y \in A_N$ such that $F_{u,x}(1/K) > 1-\nu$ and $F_{w,y}(1) > 1-\nu$. Thus, from(3) we have

$$F_{u,y}(M+1) \geq T(F_{u,x}(1/K), F_{x,y}(M/K)) \geq T(1-\nu, 1-\nu) > 1-\mu,$$

and moreover,

$$F_{u,w}(M+2) \geq T(F_{u,y}((M+1)/K), F_{y,w}(1/K)) \geq T(1-\mu, 1-\nu) \geq T(1-\mu, 1-\mu) > 1-\lambda.$$

Hence $\sup_{t>0} \inf_{u,w\in A} F_{u,w}(t) \geq \inf_{u,w\in A} F_{u,w}(M+2) \geq 1-\lambda$. By the arbitrariness of λ, we have $\sup_{t>0} \inf_{u,w\in A} F_{u,w}(t) = 1$, i.e. , A is probabilistically bounded. Also similar to proof of Theorem 2.2, [13] we can prove that A is closed. From Theorem (3.3) we see that A is closed. Therefore $A \in P_{cl,bd}(X)$, and so the proof is complete. □

Theorem 3.6. *Let (X, F, T) be a complete PMT space. Then $(P_{cp}(X), H, T)$ is complete PMT space.*

Proof. By Theorem (3.3), $(P_{cl,bd}(X), H, T)$ is a complete PMT space. By Corollary (3.4) we see that $P_{cp}(X)$ is PMT space. Since $P_{cp}(X) \subset P_{cl,bd}(X)$, it is enough that $P_{cp}(X)$ is closed with respect to H.

Suppose that $\{A_n\}_{n=1}^{\infty} \subset P_{cp}(X)$, $A_n \to A$ with respect to H. We shall prove that $A \in P_{cp}(X)$. Choose any $\varepsilon > 0$ and $\lambda \in (0, 1]$. By the left-continuity of T and lemma (2.3), we have $\sup_{0<a<1} T(a, a) = 1$, and then there exists $\mu \in (0, \lambda]$ such that

$$T(1-\mu, 1-\mu) > 1-\lambda. \tag{3.5}$$

By the convergence of $\{A_n\}$, there exists $N \in Z^+$ such that

$$H_{A,A_n}(\frac{\varepsilon}{2}) > 1-\mu \, , for\ all\ n \geq N.$$

Since by Lemma (2.6), A_N is compact, A_N is also totally bounded. Thus, A_N has a finite $(\frac{\varepsilon}{2}, \mu)$-net S_N. From this we infer that S_N is a finite (ϵ, λ)-net of A. In fact, for each $x \in A$, it follows the existence of

$y \in A_N$ such that $F_{x,y}(\frac{\varepsilon}{2K}) > 1-\mu$. For such y we can select $z \in S_N$ with $F_{y,z}(\frac{\varepsilon}{2K}) > 1-\mu$. Hence, from (5) we have

$$F_{x,z}(\varepsilon) \geq T(F_{x,y}(\frac{\varepsilon}{2K}), F_{y,z}(\frac{\varepsilon}{2K})) \geq T(1-\mu, 1-\mu) > 1-\lambda.$$

This shows that S_N is a finite (ε,λ)-net of A, and so A is totally bounded. From the completeness of (X,F,T) it follows that A is compact, i.e. , $A \in P_{cp}(X)$. □

4. Stability of KKM points

Stability of solution maps has been intensively investigated recently [15, 16, 17]. In this section, we first give some Lemmas and concepts, then we investigate on exictence of essential components and the stability of the set of KKM points in PMT space.

For a set A, we denote the set of all nonempty finite subsets of A by $\langle A\rangle$. Let A be a nonempty p-bounded subset of PMT space (X,F,T). Then:

(1) $co\,(A) = \bigcap\{B \subset X,\ B \text{ is a closed ball in } X \text{ such that } A \subset B\}$;
(2) $\mathbb{A}(X) = \{A \subset X,\ A = co\,(A)\}$, i.e. $A \in \mathbb{A}(X)$ if and only if A is an intersection of all closed balls containing A. In this case, we say that A is an admissible set in X;
(3) A is called subadmissible, if for each $D \subset< A >$, $co(D) \subset A$. Obviously, if A is an admissible subset of X, then A must be subadmissible.

Recall that closed and open balls of X are defined as

$$B_x[r,t] = \{y \in X,\ F_{x,y}(t) \geq 1-r\}, \quad B_x(r,t) = \{y \in X,\ F_{x,y}(t) > 1-r\},$$

for any $x \in X$ and $0 < r < 1$ and $t > 0$. Let (X,F,T) be a PMT space and A a subadmissible subset of X and $P_{cp}(X)$ the set of all nonempty compact subsets of X. $G : X \to P_{cp}(X)$ is called a KKM mapping, if for each $A \in< X >$, we have $co(A) \subset G(A)$. More generally, if $G : X \to P_{cp}(X)$, $S : X \to P_{cp}(X)$ are two set-valued functions such that for any $A \in< X >$, $S(co(A)) \subseteq G(A)$, then G is called a generalized KKM mapping with respect to S. If the set-valued function $S : X \to P_{cp}(X)$ satisfies the requirement that for any generalized KKM mapping $G : X \to P_{cp}(X)$ with respect to S the family $\{\overline{G(x)} :\ x \in X\}$ has the finite intersection property, then S is said to have the KKM property. We define

$$KKM(X, P_{cp}(X)) := \{S : X \to P_{cp}(X) \ :\ S \textit{ has the KKM property}\}.$$

Thus if $S \in KKM(X, P_{cp}(X))$, then for any generalized KKM mapping $G : X \to P_{cp}(X)$ with respect to S we have $\bigcap_{x\in X} G(x) \neq \emptyset$. then such a point $x^* \in \bigcap_{x\in X} G(x)$, is called the KKM point of G and denote by $\mathcal{K}(G)$ the set of all generalized KKM points of G. Let M be the collection of all KKM mappings $G : X \to P_{cp}(X)$ with respect to S. For each $G_1, G_2 \in M$ define

$$\tilde{H}_{G_1,G_2}(t) = \inf_{x\in X} H_{G_1(x),G_2(x)}(t),$$

where H is the probabilistic Hausdorff distance type defined on all compact subsets of X. Clearly $(M, \tilde{H}, T)$ is a PMT space.

Lemma 4.1. *$(M, \tilde{H}, T)$ is a complete PMT space.*

Proof. Let $\{G_n\}_{n=1}^{\infty}$ be any Cauchy sequence in M, then for any $t > 0$ and $r \in (0,1)$, there exists a positive integer k such that $\tilde{H}_{G_n,G_m}(t) > 1-r$ whenever $n, m \geq k$, i.e.

$$\inf_{x\in X} H_{G_n(x),G_m(x)}(t) > 1-r \quad ,$$

for any $n, m \geq k$. It follows that for each $x \in X$, $\{G_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(P_{cp}(X), H, T)$. By Theorem (3.6), there is $G : X \to P_{cp}(X)$ such that $H_{G_n(x),G(x)}(t) > 1-r$ for each $x \in X$. And it is easy to prove that $\inf_{x\in X} H_{G_n(x),G(x)}(t) > 1-r$. Suppose that G were not generalized KKM mapping with respect to S, then there exist $\{x_1, ..., x_m\} \subset X$ and $x' \in S(co\{x_1, ..., x_m\})$ such that $x' \in \bigcup_{i=1}^m G(x_i)$. Since $\sup_{x\in X} H_{G_n(x),G(x)}(t) > 1-r$, there is n_2 such that $\bigcup_{i=1}^m G_n(x_i) \subset \bigcup_{i=1}^m B_{G(x_i)}(r,t)$ for any $n \geq n_2$. Thus $x' \notin \bigcup_{i=1}^m G_n(x_i)$ for any $n \geq n_2$ which contradicts the assumption that G_n is generalized KKM mapping

with respect to S for all $n = 1, 2, \ldots$. Hence G must be generalized KKM mapping with respect to S, and $(M, \tilde{H}, T)$ is complete. □

Now we state some definitions. A set-valued mapping S from PMT space (X, F, T), into nonempty subsets of a PMT space (Y, F^*, T^*) is said to be probabilistic upper (lower) semicontinuous at $x_0 \in X$, if for any $0 < r < 1$, there exists $0 < r^{'} < 1$ such that $S(x^{'}) \subset B_{S(x_0)}(r, t)$ $(S(x_0) \subset B_{S(x^{'})}(r, t))$ for each $x^{'} \in X$ with $F_{x_0,x^{'}}(t) > 1 - r^{'}$, for $t > 0$. S is probabilistic continuous at $x_0 \in X$ if S is both probabilistic upper semicontinuous and probabilistic lower semicontinuous at x_0. Also S is said probabilistic metric upper semicontinuous at $x_0 \in X$ if, for any $0 < r < 1$, there exists a neighborhood U of x_0 such that $S(U) \subset B_{S(x_0)}(r, t)$ for $t > 0$. It is easily verified that if $S(x_0)$ is compact, then S is probabilistic metric upper semicontinuous at x_0 if and only if S is probabilistic upper semicontinuous at x_0. In general probabilistic metric upper semicontinuity is a weaker notion than probabilistic upper semicontinuity. On the other hand, the set-valued mapping S is said to be probabilistic metric lower semicontinuous at x_0 if for any $0 < r < 1$ there exists a neighborhood U of x_0 such that $S(x_0) \subset B_{S(x)}(r, t)$ for every $x \in U$ and $t > 0$. It is easy to see that if $S(x_0)$ is totally bounded, then S is probabilistic lower semicontinuous at x_0 if and only if S is probabilistic metric lower semicontinuous at x_0. However, we can also show that in general probabilistic lower semicontinuity is a weaker notion than probabilistic metric lower semicontinuity.

Also a subset Q in X is called a residual set if it contains a countable intersection of open dense subsets of X. A set Q is called nowhere dense in X if $int(\overline{Q}) = \emptyset$. If there exists a dense residual set Q of X such that S is continuous at each point of Q then we say that S is continuous at most point of X. In this case we shall also say that S is generically continuous on X. Result concerning generic continuity of set-valued mappings were first considered by Fort in [18]. After Fort's theorems were published there have been several extensions of his original results; see [19, 20]. In the following we will extend Fort's theorem in PMT space.

Theorem 4.2. *Let (X, F, T) be a complete PMT space, (Y, F^*, T^*) be a PMT space and $S : X \to 2^Y$ be a probabilistic metric upper semicontinuous. Then there exists a dense residual set $Q \subset X$ such that S is probabilistic metric lower semicontinuous at each $x \in Q$.*

Proof. For each $0 < r < 1$ let

$$C(r) = \{x \in X : \forall\, 0 < r_0 < r \,and\, 0 < r^{'} < 1, \exists\, y \in B_x(r^{'}, t), \,such\, that\, B_{S(y)}(r_0, t) \not\supseteq S(x)\, for\, t > 0\}$$

First, we prove that $C(r)$ is a closed set. For any $0 < k < r$ and $0 < r^{'} < 1$, let $r_0 < r_{00} < r$ and $r_{00} - r_0 = \eta$. Due to the probabilistic metric upper semicontinuity of S, for each $z \in \overline{C(r)}$, there exists $0 < r^{''} < r^{'}$ such that $S(x) \subset B_{S(z)}(\eta, t)$ for all $x \in B_z(r^{''}, t)$. Then there exists $x \in C(r) \cap B_z(r^{''}, t)$ such that $S(x) \subset B_{S(z)}(\eta, t)$. From $x \in B_z(r^{''}, t)$, Choose $0 < r^{'''} < r^{''}$ such that $B_x(r^{'''}, t) \subset B_z(r^{''}, t)$. As $x \in C(r)$, it is easy to see that there exists $y \in B_x(r^{'''}, t) \subset B_z(r^{''}, t) \subset B_z(r^{'}, t)$ such that $B_{S(y)}(r_{00}, t) \not\supseteq S(x)$. Thus, it follows that $B_{S(y)}(r_0, t) \supset S(z)$. In fact, if $B_{S(y)}(r_0, t) \supset S(z)$, then $B_{B_{S(y)}(r_0,t)}(\eta, t) \supset B_{S(y)}(\eta, t)$, so that $B_{S(y)}(r_{00}, t) \not\supseteq S(x)$. Thus, it follows that $B_{S(y)}(r_0, t) \not\supseteq S(z)$. In fact, if $B_{S(q)}(r_0, t) \supset S(z)$, then $B_{B_{S(y)}(r_0,t)}(\eta, t) \supset B_{S(z)}(\eta, t)$, so that $B_{S(y)}(r_{00}, t) \supset S(x)$ which contradicts $x \in C(r)$. Thus, it is proved that $z \in C(r)$ and $C(r)$ is a closed set. Next, we will prove that $C(r)$ is a nowhere dense set, that is, to prove that $C(r)$ contains no interior point. If not, let $x_1 \in C(r)$ be a interior point of $C(r)$. For any $0 < r_0 < r$, choose $r_0 < r_{00} < r$ and set $r_{00} - r_0 = \eta$. Then there exists $0 < r_1 < 1$ such that $B_{x_1}(r_1, t) \subset C(r)$ and $S(x^{'}) \subset B_{S(x_1)}(\eta, t)$, $\forall\, x^{'} \in B_{x_1}(r_1, t)$. From $x_1 \in C(r)$, it is known that there exists $x_2 \in B_{x_1}(r_1, t)$ such that $B_{S(x_2)}(r_{00}, t) \not\supseteq S(x_1)$. For $x_2 \in B_{x_1}(r_1, t) \subset C(r)$, choose $0 < r_2 < \frac{r_1}{2}$ such that $B_{x_2}(r_2, t) \subset B_{x_1}(r_1, t) \subset C(r)$ and $S(x^{'}) \subset B_{S(x_2)}(\eta, t)$ for all $x^{'} \in B_{x_2}(r_2, t)$. From $x_2 \in C(r)$, there exists $x_3 \in B_{x_2}(r_2, t)$ such that $B_{S(x_3)}(r_{00}, t) \not\supseteq S(x_2)$. The rest may be deduced by analogy, thus there exists $r_1, r_2, ..., r_{n-1}, r_n, ...$ such that $0 < r_n < \dfrac{r_{n-1}}{2}$,

$$B_{x_n}(r_n, t) \subset B_{x_{n-1}}(r_{n-1}, t) \subset \ldots \subset B_{x_2}(r_2, t) \subset B_{x_1}(r_1, t) \subset C(\varepsilon)$$

and

$$S(x^{'}) \subset B_{S(x_n)}(\eta, t)\,, \forall x^{'} \in B_{x_n}(r_n, t)\,.$$

We also have

$$B_{S(x_{n+1})}(r_{00},t) \not\supseteq S(x_n)\,,\ n=1,2,\ldots\,.$$

From the completeness of (X,F,T) and the closedness of $C(r)$, it is known that there exists $x^*\in C(r)$ and $x_n\to x^*$. As $x^*\in B_{x_n}(r_n,t)$ for each $n=1,2,...$, we have $S(x^*)\subset B_{S(x_n)}(\eta,t)$. Therefore,$B_{S(p^*)}(r_0,t)\subset B_{B_{S(x_n)}(\eta,t)}(r_0,t)=B_{S(x_n)}(r_{00},t)$. It follows from $B_{S(x_n)}(r_{00},t)\not\supseteq S(x_{n-1})$ that $B_{S(x^*)}(r_0,t)\not\supseteq S(x_{n-1})$. In addition, from $x_n\to x^*$ and the upper semicontinuity of S at x^*, for given $r_0>0$, it is known that $S(x_{n-1})\subset B_{S(x^*)}(r_0,t)$ when n is sufficiently large, which is a contradiction. Thus, we can prove that $C(r)$ is nowhere dense.

Let $(0,1)_R$ be the rational number set in $[0,1]$, $C=\bigcup_{r\in(0,1)_R}C(r)$ and $Q=X\setminus C$. From the completeness of X and nowhere density of $C(r)$, it is easy to see that C is of first category. Hence, Q is a dense residual set and of second category for any $0<r<1$, choose $r^{'}\in(0,1)_R$ such that $0<r_0<r$. For each $x\in Q$, by the definition of Q we have $x\notin C(r_0)$. Also by the definition of $C(r_0)$, there exists $0<r^*<r_0$ and $r^{'}>0$ such that $B_{S(y)}(r^*,t)\supset S(x)$ for all $y\in B_x(r^{'},t)$, and hence $B_{S(y)}(r,t)\supset S(x)$. From the arbitrariness of $0<r<1$, it is known that S is probabilistic metric lower semicotinuous at x. Therefore, S is probabilistic metric lower semicontinuous at each $p\in Q$.

□

Because in general probabilistic metric upper semi-continuity is a weaker notion than probabilistic upper semicontinuity, the following corollary is obvious.

Corollary 4.3. *Let (X,F,T) be a complete PMT space, (Y,F^*,T^*) be a PMT space and $S:X\to 2^Y$ be probabilistic upper semicontinuous. Then there exists a dense residual set $Q\subset X$ such that S is probabilistic metric lower semicontinuous at each $x\in Q$, and hence S is also probabilistic lower semi-continuity at each $x\in Q$.*

For each $G\in M$, $\mathcal{K}(G)$ is the set of all KKM points of G, $G\to\mathcal{K}(G)$ indeed defines a set-valued mapping $\mathcal{K}:M\to 2^X$

Lemma 4.4. *$\mathcal{K}:M\to 2^X$ is a probabilistic upper semicontinuous and compact-valued (pusco) mapping.*

Proof. For any $G\in M$, for any sequence $\{x_n\}_{m=1}^{\infty}$ in $\mathcal{K}(G)$ with $x_n\to x^*$, then $x_n\in G(x)$ for each $x\in X$. Since $G(x)$ is compact, then $x^*\in G(x)$ for each $x\in X$ and $x^*\in\bigcap_{x\in X}G(x)$, $x^*\in\mathcal{K}(G)$. Hence $\mathcal{K}(G)$ is closed, $\mathcal{K}(G)\subseteq G(x)$ must be compact. Fix $t>0$, suppose that $\mathcal{K}$ were not probabilitic upper semicontinuous at $G\in M$, then there exist $0<r_0<1$ and a sequence $\{G_n\}_{n=1}^{\infty}$ in M with $G_n\to G$ such that for each $n=1,2,...$, there is $x_n\in\mathcal{K}(G_n)$ with $x_n\notin B_{\mathcal{K}(G)}(r_0,t)$. Since $x_n\in\mathcal{K}(G_n)$, we have $x_n\in\bigcap_{x\in X}G_n(x)$. For any $x\in X$, since $G_n(x)\to G(x)$, $G_n(x)\,(n=1,2,...)$ *and* $G(x)$ is compact, thus $\bigcap_{n=1}^{\infty}G_n(x)\bigcup G(x)$ is compact. $x_n\in G_n(x)$, we may assume that $x_n\to x^*$, we obtain $x^*\in G(x)$. Thus $x^*\in\bigcap_{x\in X}G_n(x)$ and $x^*\in\mathcal{K}(G)\subset B_{K(G)}(r_0,t)$ which contradicts the assumption that $x_n\to x^*$ and $x_n\notin B_{\mathcal{K}(G)}(r_0,t)$ for each $n=1,2,...$. Therefore, $\mathcal{K}$ must be probabilistic upper semicontinuous on M. □

Definition 4.5. $G\in M$, (1) $x\in\mathcal{K}(G)$ is essential if for any $0<r<1$, there exists $0<r^{'}<1$ such that for each $G^{'}\in M$ with $\tilde{H}_{G,G'}(t)>1-r^{'}$, there exists $x^{'}\in\mathcal{K}(G^{'})$, with $F_{x,x'}(t)>1-r$, (2) G is weakly essential if there exists $x\in\mathcal{K}(G)$ which is essential and (3) G is essential if every $x\in\mathcal{K}(G)$ is essential.

Theorem 4.6. *$\mathcal{K}:M\to P_{cp}(X)$ is probabilistic lower semicontinuous at $G\in M$ if and only if G is essential.*

Proof. If $\mathcal{K}$ is probabilistic lower semicontinuous at $G\in M$, then for any $0<r<1$, there exists $0<r^{'}<1$ such that $\mathcal{K}(G)\subset B_{\mathcal{K}(G')}(r,t)$ for each $G^{'}\in M$ with $\tilde{H}_{G,G'}(t)>1-\delta$. For each $x\in\mathcal{K}(G)$ there exists $x^{'}\in\mathcal{K}(G^{'})$ with $F_{x,x'}(t)>1-r$, x is essential and G is essential. Conversely, suppose that G is essential. If $\mathcal{K}$ were not probabilistic lower semicontinuous at G, then there exist $0<r_0<1$ and a sequence $\{G_n\}_{n=1}^{\infty}$ in M with $G_n\to G$ such that for each $n=1,2,...$, there is an $x_n\in\mathcal{K}(G)$ with $x_n\in B_{\mathcal{K}(G_n)}(r_0,t)$. Since $\mathcal{K}(G)$ is compact, we map assume that $x_n\to x\in\mathcal{K}(G)$. Since x is essential, $G_n\to G$, $x_n\to x$, there is an N such that $F_{x_n,x}(t)>1-\frac{r_1}{2}$ and $B_{\mathcal{K}(G_n)}(\frac{r_1}{2},t)$ for all $n\geq N$. Hence $x_n\in B_{\mathcal{K}(G_n)}(r_1,t)$ for all $n\geq N$ which

contracts the assumption that $x_n \notin B_{\mathcal{K}(G_n)}(r_0,t)$ for all $n=1,2,\ldots$. Hence $\mathcal{K}$ must be probabilistic lower semicontinuous at G. □

Theorem 4.7. *There exists a dense residual subset Q of M such that for each $G \in Q$, G is essential.*

Proof. by Lemma (4.4), $\mathcal{K} : M \to P_{cp}(X)$ is an pusco mapping. By Corollary (4.3), there exists a dense residual subset Q of M such that for each $G \in Q$, $\mathcal{K}$ is probabilistic lower semicontinuous at G. By Theorem (4.6), for each $G \in Q$, G is essential. □

Remark 4.8. *If $G \in Q$, by Lemma* (4.4) *and Theorem* (4.6)*, $\mathcal{K}$ is probabilistic continuous, then for any $0<r<1$, there exists $0<r^{'}<1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1-r^{'}$, $H_{\mathcal{K}(G),\mathcal{K}(G')}(t) > 1-r$, G is stable.*

Now we shall introduce some definitions. For each $G \in M$, the component of a point $x \in \mathcal{K}(G)$ is the union of all connected subsets of $\mathcal{K}(G)$ which contain the point x. Note that components are connected closed subsets of $\mathcal{K}(G)$ and are also connected compact. It is easy to see that the components of two distinct points $\mathcal{K}(G)$ either coincide or are disjoint, so that all components constitute a decomposition of $\mathcal{K}(G)$ into connected pairwise disjoint compact subsets, i.e. ,

$$\mathcal{K}(G) = \bigcup_{\alpha\in\Lambda} C_\alpha(G)$$

where Λ is an index set; for any $\alpha \in \Lambda$, $C_\alpha(G)$ is a nonempty connected compact and for any $\alpha,\beta \in \Lambda(\alpha \neq \beta)$, $C_\alpha(G)\bigcap C_\beta(G) = \emptyset$.

Definition 4.9. For each $G \in M$, let $e(G)$ be a nonempty closed subset of $\mathcal{K}(G)$. Fix $t>0$, $e(G)$ is called an essential set of $\mathcal{K}(G)$ if for eny $0<r<1$, there exists $0<r^{'}<1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1-r^{'}$, $\mathcal{K}(G^{'})\bigcap B_{e(G)}(r,t) \neq \emptyset$. If $C_\alpha(G)$, the component of $\mathcal{K}(G)$ is essential, then $C_\alpha(G)$ is called an essential component of $\mathcal{K}(G)$.

Following theorem is the main result .

Theorem 4.10. *For each $G \in M$, there exists at least one essential component of $\mathcal{K}(G)$.*

Proof. By Lemma (4.4), $\mathcal{K} : M \to P_{cp}(X)$ is probabilistic upper semicontinuous, that is for any $0<r<1$, there exists $0<r^{'}<1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1-r^{'}$, $\mathcal{K}(G^{'}) \subset B_{\mathcal{K}(G)}(r,t)$. Hence $\mathcal{K}(G^{'})\bigcap B_{\mathcal{K}(G)}(r,t) \neq \emptyset$, $\mathcal{K}(G)$ is essential set of itself. Let Φ denote the family of all essential sets of $\mathcal{K}(G)$ ordered by set inclusion. Thus Φ is nonempty and every decreasing chain of elements in Φ has a lower bound (because by the compactness the intersection is in Φ); therefore by Zorn's Lemma, Φ has a minimal element $m(G)$ and $m(G)$ is essential. Suppose that $m(G)$ were not connected. Then there exist two nonempty closed sets $c_1(G)$, $c_2(G)$ and two open sets V_1 and V_2 such that $m(G) = c_1(G)\bigcup c_2(G)$, $c_1(G) \subset V_1$, $c_2(G) \subset V_2$, $V_1\bigcap V_2 = \emptyset$. Since $m(G)$ is minimal, neither $c_1(G)$ nor $c_2(G)$ is essential, there exist $0<r_1<1$, $0<r_2<1$ such that for any $0<r^{'}<1$, there exists $G_1, G_2 \in M$ such that $\tilde{H}_{G_1,G_2}(t) > 1-r^{'}$, $\tilde{H}_{G,G_2}(t) > 1-r^{'}$ with $\mathcal{K}(G_1)\bigcap[B_{c_1(G)}(r_1,t)] = \emptyset$, $\mathcal{K}(G_2)\bigcap[B_{c_2(G)}(r_2,t)] = \emptyset$. Denote $W_1 = V_1\bigcap[B_{c_1(G)}(r_1,t)]$, $W_2 = V_2\bigcap[B_{c_2(G)}(r_2,t)]$, then W_1, W_2 are open. Since $c_1(G) \subset W_1$, $c_2(G) \subset W_2$, there exists $0<r_0<1$ such that $[B_{c_1(G)}(r_1,t)] \subset W_1$, $[B_{c_2(G)}(r_2,t)] \subset W_2$. Denote $W_1^{'} = B_{c_1(G)}(r^{'},t)$, $W_2^{'} = B_{c_2(G)}(r^{'},t)$, we know that $W_1^{'}$, $W_2^{'}$ are open. Since $W_1^{'}\bigcup W_2^{'} = B_{c_1(G)\cup c_2(G)}(r^{'},t) = B_{m(G)}(r^{'},t) \supset m(G)$, then there exists $0<r^*<1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1-r^*$, $K(G^{'})\bigcap(W_1\bigcup W_2) \neq \emptyset$. We may suppose that $r^* > a$, where $a = F_{W_1',W_2'}(t) > 0$. For this r^* we can find an $r^\dagger > r^*$ such that $T(1-r^\dagger, 1-r^\dagger) \geq 1-r^*$. Thus for $0<r^\dagger<1$ there exist $G_1', G_2' \in M$ such that $\tilde{H}_{G,G_1'}(\frac{t}{2K}) > 1-r^\dagger$, $\tilde{H}_{G,G_2'}(\frac{t}{2K}) > 1-r^\dagger$, $\mathcal{K}(G_1')\bigcap W_1^{'} = \emptyset$, $\mathcal{K}(G_2')\bigcap W_2^{'} = \emptyset$. Note that

$$\tilde{H}_{G_1',G_2'}(t) > T(\tilde{H}_{G,G_1'}(\frac{t}{2K}))\ \tilde{H}_{G,G_1'}(\frac{t}{2K}) > T(1-r^\dagger, 1-r^\dagger) \geq 1-r^* > a\,.$$

Now define $G^{'}: X \to P_{cp}(X)$ as follows:

$$G^{'}(x) = [G_1^{'} \setminus W_2^{'}] \bigcap [G_2^{'} \setminus W_1^{'}], \quad x \in X$$

Suppose that $G^{'}$ were not a generalized KKM mapping with respect to S, then there exist $\{x_1, ..., x_m\} \subset X$ and $x^{'} \in S(co\{x_1, ..., x_m\})$ such that $x^{'} \notin \bigcup_{i=1}^{m} G^{'}(x_i)$. Since $W_1^{'} \bigcap W_2^{'} = \emptyset$, then $x^{'} \notin W_1^{'}$ or $x^{'} \notin W_2^{'}$. Without loss of generality, we may assume that $x^{'} \notin W_1^{'}$. Since $x^{'} \notin G_2(x_i) \setminus W_1^{'}$, then $x^{'} \notin G_2(x_i)$ for each $i = 1, ..., m$, $x^{'} \notin \bigcup_{i=1}^{m} G_2(x_i)$ which contradicts that $G_2 \in M$. Thus $G^{'} \in M$.

Next we are going to prove that $\tilde{H}_{G^{'},G_1^{'}}(t) \geq \tilde{H}_{G_1^{'},G_2^{'}}(t)$. Note that

$$\tilde{H}_{G^{'},G_1^{'}}(t) = \inf_{x \in X} H_{G^{'}(x),G_1^{'}(x)}(t) = \inf_{x \in X} \{\sup_{s<t} \min\{\inf_{y \in G^{'}(x)} F_{y,G_1^{'}(x)}(s), \inf_{y \in G_1^{'}(x)} F_{y,G^{'}(x)}(s)\}\}$$

and

$$\tilde{H}_{G_1^{'},G_2^{'}}(t) = \inf_{x \in X} H_{G_1^{'}(x),G_2^{'}(x)}(t) = \inf_{x \in X} \{\sup_{s<t} \min\{\inf_{y \in G_1^{'}(x)} F_{y,G_2^{'}(x)}(s), \inf_{y \in G_2^{'}(x)} F_{y,G_1^{'}(x)}(s)\}\}.$$

For any $y \in G_1^{'}(x)$, if $y \in G_1^{'}(x) \setminus W_2^{'}$, then $F_{y,G_1^{'}}(s) = 1$; if $y \in G_1^{'}(x) \setminus W_1^{'}$, then

$$F_{y,G_1^{'}(x)}(s) \geq \inf_{y \in G_2^{'}(x)} F_{y,G_1^{'}(x)}(s),$$

and

$$\inf_{y \in G^{'}(x)} F_{y,G_1^{'}(x)}(s) \geq \inf_{y \in G_2^{'}(x)} F_{y,G_1^{'}(x)}(s) \geq H_{G_1^{'}(x),G_2^{'}(x)}(t).$$

For any $y \in G_1^{'}(x)$, since $W_1^{'} \bigcap W_2^{'} = \emptyset$, then $y \notin W_1^{'}$ or $y \notin W_2^{'}$. If $y \notin W_2^{'}$, then $y \in G_1^{'}(x) \setminus W_2^{'}$, $F_{y,G^{'}(x)}(s) = 1$; if $y \in W_2^{'}$, then $y \notin W_1^{'}$,

$$F_{y,G^{'}(x)}(s) \geq F_{y,G_2^{'}(x)}(s) \setminus W_1^{'} = \sup_{z \in G_2^{'}(x) \setminus W_1^{'}} F_{y,z}(s).$$

Since $F_{y,G_2^{'}(x)}(s) \geq F_{G_1^{'}(x),G_2^{'}(x)}(s) \geq \tilde{H}_{G_1^{'},G_2^{'}}(t) > 1 - r^*$, and when $y \in W_2^{'}$, $z \in W_1^{'}$, $F_{y,z}(s) \leq FW_1^{'},W_2^{'}(s) \leq a \leq r^*$, thus $F_{y,G^{'}(x)}(s) = \sup_{z \in G_2^{'}(x)} F_{y,z}(s)$, $F_{y,G^{'}(x)}(s) \geq F_{y,G_2^{'}(x)}(s) \geq \inf_{y \in G_1^{'}(x)} F_{y,G_2^{'}(x)}(s)$, $\inf_{y \in G_1^{'}(x)} F_{y,G^{'}(x)}(s) \geq \inf_{y \in G_1^{'}(x)} F_{y,G_2^{'}(x)}(s) \geq H_{G_1^{'}(x),G_2^{'}(x)}(s)$.

Thus $H_{G_1^{'}(x),G_2^{'}(x)}(s) \leq H_{G^{'}(x),G_1^{'}(x)}(t)$, for any $x \in X$, $\tilde{H}_{G_1^{'},G_2^{'}}(t) \leq \tilde{H}_{G^{'},G_1^{'}}(t)$. Since

$$\begin{aligned} \tilde{H}_{G^{'},G}(2Kt) &\geq T(\tilde{H}_{G^{'},G_1^{'}}(t), \tilde{H}_{G_1^{'},G}(t)) \\ &\geq T(\tilde{H}_{G_1^{'},G_2^{'}}(t), \tilde{H}_{G_1^{'},G}(t)) \\ &\geq T(1 - r^*, 1 - r^{\dagger}) \geq T(1 - r^{\dagger}, 1 - r^{\dagger}) \geq 1 - r^*, \end{aligned}$$

$\mathcal{K}(G^{'}) \bigcap [W_1^{'} \bigcup W_2^{'}] \neq \emptyset$. Take $x^* \in \mathcal{K}(G^{'}) \bigcap [W_1^{'} \bigcup W_2^{'}]$, without losing the generality, we may assume that $x^* \in \mathcal{K}(G^{'}) \bigcap W_1^{'}$, then $x^* \notin G_2^{'}(x) \setminus W_1^{'}$ for any $x \in X$. Since $\mathcal{K}(G_1^{'}) \bigcap W_1^{'} = \emptyset$, then $x^* \notin \mathcal{K}(G_1^{'})$, that is $x^* \notin \bigcap_{x \in X} G_1^{'}(x)$. There is $\tilde{x} \in X$ such that $x^* \notin G_1^{'}(\tilde{x})$, so $x^* \notin G_1^{'}(\tilde{x}) \setminus W_2^{'}$, $x^* \notin G^{'}(\tilde{x})$, $x^* \notin \mathcal{K}(G^{'})$ which is a contradiction. Thus, $m(G)$ must be connected. Thus there is a component $c_{\alpha_0}(G)$ of $\mathcal{K}(G)$ such that $m(G) \subset c_{\alpha_0}(G)$. It is obvious that $c_{\alpha_0}(G)$ is essential which complete the proof. □

5. Conclusions

In this paper, at first we studied probabilistic metric type spaces. Next, we presented a result concerning generic continuity of set-valued mappings based upon extensions of Fort's theorems in these spaces.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally and significantly in writing this paper.

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Mahboobeh Tatari and S. M. Vaezpour,
Department of Mathematics and Computer Science, Amirkabir University of Technology, Hafez Ave., P. O. Box 15914, Tehran, Iran
E-mail address: mtatari@gmx.us
E-mail address: vaezpour@aut.ac.ir

Reza Saadati,
Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
E-mail address: rsaadati@iust.ac.ir

The essential norm of the generalized integration operator *

Yongmin Liu [1†] Yanyan Yu [2]

1. *School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China*

2. *School of Mathematics and Physics Science, Xuzhou Institute of Technology, Xuzhou 221018 , China*

Abstract: We provide a function-theoretic estimate for the essential norm of the generalized integration operator $I_{g,\varphi}^{(n)}$ from mixed-norm spaces $H(p,\, q,\, \phi)$ to Zygmund-type spaces $\mathcal{Z}_\mu$ by means of the definition of the essential norm of an operator and the properties of the analytic function.

Keywords Essential norm; generalized integration operator; mixed-norm spaces; Zygmund-type space; mean value theorem

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the linear space of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps on $\mathbb{D}$. Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. The following generalized integration operator:

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\xi)) g(\xi) d\xi, \;\; z \in \mathbb{D},$$

has been recently introduced in [23] and considerably studied (see, e.g., [10, 11, 21, 26, 36]). One of the reasons is that it provides connections between operator theorem and complex analysis and helps us to gain a deep understanding of both areas. Recently, there has a lot of work on some related operators on different spaces of analytic functions on the unit disc, see also [1, 6, 20, 28, 29, 30, 31, 32] and the related references therein.

Recall that, for $0 < p,\, q < \infty$ and ϕ normal, let $H(p, q, \phi)$ denote the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{p,q,\phi} = \left(\int_0^1 M_q^p(f, r) \frac{\phi^p(r)}{1-r} dr \right)^{1/p} < \infty,$$

where the integral means $M_p(f, r)$ are defined by

$$M_q(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta \right)^{1/q}, \quad 0 \le r < 1.$$

For $1 \le p < \infty$, the mixed-norm space $H(p, q, \phi)$ equipped with the norm $\|\cdot\|_{p,q,\phi}$ is a Banach space (see [13, 34]). For related results in the setting of the unit ball, see, for example, [12, 24] and the

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†Email: minliu@jsnu.edu.cn

references therein. Given a bounded, continuous and strictly positive function μ on $\mathbb{D}$. Let $\mathcal{Z}_\mu$ denote the space of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Under the norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|, \tag{1}$$

it is easy to see that $\mathcal{Z}_\mu$ is a Banach space.

Let X and Y be two Banach spaces. T is a bounded linear operator from X to Y. The essential norm of T is defined by

$$\|T\|_{e,X\to Y} = \inf_S \{\|T - S\|_{X\to Y}\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm and the infimum is taken over all admissible compact operator S from X to Y. For results on the essential norms of composition, weighted composition and some integral operators, see, for instance, [2, 3, 4, 5, 7, 8, 9, 14, 15, 16, 17, 18, 19, 22, 27, 33, 35, 37, 38], and the references therein.

Inspired by the above results, for $1 \le p < \infty$ and $0 < q < \infty$, we provide a function-theoretic estimate for the essential norms of the generalized integration operator $I^{(n)}_{g,\varphi}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ on the basis of the characterizations of the boundedness and compactness of the generalized integration operator ([10]). The main result is Theorem 3.1.

Throughout the paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables. For two functions A and B, we use the abbreviation $A \preceq B$ if there is a positive constant C independent of A and B such that $A \le CB$. We write $A \asymp B$ if $A \preceq B \preceq A$.

2 Some Lemmas

To prove our main results, we need several lemmas. The next Schwartz-type lemma is proved in a standard way, hence we omit its proof.

Lemma 2.1 *Assume that $0 < p, q < \infty$. Then $I^{(n)}_{g,\varphi}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact if and only $I^{(n)}_{g,\varphi}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded and for every bounded sequence $\{f_m\}$ in $H(p, q, \phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$, we have $\|I^{(n)}_{g,\varphi} f_m\|_{\mathcal{Z}_\mu} \to 0$ as $m \to \infty$.*

The following characterizations of boundedness and compactness of the operator $I^{(n)}_{g,\varphi}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ have been essentially proved in [10]. The parts of this work which are relevant here are given in the following two lemmas.

Lemma 2.2 *([10]) Assume that $0 < p, q < \infty$. Then $I^{(n)}_{g,\varphi}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded if and only if*

$$M_1 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+n+1}} < \infty, \tag{2}$$

and

$$M_2 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+n}} < \infty. \tag{3}$$

Lemma 2.3 *([10]) Assume that $0 < p, q < \infty$. Then $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact if and only if $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} = 0, \tag{4}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n}} = 0. \tag{5}$$

On the basis of the compactness criterion, the following lemma provides a sufficient condition of the compactness of the operator $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$.

Lemma 2.4 *Assume that $1 \le p < \infty$, $0 < q < \infty$. Then $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact, if $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded and $\|\varphi\|_\infty < 1$.*

Proof. Since $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded and $\|\varphi\|_\infty < 1$, the conditions (4) and (5) hold, Lemma 2.3 implies that the operator $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact.

The last auxiliary results needed are the following two lemmas.

Lemma 2.5 *([25]) Let $0 < p, q < \infty$ and $f \in H(p, q, \phi)$. Then for every $m \in \mathbb{N}_0$, there is a positive constant C independent of f such that*

$$|f^{(m)}(z)| \le C \frac{\|f\|_{p,q,\phi}}{\phi(|z|)\,(1-|z|^2)^{m+1/q}},\ z \in \mathbb{D}.$$

Lemma 2.6 *([10, P. 384]) Let $0 < p, q < \infty$, $\omega \in \mathbb{D}$ and*

$$f_{j,\omega}(z) = \frac{(1-|\omega|^2)^{j+t}}{\phi(|\omega|)(1-\overline{\omega}z)^{j+t+1/q}},\ z \in \mathbb{D}.$$

Then $f_{j,\omega} \in H(p, q, \phi)\,(j \in \mathbb{N})$, where the constant t is from the definition of the normality of the function ϕ.

3 The essential norm of the operator $I_{g,\varphi}^{(n)} : H(p, q, \phi) \to \mathcal{Z}_\mu$

The following theorem is the main result of the paper.

Theorem 3.1 *Assume that $1 \le p < \infty$, $0 < q < \infty$, $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded. Then*

$$\begin{aligned}&\|I_{g,\varphi}^{(n)}\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu}\\ &\asymp \max\left\{\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}}, \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}\right\}.\end{aligned}$$

Proof. If $\|\varphi\|_\infty < 1$, by Lemma 2.4, $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact, so $\|I_{g,\varphi}^{(n)}\|_{e,H(p,q,\phi)\to\mathcal{Z}_\mu} = 0$. The result holds.

Now assume that $\|\varphi\|_\infty = 1$. We first give the lower estimate of $\|I^{(n)}_{g,\varphi}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}$. Let $C_j(q,t) = 1/q + j + t$, $j = 1, 2, \ldots, n+2$. For a fixed $\omega \in \mathbb{D}$, set the test function

$$\begin{aligned} f_\omega(z) &= C_{n+1}(q,t) f_{1,\omega}(z) - C_1(q,t) f_{2,\omega}(z) \\ &= C_{n+1}(q,t)\frac{(1-|\omega|^2)^{1+t}}{\phi(|\omega|)(1-\overline{\omega}z)^{1+t+1/q}} - C_1(q,t)\frac{(1-|\omega|^2)^{2+t}}{\phi(|\omega|)(1-\overline{\omega}z)^{2+t+1/q}},\ z \in \mathbb{D}. \end{aligned}$$

By Lemma 2.6, we have $f_\omega \in H(p,\,q,\,\phi)$ and $\sup\limits_{\omega\in\mathbb{D}} \|f_\omega\|_{p,\,q,\,\phi} \le C$. By an easy calculation,

$$\begin{aligned} f^{(n)}_\omega(z) &= \frac{C_{n+1}(q,t)C_1(q,t)C_2(q,t)\cdots C_n(q,t)(1-|\omega|^2)^{1+t}\,(\overline{\omega})^n}{\phi(|\omega|)(1-\overline{\omega}z)^{n+1+t+1/q}} \\ &\quad - \frac{C_1(q,t)C_2(q,t)C_3(q,t)\cdots C_{n+1}(q,t)(1-|\omega|^2)^{2+t}\,(\overline{\omega})^n}{\phi(|\omega|)(1-\overline{\omega}z)^{n+2+t+1/q}}, \end{aligned} \tag{6}$$

$$\begin{aligned} f^{(n+1)}_\omega(z) &= \frac{C^2_{n+1}(q,t)C_1(q,t)C_2(q,t)\cdots C_n(q,t)(1-|\omega|^2)^{1+t}\,(\overline{\omega})^{n+1}}{\phi(|\omega|)(1-\overline{\omega}z)^{n+2+t+1/q}} \\ &\quad - \frac{C_1(q,t)C_2(q,t)\cdots C_{n+2}(q,t)(1-|\omega|^2)^{2+t}\,(\overline{\omega})^{n+1}}{\phi(|\omega|)(1-\overline{\omega}z)^{n+3+t+1/q}}, \end{aligned} \tag{7}$$

Let $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$, satisfying $\lim\limits_{k\to\infty} |\varphi(z_k)| = 1$. Let

$$\begin{aligned} &f_k(z) = f_{\varphi(z_k)}(z) \\ &= \frac{C_{n+1}(q,t)(1-|\varphi(z_k)|^2)^{1+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{1+t+1/q}} - \frac{C_1(q,t)(1-|\varphi(z_k)|^2)^{2+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{2+t+1/q}},\ k \in \mathbb{N}. \end{aligned}$$

Then $f_k \in H(p,\,q,\,\phi)$, $\sup\limits_{k\in\mathbb{N}} \|f_k\|_{p,\,q,\,\phi} \le C$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, so for every compact operator $K\colon\ H(p,\,q,\,\phi) \to \mathcal{Z}_\mu$, by Lemma 2.1 , one has $\lim\limits_{k\to\infty} \|Kf_k\|_{\mathcal{Z}_\mu} = 0$. Using (6) and (7), we have

$$f^{(n)}_k(\varphi(z_k)) = 0,\ f^{(n+1)}_k(\varphi(z_k)) = \frac{-C_1(q,t)C_2(q,t)\cdots C_{n+1}(q,t)\left(\overline{\varphi(z_k)}\right)^{n+1}}{\phi(|\varphi(z_k)|)\,(1-|\varphi(z_k)|^2)^{n+1+1/q}}.$$

Therefore

$$\begin{aligned} &C\|I^{(n)}_{g,\varphi} - K\|_{H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \ge \limsup_{k\to\infty} \|\left(I^{(n)}_{g,\varphi} - K\right) f_k\|_{\mathcal{Z}_\mu} \\ &\ge \limsup_{k\to\infty} \left(\|I^{(n)}_{g,\varphi} f_k\|_{\mathcal{Z}_\mu} - \|Kf_k\|_{\mathcal{Z}_\mu}\right) \\ &\ge \limsup_{k\to\infty} \|I^{(n)}_{g,\varphi} f_k\|_{\mathcal{Z}_\mu} \\ &\ge C_1(q,t)C_2(q,t)\cdots C_{n+1}(q,t) \limsup_{k\to\infty} \frac{\mu(z_k)|\varphi(z_k)|^{n+1}\cdot|\varphi'(z_k)g(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n+1+1/q}}, \end{aligned}$$

thus

$$\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \geq C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)\,|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1+1/q}}. \tag{8}$$

Further, let

$$h_k(z) = \frac{C_{n+2}(q,t)(1-|\varphi(z_k)|^2)^{1+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{1+t+1/q}} - \frac{C_1(q,t)(1-|\varphi(z_k)|^2)^{2+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{2+t+1/q}},\ k\in\mathbb{N}.$$

Similarly $h_k \in H(p,\,q,\,\phi)$, $\sup\limits_{k\in\mathbb{N}}\|h_k\|_{p,\,q,\,\phi}\leq C$, and

$$h_k^{(n)}(\varphi(z_k)) = \frac{C_1(q,t)C_2(q,t)\cdots C_n(q,t)\left(\overline{\varphi(z_k)}\right)^n}{\phi(|\varphi(z_k)|)\,(1-|\varphi(z_k)|^2)^{n+1/q}},\ h_k^{(n+1)}(\varphi(z_k))=0.$$

It is clear that $\{h_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, Lemma 2.1 implies that $\lim\limits_{k\to\infty}\|Kh_k\|_{\mathcal{Z}_\mu}=0$ and consequently

$$\begin{aligned}
&C\|I_{g,\varphi}^{(n)}-K\|_{H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \geq \limsup_{k\to\infty}\|\left(I_{g,\varphi}^{(n)}-K\right)h_k\|_{\mathcal{Z}_\mu}\\
&\geq \limsup_{k\to\infty}\left(\|I_{g,\varphi}^{(n)}h_k\|_{\mathcal{Z}}-\|Kh_k\|_{\mathcal{Z}_\mu}\right)\\
&\geq \limsup_{k\to\infty}\|I_{g,\varphi}^{(n)}h_k\|_{\mathcal{Z}_\mu}\\
&\geq C_1(q,t)C_2(q,t)\cdots C_n(q,t)\limsup_{k\to\infty}\frac{\mu(z_k)|\varphi(z_k)|^n\cdot|g'(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n+1/q}}\\
&= C_1(q,t)C_2(q,t)\cdots C_n(q,t)\limsup_{k\to\infty}\frac{\mu(z_k)\cdot|g'(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n+1/q}},
\end{aligned}$$

thus

$$\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \geq C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)\,|g'(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1/q}}. \tag{9}$$

By (8) and (9), we get

$$\begin{aligned}
&\max\left\{\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1+1/q}},\lim_{|\varphi(z)|\to 1}\frac{\mu(z)\,|g'(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1/q}}\right\}\\
&\preceq \|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu},
\end{aligned}$$

as desired.

Next, we intend to give the upper estimate of $\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}$. Since $I_{g,\varphi}^{(n)}\colon\ H(p,\,q,\,\phi)\to\mathcal{Z}_\mu$ is bounded, then for any $f\in H(p,\,q,\,\phi)$, there is a constant C such that $\|I_{g,\varphi}^{(n)}f\|_{\mathcal{Z}_\mu}\leq C\|f\|_{p,\,q,\,\phi}$. Taking test function $f(z)=\frac{z^n}{n}\in H(p,\,q,\,\phi)$, we obtain

$$K_1 := \sup_{z\in\mathbb{D}}\mu(z)\,|g'(z)|<\infty. \tag{10}$$

If take $f(z)=\frac{z^{n+1}}{n+1}\in H(p,\,q,\,\phi)$, we have

$$K_2:=\sup_{z\in\mathbb{D}}\mu(z)|\varphi'(z)g(z)|<\infty. \tag{11}$$

For a fixed $\rho\in(0,1)$, using the normality of ϕ we have

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)\,|\rho\varphi'(z)g(z)|}{\phi(\rho|\varphi(z)|)\,(1-|\rho\varphi(z)|^2)^{n+1+1/q}}\le\sup_{z\in\mathbb{D}}\frac{\mu(z)\,|\rho\varphi'(z)g(z)|}{\phi(\rho)\,(1-|\rho|^2)^{n+1+1/q}}<\infty,$$

and

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|g'(z)|}{\phi(\rho|\varphi(z)|)\,(1-|\rho\varphi(z)|^2)^{n+1/q}}\le\sup_{z\in\mathbb{D}}\frac{\mu(z)\,|g'(z)|}{\phi(\rho)\,(1-|\rho|^2)^{n+1/q}}<\infty.$$

By Lemma 2.2, we get $I^{(n)}_{g,\rho\varphi}\colon\ H(p,\,q,\,\phi)\to\mathcal{Z}_\mu$ is bounded. Since $|\rho\varphi(z)|\le\rho<1$, by Lemma 2.6, $I^{(n)}_{g,\rho\varphi}\colon\ H(p,\,q,\,\phi)\to\mathcal{Z}_\mu$ is compact, thus

$$\begin{aligned}
&\|I^{(n)}_{g,\varphi}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}\le\left\|I^{(n)}_{g,\varphi}-I^{(n)}_{g,\rho\varphi}\right\|_{H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}\\
&=\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\left\|\left(I^{(n)}_{g,\varphi}-I^{(n)}_{g,\rho\varphi}\right)f\right\|_{\mathcal{Z}_\mu}\\
&=\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{z\in\mathbb{D}}\mu(z)\left|\left(I^{(n)}_{g,\varphi}f-I^{(n)}_{g,\rho\varphi}f\right)''(z)\right|\\
&\le\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{z\in\mathbb{D}}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{z\in\mathbb{D}}\mu(z)|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))||g'(z)|\\
&\le\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{|\varphi(z)|\le r}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{r<|\varphi(z)|<1}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{|\varphi(z)|\le r}\mu(z)|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))||g'(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{r<|\varphi(z)|<1}\mu(z)|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))||g'(z)|\\
&:=I^{(n)}_{1,\rho,r}+I^{(n)}_{2,\rho,r}+I^{(n)}_{3,\rho,r}+I^{(n)}_{4,\rho,r},
\end{aligned}\tag{12}$$

Using the mean value theorem, the triangle inequality, (10), (11) and Lemma 2.5 we have

$$\begin{aligned}
&I^{(n)}_{1,\rho,r}=\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{|\varphi(z)|\le r}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&=\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{|\varphi(z)|\le r}|\mu(z)|\varphi'(z)g(z)|\left|f^{(n+1)}(\varphi(z))-f^{(n+1)}(\rho\varphi(z))+(1-\rho)f^{(n+1)}(\rho\varphi(z))\right|\\
&\le\sup_{\|f\|_{H(p,\,q,\,\phi)\le1}}\sup_{|\varphi(z)|\le r}(1-\rho)\mu(z)|\varphi'(z)g(z)|\left(\sup_{|w|\le r}|f^{(n+2)}(w)|+\sup_{|w|\le r}|f^{(n+1)}(w)|\right)\\
&\le C(1-\rho)\left(\frac{1}{\phi(r)(1-r^2)^{n+2+1/q}}+\frac{1}{\phi(r)(1-r^2)^{n+1+1/q}}\right)\to0\ (\rho\to1),
\end{aligned}\tag{13}$$

$$I_{3,\rho,r}^{(n)} = \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{|\varphi(z)|\le r} \mu(z)|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))||g'(z)|$$
$$\le \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{|\varphi(z)|\le r} (1-\rho)\mu(z)|g'(z)| \sup_{|w|\le r} |f^{(n+1)}(w)|$$
$$\le C(1-\rho)\frac{1}{\phi(r)(1-r^2)^{n+1+1/q}} \to 0 \ (\rho\to 1), \tag{14}$$

$$I_{2,\rho,r}^{(n)} =$$
$$\sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \mu(z)\left|f^{(n+1)}(\varphi(z)) - \rho f^{(n+1)}(\rho\varphi(z))\right| |\varphi'(z)g(z)|$$
$$\le \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \mu(z)|\varphi'(z)g(z)| \left(\left|f^{(n+1)}(\varphi(z))\right| + \left|\rho f^{(n+1)}(\rho\varphi(z))\right|\right)$$
$$\le \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \mu(z)|\varphi'(z)g(z)|\frac{C\|f\|_{p,\,q,\,\phi}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}}$$
$$+ \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \mu(z)|\varphi'(z)g(z)|\frac{C\|f\|_{p,\,q,\,\phi}}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1+1/q}}$$
$$\le \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}} + \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|\varphi'(z)g(z)|}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1+1/q}}$$
$$\to 2C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}} \ (\rho\to 1), \tag{15}$$

and

$$I_{4,\rho,r}^{(n)} = \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \mu(z)|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))||g'(z)|$$
$$\le \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}\|f\|_{p,\,q,\,\phi}$$
$$+ \sup_{\|f\|_{H(p,\,q,\,\phi)\le 1}} \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|g'(z)|}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1/q}}\|f\|_{p,\,q,\,\phi}$$
$$\le C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}} + C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|g'(z)|}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1/q}}$$
$$\to 2C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}} \ (\rho\to 1). \tag{16}$$

Using the conditions (13-16), let $\rho\to 1$ in (12), then let $r\to 1$, we have

$$\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}$$
$$\le 2C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}} + 2C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}$$
$$\preceq \max\left\{\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}}, \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}\right\},$$

we get the desired result, so we are done.

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Solution of a third order fractional system of difference equations

Asim Asiri[1], M. M. El-Dessoky[1,2] and E. M. Elsayed[1,2]

[1]King Abdulaziz University, Faculty of Science,Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mail: amkasiri@kau.edu.sa, dessokym@mans.edu.eg,
emmelsayed@yahoo.com.

ABSTRACT

This paper is devoted to study the form of the solutions and the periodicity of the following third order systems of rational difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm1\pm x_{n-2}y_{n-1}x_n},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ and y_0 are real numbers.

Keywords: System of difference equations, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Recently, many researchers have studied systems of difference equations. This concern is due to the importance of some systems that can be used in some mathematical models in biology, economics, genetics, psychology, populations etc [1-37]. Many articles discuss difference equations systems, for example, Cinar [1] investigated the following system

$$x_{n+1}=\frac{m}{y_n},\ \ y_{n+1}=\frac{py_n}{x_{n-1}y_{n-1}}.$$

Elsayed et al. [2] dealt with the solutions of the system of the difference equations

$$x_{n+1}=\frac{1}{x_{n-p}y_{n-p}},\ \ y_{n+1}=\frac{x_{n-p}y_{n-p}}{x_{n-q}y_{n-q}},$$

and

$$x_{n+1}=\frac{1}{x_{n-p}y_{n-p}z_{n-p}},\ \ y_{n+1}=\frac{x_{n-p}y_{n-p}z_{n-p}}{x_{n-q}y_{n-q}z_{n-q}},\ z_{n+1}=\frac{x_{n-q}y_{n-q}z_{n-q}}{x_{n-r}y_{n-r}z_{n-r}}.$$

The behavior of positive solutions of the following system

$$x_{n+1}=\frac{x_{n-1}}{1+x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-1}}{1+y_{n-1}x_n}.$$

has been discussed by Kurbanli et al. [3].

In [7] Yalçınkaya studied the sufficient condition for the global asymptotic stability of the following system

$$z_{n+1}=\frac{t_nz_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_nt_{n-1}+a}{z_n+t_{n-1}}.$$

In [5] El-Dessoky considered the solutions of the following system

$$x_{n+1}=\frac{y_{n-1}y_{n-2}}{x_n\left(\pm1\pm y_{n-1}y_{n-2}\right)},\ y_{n+1}=\frac{x_{n-1}x_{n-2}}{y_n\left(\pm1\pm x_{n-1}x_{n-2}\right)}.$$

Touafek et al. [6] examined the periodic nature and found the form of the solutions of the following system

$$x_{n+1}=\frac{x_{n-3}}{\pm1\pm x_{n-3}y_{n-1}},\ y_{n+1}=\frac{y_{n-3}}{\pm1\pm y_{n-3}x_{n-1}}.$$

Kurbanli [7] investigated solutions' behavior for the following

$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\ z_{n+1}=\frac{x_n}{z_{n-1}y_n}.$$

El-Dessoky et al. [8] obtained the form of the solutions of the following system

$$x_{n+1}=\frac{x_{n-1}}{1+y_nx_{n-1}},\ y_{n+1}=\frac{y_{n-1}}{1+x_ny_{n-1}},\ z_{n+1}=\frac{z_{n-m}}{x_ny_n}.$$

We consider in this paper, the solution of the systems of difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm1\pm x_{n-2}y_{n-1}x_n},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ and y_0 are real numbers.

2. ON THE SYSTEM: $X_{N+1}=\frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N},\ Y_{N+1}=\frac{X_{N-2}}{1+X_{N-2}Y_{N-1}X_N}$

In this section, we investigate the solutions of the system of two difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{1+x_{n-2}y_{n-1}x_n},\tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions are real numbers with $y_{-2}x_{-1}y_0\neq1$ and $x_{-2}y_{-1}x_0\neq-1$.

The following theorem is concerned with the form of the solutions of system (1).

Theorem 1. Suppose that $\{x_n,y_n\}$ are solutions of system (1). Then every solution of system (1) are periodic with period six and for $n=0,1,2,...,$

$$\begin{aligned}
x_{6n-2} &= x_{-2},\quad x_{6n-1}=x_{-1},\quad x_{6n}=x_0,\quad x_{6n+1}=\frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)},\\
x_{6n+2} &= y_{-1}(1+x_{-2}y_{-1}x_0),\quad x_{6n+3}=\frac{y_0}{(1-y_{-2}x_{-1}y_0)},\\
\text{and } y_{6n-2} &= y_{-2},\quad y_{6n-1}=y_{-1},\quad y_{6n}=y_0,\quad y_{6n+1}=\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)},\\
y_{6n+2} &= x_{-1}(1-y_{-2}x_{-1}y_0),\quad y_{6n+3}=\frac{x_0}{(1+x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Proof: If $n=0$, the result holds. We now assume that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-8} &= x_{-2},\quad x_{6n-7}=x_{-1},\quad x_{6n-6}=x_0,\quad x_{6n-5}=\frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)},\\
x_{6n-4} &= y_{-1}(1+x_{-2}y_{-1}x_0),\quad x_{6n-3}=\frac{y_0}{(1-y_{-2}x_{-1}y_0)},\\
\text{and } y_{6n-8} &= y_{-2},\quad y_{6n-7}=y_{-1},\quad y_{6n-6}=y_0,\quad y_{6n-5}=\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)},\\
y_{6n-4} &= x_{-1}(1-y_{-2}x_{-1}y_0),\quad y_{6n-3}=\frac{x_0}{(1+x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Now it follows from Eq.(1) that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-5}}{1-y_{6n-5}x_{6n-4}y_{6n-3}} \\
&= \frac{\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}}{1-\left(\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}y_{-1}(1+x_{-2}y_{-1}x_0)\frac{x_0}{(1+x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)\left(1-\frac{x_{-2}y_{-1}x_0}{(1+x_{-2}y_{-1}x_0)}\right)} = x_{-2}, \\
y_{6n-1} &= \frac{x_{6n-4}}{1+x_{6n-4}y_{6n-3}x_{6n-2}} \\
&= \frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{1+\left(y_{-1}(1+x_{-2}y_{-1}x_0)\frac{x_0}{(1+x_{-2}y_{-1}x_0)}x_{-2}\right)} \\
&= \frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{1+y_{-2}y_{-1}x_0} = y_{-1}, \\
x_{6n} &= \frac{y_{6n-3}}{1-y_{6n-3}x_{6n-2}y_{6n-1}} = \frac{\frac{x_0}{(1+x_{-2}y_{-1}x_0)}}{1-\left(\frac{x_0}{(1+x_{-2}y_{-1}x_0)}x_{-2}y_{-1}\right)} = x_0, \\
y_{6n+1} &= \frac{x_{6n-2}}{1+x_{6n-2}y_{6n-1}x_{6n}} = \frac{x_{-2}}{1+x_{-2}y_{-1}x_0}.
\end{aligned}$$

Also, we can prove the other relations and so the proof is completed.

Example 1. We look at interesting numerical example for the system (1) with the initial conditions $x_{-2} = 0.5,\ x_{-1} = 0.9,\ x_0 = 0.2,\ y_{-2} = 1.1,\ y_{-1} = 0.7$ and $y_0 = 0.3$. (See Fig. 1).

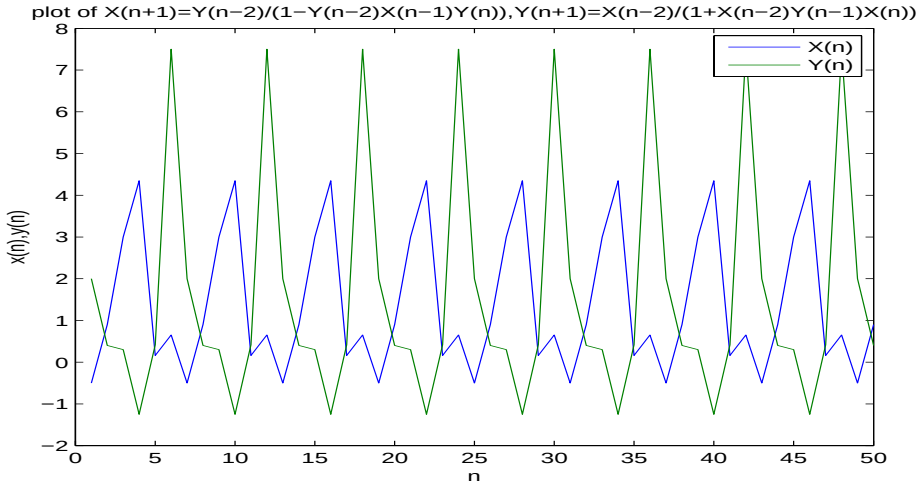


Figure 1.

3. ON THE SYSTEM: $X_{N+1} = \frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N}$, $Y_{N+1} = \frac{X_{N-2}}{1-X_{N-2}Y_{N-1}X_N}$

In this section, we get the form of the solutions of the system of two difference equations

$$x_{n+1} = \frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}x_n}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Theorem 2. Suppose that $\{x_n, y_n\}$ are solutions of system (2). Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{6n-2} &= x_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)},\\
x_{6n-1} &= x_{-1}\prod_{i=0}^{n-1}\frac{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)}{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)},\\
x_{6n} &= x_0\prod_{i=0}^{n-1}\frac{(1-(6i+2)x_{-2}y_{-1}x_0)(1-(6i+5)x_{-2}y_{-1}x_0)}{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)},\\
x_{6n+1} &= \frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)}{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)},\\
x_{6n+2} &= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)},\\
x_{6n+3} &= \frac{y_0(1-2y_{-2}x_{-1}y_0)}{(1-3y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)}{(1-(6i+6)y_{-2}x_{-1}y_0)(1-(6i+9)y_{-2}x_{-1}y_0)},\\
y_{6n-2} &= y_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)y_{-2}x_{-1}y_0)(1-(6i+3)y_{-2}x_{-1}y_0)}{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)},\\
y_{6n-1} &= y_{-1}\prod_{i=0}^{n-1}\frac{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}{(1-(6i+2)x_{-2}y_{-1}x_0)(1-(6i+5)x_{-2}y_{-1}x_0)},\\
y_{6n} &= y_0\prod_{i=0}^{n-1}\frac{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)}{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)},\\
y_{6n+1} &= \frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)},\\
y_{6n+2} &= \frac{x_{-1}(1-y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)}{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)},\\
y_{6n+3} &= \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)}.
\end{aligned}$$

where $\prod_{i=0}^{-1} C_i = 1$

Proof: The result holds for $n = 0$. Now suppose that $n > 1$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned}
x_{6n-5} &= \frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)}{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)},\\
x_{6n-4} &= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)},\\
x_{6n-3} &= \frac{y_0(1-2y_{-2}x_{-1}y_0)}{(1-3y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)}{(1-(6i+6)y_{-2}x_{-1}y_0)(1-(6i+9)y_{-2}x_{-1}y_0)},\\
y_{6n-5} &= \frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)},
\end{aligned}$$

$$y_{6n-4} = \frac{x_{-1}(1-y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)}{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)},$$

$$y_{6n-3} = \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)}.$$

It follows from Eq.(2) that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-5}}{1-y_{6n-5}x_{6n-4}y_{6n-3}} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(\begin{array}{c} 1-\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)} \\ \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)} \\ \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)} \end{array}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(1-\frac{x_0y_{-1}x_{-2}}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+9)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(1-\frac{x_{-2}y_{-1}x_0}{(1-(6n-3)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(\frac{1-(6n-2)x_{-2}y_{-1}x_0}{1-(6n-3)x_{-2}y_{-1}x_0}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(\frac{1-(6n-2)x_{-2}y_{-1}x_0}{1-(6n-3)x_{-2}y_{-1}x_0}\right)}.
\end{aligned}$$

Then we see that

$$x_{6n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}.$$

Also, we see from Eq.(2) that

$$\begin{aligned}
y_{6n-1} &= \frac{x_{6n-4}}{1-x_{6n-4}y_{6n-3}x_{6n-2}} \\
&= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(\begin{array}{c} \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)} \\ \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)} \\ x_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)} \end{array}\right)}
\end{aligned}$$

$$
\begin{aligned}
y_{6n-1} &= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(\begin{array}{c}\frac{x_{-2}x_0y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)}\\ \prod_{i=0}^{n-1}\frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}\end{array}\right)}\\
&= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(x_{-2}x_0y_{-1}\prod_{i=0}^{n-2}(1-(6i+4)x_{-2}y_{-1}x_0)\prod_{i=0}^{n-1}\frac{1}{(1-(6i+4)x_{-2}y_{-1}x_0)}\right)}\\
&= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(\frac{x_{-2}x_0y_{-1}}{1-(6n-2)x_{-2}y_{-1}x_0}\right)}\\
&= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{\left(\frac{1-(6n-1)x_{-2}y_{-1}x_0}{1-(6n-2)x_{-2}y_{-1}x_0}\right)}.
\end{aligned}
$$

Then

$$
y_{6n-1} = y_{-1}\prod_{i=0}^{n-1}\frac{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}{(1-(6i+2)x_{-2}y_{-1}x_0)(1-(6i+5)x_{-2}y_{-1}x_0)}.
$$

Similarly we can prove the other relations. Hence, the proof is completed.

Lemma 1. If $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$ and y_0 are arbitrary real numbers and $\{x_n, y_n\}$ are solutions of system (2), then the following statements are true:

(i) If $x_{-2} = 0,\ y_{-1} \neq 0,\ x_0 \neq 0$, then we have $x_{6n-2} = y_{6n+1} = 0$ *and* $x_{6n} = y_{6n+3} = x_0,\ x_{6n+2} = y_{6n-1} = y_{-1}$.

(ii) If $x_{-1} = 0,\ y_{-2} \neq 0,\ y_0 \neq 0$, then we have $x_{6n-1} = y_{6n+2} = 0$ *and* $x_{6n+1} = y_{6n-2} = y_{-2},\ x_{6n+3} = y_{6n} = y_0$.

(iii) If $x_0 = 0,\ y_{-1} \neq 0,\ x_{-2} \neq 0$, then we have $x_{6n} = y_{6n+3} = 0$ *and* $x_{6n-2} = y_{6n+1} = x_{-2},\ x_{6n+2} = y_{6n-1} = y_{-1}$.

(iv) If $y_{-2} = 0,\ x_{-1} \neq 0,\ y_0 \neq 0$, then we have $y_{6n-2} = x_{6n+1} = 0$ *and* $x_{6n-1} = y_{6n+2} = x_{-1},\ x_{6n+3} = y_{6n} = y_0$.

(v) If $y_{-1} = 0,\ x_0 \neq 0,\ x_{-2} \neq 0$, then we have $y_{6n-1} = x_{6n+2} = 0$ *and* $x_{6n-2} = y_{6n+1} = x_{-2},\ x_{6n} = y_{6n+3} = x_0$.

(vi) If $y_0 = 0,\ y_{-2} \neq 0,\ x_{-1} \neq 0$, then we have $y_{6n} = x_{6n+3} = 0$ *and* $x_{6n-1} = y_{6n+2} = x_{-1},\ x_{6n+1} = y_{6n-2} = y_{-2}$.

Proof: The proof follows from the form of the solutions of system (2).

Example 2. We assume the initial conditions $x_{-2} = -0.5,\ x_{-1} = 0.9,\ x_0 = 3,\ y_{-2} = 2,\ y_{-1} = 0.4$ *and* $y_0 = 0.3$, for the difference system (2), see Fig. 2.

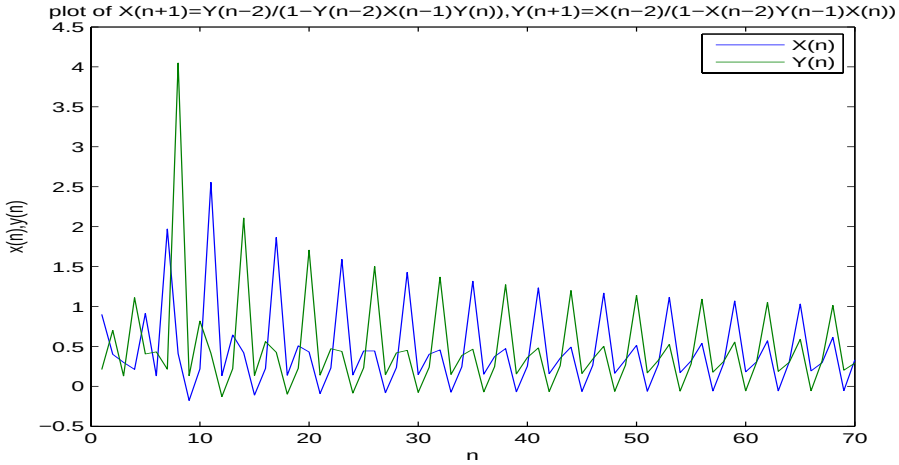


Figure 2.

The following cases can be proved similarly.

4. ON THE SYSTEM: $X_{N+1} = \frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N}$, $Y_{N+1} = \frac{X_{N-2}}{-1+X_{N-2}Y_{N-1}X_N}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers such that $x_{-2}y_{-1}x_0 \neq \pm 1$ and $y_{-2}x_{-1}y_0 \neq 1, \neq \frac{1}{2}$.

Theorem 3. If $\{x_n, y_n\}$ are solutions of difference equation system (3). Then every solution of system (3) are periodic with period twelve and for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{12n-2} &= x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{y_{-2}}{(1 - y_{-2}x_{-1}y_0)}, \\
x_{12n+2} &= -y_{-1}(-1 + x_{-2}y_{-1}x_0), \quad x_{12n+3} = \frac{y_0(-1 + 2y_{-2}x_{-1}y_0)}{(-1 + y_{-2}x_{-1}y_0)}, \\
x_{12n+4} &= \frac{x_{-2}(1 + x_{-2}y_{-1}x_0)}{(-1 + x_{-2}y_{-1}x_0)}, \; x_{12n+5} = \frac{x_{-1}}{(-1 + 2y_{-2}x_{-1}y_0)}, \\
x_{12n+6} &= \frac{x_0(-1 + x_{-2}y_{-1}x_0)}{(1 + x_{-2}y_{-1}x_0)}, \quad x_{12n+7} = \frac{-y_{-2}(-1 + 2y_{-2}x_{-1}y_0)}{(-1 + y_{-2}x_{-1}y_0)}, \\
x_{12n+8} &= -y_{-1}(1 + x_{-2}y_{-1}x_0), \quad x_{12n+9} = \frac{-y_0}{(1 - y_{-2}x_{-1}y_0)}, \\
y_{12n-2} &= y_{-2}, \quad y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{x_{-2}}{(-1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+2} &= \frac{-x_{-1}(-1 + y_{-2}x_{-1}y_0)}{(-1 + 2y_{-2}x_{-1}y_0)}, \; y_{12n+3} = \frac{-x_0}{(1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+4} &= -y_{-2}, \; y_{12n+5} = -y_{-1}, \; y_{12n+6} = -y_0, \; y_{12n+7} = \frac{-x_{-2}}{-1 + x_{-2}y_{-1}x_0}, \\
y_{12n+8} &= \frac{x_{-1}(-1 + y_{-2}x_{-1}y_0)}{(-1 + 2y_{-2}x_{-1}y_0)}, \quad y_{12n+9} = \frac{x_0}{(1 + x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Example 3. Figure (3) shows the behavior of the solution of the difference system (3) with the initial conditions $x_{-2} = -0.7,\ x_{-1} = 1.6,\ x_0 = -0.3,\ y_{-2} = -0.2,\ y_{-1} = -1.9$ and $y_0 = 1.1$.

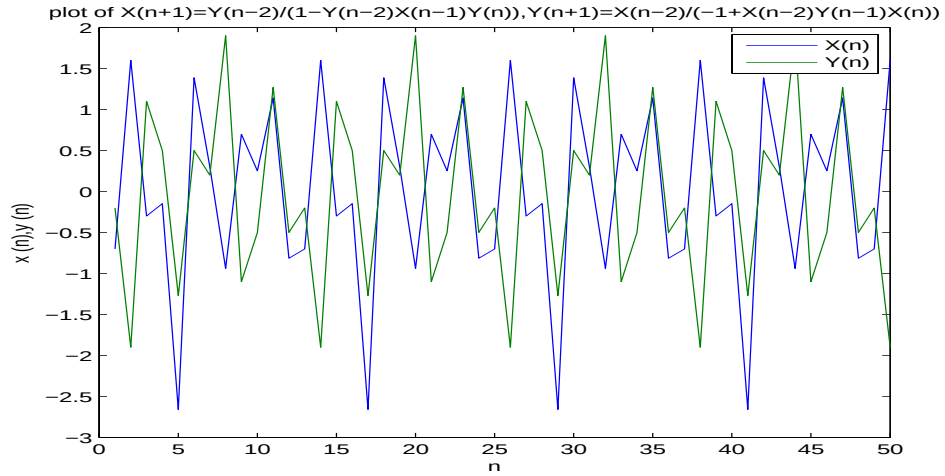


Figure 3.

5. ON THE SYSTEM $X_{N+1} = \frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N},\ Y_{N+1} = \frac{X_{N-2}}{-1-X_{N-2}Y_{N-1}X_N}$

In this section, we discuss the solutions of the following system of the difference equations

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \tag{4}$$

where $n \in \mathbb{N}_0$ and the initial conditions are real numbers with $x_{-2}y_{-1}x_0 \neq -1, \neq -\frac{1}{2}$ and $y_{-2}x_{-1}y_0 \neq \pm 1$.

Theorem 4. Suppose that $\{x_n, y_n\}$ are solutions of system (4). Then every solution of system (4) are periodic with period twelve and for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{12n-2} &= x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{y_{-2}}{(1 - y_{-2}x_{-1}y_0)},\\
x_{12n+2} &= \frac{y_{-1}(1 + x_{-2}y_{-1}x_0)}{(1 + 2x_{-2}y_{-1}x_0)},\ x_{12n+3} = \frac{y_0}{(1 + y_{-2}x_{-1}y_0)},\ x_{12n+4} = -x_{-2},\\
x_{12n+5} &= -x_{-1}, \quad x_{12n+6} = -x_0, \quad x_{12n+7} = \frac{-y_{-2}}{(1 - y_{-2}x_{-1}y_0)},\\
x_{12n+8} &= \frac{-y_{-1}(1 + x_{-2}y_{-1}x_0)}{(1 + 2x_{-2}y_{-1}x_0)}, \quad x_{12n+9} = \frac{-y_0}{(1 + y_{-2}x_{-1}y_0)},\\
y_{12n-2} &= y_{-2}, \quad y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{-x_{-2}}{(1 + x_{-2}y_{-1}x_0)},\\
y_{12n+2} &= -x_{-1}(1 - y_{-2}x_{-1}y_0), \quad y_{12n+3} = \frac{-x_0(1 + 2x_{-2}y_{-1}x_0)}{(1 + x_{-2}y_{-1}x_0)},\\
y_{12n+4} &= \frac{-y_{-2}(1 + y_{-2}x_{-1}y_0)}{(1 - y_{-2}x_{-1}y_0)},\ y_{12n+5} = \frac{-y_{-1}}{(1 + 2x_{-2}y_{-1}x_0)},\\
y_{12n+6} &= \frac{-y_0(1 - y_{-2}x_{-1}y_0)}{(1 + y_{-2}x_{-1}y_0)}, \quad y_{12n+7} = \frac{x_{-2}(1 + 2x_{-2}y_{-1}x_0)}{(1 + x_{-2}y_{-1}x_0)},\\
y_{12n+8} &= x_{-1}(1 + y_{-2}x_{-1}y_0), \quad y_{12n+9} = \frac{x_0}{(1 + x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Example 4. Figure (4) shows the periodicity of the solution of the difference system (4) when we put the initial conditions $x_{-2} = -0.7,\ x_{-1} = .6,\ x_0 = 5,\ y_{-2} = -.5,\ y_{-1} = .18$ and $y_0 = 6$.

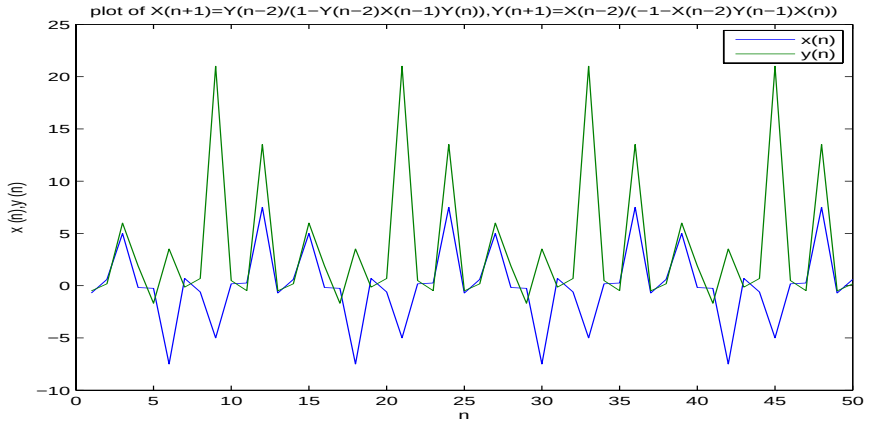


Figure 4.

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Inequalities For Orlicz mixed Harmonic Quermassintegrals *

Lewen Ji, Zhenbing Zeng †
Department of Mathematics, Shanghai University, Shanghai, 200444, China

Abstract: In this paper, we introduce the notion of Orlicz mixed harmonic quermassintegrals. The variational formula of harmonic quermassintegrals with respect to the Orlicz combination is proved. The Minkowski type inequality and the Brunn-Minkowski type inequality are established for Orlicz mixed harmonic quermassintegrals.
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1. Introduction

Associated with a body $K \in \mathcal{K}_0^n$ are its harmonic quermassintegrals, $\widehat{W}_0, \widehat{W}_1, \cdots, \widehat{W}_n$. These quermassintegrals were introduced by Hadwiger [6,section 6.4.8], and can be defined by letting $\widehat{W}_0 = V(K), \widehat{W}_n = \omega_n$, and for $j = 1, 2, \cdots, n-1$,

$$\widehat{W}_{n-j} = \frac{\omega_n}{\omega_j}[\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)]^{-1}. \tag{1.1}$$

Where $K|\xi$ denote the image of the orthogonal projection of K onto ξ. From the Schwarz or Höder inequality, we have

$$\widehat{W}_i(K) \leq W_i(K).$$

Here $0 < i < n$, with equality hold if and only if K has constant outer $n-i$-measure. For more details, please see [8,10].

Recently, the Orlicz-Brunn Minkowski theory originated with the work of Lutwak, Yang and Zhang [11,12]and Haberl [5]. This theory is much more general than the L_p Brunn-Minkowski theory(see [2,9,13]), for the development of the Orlicz Brunn Minkowski theory, see[3-5,7,11-12,15-16]. Gardner [3] extended the L_p Brunn-Minkowski theory to a Orlicz Brunn-Minkowski theory. As the same time, Xi, jin and leng[15] defined the Orlicz addition and given the Orlicz Brunn-Minkowski inequality. Note that xi use a completely different approach technique of Steiner symmetrization, although these results coincide with Gardner.

Following the spirit of Hadwiger, we introduce the Orlicz mixed harmonic quermassintegrals as follow: Let $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$, we have, for $j = 1, 2, \cdots, n-1$,

$$\widehat{W}_{\phi,n-j}(K, L) = \frac{\omega_n}{\omega_j}[\int_{Gr(n,j)} \frac{V_\phi^{\{j\}}(K|\xi, L|\xi)}{V^{\{j\}}(K|\xi)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)]^{-1}. \tag{1.2}$$

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†Corresponding author:zbzeng@shu.edu.cn; jilewen2008@163.com.

Note that, taking $K = L, \phi(1) = 1$, we have $\widehat{W}_{\phi,n-j}(K,K) = \widehat{W}_{n-j}(K)$. where $V^{\{j\}}(K|\xi)$ denotes the $j-$dimensional volume of intersection of K with an $j-$ dimensional subspace $\xi \subset R^n$. The Grassmann manifold $Gr(n,j)$ is endowed with the normalized Haar measure.

In Section 3. Using the Orlicz combination, we give the variational formula of harmonic quermassintegrals. That is, if $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$, then for each $j = 1, 2, \cdots, n-1$, $\xi \in Gr(n,j)$,

$$\lim_{\varepsilon \to 0^+} \frac{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L) - \widehat{W}_{n-j}(K)}{\varepsilon} = \frac{j}{\phi_l'(1)} \widehat{W}_{\phi,n-j}(K,L)^{-1} \widehat{W}_{n-j}(K)^2.$$

Orlicz mixing homogeneous quermassintegrals as a generalization of the harmonic homogeneous integration. A nature question is whether there is a Minkowski type isoperimetric inequality for Orlicz mixing homogeneous quermassintegrals, we give a definite answer.

In the section 4, we prove the following Minkowski type inequality: Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \phi \in \mathcal{C}^+$. Then

$$\widehat{W}_{\phi,n-j}(K,L)^{-1} \geq \widehat{W}_{n-j}(K)^{-1} \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j}).$$

If ϕ is strictly convex, with equality holds if and only if K and L are dilations.

We also establish the Brunn-Minkowski type inequality for convex bodies. The following Brunn-Minkowski type inequality: Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \varepsilon > 0$. Then for any $\phi \in \mathcal{C}^+$,

$$\phi(1) \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}) + \varepsilon \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}).$$

If ϕ is strictly convex, with equality holds if and only if K and L are dilations.

In the section 5, We obtain the Brunn-Minkowski type inequality for Orlicz sum $+_\phi(K_1, K_2, \cdots, K_n)$. Let $K_1, K_2, \cdots, K_n \in \mathcal{K}_0^n \in \mathcal{K}_0^n$, ϕ is $n \geq 2$ variate functions, $j = 1, 2, \cdots, n-1$. Then following inequality hold,

$$1 \geq \phi((\frac{\widehat{W}_{n-j}(K_1)}{\widehat{W}_{n-j}(+_\phi(K_1, K_2, \cdots, K_n))})^{1/j}, \cdots, (\frac{\widehat{W}_{n-j}(K_n)}{\widehat{W}_{n-j}(+_\phi(K_1, K_2, \cdots, K_n))})^{1/j}).$$

If ϕ is strictly convex, with equality holds if and only if $K_1, K_2, \cdots, K_n$ are dilations.

2. Notation and background material

We collect some basic facts about convex bodies that are needed in our paper.

2.1. Mixed volumes

The unit ball in R^n and unit sphere denoted by B and S^{n-1}, respectively. Let $Gr(n,j)$ denote the Grassmann manifold of $j-$dimensional subspaces ξ through the origin in R^n. $d\mu_j(\xi)$ is the normalized rotation invariant measure on $Gr(n,j)$ and to emphasize the dependence of j.

The Minkowski addition and scalar product of sets K and L in R^n is defined by(see[1])

$$aK + bL = \{ax + by : x \in K, y \in L\}, \text{for all } \ a, b \in R. \tag{2.1}$$

If $K, L \in \mathcal{K}_0^n$ can be defined as a convex body such that

$$h_{aK+bL}(u) = ah_K(u) + bh_L(u), \text{for all } \ u \in S^{n-1}.$$

The volume of a Minkowski combination $\lambda_1 L_1 + \cdots + \lambda_m L_m$ of convex bodies $L_1, \ldots, L_m$ can be expressed as a homogeneous polynomial of degree n (see [1]):

$$V(\lambda_1 L_1 + \cdots + \lambda_m L_m) = \sum_{i_1,\ldots,i_n} V(L_{i_1}, \ldots, L_{i_n})\lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $V(L_{i_1}, \ldots, L_{i_n})$ are called mixed volumes of $L_{i_1}, \ldots, L_{i_n}$.

2.2. Orlicz mixed volumes

Consider convex function $\phi : (-\infty, 0) \cup (0, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = \infty, \lim_{t\to 0} \phi(t) = 0$. we assume that $\mathcal{C}$ be the class of convex function $\phi : (0, \infty) \to (0, \infty)$. Let $\mathcal{C}^+$ be the class of convex and strictly increasing functions $\phi : [0, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = \infty, \lim_{t\to 0} \phi(t) = 0$. It is easy to conclude from [14, pp. 23-24] that $\phi \in \mathcal{C}$ is continuous on $[0, \infty)$, and the left derivative ϕ'_l and right derivative ϕ'_r exists and ϕ'_l are positive on $(0, \infty)$.

The Orlicz combination were defined by Gardner [3] and xi [15]. Let $\alpha, \beta > 0$(not both zero) and $\phi \in \mathcal{C}^+$, the Orlicz combination $\alpha \cdot K +_\phi \beta \cdot L$ given by, for all $u \in S^{n-1}$

$$h_{\alpha\cdot K+_\phi\beta\cdot L}(u) = \inf\{t > 0 : \alpha\phi\Big(\frac{h_K(u)}{t}\Big) + \beta\phi\Big(\frac{h_L(u)}{t}\Big) \le \phi(1)\}. \tag{2.2}$$

By the definition of Orlicz combination and Orlicz mixed volume, we derive

$$\phi(1) = \alpha\phi(\frac{h_K(u)}{h_{\alpha\cdot K+_\phi\beta\cdot L}(u)}) + \beta\phi(\frac{h_K(u)}{h_{\alpha\cdot L+_\phi\beta\cdot L}(u)}). \tag{2.3}$$

$$V_\phi(K, K) = \phi(1)V(K)$$

$$V(K +_\phi \varepsilon \cdot L)\phi(1) = V_\phi(K +_\phi \varepsilon \cdot L, K) + \varepsilon V_\phi(K +_\phi \varepsilon \cdot L, L) \text{ for all } \varepsilon > 0. \tag{2.4}$$

If $\phi(t) = t^p$, $p \ge 1$, then the Orlicz combination reduces to the L_p combination. Xi[15] defined the Orlicz mixed volume $V_\phi(K, L)$ of $K, L \in \mathcal{K}_0^n$ by

$$V_\phi(K, L) = \frac{\phi'_l(1)}{n} \lim_{\varepsilon\to 0^+} \frac{V(K +_\phi \varepsilon \cdot L) - V(K)}{\varepsilon}$$

and obtain the following integral formula of the Orlicz mixed volume:

$$V_\phi(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi(\frac{h_L(u)}{h_K(u)})h_K(u)dS(K, u).$$

Jensen' inequality: Let μ be a probability measure in a space X, let U be an open convex set in R^n, and let φ be a convex real-valued function on U. Assume that $g : X \to U$ is measurable and component-wise μ-integrable, and that $\varphi \circ g$ is μ-integrable. Let $z_0 = \int_X g(x)d\mu(x)$. Then $z_0 \in U$ and

$$\int_X \varphi(g(x))d\mu(x) \ge \varphi(\int_X g(x)d\mu(x)).$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for $\mu-$almost all $x \in X$.(*see*[4])

3. Orlicz mixed harmonic quermassintegrals

In this section, We are ready to derive the variational formula of harmonic quermassintegrals with respect to Orlicz combination. For this aim, the following Lemma is needed.

According to the proof of lemma 5.2 and theorem 2(see[15]) , we give the following Lemma :

Lemma 3.1 Let $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+, 0 < \varepsilon_0 < 1$ and $K_\varepsilon = K +_\phi \varepsilon \cdot L$. Then for $\varepsilon \in (0, \varepsilon_0]$, the family of functions $\{\frac{V^{\{j\}}(K_\varepsilon|\cdot) - V^{\{j\}}(K|\cdot)}{\varepsilon}\}$ uniformly bounded on $Gr(n, j)$.

Lemma 3.2[7] Let $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$ and $j = 1, 2, \cdots, n-1$. Then for each $\xi \in Gr(n, j)$ and $\varepsilon > 0$, we have

$$(K +_\phi \varepsilon \cdot L)|\xi = (K|\xi) +_\phi \varepsilon \cdot (L|\xi).$$

Theorem 3.3 If $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$, then for each $j = 1, 2, \cdots, n-1$,

$$\lim_{\varepsilon \to 0^+} \frac{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L) - \widehat{W}_{n-j}(K)}{\varepsilon} = \frac{j}{\phi_l'(1)} \widehat{W}_{\phi, n-j}(K, L)^{-1} \widehat{W}_{n-j}(K)^2.$$

Proof. By the Lemma 3.1, there exist a positive constant c, such that for all $\xi \in Gr(n, j)$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\frac{V^{\{j\}}(K_\varepsilon|\xi) - V^{\{j\}}(K|\xi)}{\varepsilon} < c.$$

Therefore,

$$\begin{aligned} &\left| \frac{V^{\{j\}}(K_\varepsilon|\xi)^{-1} - V^{\{j\}}(K|\xi)^{-1}}{\varepsilon} \right| \\ &= \frac{V^{\{j\}}(K_\varepsilon|\xi) - V^{\{j\}}(K|\xi)}{\varepsilon V^{\{j\}}(K_\varepsilon|\xi) V^{\{j\}}(K|\xi)} \leq \frac{c}{\min_{\xi \in Gr(n,j)} V^{\{j\}}(K|\xi)^2}. \end{aligned}$$

Thus, we know that the family of functions $\{\frac{V^{\{j\}}(K_\varepsilon|\cdot)^{-1} - V^{\{j\}}(K|\cdot)^{-1}}{\varepsilon}\}_{0<\varepsilon\leq\varepsilon_0}$ uniformly bounded on $Gr(n, j)$ and calculating the limitation

$$\lim_{\varepsilon \to 0^+} \frac{V^{\{j\}}(K_\varepsilon|\xi)^{-1} - V^{\{j\}}(K|\xi)^{-1}}{\varepsilon} = -\frac{j}{\phi_l'(1)} V_\phi^{\{j\}}(K|\xi, L|\xi) V^{\{j\}}(K|\xi)^{-2},$$

pointwise on $Gr(n, j)$. It follows from Lebesgue dominated convergence theorem, Lemma 3.2 that

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \frac{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L) - \widehat{W}_{n-j}(K)}{\varepsilon} \\ &= -\frac{\omega_n}{\omega_j} \left[\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)\right]^{-2} \lim_{\varepsilon \to 0^+} \int_{Gr(n,j)} \frac{V^{\{j\}}((K +_\phi \varepsilon \cdot L)|\xi)^{-1} - V^{\{j\}}(K|\xi)^{-1}}{\varepsilon} d\mu_j(\xi) \\ &= \frac{\omega_n}{\omega_j} \left[\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)\right]^{-2} \frac{j}{\phi_l'(1)} \int_{Gr(n,j)} V_\phi^{\{j\}}(K|\xi, L|\xi) V^{\{j\}}(K|\xi)^{-2} d\mu_j(\xi) \\ &= \frac{j}{\phi_l'(1)} \widehat{W}_{n-j}(K)^2 \widehat{W}_{\phi, n-j}(K, L)^{-1}, \end{aligned}$$

as desired. □

4. Brunn-Minkowski type inequalities

In the section. We establish the Minkowski type isoperimetric inequality for Orlicz mixed affine quermassintegrals, on this basis, we obtain the Brunn-Minkowski type inequality with respect to Orlicz combination for Orlicz mixed harmonic quermassintegrals.

Theorem 4.1 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \phi \in \mathcal{C}^+$. Then

$$\widehat{W}_{\phi,n-j}(K,L)^{-1} \geq \widehat{W}_{n-j}(K)^{-1}\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j}). \tag{4.1}$$

If K and L are dilations, then equality holds in (4.1). Conversely, if ϕ is strictly convex, then K and L are dilations.

Proof. The condition on K guarantees that $\widehat{W}_{n-j}(K) > 0$. Since

$$\widehat{W}_{n-j}(K) = \frac{\omega_n}{\omega_j}[\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1}d\mu_j(\xi)]^{-1}.$$

$\frac{(\frac{\omega_n}{\omega_j})^{-1}V^{\{j\}}(K|\xi)^{-1}d\mu_j(\xi)}{\widehat{W}_{n-j}^{-1}(K)}$ is a probability measure on $Gr(n,j)$. It follows from Jensen's inequality and Hölder inequality that

$$\begin{aligned}(\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)})^{-1} &= \frac{(\frac{\omega_n}{\omega_j})^{-1}\int_{Gr(n,j)} V_\phi^{\{j\}}(K|\xi,L|\xi)V^{\{j\}}(K|\xi)^{-2}d\mu_j(\xi)}{\widehat{W}_{n-j}(K)^{-1}} \\ &\geq \phi(\frac{(\frac{\omega_n}{\omega_j})^{-1}\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1-1/j}V^{\{j\}}(L|\xi)^{1/j}d\mu_j(\xi)}{\widehat{W}_{n-j}(K)^{-1}}) \\ &\geq \phi(\frac{\widehat{W}_{n-j}(K)^{-(1+1/j)}\widehat{W}_{n-j}(L)^{1/j}}{\widehat{W}_{n-j}(K)^{-1}}) = \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j}).\end{aligned}$$

If K and L are dilations, taking $K = aL, a > 0$, then by Orlicz mixed harmonic quermassintegrals and harmonic quermassintegrals, we have following equality holds,

$$[\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)}]^{-1} = \phi(a) = \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j})$$

conversely, we will show K and L are dilations when ϕ is strictly convex. We are divided into two cases to prove.

Firstly, we suppose $j = 1$. By the define of Orlicz mixed volume, we derive

$$\frac{V_\phi^{\{1\}}(K|u,L|u)}{V^{\{1\}}(K|u)} = \phi(\frac{h_L(u)}{h_K(u)})\frac{h_K(u)}{h_K(u)+h_K(-u)} + \phi(\frac{h_L(-u)}{h_K(-u)})\frac{h_K(-u)}{h_K(u)+h_K(-u)}.$$

Note that,

$$\frac{h_L(u)}{h_K(u)}\frac{h_K(u)}{h_K(u)+h_K(-u)} + \frac{h_L(-u)}{h_K(-u)}\frac{h_K(-u)}{h_K(u)+h_K(-u)} = \frac{V^{\{1\}}(L|u)}{V^{\{1\}}(K|u)}.$$

Since equality hold in inequality (4.1), we know that $\frac{V_\phi^{\{1\}}(K|u,L|u)}{V^{\{1\}}(K|u)} = \frac{V^{\{1\}}(L|u)}{V^{\{1\}}(K|u)}$. It follows from ϕ is strictly convex that

$$\frac{h_L(u)}{h_K(u)} = \frac{h_L(-u)}{h_K(-u)}.$$

And by Hölder equality condition, we conclude, exist a positive constant $b > 0$, for any $u \in S^{n-1}$, such that $V^{\{1\}}(K|u) = bV^{\{1\}}(L|u)$. Consequently, we obtain K and L are dilations.

Now, we suppose $2 \leq j \leq n-1$. For any $\xi \in Gr(n,j)$, we have $\frac{V_\phi^{\{j\}}(K|\xi,L|\xi)}{V^{\{j\}}(K|\xi)} = \phi((\frac{V^{\{j\}}(L|\xi)}{V^{\{j\}}(K|\xi)})^{1/j})$ according to equality hold in inequality (4.1). By the Mimkowski isoperimetric inequality of Orlicz mixed volume, we have, for $\xi \in Gr(n,j)$, $K|\xi$ and $L|\xi$ are dilations of each. By Lemma 3.12(see[7]), we derive K and L are dilations for each $\xi \in Gr(n,j)$. □

In Theorem 4.1, we derive the following corollary:

Corollary 4.2 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, p \geq 1$. Then

$$\widehat{W}_{p,n-j}(K,L)^{-1} \geq \widehat{W}_{n-j}(K)^{-1}(\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{p/j}. \tag{4.2}$$

When $p = 1$, the equality holds for $2 \leq j \leq n-1$ if and only if K and L are homothetic; the equality holds for $j = 1$ if and only if exist a positive constant a such that for $u \in S^{n-1}$, $V^{\{1\}}(K|u) = aV^{\{1\}}(L|u)$. When $p > 1$, the equality holds if and only if K and L are dilations.

Now, we obtain the following Brunn-Minkowski type inequality for Orlicz mixed harmonic quermassintegrals as follows:

Theorem 4.3 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \varepsilon > 0$. Then for any $\phi \in \mathcal{C}^+$,

$$\phi(1) \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}). \tag{4.3}$$

If ϕ is strictly convex, equality holds in (4.3) if and only if K and L are dilations.

Proof. Let $K_\phi = K +_\phi \varepsilon \cdot L$. From Orlicz mixed harmonic quermassintegrals, Orlicz combination, the Theorem 4.1 and (2.4), it follows that

$$\begin{aligned}\phi(1) &= \frac{(\frac{\omega_n}{\omega_j})^{-1}\int_{Gr(n,j)}\phi(1)V^{\{j\}}(K_\phi|\xi)^{-1}d\mu_j(\xi)}{\widehat{W}_{n-j}^{-1}(K_\phi)} \\ &= \frac{\widehat{W}_{n-j}^{-1}(K_\phi,K) + \varepsilon\widehat{W}_{n-j}(K_\phi,L)^{-1}}{\widehat{W}_{n-j}(K_\phi)^{-1}} \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}).\end{aligned}$$

The equality holds in theorem 4.1, we know that, if K and L are dilations, then equality holds in (4.3). Conversely, if ϕ is strictly convex, then K and L are dilations.

□

Using Theorem 4.3, we give the following Corollary:

Corollary 4.4 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \varepsilon > 0$. Then for any $p \geq 1$,

$$\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)^{p/j} \geq (\widehat{W}_{n-j}(K))^{p/j} + \varepsilon(\widehat{W}_{n-j}(L))^{p/j}. \tag{4.4}$$

When $p = 1$, the equality holds for $2 \leq j \leq n-1$ if and only if K and L are homothetic; the equality holds for $j = 1$ if and only if exist a positive constant a such that for $u \in S^{n-1}$, $V^{\{1\}}(K|u) = aV^{\{1\}}(L|u)$. When $p > 1$, the equality holds if and only if K and L are dilations.

We now derive the equivalence between the inequality (4.1) and the inequality (4.3). We have proved Theorem 4.3 by Theorem 4.1. Thus, we only need to prove the inequality (4.1) by the inequality (4.3).

Proof. Let $K_\phi = K +_\phi \varepsilon \cdot L$, by (4.3), the following function

$$f(\varepsilon) = \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)$$

is non-positive, then

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{f(\varepsilon)-f(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{\varepsilon} + \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j})\\ &= \lim_{\varepsilon\to 0^+}\frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1}\lim_{\varepsilon\to 0^+}\frac{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1}{\varepsilon}\\ &+ \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}).\end{aligned}$$

Let $a = (\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}$ and $a \to 1^-$ as $\varepsilon \to 0^+$, consequently,

$$\lim_{\varepsilon\to 0^+}\frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1} = \lim_{a\to 1^+}\frac{\phi(a)-\phi(1)}{a-1} = \phi_l^{'}(1)$$

and by theorem 3.3, we have

$$\lim_{\varepsilon\to 0^+}\frac{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1}{\varepsilon} = -\frac{1}{\phi_l^{'}(1)}(\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)})^{-1}.$$

Thus,

$$\lim_{\varepsilon\to 0^+}\frac{f(\varepsilon)-f(0)}{\varepsilon} = -(\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)})^{-1} + \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) \le 0,$$

which finishes the proof. □

Obviously, We derive the equivalence between the inequality (4.2) and the inequality (4.4).

5. Brunn-Minkowski inequality with respect to n variables

Gardner in [3] introduced the notion of Orlicz sum: consider a young function $\phi : [0,\infty)^n \to [0,\infty)$, that is for every $u \in S^{n-1} \cap [0,\infty)^n, \phi(tu)$ is convex and strictly increasing with $\phi(0) = 0$. For $K_1, K_2, \cdots, K_n \in \mathcal{K}_0^n$, exist a unique convex body $+_\phi(K_1, K_2, \cdots, K_n) \in \mathcal{K}_0^n$, the support function defined by

$$h_{+_\phi(K_1,K_2,\cdots,K_n)}(u) = \inf\{\lambda > 0 : \phi(\frac{h_{K_1}(u)}{\lambda}, \frac{h_{K_2}(u)}{\lambda}, \cdots, \frac{h_{K_n}(u)}{\lambda}) \le 1\}, u \in R^n.$$

The convex body $+_\phi(K_1, K_2, \cdots, K_n)$ is called Orlicz sum of $K_1, K_2, \cdots, K_n$.

We obtain the Brunn-Minkowski type inequality on above Orlicz sum.

Theorem 5.1 Let $K_1, K_2, \cdots, K_n \in \mathcal{K}_0^n \in \mathcal{K}_0^n$, ϕ is $n \geq 2$ variate young functions, $j = 1, 2, \cdots, n-1$. Then following inequality hold,

$$1 \geq \phi((\frac{\widehat{W}_{n-j}(K_1)}{\widehat{W}_{n-j}+_\phi(K_1, K_2, \cdots, K_n)})^{1/j}, \cdots, (\frac{\widehat{W}_{n-j}(K_n)}{\widehat{W}_{n-j}+_\phi(K_1, K_2, \cdots, K_n)})^{1/j}).$$

If $K_1, K_2, \cdots, K_n$ are dilations, then the above equality holds . Conversely, if ϕ is strictly convex, then $K_1, K_2, \cdots, K_n$ are dilations.

Proof. Considering a probability measure ν on $Gr(n, j)$, is given by

$$d\nu(\xi) = \frac{V^{\{j\}}(K|\xi)^{-1}}{\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)} d\mu_j(\xi).$$

Using the same method of theory 4.1 . We have , for all $\xi \in Gr(n, j)$

$$1 \geq \phi((\frac{V^{\{j\}}(K_1|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}, \cdots, (\frac{V^{\{j\}}(K_n|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}),$$

here $K = +_\phi(K_1, K_2, \cdots, K_n)$.

It follows from Jensen's inequality that

$$\begin{aligned}
1 &\geq \int_{Gr(n,j)} \phi((\frac{V^{\{j\}}(K_1|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}, \cdots, (\frac{V^{\{j\}}(K_n|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}) d\nu(\xi) \\
&\geq \phi(\int_{Gr(n,j)} ((\frac{V^{\{j\}}(K_1|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}, \cdots, (\frac{V^{\{j\}}(K_n|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}) d\nu(\xi)) \\
&= \phi(\frac{\int_{Gr(n,j)} V^{\{j\}}(K_1|\xi)^{1/j} V^{\{j\}}(K|\xi)^{-1/j+1} d\mu_j(\xi)}{\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)}, \\
&\cdots, \frac{\int_{Gr(n,j)} V^{\{j\}}(K_n|\xi)^{1/j} V^{\{j\}}(K|\xi)^{-1/j+1} d\mu_j(\xi)}{\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)}) \\
&\geq \phi((\frac{\widehat{W}_{n-j}(K_1)}{\widehat{W}_{n-j}(K)})^{1/j}, \cdots, (\frac{\widehat{W}_{n-j}(K_n)}{\widehat{W}_{n-j}(K)})^{1/j}).
\end{aligned}$$

Here equality holds according to the equality holds of theorem 4.1, we obtain, if $K_1, K_2, \cdots, K_n$ are dilations, then the above equality holds . Conversely, if ϕ is strictly convex, then $K_1, K_2, \cdots, K_n$ are dilations. □

Let $\phi_1, \phi_2 \in \mathcal{C}^+$ and $\phi(t_1, t_2) = \phi_1(t_1) + \phi_2(t_2)$, Then for every $K, L \in \mathcal{K}_o^n$, we have

$$1 \geq \phi_1(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(+_\phi(K, L)}^{1/j}) + \phi_2(\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(+_\phi(K, L)}^{1/j}).$$

More generally, taking $\phi(t_1, t_2) = \frac{\phi_1(t_1)+\varepsilon\phi_2(t_2)}{\phi(1)}$, then, for $+_\phi(K, L) = K +_\phi \varepsilon \cdot L$, we have,

$$\phi(1) \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)}^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)}^{1/j}).$$

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ROUGHNESS IN $(\in_\gamma, \in_\gamma \vee q_\delta)$-FUZZY SUBSTRUCTURES OF SEMIGROUPS BASED ON SET VALUED MAPPING

NOOR REHMAN, SYED INAYAT ALI SHAH, ABBAS ALI, AND RABIA ASLAM

Abstract. Study of generalized roughness for fuzzy algebraic substrucures of semigroups has been initiated. Many different kinds of set valued maps are needed to preserve an algebraic substrucure while considering its lower and upper approximations. In the present paper generalized lower and upper approximations in $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of semigroups have been investigated. An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of a subsemigroup have two parts viz. lower and upper parts. Several properties of lower and upper approximations have been given for these. To conclud this paper, lower and upper approximations for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals have been discussed in semigroups.

1. INTRODUCTION

A semigroup is an algebraic structure consisting of a nonempty set S together with an associative binary operation. Semigroups are important in many areas of mathematics, for example coding and language theory, automata theory, combinatorics and mathematical analysis. Zadeh introduced the notion of fuzzy subset in [24].

The idea of quasi coincidence of a fuzzy point with a fuzzy set which is mentioned in [17], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bakat and Das (see [2]-[4]) . Fuzzy point play a vital role in the study of (α, β)-fuzzy subgroups initiated by Bhakat and Das [3], using the combined notions of *" belongingness"* and *"quasi-coincidence"* of a fuzzy point with a fuzzy set. Shabir et al. have applied this concept in semigroup [20]. Rehman and Shabir initiated the study of (α, β)-fuzzy substructures in ternary semigroups [18]. Jun introduced $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK /BCI algebras [11]. Shabir et al. introduced $(\in, \in \vee q_k)$-fuzzy ideals in semigroup [21]. Rehman and Shabir initiated the study of $(\in, \in \vee q_k)$-fuzzy substructures in ternary semigroups [22]. Shabir and Ali introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal in semigroup [23]. Rehman and Shabir initiated the study of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy substructures in ternary semigroups [19].

Pawlak was the first to discuss rough set with the help of equivalence relation among the elements of a set which is a key point to discuss the uncertainty [16]. There are at least two methods for the development of rough set theory, the constructive and axiomatic approaches. In rough sets, equivalence classes play an important role in the construction of both lower and upper approximations. But some times in algebraic structures, it is difficult to find equivalence relations. Many researchers have worked on this to initiate rough set without equivalence relations.

Key words and phrases. Semigroup, Generalized fuzzy roughness, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

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Couso and Dubois in [7] initiated generalized rough set or "T-rough set" with the help of a set valued mapping, which is a more generalized rough form of the Pawlak rough set. The notion of roughness in fuzzy set introduced by Dubois and Prade in [9]. Some researchers applied this concept in [1] and [6].

Many researchers have taken interest to apply the concept of roughness in different algebraic structures (see [5],[8],[12],[15]). Hossini has applied generalized rough set in fuzzy algebraic structures (See [13],[14]). However, in the case of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy algebraic structures much attention has not been paid. Therefore it is important to study the roughness in generalized fuzzy algebraic structures such as in $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy structures.

2. Preliminaries

In this section some basic concepts of fuzzy set, fuzzy point, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy substructures of semigroup, different types of set valued homomorphism are given. Throughout this paper S will denote a semigroup unless specified otherwise

A fuzzy subset μ of S is a function $\mu : S \to [0,1]$. A fuzzy subset μ of S is called fuzzy left (right) ideal of S if $\mu(ab) \geq \mu(b)$ $(\mu(ab) \geq \mu(a))$ for all $a, b \in S$. μ is fuzzy ideal of S if it is both fuzzy left and fuzzy right ideal of S.

Definition 1. [17] *A fuzzy subset μ of S of the form*

$$\mu(x) = \begin{cases} t \in (0,1] & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2. [23] *Let μ be a fuzzy subset of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if the following condition hold:*

$$(\forall x, y \in S)(\forall t_1, t_2 \in (\gamma, 1])\left(x_{t_1}, y_{t_2} \in_\gamma \mu \to (xy)_{\min\{t_1,t_2\}} \in_\gamma \vee q_\delta \mu\right)$$

Theorem 1. [23] *Let μ be a fuzzy subset of S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $\mu(xy) \vee \gamma \geq \min\{\mu(x), \mu(y), \delta\}$ for all $x, y \in S$.*

Definition 3. [23] *Let μ be a fuzzy subset of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S if the following condition holds:*

$$(\forall x, y \in S)(\forall t \in (\gamma, 1])(y_t \in \mu \to (xy)_t \in_\gamma \vee q_\delta \mu)$$

Theorem 2. [23] *Let μ be a fuzzy subset of a semigroup S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S if and only if $\mu(ab) \vee \gamma \geq \min\{\mu(b), \delta\}$ for all $a, b \in S$.*

Definition 4. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if the following condition holds:*

$$(\forall x, a, y \in S)(\forall t \in (0,1])(a_t \in_\gamma \mu \to (xay)_t \in_\gamma \vee q_\delta \mu)$$

Theorem 3. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if and only if $\mu(xay) \vee \gamma \geq \min\{\mu(a), \delta\}$ for all x, y and $a \in S$.*

Definition 5. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if the following condition holds:*

$$(\forall x, a, y \in S)(\forall t_1, t_2 \in (\gamma, 1])\left(x_{t_1}, y_{t_2} \in_\gamma \mu \to (xay)_{\min\{t_1,t_2\}} \in_\gamma \vee q_\delta \mu\right)$$

Theorem 4. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if $\mu(xay) \vee \gamma \geq \min\{\mu(x), \mu(y), \delta\}$ for all x, y and $a \in S$.*

Definition 6. [23] *Let λ, μ be fuzzy subsets of a semigroup S, we define the fuzzy subsets $\lambda^*, (\lambda \wedge \mu)^*, (\lambda \vee \mu)^*$ of S as follows:*

$$\lambda^*(x) = (\lambda(x) \vee \gamma) \wedge \delta, (\lambda \wedge \mu)^*(x) = ((\lambda \wedge \mu)(x) \vee \gamma) \wedge \delta$$

$$\text{and } (\lambda \vee \mu)^*(x) = ((\lambda \vee \mu)(x) \vee \gamma) \wedge \delta.$$

Definition 7. *Let λ, μ be fuzzy subsets of S, we define the fuzzy subsets $\lambda^\diamond, (\lambda \wedge \mu)^\diamond, (\lambda \vee \mu)^\diamond$ of S as follows:*

$$\lambda^\diamond(x) = \lambda(x) \vee \delta, (\lambda \wedge \mu)^\diamond(x) = ((\lambda \wedge \mu)(x) \vee \delta)$$

$$\text{and } (\lambda \vee \mu)^\diamond(x) = ((\lambda \vee \mu)(x) \vee \delta).$$

Definition 8. *Let $T : S \to \mathcal{P}(S)$ be a set valued (SV) mapping. Then T is called an SV-homomorphism, if $T(a)T(b) \subseteq T(ab)$ for all $a, b \in S$.*

Definition 9. *Let $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. Then T is called reflexive if $a \in T(a)$ for all $a \in S$. In this paper reflexive set valued homomorphism will be denoted by RSV-homomorphism.*

Definition 10. *Let $T : S \to \mathcal{P}(S)$ be a set valued (SV) mapping. Then T is called a strong set valued (SSV) homomorphism, if $T(a)T(b) = T(ab)$ for all $a, b \in S$.*

3. Generalized roughness in $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups

This section deals with generalized roughness in fuzzy sets and the approximation of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups. We provide an example to show that the lower approximation of an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of a semigroup S is not an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of a semigroup S under an SV-homomorphism.

Definition 11. *Let S be a semigroup and $T : S \to P(S)$ be an SV-mapping. Let μ be a fuzzy subset of S. For every $x \in S$, we define T-rough lower and T-rough upper fuzzy subsets of S by*

$$\underline{T}(\mu)(x) = \bigwedge_{a \in T(x)} \mu(a) \text{ and } \overline{T}(\mu)(x) = \bigvee_{a \in T(x)} \mu(a).$$

Proposition 1. *Let $T : S \to P(S)$ be an SV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.*

Proof. Let $a, b \in S$ and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Let $a_{t_1}, b_{t_2} \in_\gamma \overline{T}(\mu)$. Then $\overline{T}(\mu)(a) \geq t_1 > \gamma$ and $\overline{T}(\mu)(b) \geq t_2 > \gamma$, where $t_1, t_2 \in (\gamma, 1]$. It

follows that

$$\begin{aligned}
\min\{t_1,t_2,\delta\} &\leq \overline{T}(\mu)(a)\wedge\overline{T}(\mu)(b)\wedge\delta\\
&= \bigvee_{x\in T(a)}(\mu(x))\wedge\bigvee_{y\in T(b)}(\mu(y))\wedge\delta\\
&= \bigvee_{x\in T(a),y\in T(b)}(\mu(x)\wedge\mu(y)\wedge\delta)\\
&\leq \bigvee_{x\in T(a),y\in T(b)}\mu(xy)\vee\gamma=\left(\bigvee_{xy\in T(a)T(b)}\mu(xy)\right)\vee\gamma\\
&= \left(\bigvee_{z\in T(a)T(b)}\mu(z)\right)\vee\gamma \quad (\text{where } z=xy)\\
&\leq \left(\bigvee_{z\in T(ab)}\mu(z)\right)\vee\gamma=\overline{T}(\mu)(ab)\vee\gamma.
\end{aligned}$$

This implies that $\min\{t_1,t_2,\delta\}\leq\overline{T}(\mu)(ab)\vee\gamma$.

If $\min\{t_1,t_2\}>\delta$, then $\overline{T}(\mu)(ab)>\delta$. This implies that $\overline{T}(\mu)(ab)+\min\{t_1,t_2\}>\delta+\delta=2\delta$. Which implies $(ab)_{\min\{t_1,t_2\}}\,q_\delta\overline{T}(\mu)$. If $\min\{t_1,t_2\}\leq\delta$, then $\overline{T}(\mu)(ab)\geq\min\{t_1,t_2\}>\gamma$. This implies that $(ab)_{\min\{t_1,t_2\}}\in_\gamma\overline{T}(\mu)$. Hence $(ab)_{\min\{t_1,t_2\}}\in_\gamma\vee q_\delta\overline{T}(\mu)$. Therefore $\overline{T}(\mu)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. □

Proposition 2. *Let $T:S\to P(S)$ be an SSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S, then $\underline{T}(\mu)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S.*

Proof. Let $a,b\in S$ and let μ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. Let $a_{t_1},b_{t_2}\in\underline{T}(\mu)$. Then $\underline{T}(\mu)(a)\geq t_1>\gamma$ and $\underline{T}(\mu)(b)\geq t_2>\gamma$, where $t_1,t_2\in(\gamma,1]$. It follows that

$$\begin{aligned}
\{\underline{T}(\mu)(ab)\vee\gamma\} &= \bigwedge_{z\in T(ab)}\mu(z)\vee\gamma=\bigwedge_{z\in T(a)T(b)}\mu(z)\ \vee\gamma\\
&= \bigwedge_{xy\in T(a)T(b)}\mu(xy)\vee\gamma \qquad (\text{where } z=xy)\\
&\geq \bigwedge_{x\in T(a),y\in T(b)}(\mu(x)\wedge\mu(y)\wedge\delta)\\
&= \left(\bigwedge_{x\in T(a)}(\mu(x))\right)\wedge\left(\bigwedge_{y\in T(b)}(\mu(y))\right)\wedge\delta\\
&= \underline{T}(\mu)(a)\wedge\underline{T}(\mu)(b)\wedge\delta\geq\min\{t_1,t_2,\delta\}.
\end{aligned}$$

This implies that $\{\underline{T}(\mu)(ab)\vee\gamma\}\geq\min\{t_1,t_2,\delta\}$.

If $\min\{t_1,t_2\}>\delta$, then$\underline{T}(\mu)(ab)>\delta$. This implies that $\underline{T}(\mu)(ab)+\min\{t_1,t_2\}>\delta+\delta=2\delta$. Which implies $(ab)_{\min\{t_1,t_2\}}\,q_\delta\underline{T}(\mu)$. If $\min\{t_1,t_2\}\leq\delta$, then $\underline{T}(\mu)(ab)\geq\min\{t_1,t_2\}>\gamma$. This implies that $(ab)_{\min\{t_1,t_2\}}\in_\gamma\underline{T}(\mu)$. Hence $(ab)_{\min\{t_1,t_2\}}\in_\gamma\vee q_\delta\underline{T}(\mu)$. Therefore $\underline{T}(\mu)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. □

Remark 1. *If T is an SV-homomorphism, then $\underline{T}(\mu)$ may not be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup as seen in the following example.*

Example 1. *Consider the semigroup S with the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define an SV-mapping $T : S \to P(S)$ by $T(a) = \{a, b, c, d\}, T(b) = \{a, b, c\}$, and $T(c) = T(d) = \{a, b\}$. Then T is an SV homomorphism. Let μ be a fuzzy subset of S define by $\mu(a) = 0.5, \mu(b) = 0.6, \mu(c) = 0.6, \mu(d) = 0.2$. Also $\underline{T}(\mu)(d) = \underline{T}(\mu)(b) = \underline{T}(\mu)(c) = 0.5$ and $\underline{T}(\mu)(a) = 0.2$. Let $a_{0.2}, b_{0.3}, c_{0.36}, d_{0.15} \in \underline{T}(\mu)$. Then μ is an $(\in_{0.1}, \in_{0.1} \vee q_{0.3})$-fuzzy subsemigroup of S. But $(bb)_{\min\{0.3,0.3\}} = (a)_{0.3}$. Since $\underline{T}(\mu)(a) = 0.2 \ngeq 0.3$. This implies that $(a)_{0.3} \overline{\in}_{0.1} \underline{T}(\mu)$. Also $\underline{T}(\mu)(bb) + \min\{0.3, 0.3\} \not> 2\delta$. This implies that $(a)_{0.3} \overline{q}_{0.3} \underline{T}(\mu)$. Hence $\underline{T}(\mu)$ is not an $(\in_{0.1}, \in_{0.1} \vee q_{0.3})$-fuzzy subsemigroup of S.

4. Approximation of some types of fuzzy subsets

The notion of different types of fuzzy subsets related to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets of S is given in [23]. In this section we discuss the lower and upper approximations of these subsets.

Proposition 3. *Let $T : S \to P(S)$ be an SV-homomorphism and let λ, μ be fuzzy subsets of S. Then the following hold:*

(i) $\overline{T}(\lambda \wedge \mu)^\diamond = \overline{T}(\lambda)^\diamond \wedge \overline{T}(\mu)^\diamond$
(ii) $\overline{T}(\lambda \vee \mu)^\diamond = \overline{T}(\lambda)^\diamond \vee \overline{T}(\mu)^\diamond$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\overline{T}(\lambda \wedge \mu)(x)^\diamond &= \bigvee_{z \in T(x)} (\lambda \wedge \mu)^\diamond(z) = \bigvee_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge \bigvee_{z \in T(x)} ((\mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} \lambda^\diamond(z) \wedge \bigvee_{z \in T(x)} \mu^\diamond(z) = \overline{T}(\lambda)^\diamond(x) \wedge \overline{T}(\mu)^\diamond(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 7. □

Proposition 4. *Let $T : S \to P(S)$ be an SV-homomorphism and let λ,μ be fuzzy subsets of S. Then the following hold:*

(i) $\underline{T}(\lambda \wedge \mu)^\diamond = \underline{T}(\lambda)^\diamond \wedge \underline{T}(\mu)^\diamond$
(ii) $\underline{T}(\lambda \vee \mu)^\diamond = \underline{T}(\lambda)^\diamond \vee \underline{T}(\mu)^\diamond$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\underline{T}(\lambda \wedge \mu)^{\diamond}(x) &= \bigwedge_{z \in T(x)} (\lambda \wedge \mu)^{\diamond}(z) = \bigwedge_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge \bigwedge_{z \in T(x)} ((\mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} \lambda^{\diamond}(z) \wedge \bigwedge_{z \in T(x)} \mu^{\diamond}(z) = \underline{T}(\lambda)^{\diamond}(x) \wedge \underline{T}(\mu)^{\diamond}(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 7. □

Proposition 5. *Let $T : S \to P(S)$ be an SV homomorphism and let λ, μ be fuzzy subsets of S. Then the following hold:*

(i) $\overline{T}(\lambda \wedge \mu)^{*} = \overline{T}(\lambda)^{*} \wedge \overline{T}(\mu)^{*}$
(ii) $\overline{T}(\lambda \vee \mu)^{*} = \overline{T}(\lambda)^{*} \vee \overline{T}(\mu)^{*}$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\overline{T}(\lambda \wedge \mu)^{*}(x) &= \bigvee_{z \in T(x)} (\lambda \wedge \mu)^{*}(z) = \bigvee_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \gamma) \wedge \delta \\
&= \bigvee_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge ((\mu)(z) \vee \gamma)) \wedge \delta \\
&= \bigvee_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge (((\mu)(z) \vee \gamma) \wedge \delta) \\
&= \bigvee_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge \bigvee_{z \in T(x)} (((\mu)(z) \vee \gamma) \wedge \delta) \\
&= \bigvee_{z \in T(x)} \lambda^{*}(z) \wedge \bigvee_{z \in T(x)} \mu^{*}(z) = \overline{T}(\lambda)^{*}(x) \wedge \overline{T}(\mu)^{*}(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 6. □

Proposition 6. *Let S be a semigroup and $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. If λ and μ are fuzzy ideals of S, then*

$$\overline{T}(\mu \circ \lambda)^{*}(y) \leq \overline{T}(\mu)^{*}(y) \wedge \overline{T}(\lambda)^{*}(y).$$

Proof. Let μ and λ be fuzzy ideals of S. Then

$$\begin{aligned}
(\mu \circ \lambda)(y) &\leq (\mu \circ S)(y) = \bigvee_{y=ab} \{\mu(a) \wedge S(b)\} \\
&= \bigvee_{y=ab} \{\mu(a) \wedge 1\} = \mu(a) \leq \mu(ab) = \mu(y).
\end{aligned}$$

That is $(\mu \circ \delta)(y) \leq \mu(y)$. Now $((\mu \circ \delta)(y) \vee \gamma) \wedge \delta \leq (\mu(y) \vee \gamma) \wedge \delta$. This implies that $(\mu \circ \lambda)^{*}(y) \leq (\mu)^{*}(y)$. Similarly $(\mu \circ \lambda)^{*}(y) \leq (\lambda)^{*}(y)$. Hence $\overline{T}(\mu \circ \lambda)^{*}(y) \leq \overline{T}(\mu)^{*}(y)$ and $\overline{T}(\mu \circ \lambda)^{*}(y) \leq \overline{T}(\lambda)^{*}(y)$.

Therefore $\overline{T}(\mu \circ \lambda)^*(y) \leq \left\{\overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y)\right\}$. □

In general equality does not hold in above proposition. Following example makes the situation clear.

Example 2. *Consider the semigroup S of Example 1. Define an SV-mapping $T : S \to \mathcal{P}(S)$ by $T(a) = \{a, b, c\}, T(b) = \{b, c\}, T(c) = \{c\}$ and $T(d) = \{d\}$. Then T is an SV-homomorphism. Let λ, μ be fuzzy subsets of S defined by $\mu(a) = 0.4, \mu(b) = 0.35, \mu(c) = 0.2$ and $\mu(d) = 0.2$ also $\lambda(a) = 0.38, \lambda(b) = 0.3, \lambda(c) = 0.1 = \lambda(d)$. Then clearly λ and μ are fuzzy ideals of S. Also $(\mu \circ \lambda)^*(a) = 0.36$, $(\mu \circ \lambda)^*(b) = (\mu \circ \lambda)^*(c) = (\mu \circ \lambda)^*(d) = 0.1$. This implies that $\overline{T}(\mu \circ \lambda)^*(a) = 0.36, \overline{T}(\mu \circ \lambda)^*(b) = \overline{T}(\mu \circ \lambda)^*(c) = \overline{T}(\mu \circ \lambda)^*(d) = 0.1$. Also $\overline{T}(\mu)^*(b) = 0.35$ and $\overline{T}(\lambda)^*(b) = 0.3$. Therefore $\overline{T}(\mu \circ \lambda)^*(b) \not\geq \left\{\overline{T}(\mu)^*(b) \wedge \overline{T}(\lambda)^*(b)\right\}$, where $\gamma = 0.1$ and $\delta = 0.36$.*

However in case of an idempotent semigroup equality can be shown.

Proposition 7. *Let S be an idempotent semigroup and $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. If μ and λ are fuzzy ideals of S, then*

$$\overline{T}(\mu \circ \lambda)^*(y) = \overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y).$$

Proof. From Proposition 6, it is obvious that $\overline{T}(\mu \circ \lambda)^*(y) \leq \overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y)$. For the reverse inequality, let $y \in S$. It follows that

$$\begin{aligned}
\overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y) &= \bigvee_{a \in T(y)} (\mu)^*(a) \wedge \bigvee_{b \in T(x)} (\lambda)^*(b) \\
&= \bigvee_{a \in T(y)} (((\mu)(y) \vee \gamma) \wedge \delta) \wedge \bigvee_{b \in T(y)} (((\lambda)(b) \vee \gamma) \wedge \delta) \\
&= \bigvee_{a \in T(y), b \in T(y)} (((\mu)(a) \vee \gamma) \wedge \delta) \wedge (((\lambda)(b) \vee \gamma) \wedge \delta) \\
&= \bigvee_{ab \in T(y)T(y)} (((\mu)(a) \vee \gamma) \wedge ((\lambda)(b) \vee \gamma)) \wedge \delta \\
&= \bigvee_{ab \in T(y)T(y)} (((\mu)(a) \wedge (\lambda)(b)) \vee \gamma) \wedge \delta \\
&\leq \bigvee_{z = ab \in T(yy)} (((\mu)(a) \wedge (\lambda)(b)) \vee \gamma) \wedge \delta \\
&= \bigvee_{z \in T(y)} \left(\bigvee_{z = ab} ((\mu)(a) \wedge (\lambda)(b)) \vee \gamma \right) \wedge \delta \\
&= \bigvee_{z \in T(y)} ((\mu \circ \lambda)(z) \vee \gamma) \wedge \delta = \\
&= \bigvee_{z \in T(y)} (\mu \circ \lambda)^*(z) = \overline{T}(\mu \circ \lambda)^*(y).
\end{aligned}$$

This implies that $\overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y) \leq \overline{T}(\mu \circ \lambda)^*(y)$. Hence $\overline{T}(\mu \circ \lambda)^*(y) = \overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y)$. □

Proposition 8. *Let $T : S \to P(S)$ be an SV-homomorphism and let λ,μ be fuzzy subsets of S. Then the following hold:*

(i) $\underline{T}(\lambda \wedge \mu)^* = \underline{T}(\lambda)^* \wedge \underline{T}(\mu)^*$
(ii) $\underline{T}(\lambda \vee \mu)^* = \underline{T}(\lambda)^* \vee \underline{T}(\mu)^*$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\underline{T}(\lambda \wedge \mu)^*(x) &= \bigwedge_{z \in T(x)} (\lambda \wedge \mu)^*(z) = \bigwedge_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \gamma) \wedge \delta \\
&= \bigwedge_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge ((\mu)(z) \vee \gamma)) \wedge \delta \\
&= \bigwedge_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge (((\mu)(z) \vee \gamma) \wedge \delta) \\
&= \bigwedge_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge \bigwedge_{z \in T(x)} (((\mu)(z) \vee \gamma) \wedge \delta) \\
&= \bigwedge_{z \in T(x)} \lambda^*(z) \wedge \bigwedge_{z \in T(x)} \mu^*(z) = \underline{T}(\lambda)^*(x) \wedge \underline{T}(\mu)^*(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 6. □

Proposition 9. *Let S be an idempotent semigroup and $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. If μ and λ are fuzzy ideals of S, then*

$$\underline{T}(\mu \circ \lambda)^*(y) = \left\{\underline{T}(\mu)^*(y) \wedge \underline{T}(\lambda)^*(y)\right\}.$$

Proof. The proof is similar to the proof of Proposition 7. □

5. Upper and lower approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of semigroups

In this section some properties of upper an lower approximation for fuzzy ideals of semigroups are studied.

Proposition 10. *Let $T : S \to P(S)$ be an RSV-homomorphism. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. Let $a, b \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Let $b_t \in_\gamma \overline{T}(\mu)$. Then $\overline{T}(\mu)(b) \geq t > \gamma$, where $t \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\min\{t, \delta\} &\leq \overline{T}(\mu)(b) \wedge \delta = \bigvee_{y \in T(b)} (\mu(y)) \wedge \delta \\
&= \bigvee_{y \in T(b)} (\mu(y) \wedge \delta) \leq \bigvee_{a \in T(a), y \in T(b)} \mu(ay) \vee \gamma \\
&= \left(\bigvee_{ay \in T(a)T(b)} \mu(ay)\right) \vee \gamma = \left(\bigvee_{z \in T(a)T(b)} \mu(z)\right) \vee \gamma \quad (z = ay) \\
&\leq \left(\bigvee_{z \in T(ab)} \mu(z)\right) \vee \gamma = \overline{T}(\mu)(ab) \vee \gamma.
\end{aligned}$$

This implies that $\min\{t, \delta\} \leq \left\{\overline{T}(\mu)(ab) \vee \gamma\right\}$.

Hence by Theorem 2, $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. □

Proposition 11. *Let $T : S \to P(S)$ be an SSV-homomorphism and and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, then $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. Let $a, b \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Let $b_t \in \underline{T}(\mu)$. Then $\underline{T}(\mu)(b) \geq t > \gamma$, where $t \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\underline{T}(\mu)(ab) \vee \gamma &= \bigwedge_{z \in T(ab)} \mu(z) \vee \gamma = \bigwedge_{z \in T(a)T(b)} \mu(z) \vee \gamma \\
&= \bigwedge_{xy \in T(a)T(b)} \mu(xy) \vee \gamma \quad (\text{where } z = xy) \\
&\geq \bigwedge_{y \in T(b)} (\mu(y) \wedge \delta) = \left(\bigwedge_{y \in T(b)} (\mu(y)) \right) \wedge \delta \\
&= \underline{T}(\mu)(b) \wedge \delta \geq \min\{t, \delta\}.
\end{aligned}$$

This implies that $\underline{T}(\mu)(ab) \vee \gamma \geq \min\{t, \delta\}$. Hence by Theorem 2, $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. □

Proposition 12. *Let $T : S \to P(S)$ be an RSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.*

Proof. It is straightforward. □

Proposition 13. *Let $T : S \to P(S)$ be an SSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S, then $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.*

Proof. It is straightforward. □

Proposition 14. *Let $T : S \to P(S)$ be an RSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. Let $a, x, y \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Let $x_{t_s}, y_{t_p} \in_\gamma \overline{T}(\mu)$. Then $\overline{T}(\mu)(x) \geq t_s > \gamma$ and $\overline{T}(\mu)(y) \geq t_p > \gamma$, where $t_s, t_p \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\min\{t_s, t_p, \delta\} &\leq \left\{\overline{T}(\mu)(x) \wedge \overline{T}(\mu)(y) \wedge \delta\right\} = \bigvee_{b \in T(x)} \mu(b) \wedge \bigvee_{d \in T(y)} \mu(d) \wedge \delta \\
&= \bigvee_{b \in T(x), d \in T(y)} (\mu(b) \wedge \mu(d) \wedge \delta) \leq \bigvee_{b \in T(x), a \in T(a), d \in T(y)} \mu(bad) \vee \gamma \\
&= \bigvee_{ba \in T(x)T(a), d \in T(y)} \mu(bad) \vee \gamma \leq \bigvee_{bc \in T(xa), d \in T(y)} \mu(bad) \vee \gamma \\
&= \bigvee_{(ba)d \in T(xa)T(y)} \mu(bad) \vee \gamma \leq \bigvee_{(ba)d \in T((xa)y)} \mu(bad) \vee \gamma \\
&= \bigvee_{z \in T((xa)y)} \mu(z) \vee \gamma \quad (\text{where } z = bad) \\
&= \left\{\overline{T}(\mu)(xay) \vee \gamma\right\}.
\end{aligned}$$

This implies that $\min\{t_s, t_p, \delta\} \leq \overline{T}(\mu)((xa)y) \vee \gamma$.

Hence by Theorem 4, $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-deal of S. □

Proposition 15. *Let $T : S \to P(S)$ be an SSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, then $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. Let $a, x, y \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Let $x_{t_s}, y_{t_p} \in_\gamma \underline{T}(\mu)$. Then $\underline{T}(\mu)(x) \geq t_s > \gamma$ and $\underline{T}(\mu)(y) \geq t_p > \gamma$, where $t_s, t_p \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\underline{T}(\mu)((xa)y) \vee \gamma &= \bigwedge_{z \in T((xa)y)} \mu(z) \vee \gamma = \bigwedge_{z \in T(xa)T(y)} \mu(z) \vee \gamma \\
&= \bigwedge_{(bc)d \in T((xa)y)} \mu((bc)d) \vee \gamma \quad (\text{where } z = (bc)d) \\
&= \bigwedge_{bc \in T(xa), d \in T(y)} \mu((bc)d) \vee \gamma = \bigwedge_{bc \in T(x)T(a), d \in T(y)} \mu((bc)d) \vee \gamma \\
&= \bigwedge_{b \in T(x), c \in T(a), d \in T(y)} \mu((bc)d) \vee \gamma \\
&\geq \bigwedge_{b \in T(x), d \in T(y)} (\mu(b) \wedge \mu(d) \wedge \delta) \\
&= \bigwedge_{b \in T(x)} \mu(b) \wedge \bigwedge_{d \in T(y)} \mu(d) \wedge \delta = \underline{T}(\mu)(x) \wedge \underline{T}(\mu)(y) \wedge \delta.
\end{aligned}$$

This implies that $\underline{T}(\mu)((xa)y) \vee \gamma \geq \min\{t_1, t_2, \delta\}$.

Hence by Theorem 4, $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. □

Conclusion; Associative algebras are being studied all over the globe, in particular semigroups have attracted many authors and researchers. The $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy algebraic substructures are generalizations of fuzzy algebraic substructures and $(\in, \in \vee q_k)$-fuzzy algebraic substructures. In this paper, generalized roughness have been studied for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy algebraic substructures of semigroups. It is seen that, in order to preserve a particular algebraic substructure in case of its approximations, many types of set valued homomorphisms are required. This aspect of roughness study in semigroups makes this study more interesting.

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Corresponding author: Department of Mathematics and Statistics, Riphah International University Islamabad Pakistan
E-mail address: noorrehman82@yahoo.com

Department of Mathematics, Islamia College University Peshawar KPK Pakistan

Department of Mathematics and Statistics, Riphah International University Islamabad Pakistan
E-mail address: abbasali5068@gmail.com

Department of Mathematics and Statistics, Riphah International University Islamabad Pakistan

Equivalence between some iterations in $CAT(0)$ spaces

Kyung Soo Kim

Graduate School of Education, Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 51767, Republic of Korea
e-mail: kksmj@kyungnam.ac.kr

Abstract. We obtain some equivalence conditions for the convergence of iterative sequences for set-valued contraction mapping in $CAT(0)$ spaces.

1. Introduction

Let (X, d) be a metric space. One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [32] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0, 1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. A metric space together with a convex structure W is known as a convex metric space. A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K$$

for all $x, y \in K$ and $\lambda \in [0, 1]$. In fact, every normed space and its convex subsets are convex metric spaces but the converse is not true, in general (see, [32]).

Example 1.1. ([15, 16]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \ , \ x_2 > 0\}$. For all $x = (x_1, x_2), y = (y_1, y_2) \in X$ and $\lambda \in [0, 1]$. We define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1 - \lambda)y_1, \frac{\lambda x_1 x_2 + (1 - \lambda)y_1 y_2}{\lambda x_1 + (1 - \lambda)y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (X, d, W) is a convex metric space but not a normed linear space.

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A metric space X is a $CAT(0)$ space. This term is due to M. Gromov [10] and it is an acronym for E. Cartan, A.D. Aleksandrov and V.A. Toponogov. If it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane(see, *e.g.*, [2], p.159). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [2] or Burago *et al.* [1].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists(see, [2]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,
>
> $$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces*(see [22]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [2], p.163). The above inequality has been extended by Khamsi and Kirk [12] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\leq \alpha d^2(z,x) + (1-\alpha)d^2(z,y) - \alpha(1-\alpha)d^2(x,y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$. The inequality (CN*) also appeared in [5].

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality(see, [2], p.163). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z,x) + (1-\alpha)d(z,y)$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$. In view of the above inequality, $CAT(0)$ space have Takahashi's convex structure $W(x,y,\alpha) = \alpha x \oplus (1-\alpha)y$. It is easy to see that for any $x, y \in X$ and $\lambda \in [0,1]$,

$$\begin{aligned} d(x, (1-\lambda)x \oplus \lambda y) &= \lambda d(x,y), \\ d(y, (1-\lambda)x \oplus \lambda y) &= (1-\lambda)d(x,y). \end{aligned}$$

As a consequence,

$$\begin{aligned} 1 \cdot x \oplus 0 \cdot y &= x, \\ (1-\lambda)x \oplus \lambda x &= \lambda x \oplus (1-\lambda)x = x. \end{aligned} \tag{1.1}$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x,y] \subset K$.

2. Preliminaries

Let D be a nonempty subset of a $CAT(0)$ space X. We shall denote by $CB(D)$ the family of nonempty bounded closed subset of D. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(D)$, *i.e.*,

$$H(A,B) = \max\left\{\sup_{a\in A} dist(a,B), \sup_{b\in B} dist(b,A)\right\}, \quad A, B \in CB(D),$$

where $dist(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from the point a to the set B.

A multivalued mapping $T : D \to CB(D)$ is said to be a *contraction* if there exists a constant $k \in (0,1)$ such that

$$H(Tx, Ty) \leq k \cdot d(x,y), \quad \forall\, x, y \in D.$$

A point x is called a *fixed point* of any mapping T if $x \in Tx$. We denote by $F(T)$ the set of all fixed points of T.

Let X be a $CAT(0)$ space, and let $\{x_n\}$ be a bounded sequence in X, for $x \in X$ we let

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a $CAT(0)$ space asymptotic center $A(\{x_n\})$ consists of exactly one point(see, *e.g.*, [6], Proposition 7).

Definition 2.1. ([23]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one must write

$$x_n \xrightarrow{\triangle} x \quad \text{or} \quad \triangle - \lim_{n\to\infty} x_n = x$$

and call x the $\triangle$-limit of $\{x_n\}$.

Remark 2.1. In a $CAT(0)$ space X, strong convergence implies $\triangle$-convergence.

Lemma 2.1. ([28]) *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Then for any given $\varepsilon > 0$ and for any given $x, y \in X$, $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \le (1 + \varepsilon)H(Tx, Ty)$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

Definition 2.2. Let D be a nonempty convex subset of a $CAT(0)$ space X, $T : D \to CB(D)$ be a multivalued mapping. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying some conditions.
(1) The sequence of Picard iterates (*cf.*, [30]) is defined by $w_0 \in D$,

$$w_{n+1} = \nu_n, \tag{$\mathbb{P}$}$$

where $\nu_n \in Tw_n$ such that

$$d(\nu_{n+1}, \nu_n) \le (1 + \varepsilon)H(Tw_{n+1}, Tw_n).$$

(2) The sequence of Mann iterates (*cf.*, [27]) is defined by $u_0 \in D$,

$$u_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n\theta_n, \tag{$\mathbb{M}$}$$

where $\theta_n \in Tu_n$ such that

$$d(\theta_{n+1}, \theta_n) \le (1 + \varepsilon)H(Tu_{n+1}, Tu_n).$$

(3) The sequence of Ishikawa iterates (*cf.*, [11]) is defined by $r_0 \in D$,

$$\begin{aligned} s_n &= (1-\beta_n)r_n \oplus \beta_n\delta_n, \\ r_{n+1} &= (1-\alpha_n)r_n \oplus \alpha_n\sigma_n, \end{aligned} \tag{$\mathbb{I}$}$$

where $\delta_n \in Tr_n$ and $\sigma_n \in Ts_n$ such that

$$\begin{aligned} d(\delta_{n+1}, \delta_n) &\leq (1+\varepsilon)H(Tr_{n+1}, Tr_n), \\ d(\sigma_{n+1}, \sigma_n) &\leq (1+\varepsilon)H(Ts_{n+1}, Ts_n). \end{aligned}$$

(4) The sequence of three-step iterates (*cf.*, [13, 14]) is defined by $x_0 \in D$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n \oplus \gamma_n\mu_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n\xi_n, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n\eta_n, \end{aligned} \tag{$\mathbb{TH}$}$$

where $\mu_n \in Tx_n$, $\xi_n \in Tz_n$ and $\eta_n \in Ty_n$ such that

$$\begin{aligned} d(\mu_{n+1}, \mu_n) &\leq (1+\varepsilon)H(Tx_{n+1}, Tx_n), \\ d(\xi_{n+1}, \xi_n) &\leq (1+\varepsilon)H(Tz_{n+1}, Tz_n), \\ d(\eta_{n+1}, \eta_n) &\leq (1+\varepsilon)H(Ty_{n+1}, Ty_n). \end{aligned}$$

Another iteration processes and other some results in $CAT(0)$ space have been studied extensively by various authors(see *e.g.* [4, 9, 17, 24, 26, 31]).

Lemma 2.2. ([7]) *Let $\{a_n\}$ be recursively generated by*

$$a_{n+1} = (1-t_n)a_n + b_n^2$$

with $n \geq 1$, $a_n \geq 0$, $\{t_n\} \subseteq [0,1]$ and

$$\sum_{n=1}^{\infty} b_n^2 < \infty, \quad \sum_{n=1}^{\infty} t_n = \infty.$$

Then

$$\lim_{n\to\infty} a_n = 0.$$

3. Main theorems

Theorem 3.1. *Let (X,d) be a $CAT(0)$ space and D be a nonempty convex subset of X. Let $T : D \to CB(D)$ be a multivalued contraction mapping with $k < \frac{1}{1+\varepsilon}$ and $F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. Let a constant L satisfying $\sup_{w\in Tx, x\in D} d(p,w) \leq L$, for all $x \in D$. Let $\{w_n\}$ and $\{x_n\}$ be the Picard and three step iterative sequence defined by ($\mathbb{P}$) and ($\mathbb{TH}$) respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n, \gamma_n \in [0,1], \quad \forall\, n \geq 0$;

(ii) $\lim_{n\to\infty}\beta_n = 0$;
(iii) $\sum_{n=0}^{\infty}\alpha_n\beta_n < \infty, \quad \sum_{n=0}^{\infty}(1-\alpha_n) = \infty$.

If $w_0 = x_0$, then the following statements are equivalent:

(1) *the Picard iterative sequence $\{w_n\}$ $\triangle$-converegs to $x^* \in F(T)$;*
(2) *the three step iterative sequence $\{x_n\}$ $\triangle$-converegs to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

Proof. From Nadler [28], there exists a fixed point $x^* \in F(T)$. Put

$$M' = L + d(p, x_0).$$

From the contractive of T, we have

$$\begin{aligned}
&d^2(x_{n+1}, p) = d^2((1-\alpha_n)x_n \oplus \alpha_n\eta_n, p)\\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(\eta_n, p) - (1-\alpha_n)\alpha_n d^2({}_n, \eta_n)\\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n(H(Ty_n, Tp))^2\\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n k^2 d^2(y_n, p)\\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n k^2(d^2((1-\beta_n)x_n \oplus \beta_n\xi_n, p))\\
&\le (1-\alpha_n)d^2(x_n, p)\\
&\quad + \alpha_n k^2((1-\beta_n)d^2(x_n, p) + \beta_n d^2(\xi_n, p) - \beta_n(1-\beta_n)d^2(x_n, \xi_n))\\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n \cdot k^2(1-\beta_n)d^2(x_n, p)\\
&\quad + \alpha_n\beta_n \cdot k^4 \cdot d^2(z_n, p) - \alpha_n\beta_n(1-\beta_n)k^2 \cdot d^2(x_n, \xi_n)\\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n(1-\beta_n) \cdot k^2 d^2(x_n, p)\\
&\quad + \alpha_n\beta_n \cdot k^4((1-\gamma_n)d^2(x_n, p) + \gamma_n d^2(\mu_n, p) - (1-\gamma_n)\gamma_n d^2(x_n, \mu_n))\\
&\quad - \alpha_n\beta_n(1-\beta_n)k^2 \cdot d^2(x_n, \xi_n)\\
&\le d^2(x_n, p) - \alpha_n\beta_n\gamma_n(1-\gamma_n) \cdot k^4 \cdot d^2(x_n, \mu_n),
\end{aligned}$$

for $p \in F(T)$. This implies

$$0 \le \alpha_n\beta_n\gamma_n(1-\gamma_n)k^4 d^2(x_n, \mu_n) \le d^2(x_n, p) - d^2(x_{n+1}, p).$$

Therefore, we have

$$d(x_{n+1}, p) \le d(x_n, p).$$

By induction, it is easy to see that

$$\sup_{n\ge 0}\{d(p, \mu_n), d(p, \eta_n), d(p, \xi_n), d(p, x_n), d(p, y_n), d(p, z_n)\} \le M',$$

for $\mu_n \in Tx_n$, $\eta_n \in Ty_n$ and $\xi_n \in Tz_n$, $n \ge 0$. By hypothesis, let

$$M'' = d(p, w_0) + d(p, w_1) < \infty, \quad \forall\, p \in F(T).$$

Put

$$M = \max\{M', M''\}.$$

From $\{w_n\}$ be the Picard iterative sequence defined by $(\mathbb{P})$, we have

$$\begin{aligned} d(\nu_n, \nu_{n+1}) &\leq (1+\varepsilon)H(Tw_n, Tw_{n-1}) \\ &\leq (1+\varepsilon)k \cdot d(w_n, w_{n-1}) \\ &= (1+\varepsilon)k \cdot d(\nu_{n-1}, \nu_{n-2}) \\ &\leq (1+\varepsilon)k(1+\varepsilon)H(Tw_{n-1}, Tw_{n-2}) \\ &\leq ((1+\varepsilon)k)^2 d(w_{n-1}, w_{n-2}) \\ &\vdots \\ &\leq ((1+\varepsilon)k)^n d(w_1, w_0) \\ &\leq ((1+\varepsilon)k)^n M \end{aligned} \tag{3.1}$$

for any given $\varepsilon > 0$. From $\{x_n\}$ be the three step iterative sequence defined by $(\mathbb{TH})$ and (3.1), for each $n \geq 0$

$$\begin{aligned} d(x_{n+1}, w_{n+1}) &= d((1-\alpha_n)x_n \oplus \alpha_n\eta_n, \nu_n) \\ &\leq (1-\alpha_n)d(x_n, \nu_n) + \alpha_n \cdot d(\eta_n, \nu_n) \\ &\leq (1-\alpha_n)\{d(x_n, w_n) + d(w_n, \nu_n)\} \\ &\quad + \alpha_n k \cdot d(y_n, w_n) \\ &\leq (1-\alpha_n)\{d(x_n, w_n) + ((1+\varepsilon)k)^n M\} \\ &\quad + \alpha_n k \cdot d(y_n, w_n), \end{aligned} \tag{3.2}$$

$$\begin{aligned} d(y_n, w_n) &= d((1-\beta_n)x_n \oplus \beta_n\xi_n, w_n) \\ &\leq (1-\beta_n)d(x_n, w_n) + \beta_n \cdot d(\xi_n, \nu_{n-1}) \\ &\leq (1-\beta_n)d(x_n, w_n) \\ &\quad + \beta_n(d(\xi_n, , \nu_n) + d(\nu_n, \nu_{n-1})) \\ &\leq (1-\beta_n)d(x_n, w_n) \\ &\quad + \beta_n k \cdot d(z_n, w_n) + \beta_n((1+\varepsilon)k)^n M \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(z_n, w_n) &= d((1-\gamma_n)x_n \oplus \gamma_n\mu_n, w_n) \\ &\leq (1-\gamma_n)d(x_n, w_n) + \gamma_n \cdot d(\mu_n, \nu_{n-1}) \\ &\leq (1-\gamma_n)d(x_n, w_n) \\ &\quad + \gamma_n k\{d(x_n, w_n) + d(w_n, w_{n-1})\} \\ &\leq (1-\gamma_n)d(x_n, w_n) \\ &\quad + \gamma_n k\{d(x_n, w_n) + ((1+\varepsilon)k)^{n-1} M\}. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned} & d(y_n, w_n) \\ & \leq (1-\beta_n)d(x_n, w_n) \\ & \quad + \beta_n k\Big[(1-\gamma_n)d(x_n, w_n) + \gamma_n k\{d(x_n, w_n) + ((1+\varepsilon)k)^{n-1}M\}\Big] \\ & \quad + \beta_n((1+\varepsilon)k)^n M \\ & \leq (1-\beta_n)d(x_n, w_n) + \beta_n((1+\varepsilon)k)^n M \\ & \quad + \beta_n k\{(1-\gamma_n(1-k))d(x_n, w_n) + \gamma_n((1+\varepsilon)k)^n M\}. \end{aligned} \tag{3.5}$$

Combining (3.5) and (3.2), we can obtain

$$\begin{aligned} & d(x_{n+1}, w_{n+1}) \\ & \leq (1-\alpha_n)\{d(x_n, w_n) + ((1+\varepsilon)k)^n M\} \\ & \quad + \alpha_n k\Big[(1-\beta_n)d(x_n, w_n) + \beta_n((1+\varepsilon)k)^n M \\ & \quad + \beta_n k\Big\{(1-\gamma_n(1-k))d(x_n, w_n) + \gamma_n((1+\varepsilon)k)^n M\Big\}\Big] \\ & = (1-\alpha_n)d(x_n, w_n) + (1-\alpha_n)((1+\varepsilon)k)^n M \\ & \quad + \alpha_n k(1-\beta_n)d(x_n, w_n) + \alpha_n\beta_n k((1+\varepsilon)k)^n M \\ & \quad + \alpha_n\beta_n k^2\Big\{(1-\gamma_n(1-k))d(x_n, w_n) + \gamma_n((1+\varepsilon)k)^n M\Big\} \\ & = \Big[1-\alpha_n+\alpha_n(1-\beta_n)k + \alpha_n\beta_n k^2(1-\gamma_n(1-k))\Big]d(x_n, w_n) \\ & \quad + (1-\alpha_n)((1+\varepsilon)k)^n M + \alpha_n\beta_n k(1+k\gamma_n)((1+\varepsilon)k)^n M \\ & \leq (1-\alpha_n(1-k))d(x_n, w_n) \\ & \quad + \{(1-\alpha_n) + \alpha_n\beta_n(1+k\gamma_n)\}((1+\varepsilon)k)^n M. \end{aligned} \tag{3.6}$$

Take

$$a_n = d(x_n, w_n), \quad t_n = \alpha_n(1-k)$$

and

$$b_n^2 = \{(1-\alpha_n) + \alpha_n\beta_n(1+k\gamma_n)\}((1+\varepsilon)k)^n M$$

in (3.6). Since $(1+\varepsilon)k < 1$, $\sum_{n=0}^{\infty}\alpha_n\beta_n < \infty$ and $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, we have

$$\sum_{n=0}^{\infty} t_n = \infty, \quad \sum_{n=0}^{\infty} b_n^2 < \infty.$$

By Lemma 2.2, we know that

$$\triangle - \lim_{n\to\infty} d(x_n, w_n) = 0.$$

If $w_n \xrightarrow{\triangle} x^* \in F(T)$ as $n \to \infty$, by Definition 2.1, we have

$$d(x_{n_k}, x^*) \leq d(x_{n_k}, w_{n_k}) + d(w_{n_k}, x^*) \xrightarrow{\triangle} 0$$

as $n \to \infty$. If $x_n \xrightarrow{\triangle} x^* \in F(T)$ as $n \to \infty$, we have

$$d(w_{n_k}, x^*) \leq d(w_{n_k}, x_{n_k}) + d(x_{n_k}, x^*) \xrightarrow{\triangle} 0$$

as $n \to \infty$. Therefore, the equivalence between the statement (1) and (2) was proved. Finally, we prove that $x^* \in X$ is the unique fixed point of T. In fact, let $x^*, y^* \in X$ be two fixed points of T. Since T is a multivalued contraction with constant $0 < k < 1$, we have

$$\begin{aligned} d(x^*, y^*) &\leq (1+\varepsilon)H(Tx^*, Ty^*) \\ &\leq (1+\varepsilon)k \cdot d(x^*, y^*). \end{aligned}$$

Since ε is arbitrary, this implies that

$$d(x^*, y^*) = 0,$$

i.e.,

$$x^* = y^*.$$

This completes the proof. □

If $\gamma_n = 0$ in $(\mathbb{TH})$, then it reduces to $(\mathbb{I})$. So we can easily prove the following corollary.

Corollary 3.1. *Let (X, d) be a $CAT(0)$ space and D be a nonempty convex subset of X. Let $T : D \to CB(D)$ be a multivalued contraction mapping with $k < \frac{1}{1+\varepsilon}$ and $F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. Let a constant L satisfying $\sup_{w \in Tx, x \in D} d(p, w) \leq L$, for all $x \in D$. Let $\{w_n\}$ and $\{r_n\}$ be the Picard and Ishikawa iterative sequence defined by $(\mathbb{P})$ and $(\mathbb{I})$ respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n \in [0, 1], \quad \forall\, n \geq 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty, \quad \sum_{n=0}^{\infty} (1 - \alpha_n) = \infty$.

If $w_0 = r_0$, then the following statements are equivalent:

(1) *the Picard iterative sequence $\{w_n\}$ $\triangle$-converegs to $x^* \in F(T)$;*
(2) *the Ishikawa iterative sequence $\{x_n\}$ $\triangle$-converegs to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

If $\beta_n = 0$ in $(\mathbb{I})$, then it reduces to $(\mathbb{M})$. So we can easily prove the following corollary.

Corollary 3.2. *Let (X, d) be a $CAT(0)$ space and D be a nonempty convex subset of X. Let $T : D \to CB(D)$ be a multivalued contraction mapping with $k < \frac{1}{1+\varepsilon}$ and $F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. Let a constant L satisfying $\sup_{w \in Tx, x \in D} d(p, w) \leq L$, for all $x \in D$. Let $\{w_n\}$ and $\{r_n\}$ be the Picard and Mann iterative sequence defined by $(\mathbb{P})$ and $(\mathbb{M})$ respectively and satisfying the following conditions:*

(i) $\alpha_n \in [0, 1], \quad \forall\, n \geq 0$;
(ii) $\sum_{n=0}^{\infty}(1 - \alpha_n) = \infty$.

If $w_0 = u_0$, then the following statements are equivalent:

(1) *the Picard iterative sequence $\{w_n\}$ $\triangle$-converegs to $x^* \in F(T)$;*
(2) *the Mann iterative sequence $\{x_n\}$ $\triangle$-converegs to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

4. Some remarks and open problem

For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding triangle in a model space with curvature κ.

For $\kappa = 0$, the 2-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the 2-dimensional sphere $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$ whose metric is length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the 2-dimensional hyperbolic space $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance. For more details about the properties of $CAT(\kappa)$ spaces, see [2], [8], [20], [21], [29].

Open Problem 1. It will be interesting to obtain a generalization of both Theorem 3.1 and Theorem 3.2 to $CAT(\kappa)$ space.

Open Problem 2. Can Theorem 3.1 be generalized to more than one contractive, or a commutative or left amenable semigroup S of mappings for which the sequence is defined by a strongly left invariant sequence (or net) of finite means on S(see [18], [19], [25])?

Competing interests
The author declares to have no competing interests.

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Strong convergence theorems for the generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces

Qian Yan

School of Mathematics Science, Chongqing Normal University, Chongqing 401331, China

Shaotao Hu*

School of Marxism, Chongqing Normal University, Chongqing 401331, China

Abstract

The purpose of this paper is to introduce the generalized viscosity implicit rules of one asymptotically nonexpansive mapping in Hilbert spaces. We obtain some strong convergence theorems under certain assumptions imposed on the parameters. We also apply our main results to solve mixed equilibrium problem in Hilbert spaces. A numerical example is also given to support our main results. The results obtained in this paper improve and extend many recent ones in this field.

Keywords:
Fixed point; Generalized implicit rule; Asymptotically nonexpansive mapping; Hilbert spaces.
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1. Introduction

Let C be a subset of real Hilbert space H. Let $F(T)$ be the set of fixed points of mapping T. We recall some basic definitions.

A mapping $f : C \to C$ is called a strict contraction, if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \ \forall \ x, y \in C. \tag{1.1}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall \ x, y \in C. \tag{1.2}$$

A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{\theta_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \theta_n) \|x - y\|, \forall \, n \geq 0, \ x, y \in C. \tag{1.3}$$

It is easy to see that asymptotically nonexpansive mapping contains strict contraction, nonexpansive mapping as a special case.

A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in C. \tag{1.4}$$

*Corresponding author.
Email addresses: qianyanmath@163.com (Qian Yan), shaotaohu@163.com (Shaotao Hu*)
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A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.5}$$

Recently, viscosity iterative algorithms for finding a common element of the set of fixed point of nonexpansive mappings, the set of solution of variational inequality problem and mixed equilibrium problems have been investigated extensively by many authors, see [1-15] and the references therein. For instance, Moudafi[1] introduced the viscosity technique for nonexpansive mappings in Hilbert spaces. Xu [2] refined the main results of [1] in Hilbert spaces and extended them to more general uniformly smooth spaces. Precisely, he proved that the suggested viscosity iterative sequence converges strongly to a fixed point of one nonexpansive mapping, which also solves some variational inequality.

Very recently, the implicit midpoint rule has become a powerful methods for solving ordinary differential equations; see [16-22] and the references therein. Xu et al. [20] considered the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \ n \geq 0. \tag{1.6}$$

By using contractions to regularize the implicit midpoint rule for nonexpansive mappings, they proved that the iterative sequence defined by (1.6) converges in norm to a fixed point of T, which also solves the variational inequality:

$$\langle (I - f)q, x - q\rangle \geq 0, \ x \in F(T). \tag{1.7}$$

On the other hand, many authors studied the Mann and Ishikawa iterations processes for asymptotically nonexpansive mapping in Hilbert spaces or Banach spaces, see [23-30] and the references therein. For example, Lou et al.[24] investigated some iterative algorithms for asymptotically nonexpansive mapping on a uniformly convex Banach space with uniformly Gâteaux differentiable norm.

In this paper, we introduce a viscosity implicit rules for an asymptotically nonexpansive mapping in Hilbert spaces. Under suitable assumptions imposed on the parameters, we obtain some strong convergence theorems for finding a fixed point of an asymptotically nonexpansive mapping. We also apply our main results to solve mixed equilibrium problem in Hilbert spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of H. For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \ \text{ for all } y \in C. \tag{2.1}$$

In this case, P is called a metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \forall\, x, y \in H. \tag{2.2}$$

Furthermore, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.3}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall\, x \in H, \ y \in C. \tag{2.4}$$

We need the following lemmas for proving our main results.

Lemma 2.1 ([2]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) either $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([21]). *Let T be an asymptotically nonexpansive mapping defined on a nonempty bounded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $Tx_n - x_n \to 0$, then $z \in F(T)$.*

3. Main results

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$ such that $F(T) \neq \emptyset$ and $f : C \to C$ a strict contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n(\frac{x_n + x_{n+1}}{2}), \tag{3.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $\sum_{n=0}^{\infty} \sup_{x\in C'} \left\|T^{n+1}x - T^n x\right\| < \infty$, where C' is a closed convex subset of C that contains sequence $\{x_n\}$.
Then $\{x_n\}$ converges strongly to a fixed point q of the asymptotically nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)q, y - q\rangle \geq 0, \text{ for all } y \in F(T).$$

Proof. First, we show that $\{x_n\}$ is bounded. Indeed, take $p \in F(T)$ arbitrarily, since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N, \frac{\theta_n}{\alpha_n} \leq \frac{1-\alpha}{2}$. Choose a constant $M_1 > 0$ sufficiently large such that

$$\|x_N - p\| \leq M_1, \|f(p) - p\| \leq \frac{1 - \alpha}{2} M_1.$$

We proceed by induction to show that $\|x_n - p\| \leq M_1, \forall n \geq 1$. Assume $\|x_n - p\| \leq M_1$, for some $n \geq N$. We show that $\|x_{n+1} - p\| \leq M_1$. We observe

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n f(x_n) + (1 - \alpha_n)T^n(\frac{x_n + x_{n+1}}{2}) - p\| \\
&= \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)[T^n(\frac{x_n + x_{n+1}}{2}) - p]\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|T^n(\frac{x_n + x_{n+1}}{2}) - p\| \\
&\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)(\theta_n + 1)\|\frac{x_n + x_{n+1}}{2} - p\| \\
&\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \frac{(1 - \alpha_n)(\theta_n + 1)}{2}\|x_n - p\| + \frac{(1 - \alpha_n)(\theta_n + 1)}{2}\|x_{n+1} - p\|
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \frac{1 - \alpha_n + 2\alpha_n\alpha + (1 - \alpha_n)\theta_n}{1 + \alpha_n - (1 - \alpha_n)\theta_n}\|x_n - p\| + \frac{\alpha_n}{1 + \alpha_n - (1 - \alpha_n)\theta_n}\|f(p) - p\| \\
&= [1 - \frac{2\alpha_n(1 - \alpha) - 2(1 - \alpha_n)\theta_n}{1 + \alpha_n - (1 - \alpha_n)\theta_n}]\|x_n - p\| + \frac{\alpha_n}{1 + \alpha_n - (1 - \alpha_n)\theta_n}\|f(p) - p\| \\
&\leq [1 - \frac{\alpha_n(1 - \alpha)}{1 + \alpha_n - (1 - \alpha_n)\theta_n}]\|x_n - p\| + \frac{\alpha_n(1 - \alpha)}{1 + \alpha_n - (1 - \alpha_n)\theta_n}\frac{\|f(p) - p\|}{1 - \alpha} \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \alpha}\} \\
&\leq M_1.
\end{aligned} \tag{3.2}$$

This implies that $\{x_n\}$ is bounded.

Next, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. It follows from (3.1) that

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&=\|\alpha_n f(x_n)+(1-\alpha_n)T^n(\frac{x_n+x_{n+1}}{2})-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})T^{n-1}(\frac{x_{n-1}+x_n}{2})\| \\
&=\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1})+(1-\alpha_n)[T^n(\frac{x_n+x_{n+1}}{2})-T^n(\frac{x_{n-1}+x_n}{2})] \\
&\quad+(1-\alpha_n)T^n(\frac{x_{n-1}+x_n}{2})-(1-\alpha_{n-1})T^n(\frac{x_{n-1}+x_n}{2}) \\
&\quad+(1-\alpha_{n-1})[T^n(\frac{x_{n-1}+x_n}{2})-T^{n-1}(\frac{x_{n-1}+x_n}{2})]\| \\
&=\|\alpha_n(f(x_n)-f(x_{n-1}))+(1-\alpha_n)[T^n(\frac{x_n+x_{n+1}}{2})-T^n(\frac{x_{n-1}+x_n}{2})]+(\alpha_n-\alpha_{n-1}) \\
&\quad\cdot[f(x_{n-1})-T^n(\frac{x_{n-1}+x_n}{2})]+(1-\alpha_{n-1})[T^n(\frac{x_{n-1}+x_n}{2})-T^{n-1}(\frac{x_{n-1}+x_n}{2})]\| \\
&\le\alpha\alpha_n\|x_n-x_{n-1}\|+(1-\alpha_n)(\theta_n+1)(\frac{\|x_{n+1}-x_n\|}{2}+\frac{\|x_n-x_{n-1}\|}{2}) \\
&\quad+|\alpha_n-\alpha_{n-1}|\,\|f(x_{n-1})-T^n(\frac{x_{n-1}+x_n}{2})\|+\sup_{x\in C'}\|T^nx-T^{n-1}x\| \\
&=\frac{2\alpha\alpha_n+(1-\alpha_n)(\theta_n+1)}{2}\|x_n-x_{n-1}\|+\frac{(1-\alpha_n)(\theta_n+1)}{2}\|x_{n+1}-x_n\| \\
&\quad+|\alpha_n-\alpha_{n-1}|\,M_2+\sup_{x\in C'}\|T^nx-T^{n-1}x\|,
\end{aligned}$$

where M_2 is a constant such that

$$M_2=\sup_{n\ge0}\|f(x_{n-1})-T^n(\frac{x_{n-1}+x_n}{2})\|.$$

It follows that

$$\begin{aligned}
\frac{2-(1-\alpha_n)(\theta_n+1)}{2}\|x_{n+1}-x_n\|\le{}&\frac{2\alpha\alpha_n+(1-\alpha_n)(\theta_n+1)}{2}\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\,M_2 \\
&+\sup_{x\in C'}\|T^nx-T^{n-1}x\|.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\le\frac{2\alpha\alpha_n+(1-\alpha_n)(\theta_n+1)}{2-(1-\alpha_n)(\theta_n+1)}\|x_n-x_{n-1}\|+\frac{2M_2}{2-(1-\alpha_n)(\theta_n+1)}\cdot \\
&\quad|\alpha_n-\alpha_{n-1}|+\frac{2}{2-(1-\alpha_n)(\theta_n+1)}\sup_{x\in C'}\|T^nx-T^{n-1}x\| \\
&=\left(1-\frac{2[1-\alpha\alpha_n-(1-\alpha_n)(\theta_n+1)]}{2-(1-\alpha_n)(\theta_n+1)}\right)\|x_n-x_{n-1}\|+\frac{2M_1}{2-(1-\alpha_n)(\theta_n+1)}\cdot \\
&\quad|\alpha_n-\alpha_{n-1}|+\frac{2}{2-(1-\alpha_n)(\theta_n+1)}\sup_{x\in C'}\|T^nx-T^{n-1}x\|. \qquad (3.3)
\end{aligned}$$

Let

$$\gamma_n=\frac{2[1-\alpha\alpha_n-(1-\alpha_n)(\theta_n+1)]}{2-(1-\alpha_n)(\theta_n+1)}.$$

We note

$$\begin{aligned}\gamma_n &= \frac{2[\alpha_n(1-\alpha)+\theta_n(\alpha_n-1)]}{1-\theta_n+\alpha_n(\theta_n+1)}\\ &\geq \frac{2[\alpha_n(1-\alpha)+\theta_n(\alpha_n-1)]}{1-\theta_n+(\theta_n+1)}\\ &= \alpha_n(1-\alpha)+\theta_n(\alpha_n-1)\\ &\geq \alpha_n(1-\alpha)-\theta_n \geq \frac{1-\alpha}{2}\alpha_n.\end{aligned}$$

By condition (i), we have $\sum_{n=0}^{\infty}\gamma_n=\infty$. Apply Lemma 2.1 to (3.3), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

Next, we prove that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. In fact, we have

$$\begin{aligned}\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\| &= \alpha_n\|f(x_n)-T^n(\frac{x_n+x_{n+1}}{2})\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.5}$$

Moreover, we get

$$\begin{aligned}&\|x_n-T^nx_n\|\\ &= \|x_n-x_{n+1}+x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})+T^n(\frac{x_n+x_{n+1}}{2})-T^nx_n\|\\ &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\|+\|T^n(\frac{x_n+x_{n+1}}{2})-T^nx_n\|\\ &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\|+\frac{\theta_n+1}{2}\|x_{n+1}-x_n\|\\ &= \frac{\theta_n+3}{2}\|x_{n+1}-x_n\|+\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\|.\end{aligned}$$

Combining (3.4) and (3.5), we can obtain

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \tag{3.6}$$

We notice

$$\begin{aligned}&\|x_n-Tx_n\|\\ &= \|x_n-T^nx_n+T^nx_n-T^{n+1}x_n+T^{n+1}x_n-Tx_n\|\\ &\leq \|x_n-T^nx_n\|+\|T^nx_n-T^{n+1}x_n\|+(1+\theta_1)\|T^nx_n-x_n\|\\ &\leq \|x_n-T^nx_n\|+\sup_{x\in C'}\|T^nx-T^{n+1}x\|+(1+\theta_1)\|T^nx_n-x_n\|.\end{aligned}$$

By condition (iv) and (3.6), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.7}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle q-f(q),q-x_n\rangle\leq 0, \tag{3.8}$$

where $q=P_{F(T)}f(q)$. Indeed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle q-f(q),q-x_n\rangle=\lim_{i\to\infty}\langle q-f(q),q-x_{n_i}\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to p. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup p$. From (3.7) and Lemma 2.2, we have $p \in F(T)$. This together with the property of the metric projection implies that

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n\rangle = \lim_{i\to\infty} \langle q - f(q), q - x_{n_i}\rangle = \langle q - f(q), q - p\rangle \le 0.$$

Then (3.8) holds. Finally, we show that $x_n \to q$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \langle \alpha_n f(x_n) + (1-\alpha_n)T^n(\frac{x_n + x_{n+1}}{2}) - q, x_{n+1} - q\rangle \\
&= \langle \alpha_n (f(x_n) - q) + (1-\alpha_n)(T^n(\frac{x_n + x_{n+1}}{2}) - q), x_{n+1} - q\rangle \\
&= \alpha_n \langle f(x_n) - f(q), x_{n+1} - q\rangle + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle + (1-\alpha_n)\langle T^n(\frac{x_n + x_{n+1}}{2}) - q, x_{n+1} - q\rangle \\
&\le \alpha\alpha_n \|x_n - q\| \cdot \|x_{n+1} - q\| + (1-\alpha_n)(\theta_n + 1)\|\frac{x_n - q}{2} + \frac{x_{n+1} - q}{2}\| \cdot \|x_{n+1} - q\| \\
&\quad + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle \\
&\le \frac{\alpha\alpha_n}{2}\|x_n - q\|^2 + \frac{\alpha\alpha_n}{2}\|x_{n+1} - q\|^2 + \frac{(1-\alpha_n)(\theta_n+1)}{4}\|x_n - q\|^2 \\
&\quad + \frac{(1-\alpha_n)(\theta_n+1)}{4}\|x_{n+1} - q\|^2 + \frac{(1-\alpha_n)(\theta_n+1)}{2}\|x_{n+1} - q\|^2 + \alpha_n\langle f(q) - q, x_{n+1} - q\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}{4}\|x_{n+1} - q\|^2 \\
&\le \frac{2\alpha\alpha_n + (1-\alpha_n)(\theta_n+1)}{4}\|x_n - q\|^2 + \alpha_n\langle f(q) - q, x_{n+1} - q\rangle.
\end{aligned}$$

That is

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\le \frac{2\alpha\alpha_n + (1-\alpha_n)(\theta_n+1)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}\|x_n - q\|^2 + \frac{4\alpha_n}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}\langle f(q) - q, x_{n+1} - q\rangle \\
&= [1 - \frac{4(\alpha_n\theta_n + \alpha_n - \alpha\alpha_n - \theta_n)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}]\|x_n - q\|^2 + \frac{4\alpha_n}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)} \cdot \\
&\quad \langle f(q) - q, x_{n+1} - q\rangle. \qquad (3.9)
\end{aligned}$$

Put

$$\gamma_n = \frac{4(\alpha_n\theta_n + \alpha_n - \alpha\alpha_n - \theta_n)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}.$$

We have

$$\begin{aligned}
\gamma_n &= \frac{4[\theta_n(\alpha_n - 1) + \alpha_n(1-\alpha)]}{1 - 2\alpha\alpha_n + 3\theta_n(\alpha_n - 1) + 3\alpha_n} \\
&\ge \frac{4[\theta_n(\alpha_n - 1) + \alpha_n(1-\alpha)]}{1 + 3\alpha_n} \\
&\ge \theta_n(\alpha_n - 1) + \alpha_n(1-\alpha) \\
&\ge \alpha_n(1-\alpha) - \theta_n \ge \frac{1-\alpha}{2}\alpha_n.
\end{aligned}$$

Apply Lemma 2.1 to (3.9), we obtain $x_n \to q$ as $n \to \infty$. This completes the proof. □

Since nonexpansive mapping is asymptotically nonexpansive, so we obtain the main results of [20].

Theorem 3.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with such that $F(T) \neq \emptyset$ and $f : C \to C$ a strict contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \tag{3.10}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} = 0$.
Then $\{x_n\}$ converges strongly to a fixed point q of nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)q, y - q\rangle \geq 0, \text{ for all } y \in F(T).$$

Now we give an example that one mapping satisfies condition (iv) in Theorem 3.1.

Example 3.3. *Let $T : C \to C$ be a strict contraction with a constant $\beta \in (0, 1)$ and let C' be a bounded subset of C. Then*

$$\left\|T^{n+1}x - T^n x\right\| \leq \beta^n \|Tx - x\| \leq \beta^n K_1, \forall\, x \in C',$$

where K_1 is a constant such that $K_1 = \sup_{x\in C'}(\|x\| + \|Tx\|)$. It follows that

$$\sum_{n=1}^{\infty} \sup_{x\in C'} \left\|T^{n+1}x - T^n x\right\| \leq \sum_{n=1}^{\infty} \beta^n K_1 = \frac{\beta K_1}{1 - \beta} < \infty.$$

Example 3.4. *Let C be a nonempty closed convex subset of a Banach space. Define mapping $T : C \to C$ as $T^n x = (1 + \frac{1}{n})x$ for any $x \in C$. It is easy to see that T is asymptotically nonexpansive mapping in the intermediate sense. Let $\{x_n\}$ be a bounded sequence in C, we observe*

$$\left\|T^{n+1}x_n - T^n x_n\right\| = \frac{1}{n(n+1)} \|x_n\| \leq \frac{1}{n^2} \|x_n\| \leq \frac{1}{n^2} K_2,$$

where K_2 is a constant such that $K_2 = \sup_{n\geq 1} \|x_n\|$. Hence we obtain

$$\sum_{n=1}^{\infty} \left\|T^{n+1}x_n - T^n x_n\right\| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} K_2 < \infty.$$

4. Applications

In this section, we apply our main results to solve mixed equilibrium problems.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $F : C \times C \to \mathbb{R}$ be a bifunction. The mixed equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \forall\, y \in C. \tag{4.1}$$

The set of solutions of (1.1) is denoted by $MEP(F, \varphi)$. If $\varphi = 0$, then problem (4.1) reduces to equilibrium problem which is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall\, y \in C. \tag{4.2}$$

We denote the set of solutions of (4.2) by $EP(F)$.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C([13]):

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for any $x,y\in C$;
(A3) for each $y\in C$, $x\mapsto F(x,y)$ is weakly upper semicontinuous;
(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex;
(A5) for each $x\in C$, $y\mapsto F(x,y)$ is lower semicontinuous;
(B1) for each $x\in H$ and $r>0$, there exists a bounded subset $D_x\subseteq C$ and $y_x\in C$ such that for any $z\in C\backslash D_x$,

$$F(z,y_x)+\varphi(y_x)+\frac{1}{r}\langle y_x-z,z-x\rangle<\varphi(z);$$

(B2) C is bounded set.

Lemma 4.1 ([13]). *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi:C\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r>0$ and $x\in H$, define a mapping $T_r^{(F,\varphi)}:H\to C$ as follows.*

$$T_r^{(F,\varphi)}(x)=\left\{z\in C:F(z,y)+\varphi(y)+\frac{1}{r}\langle y-z,z-x\rangle\geq\varphi(z),\forall\, y\in C\right\}$$

for all $x\in H$. Then the following conclusions hold:
(1) for each $x\in H$, $T_r^{(F,\varphi)}(x)\neq\emptyset$;
(2) $T_r^{(F,\varphi)}$ is single-valued;
(3) $T_r^{(F,\varphi)}$ is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\left\|T_r^{(F,\varphi)}(x)-T_r^{F,\varphi}(y)\right\|^2\leq\left\langle T_r^{(F,\varphi)}(x)-T_r^{(F,\varphi)}(y),x-y\right\rangle;$$

(4) $F(T_r^{(F,\varphi)})=MEP(F,\varphi)$;
(5) $MEP(F,\varphi)$ is closed and convex.

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A5), $\varphi:C\to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. Let $f:C\to C$ be a strict contraction with coefficient $\alpha\in[0,1)$. Pick any $x_0\in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)z_n,\\ z_n\in C\ \text{such that}\ F(z_n,y)+\varphi(y)+\frac{1}{r}\langle y-z_n,z_n-u_n\rangle\geq\varphi(z_n),\forall\, r>0,y\in C,\\ u_n=\frac{x_n+x_{n+1}}{2},\end{cases}\tag{4.3}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty}\alpha_n=0,\sum_{n=0}^{\infty}\alpha_n=\infty$;
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ or $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n-1}}=0$.
Then $\{x_n\}$ converges strongly to an element of mixed equilibrium problem (4.1), which is also the solution of the variational inequality

$$\langle(I-f)q,y-q\rangle\geq 0,\ \text{for all}\ y\in MEP(F,\varphi).$$

Proof. We can rewrite (4.3) as follows:

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_r^{(F,\varphi)}(\frac{x_n+x_{n+1}}{2}).$$

Then we obtain the desired results by Theorem 3.2 easily. □

5. Numerical Examples

Example 5.1. *Let inner product* $< \cdot, \cdot >: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *be defined by*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

and the usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ *be defined by*

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + y_1^2 + z_1^2},\ \forall\ \boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Let $T, f : \mathbb{R}^3 \to \mathbb{R}^3$ *be defined by*

$$T\boldsymbol{x} = \frac{1}{3}(\boldsymbol{x}),\ f(\boldsymbol{x}) = \frac{1}{4}\boldsymbol{x},\ \forall\ x \in \mathbb{R}.$$

Let $\alpha_n = \frac{1}{5n}, \forall\ n \in \mathbb{N}$ *and let* $\{x_n\}$ *be a sequence generated by (3.10). It is easy to see that* $F(T) = \{0\}$. *Then* $\{\boldsymbol{x}_n\}$ *converges strongly to* 0 *by Theorem 3.2.*

We can rewrite (3.10) as follows:

$$\boldsymbol{x}_{n+1} = \frac{10n+1}{50n+2}\boldsymbol{x}_n. \tag{5.1}$$

Choosing $\boldsymbol{x}_1 = (1, 3, 5)$ *in (5.1), we have the following numerical results in Figure* 1 *and Figure* 2.

Figure 1:

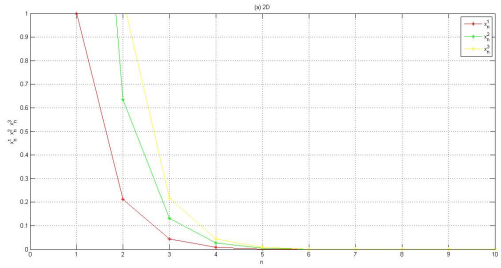
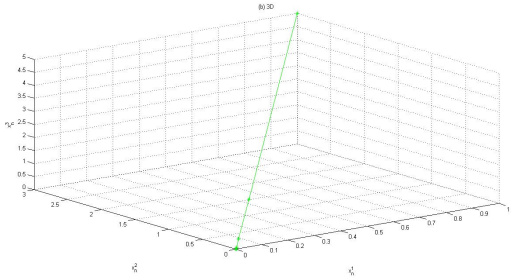


Figure 2:



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HERMITE HADAMARD TYPE INEQUALITIES FOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS FOR FUZZY INTEGRALS

M. A. LATIF [1], WAJEEHA IRSHAD [2], AND M. MUSHTAQ [3]

Abstract. In this paper we prove Hermite–Hadamard type inequalities for m-convex and (α, m)-convex functions for fuzzy integrals. Some examples are also given to illustrate the results.

1. Main Results

Let $[0, b]$, where $b > 0$, be an interval of the real line $\mathbb{R}$. A function f is said to be convex on $[0, b]$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$ and

a function f is starshaped with respect to the origin on $[0, b]$ if

$$f(tx) \leq tf(x),$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$.

In [26] G. Toader, (see also [1, 2, 4]) defined m-convexity: another intermediate between the usual convexity and starshaped convexity as follow:

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

The class of all m-convex functions on $[0, b]$ for which $f(0) \leq 0$ is denoted by $K_m(b)$. Obviously, for $m = 1$, m-convexity is the standard convexity of functions on $[0, b]$, and for $m = 0$ the concept of starshaped functions.

The following lemmas hold (see [26] see also [1, Lemma A & Lemma B, Page 2]).

Lemma 1. [1, Lemma A, Page 2] *If f is in the class $K_m(b)$, then it is starshaped.*

Lemma 2. [1, Lemma B, Page 2] *If f is in the class $K_m(b)$ and $0 < n < m \leq 1$, then f is in the class $K_n(b)$.*

From Lemma 2 and Lemma 3 it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is, $K_1(b)$ is a proper subclass of the class of

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convex functions on $[0,b]$. It is interesting to point out that for any $m \in (0,1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense (see [27]).

The notion of m-convexity was further generalized in the following definition (see also [1, 2]).

Definition 2. [11] *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0,1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

The class of all (α, m)-convex functions on $[0,b]$ for which $f(0) \leq 0$ is denoted by $K_m^{\alpha}(b)$.

If we take $(\alpha, m) = (1, m)$, it can be easily seen that (α, m)-convexity reduces to m-convexity and for $(\alpha, m) = (1, 1)$, (α, m)-convexity reduces to the concept of usual convexity defined on $[0,b], b > 0$.

For further results on inequalities related to m-convex and (α, m)-convex functions we refer the readers to [1, 2, 4].

In [4], S.S. Dragomir and G. Toader proved the following Hadamard type inequality for m-convex functions:

Theorem 1. *Let $f : [0,\infty) \to \mathbb{R}$ be an m-convex function with $m \in (0,1]$. If $0 \leq a < b < \infty$ and $f \in L^1([a,b])$ then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\} \tag{2.1}$$

We will see that this inequalitiy does not valid for fuzzy integrals in general.

To prove our assertion we consider the function $f : [0,\infty) \to [0,\infty)$, $f(x) = ax^n$, $n \in \mathbb{N}$, $n \geq 2$, $a \geq 0$, then f is m-convex on $[0,\infty)$, $m \in (0,1]$.

Example 1. *Take $X = [0,1]$ and let μ be the usual Lebesgue measure on X. Let $f : [0,\infty) \to [0,\infty)$ be defined as $f(x) = \frac{x^2}{3}$ with $m = \frac{9}{10}$. Now to calculate the Sugeno integral $\int_0^1 \frac{x^2}{3} d\mu$, consider the distribution function F associated to f on $[0,1]$ then*

$$\begin{aligned} F(\alpha) &= \mu\left([0,1] \cap \{f \geq \alpha\}\right) = \mu\left([0,1] \cap \left\{\frac{x^2}{3} \geq \alpha\right\}\right) \\ &= \mu\left([0,1] \cap \left\{x \geq \sqrt{3\alpha}\right\}\right) = 1 - \sqrt{3\alpha} \end{aligned}$$

and we solve the equation $1 - \sqrt{3\alpha} = \alpha$. It can be easily seen that the solution of this equation is $\frac{5}{2} - \frac{1}{2}\sqrt{21}$, therefore by Remark 1, we have that

$$\int_0^1 \frac{x^2}{3} d\mu = \frac{5}{2} - \frac{1}{2}\sqrt{21} \approx 0.208\,71.$$

Now

$$\frac{f(a) + mf\left(\frac{b}{m}\right)}{2} = \frac{5}{27} \approx 0.1851852$$

and on the other hand

$$\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}=\frac{1}{6}\approx 0.16666667.$$

Therefore

$$\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}=\frac{1}{6}\approx 0.16666667.$$

Which follows that (2.1) is not satisfied in the fuzzy context.

Now we prove Hdadamard type inequalites like (2.1) but for Sugeno Integral or fuzzy integral.

Theorem 2. *Let $g:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]\subset[0,\infty)$, then*

$$\int_0^1 gd\mu\leq\min\left\{1,\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)},\frac{g\left(1\right)}{1+g\left(1\right)-mg(0)}\right\}. \tag{2.2}$$

Proof. Since g is an m-convex function, therefore for $x\in[0,1]$ and $m\in(0,1]$, we have

$$g(x)=g\left(\left(1-x\right)0+1\cdot x\right)\leq\left(1-x\right)g(0)+mxg\left(\frac{1}{m}\right)=h(x)$$

and hence by (3) of Proposition 1,

$$\int_0^1 gd\mu\leq\int_0^1\left(\left(1-x\right)g(0)+mxg\left(\frac{1}{m}\right)\right)d\mu=\int_0^1 h(x)d\mu.$$

Let F be the distribution function associated to h on $[0,1]$, then

$$\begin{aligned}F(\alpha)&=\mu\left([0,1]\cap\{h\geq\alpha\}\right)=\mu\left([0,1]\cap\left\{\left(1-x\right)g(0)+mxg\left(\frac{1}{m}\right)\geq\alpha\right\}\right)\\&=\mu\left([0,1]\cap\left\{x\geq\frac{\alpha-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right\}\right)\\&=1-\frac{\alpha-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\end{aligned}$$

and as a solution of the equation $\alpha=1-\frac{\alpha-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}$, we get

$$\alpha=\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)}. \tag{2.3}$$

Analogously by the m-convexity of g, we also have

$$g(x)=g\left(\left(1-x\right)0+1\cdot x\right)\leq m\left(1-x\right)g(0)+xg(1)=h_1(x).$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[0,1]$, then

$$\alpha=\frac{g\left(1\right)}{1+g\left(1\right)-mg(0)}. \tag{2.4}$$

By (1) of Proposition 1, we have that

$$\int_0^1 h(x)d\mu=\int_0^1 h_1(x)d\mu\leq\mu\left([0,1]\right)=1. \tag{2.5}$$

The equations (2.3), (2.4), (2.5) and the definition of Sugeno integral give us the desired inequality. □

A similar results may be stated as follow, however, we leave the details for the intrested readers.

Proposition 1. *Let $g:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(0)>g(1)$ and $g(0)>mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]\subset[0,\infty)$, then*

$$\int_0^1 g d\mu \leq \min\left\{1, \frac{g(0)}{1-mg\left(\frac{1}{m}\right)+g(0)}, \frac{mg(0)}{1-g(1)+mg(0)}\right\}. \tag{2.6}$$

Remark 1. *If $m=1$, then the inequalities (2.2) and (2.6) become those inequalities proved in Theroem 1 and Theorem 2 from* [3, p. 3].

Now we give general cases of Theroem 2 and Theorem 3.

Theorem 3. *Let $f:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(\frac{a}{m})<g(b)$ and $g(a)<mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$ and $0\leq a<b<\infty$. Then*

$$\int_a^b g d\mu \leq \min\left\{1, \frac{mg\left(\frac{b}{m}\right)(b-a)}{b-a+mg\left(\frac{b}{m}\right)-g(a)}, \frac{(b-a)g(b)}{b-a+g(b)-mg\left(\frac{a}{m}\right)}\right\}. \tag{2.7}$$

Proof. Since g is an m-convex function $m\in(0,1]$, therefore for $x\in[a,b]$, $0\leq a<b<\infty$, we have

$$\begin{aligned} g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right) \\ &\leq \left(\frac{b-x}{b-a}\right)g(a)+m\left(\frac{x-a}{b-a}\right)g\left(\frac{b}{m}\right)=h(x) \end{aligned}$$

By (3) of Proposition 1, we have

$$\int_a^b g d\mu \leq \int_a^b \left[\left(\frac{b-x}{b-a}\right)g(a)+m\left(\frac{x-a}{b-a}\right)g\left(\frac{b}{m}\right)\right]d\mu=\int_a^b h(x)d\mu.$$

Let us consider the distribution function F given by

$$\begin{aligned} F(\alpha) &= \mu\left([a,b]\cap\{h\geq\alpha\}\right) \\ &= \mu\left([a,b]\cap\left\{\left(\frac{b-x}{b-a}\right)g(a)+m\left(\frac{x-a}{b-a}\right)g\left(\frac{b}{m}\right)\geq\alpha\right\}\right) \\ &= \mu\left([a,b]\cap\left\{x\geq\frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right\}\right) \\ &= b-\frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)} \end{aligned}$$

and as solution of the equation $b-\frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)}=\alpha$, we get that

$$\alpha=\frac{mg\left(\frac{b}{m}\right)(b-a)}{b-a+mg\left(\frac{b}{m}\right)-g(a)}. \tag{2.8}$$

Analogously by the m-convexity of g, we also have

$$g(x) = g\left(\left(1-\frac{x-a}{b-a}\right)a + \frac{x-a}{b-a}\cdot b\right)$$
$$\leq m\left(\frac{b-x}{b-a}\right)g(\frac{a}{m}) + \left(\frac{x-a}{b-a}\right)g(b) = h_1(x).$$

Arguing smilarly, let F_1 be the distribution accociated to h_1 on $[a,b]$, then

$$\alpha = \frac{(b-a)\,g(b)}{b-a+g(b)-mg\left(\frac{a}{m}\right)}. \tag{2.9}$$

Moreover by (1) of Proposition 1, we have

$$\int_a^b h(x)d\mu = \int_a^b h_1(x)d\mu \leq \mu([a,b]) = b-a. \tag{2.10}$$

From (2.8), (2.9), (2.10) and by the definition of fuzzy integral, we obtain (2.7). This completes the proof of the Teorem. □

Again, we state smilar results like the one proved in Theorem 4, however, the details are left to the intrested readers.

Proposition 2. *Let $f:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(\frac{a}{m}) > g(b)$ and $g(a) > mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$ and $0\leq a<b<\infty$. Then*

$$\int_a^b gd\mu \leq \min\left\{1, \frac{(b-a)\,g(a)}{b-a+g(a)-mg\left(\frac{b}{m}\right)}, \frac{m(b-a)\,g\left(\frac{a}{m}\right)}{b-a+mg\left(\frac{a}{m}\right)-g(b)}\right\}. \tag{2.11}$$

Remark 2. *If $m=1$, then the inequalities (2.7) and (2.11) become those inequalities proved in Theorem 3 from* [3, p. 4].

Example 2. *Take $X=[0,1]$ and let μ be the usual Lebesgue measure on X. Let $f:[0,\infty)\to[0,\infty)$ be defined as $f(x)=x^2$, then f is an m-convex function on $[0,1]$ with $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$, $m\in(0,1]$. Now*

$$\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)} = \frac{1}{m+1}$$

and

$$\frac{g(1)}{1+g(1)-mg(0)} = \frac{1}{2}$$

Therefore by Theorem 2, we have

$$\int_0^1 x^2 d\mu \leq \frac{1}{2}.$$

Now we give our results for (α,m)-convex functions

Theorem 4. *Let $g:[0,\infty)\to[0,\infty)$ be an (α,m)-convex function with α, $m\in(0,1]^2$ such that $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]$, then*

$$\int_0^1 gd\mu \leq \min\{1,\alpha_1,\alpha_2\}, \tag{2.12}$$

where α_1 *and* α_2 *are positive real solutions of the equations* $\alpha^{'} = 1-\left(\frac{\alpha^{'}-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}$ *and* $\alpha^{'} = 1-\left(\frac{\alpha^{'}-g(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}$ *respectively.*

Proof. Since g is an (α, m)-convex function, therefore for $x \in [a, b]$ and $\alpha,\ m \in (0, 1]^2$, we have

$$g(x) = g\left((1-x)\,0 + 1\cdot x\right) \leq (1-x^{\alpha})\,g(0) + mx^{\alpha}g\left(\frac{1}{m}\right) = h(x)$$

and hence by (3) of Proposition 1,

$$\int_0^1 g d\mu \leq \int_0^1 \left((1-x^{\alpha})\,g(0) + mx^{\alpha}g\left(\frac{1}{m}\right)\right) d\mu = \int_0^1 h(x)d\mu.$$

Let F be the distribution function associated to h on $[0, 1]$, then

$$\begin{aligned} F(\alpha^{'}) &= \mu\left([0,1]\cap\left\{h \geq \alpha^{'}\right\}\right) = \mu\left([0,1]\cap\left\{(1-x^{\alpha})\,g(0) + mx^{\alpha}g\left(\frac{1}{m}\right) \geq \alpha^{'}\right\}\right) \\ &= \mu\left([0,1]\cap\left\{x \geq \left(\frac{\alpha^{'}-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}\right\}\right) \\ &= 1-\left(\frac{\alpha^{'}-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}} \end{aligned}$$

and hence we get the equation

$$\alpha^{'} = 1-\left(\frac{\alpha^{'}-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}. \tag{2.13}$$

Analogously by the (α, m)-convexity of g, we also have

$$g(x) = g\left((1-x)\,0 + 1\cdot x\right) \leq m\,(1-x^{\alpha})\,g(0) + x^{\alpha}g(1) = h_1(x).$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[0, 1]$, then we have that the following equation:

$$\alpha^{'} = 1-\left(\frac{\alpha^{'}-g(0)}{g\,(1)-mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.14}$$

By (1) of Proposition 1, we have that

$$\int_0^1 h(x)d\mu = \int_0^1 h_1(x)d\mu \leq \mu\left([0,1]\right) = 1. \tag{2.15}$$

The equations (2.13), (2.14), (2.15) and the definition of Sugeno integral give us the required inequality. □

A similar result can be stated as follow, however, the details are left to the intrested reasers:

Proposition 3. *Let $g : [0,\infty) \to [0,\infty)$ be an (α, m)-convex function with α, $m \in (0,1]^2$ such that $mg(0) > g(1)$ and $g(0) > mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1] \subset [0,\infty)$, then*

$$\int_0^1 g d\mu \leq \min\{1, \alpha_1, \alpha_2\}, \tag{2.16}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha' = \left(\frac{g(0)-\alpha'}{g(0)-mg\left(\frac{1}{m}\right)}\right)^{\frac{1}{\alpha}}$ and $\alpha' = \left(\frac{g(0)-\alpha'}{mg(0)-g(1)}\right)^{\frac{1}{\alpha}}$ respectively.

Remark 3. *If $(\alpha, m) = (1,1)$, then the inequalities (2.12) and (2.16) become those inequalities proved in Theroem 1 and Theorem 2 from* [3, p.4].

Now in following results we give the general case of the last two results.

Theorem 5. *Let $g : [0,\infty) \to [0,\infty)$ be an (α, m)-convex function with α, $m \in (0,1]^2$ such that $mg(\frac{a}{m}) < g(b)$ and $g(a) < mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$, $0 \leq a < b < \infty$, then*

$$\int_0^1 g d\mu \leq \min\{1, \alpha_1, \alpha_2\}, \tag{2.17}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha' = (b-a)\left[1 - \left(\frac{\alpha'-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]$ and $\alpha' = (b-a)\left[1 - \left(\frac{\alpha'-g\left(\frac{a}{m}\right)}{g(b)-mg\left(\frac{a}{m}\right)}\right)^{\frac{1}{\alpha}}\right]$ respectively.

Proof. Since g is an (α, m)-convex function, therefore for $x \in [0,1]$ and α, $m \in (0,1]^2$, we have

$$\begin{aligned} g(x) &= g\left(\left(1 - \frac{x-a}{b-a}\right)a + \frac{x-a}{b-a} \cdot b\right) \\ &\leq \left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right) = h(x) \end{aligned}$$

and hence by (3) of Proposition 1,

$$\int_0^1 g d\mu \leq \int_0^1 \left[\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right)\right] d\mu = \int_0^1 h(x) d\mu.$$

Let F be the distribution function associated to h on $[0,1]$, then

$$\begin{aligned}
F(\alpha^{'}) &= \mu\left([a,b]\cap\left\{h\geq\alpha^{'}\right\}\right)\\
&= \mu\left([a,b]\cap\left\{\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g(a)+m\left(\frac{x-a}{b-a}\right)^{\alpha}g\left(\frac{b}{m}\right)\geq\alpha^{'}\right\}\right)\\
&= \mu\left([a,b]\cap\left\{x\geq a+(b-a)\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right\}\right)\\
&= (b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]
\end{aligned}$$

and hence we get the equation

$$\alpha^{'}=(b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]. \tag{2.18}$$

Analogously by the (α,m)-convexity of g, we also have

$$\begin{aligned}
g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right)\\
&\leq m\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g\left(\frac{a}{m}\right)+\left(\frac{x-a}{b-a}\right)^{\alpha}g(b)=h_1(x).
\end{aligned}$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[0,1]$, then we have that the following equation:

$$\alpha^{'}=(b-a)\left[1-\left(\frac{\alpha^{'}-g\left(\frac{a}{m}\right)}{g(b)-mg\left(\frac{a}{m}\right)}\right)^{\frac{1}{\alpha}}\right]. \tag{2.19}$$

By (1) of Proposition 1, we have that

$$\int_0^1 h(x)d\mu=\int_0^1 h_1(x)d\mu\leq\mu([a,b])=b-a. \tag{2.20}$$

The equations (2.18), (2.19), (2.20) and the definition of Sugeno integral give us the required inequality. □

A similar resutl is stated below, however, the details are left:

Theorem 6. *Let $g:[0,\infty)\to[0,\infty)$ be an (α,m)-convex function with α, $m\in(0,1]^2$ such that $mg(\frac{a}{m})>g(b)$ and $g(a)>mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$, $0\leq a<b<\infty$, then*

$$\int_0^1 gd\mu\leq\min\{1,\alpha_1,\alpha_2\}, \tag{2.21}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha^{'}=(b-a)\left[\left(\frac{g(a)-\alpha^{'}}{g(a)-mg\left(\frac{b}{m}\right)}\right)^{\frac{1}{\alpha}}\right]$ and $\alpha^{'}=(b-a)\left[\left(\frac{g\left(\frac{a}{m}\right)-\alpha^{'}}{mg\left(\frac{a}{m}\right)-g(b)}\right)^{\frac{1}{\alpha}}\right]$ respectively.

Remark 4. *If $(\alpha, m) = (1, 1)$, then the inequalities (2.17) and (2.21) become those inequalities proved in Theroem 3 from* [3, p.4].

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College of Science, Department of Mathematics, University of Hail, Hail 2440, Saudi Arabia.

E-mail address: m_amer_latif@hotmail.com

E-mail address: wchattah@hotmail.com

E-mail address: mmushtaq@uet.edu.pk

[2,3]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan

Very true operators on equality algebras

Jun Tao Wang[a], Xiao Long Xin[a,*],Young Bae Jun[b]

[a]*School of Mathematics, Northwest University, Xi'an, 710127, China*
[b]*Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea*

Abstract

In this paper, we introduce the concept of very true operators on equality algebras and investigate some related properties of such operators. As an application of properties of very true operators on equality algebras, we give a characterization of prelinear equality algebras, and discuss the relation between very true operator and internal state operator on equality algebras. Moreover, we put forth the notion of very true filter on equality algebras and obtain some important results. In particular, using very true filter, we characterize the simple very true equality algebras and establish the uniform structures on very true equality algebras.

Keywords: fuzzy higher logic; equality algebras; very true operators; very true filters; uniform structures.

1. Introduction

Fuzzy type theory [13, 14, 15, 16], whose basic connective is a fuzzy equality $\sim$, was developed as a fuzzy counterpart of the classical higher-order logic (type theory in which identity is a basic connective, see [9]). Since the truth values for algebra of fuzzy type theory is no longer a residuated lattice, a specific algebra called an EQ-algebra [17] was proposed. Viewing the axioms of EQ-algebras with a purely algebraic eye it appears that unlike in the case of residuated lattices where the adjointness condition ties product with implication. By contrast, the product in EQ-algebras is quite loosely related to the other connectives, which lead to the product in EQ-algebra may be replaced by other similar connectives. Furthermore, the freedom in choosing the product might prohibit to find deep related algebraic results. For this reason, a new algebraic structure was introduced in [11], called equality algebra, which consisting of two binary operations-meet and equivalence, and constant 1. It was proved in [12] that any equality algebra has a corresponding BCK-meet-semilattice and any BCK-meet-semilattice with (D) has a corresponding equality algebras. Apart from their algebraic interest, the general motivation for equality algebras from the side of logic was to define an algebraic structure which (with appropriate extensions) is suitable to axiomatize a large class of substructural logics based on an equivalence connective rather than implication. The very first step toward this aim has been done in [11]. Indeed, the equality algebras could also be candidates for a possible algebraic semantics for fuzzy type theory, which lead to the study of equality algebra is highly motivated.

*Corresponding author.
Email addresses: wjt@stumail.nwu.edu.cn(J.T. Wang), xlxin@nwu.edu.cn (X.L. Xin), skywine@gmail.com(Y.B. Jun).

In the sense of Zadeh [5](1975), fuzzy logic distinguish in broad and narrow sense. In narrow sense, fuzzy logic deals with some many valued logic but asks questions different from those asked by logicians, who devote to completeness and soundness, etc. Compare to narrow sense, Zadeh stress the importance of fuzzy truth values as very true, quite true and so on, that themselves are fuzzy subsets of all truth degrees. He always gives some examples of handling these fuzzy truth values but seems uninterested in any sort of axiomatization. In order to answer the question " if any natural axiomatization is possible and how far can even this sort of fuzzy logic be captured by standard methods of mathematical logic", Hájek[8] introduced the concept of very true operator on BL-algebras as a tool for reducing the number of possible logical values in many valued fuzzy logic. Consequently, the notion of very true operator has been extended to other logical algebras such as MV-algebra [2], Rℓ-monoid[3], residuated lattices[10, 20] and provided an algebraic foundation for reasoning about fuzzy truth valued of events in many valued logic system, which belong to a subclass of substructural logic based on an fuzzy implication.

As for BL-algebras, MV-algebras Rℓ-monoid and residuated lattices, we observed that although they are different algebras they all are essentially particular types of equality algebras. Thus, it is natural to generalized the concept of very true operators to equality algebras for studying the most general results regarding very true operators in the above-mentioned algebras. On the other hand, BL, Łukasiewicz, ML, they are all many valued logic system belong to a subclass of substructural logic based on an fuzzy implication. However, the logic system corresponding to equality algebra is a fuzzy higher logic, which belong to subclass of substructural logic based on fuzzy equality and is different from above common many valued logic system that based on fuzzy implication. Moreover, all results of this paper may be considered providing an algebraic foundation for reasoning about probabilities of fuzzy events for higher fuzzy logic. This is the motivation for us to investigate very true operators on equality algebras.

Based on above consideration, we enrich the language of equality algebras by adding a very true operator to get algebras named very true equality algebras. This paper is structured in five sections. In order to make the paper as self-contained as possible, we recapitulate in Section 2 the definition of equality algebras, and review their basic properties that will be used in the remainder of the paper. In Section 3, we introduce very true operator on equality algebras and study some properties of them. Also, we give a characterizations of a prelinear equality algebras and discuss the relation between very true operator and internal state on equality algebras. In Section 4, we investigate very true filter of very true equality algebras. In particular, by using very true filter, simple very true equality algebras are characterized and the uniform structures on very true equality algebras are established.

2. Preliminaries

In this section, we summarize some definitions and results about equality algebras which will be used in the following and we shall not cite them every time they are used.

Definition 2.1. [11] An algebra $(\mathcal{E}, \sim, \wedge, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if it satisfies the following conditions:

(E1) $(\mathcal{E}, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., $\wedge$-semilattice with top element 1),

(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$,

for all $x, y, z \in \mathcal{E}$.

In what follows, by $\mathcal{E}$ we denote the universe of $(\mathcal{E}, \sim, \wedge, 1)$. For any $x, y \in \mathcal{E}$, we define fuzzy implication as $x \to y = x \sim (x \wedge y)$ and agree that $\sim$ and $\to$ have higher priority than $\wedge$.

On an equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ we define $x \leq y$ iff $x \wedge y = x$. It is easy to check that $\leq$ is a partial order relation on $\mathcal{E}$ and for all $x \in \mathcal{E}$, $x \leq 1$.

Definition 2.2. [6, 7] Let $(\mathcal{E}, \sim, \wedge, 1)$ be an equality algebra. Then $\mathcal{E}$ is called:

(1) *bounded* if there exists an element $0 \in \mathcal{E}$ such that $0 \leq x$ for all $x \in \mathcal{E}$,
(2) *prelinear* if for all $x, y \in \mathcal{E}$, 1 is the unique upper bounded in $\mathcal{E}$ of the set $\{(y \to x), (x \to y)\}$.

Proposition 2.3. [6, 7] If $(\mathcal{E}, \sim, \wedge, 1)$ is a prelinear equality algebra, then $(\mathcal{E}, \leq)$ is a lattice, where the join operation is given by $x \vee y = ((x \to y) \to y) \wedge (y \to x) \to x)$, for any $x, y \in \mathcal{E}$.

Proposition 2.4. [6, 7] An equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ is prelinear if and only if $(x \to y) \to z \leq ((y \to x) \to z) \to z$, for all $x, y, z \in \mathcal{E}$.

Proposition 2.5. [11, 12] In every equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ the following properties hold for all $x, y, z \in \mathcal{E}$:

(1) $x \sim y \leq x \leftrightarrow y \leq x \to y$,
(2) $x \sim y = 1$ iff $x = y$,
(3) $x \to y = 1$ iff $x \leq y$,
(4) $x \to y = 1$ and $y \to x = 1$ imply $x = y$,
(5) $1 \to x = x$, $x \to 1 = 1$, $x \to x = 1$,
(6) $x \leq y \to x$,
(7) $x \leq (x \to y) \to y$,
(8) $x \to y \leq (y \to z) \to (x \to z)$,
(9) $x \leq y \to z$ iff $y \leq x \to z$,
(10) if $x \leq y$, then $x \leq x \sim y$,
(11) $x \leq y$ imply $y \to x = y \sim x$,
(12) $x \leq y$ imply $y \to z \leq x \to z$, $z \to x \leq z \to y$,
(13) If $\mathcal{E}$ is a prelinear equality algebra, then $\bigwedge_{i \in I}(x_i \to y) = \bigvee_{i \in I} x_i \to y$, provided that both infimum as well as supremum exist.

Definition 2.6. [11] Let $(\mathcal{E}, \sim, \wedge, 1)$ and $(\mathcal{E}', \rightsquigarrow, \sqcap, 1')$ be two equality algebras and $f : \mathcal{E} \longrightarrow \mathcal{E}'$ be a mapping. We call f a *homomorphism* if the following conditions hold for all $x, y \in \mathcal{E}$:

(1) $f(x \wedge y) = f(x) \sqcap f(y)$,
(2) $f(x \sim y) = f(x) \rightsquigarrow f(y)$.

The following theorem provides a connection of equality algebras with a special class of BCK-algebras with meet.

Theorem 2.7. [4, 11, 12] The following two statements hold:

(1) For any equality algebra $(\mathcal{E}, \sim, \wedge, 1)$, the structure $\Psi(\mathcal{E}) = \{\mathcal{E}, \wedge, \rightarrow, 1\}$ is a BCK-meet-semilattice, where $\rightarrow$ denotes the implication of $\mathcal{E}$,
(2) For any BCK(D)-meet-semilattice $(B, \wedge, \rightarrow, 1)$, the structure $\Phi(B) = \{B, \wedge, \leftrightarrow, 1\}$ is an equality algebra, where $\leftrightarrow$ denotes the equivalence operation of B. Moreover, the implication of $\Phi(B)$ coincide with $\rightarrow$, that is, $x \rightarrow y = x \leftrightarrow (x \wedge y)$.

Let $(\mathcal{E}, \sim, \wedge, 1)$ be an equality algebra. A nonempty set F is called a *filter* of $\mathcal{E}$ if it satisfies: (1) $x \in F$, $x \leq y$ implies $y \in F$, (2) $x \in F$, $x \sim y \in F$ implies $y \in F$. One can prove that the set of filters of an equality algebra coincide with the set of filter of its underlying BCK-algebra. A filter F of an equality algebra $\mathcal{E}$ is *proper* if $F \neq \mathcal{E}$. A proper filter is called *maximal* if it is not strictly contained in any other proper filter of $\mathcal{E}$. We will denote by $F(\mathcal{E})$ the set of all filter of $\mathcal{E}$. Clearly, $\{1\}$, $\mathcal{E} \subseteq F(\mathcal{E})$ and $F(\mathcal{E})$ is closed under arbitrary intersections. As a consequence, $(F(\mathcal{E}), \subseteq)$ forms a complete lattice. An equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ is calle a *simple* if $F(\mathcal{E}) = \{\{1\}, \mathcal{E}\}$. (see [4, 12])

Definition 2.8. [4, 11, 12] Let $(\mathcal{E}, \sim, \wedge, 1)$ be an equality algebra. A subset $\theta \subseteq \mathcal{E} \times \mathcal{E}$ is called a *congruence* of $\mathcal{E}$ if it is an equivalence relation on $\mathcal{E}$ and for all $x_1, y_1, x_2, y_2 \in \mathcal{E}$ such that $(x_1, y_1), (x_2, y_2) \in \theta$ the following hold:

(1) $(x_1 \wedge x_2, y_1 \wedge y_2) \in \theta$,
(2) $(x_1 \sim x_2, y_1 \sim y_2) \in \theta$.

Let F be a filter of $\mathcal{E}$. Define the congruence relation $\equiv_F$ on $\mathcal{E}$ by $x \equiv_F y$ if and only if $x \sim y \in F$. The set of all congruence class is denote by $\mathcal{E}/F$, i.e. $\mathcal{E}/F = \{[x] | x \in \mathcal{E}\}$, where $[x] = \{x \in \mathcal{E} | x \equiv_F y\}$. Define $\bullet$, $\rightharpoonup$ on $\mathcal{E}/F$ as follows: $[x] \bullet [y] = [x \wedge y]$, $[x] \rightharpoonup [y] = [x \sim y]$. Therefore, $(\mathcal{E}/F, \bullet, \rightharpoonup, [1])$ is an equality algebra which is called a *quotient equality algebra* of $\mathcal{E}$ with respect to F. (see [4])

In what follows, we review some notions about uniformity which will be necessary in the following section.

Let X be a nonempty set and U, V be any subset of $X \times X$. Defined $U \circ V = \{(x, y) \in X \times X |$ for some $z \in X, (z, y) \in U$ and $(x, z) \in V\}$, $U^{-1} = \{(x, y) \in X \times X | (y, x) \in U\}$, $\triangle = \{(x, x) \in X \times X | x \in X\}$.

Definition 2.9. [1, 18, 19] By an *uniformity* on X we shall mean a nonempty collection K of subsets of $X \times X$ which satisfies the following conditions:

(U1) $\triangle \subseteq U$ for any $U \in K$,
(U2) If $U \in K$, then $U^{-1} \in K$,
(U3) If $U \in K$, then there exists a $V \in K$ such that $V \circ V \subseteq U$,
(U4) If $U, V \in K$, then $U \cap V \in K$,
(U5) If $U \in K$ and $U \subseteq V \subseteq X \times X$ then $V \in K$,

The pair (X, K) is called an *uniform structure* (uniform space).

3. Very true operators on equality algebras

In this section, we introduce the notion of very true operators in an equality algebras and investigate some related properties of such operators. Also, we give characterizations of prelinea equality algebras and discuss relations between very true operators and internal state operators on equality algebras.

Definition 3.1. Let $\mathcal{E}$ be an equality algebra. The mapping $\tau : \mathcal{E} \longrightarrow \mathcal{E}$ is called a *very true operator* if it satisfies the following conditions:

(VE1) $\tau(1) = 1$,
(VE2) $\tau(x) \leq x$,
(VE3) $\tau(x) \leq \tau\tau(x)$,
(VE4) $\tau(x \sim y) \leq \tau(x) \sim \tau(y)$,
(VE5) $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$.

The pair $(\mathcal{E}, \tau)$ is said to be *very true equality algebra.*

Note. (1) Such a proliferation of logics deserves some explanation. Since 1 is considered as the logical value absolutely true. First note that (VE1) means that absolutely true is very true, which is the standard axiom obtain the classical logic. (VE2) means that if φ is very true then it is true. (VE3) says that very true of very true is very true, which is a kind of necessitation with respect to very true connective. (VE4) means that if both φ and $\varphi \sim \psi$ are very true then so is ψ, that means the connective τ preserve modus ponens based on fuzzy equality $\sim$. (VE5) says that if both φ and ψ are very true then so is conjunction $\varphi \wedge \psi$, one can easily to check that (VE5) is sound for each natural interpretation in equality algebra. Indeed, the order in equality algebra is lattice order, that is the conjunction $\wedge$ is interpreted as the lattice meet $\wedge$.

(2) Although equality algebras belong to some subclasses of substructural logics based on fuzzy equality rather than on fuzzy implication, we define very true operators on them in the way which is in accordance with traditional definitions in residuated lattices. In fact, a very true operator on residuated lattice was introduced by Vychodil [10] in 2005 as a mapping $\tau : \mathcal{E} \to \mathcal{E}$ satisfying conditions (VE1)-(VE3) in Definition 3.1 and (VE4') $\tau(x \to y) \leq \tau(x) \to \tau(y)$. We know that residuated lattices are special cases of equality algebras satisfying the residuated law. In residuated lattice, fuzzy equality $\sim$ can be defined by $x \sim y = (x \to y) \wedge (y \to x)$. From (VE4) and (VE5), one can obtain that the connective τ always preserve modus ponens based on fuzzy implication. Based on (VE4') and the isotonicity of very true connective τ, one can easily obtain $\tau(x \sim y) = \tau((x \to y) \wedge (y \to x)) \leq \tau(x \to y) \wedge \tau(y \to x) \leq (\tau(x) \to \tau(y)) \wedge (\tau(y) \to \tau(x)) = \tau(x) \sim \tau(y)$, thus the (VE5) hold. From this point of view, the very true equality algebra essentially generalize residuated lattice with very true operator. Thus, it is the most general logic algebras with very true in the existing ones founded in the literature, at least to the authors knowledge.

Now, we will give some important examples to illustrate above definition is existing and meaningful.

Example 3.2. For every equality algebra $\mathcal{E}$ there exist at least two very true operator. One is the identical mapping $\tau_0(x) = x$ for any $x \in \mathcal{E}$, and the other is defined by $\tau_1(1) = 1$ and $\tau_1(x) = 0$ for any $x < 1$. It is evident that if τ is a very true operator on $\mathcal{E}$, we have $\tau_1(x) \leq \tau(x) \leq \tau_0(x)$. Thus these τ_0 ,τ_1 are extremal.

Example 3.3. Let $\mathcal{E} = \{0, a, b, 1\}$ with $0 \leq a \leq b \leq 1$. Consider the operation $\sim$ and $\rightarrow$ given by the following tables:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(\mathcal{E}, \sim, \wedge, 1)$ is an equality algebra in [4]. Now,we define $\tau(0) = 0$, $\tau(a) = a$, $\tau(b) = a$, $\tau(1) = 1$. One can easily check that τ is a very true operator on $\mathcal{E}$. However, τ is not a endhomomorphism on $\mathcal{E}$ since $\tau(a \sim b) = a \neq 1 = \tau(a) \sim \tau(b)$.

Next, we present some useful properties of very true operator on equality algebras.

Proposition 3.4. Let $(\mathcal{E}, \tau)$ be a very true equality algebra, then for any $x, y, z \in \mathcal{E}$ we have,

(1) If $\mathcal{E}$ is a bounded equality algebra, then $\tau(0) = 0$,
(2) $\tau(x) = 1$ if and only if $x = 1$,
(3) $\tau\tau(x) = \tau(x)$,
(4) $\tau(x \rightarrow y) \leq \tau(x) \rightarrow \tau(y)$,
(5) $x \leq y$ implies $\tau(x) \leq \tau(y)$,
(6) $\tau(x) \leq y$ if and only if $\tau(x) \leq \tau(y)$,
(7) $\tau(\mathcal{E}) = Fix_\tau$,where $Fix_\tau = \{x \in \mathcal{E} | \tau(x) = x\}$,
(8) Fix_τ is closed under $\wedge$,
(9) If $y \leq x$, then $\tau(x) \rightarrow \tau(y) = \tau(x) \sim \tau(y)$,
(10) $\tau(x \sim y) \leq \tau(x) \rightarrow \tau(y)$,
(11) $\tau(x \sim y) \leq (x \wedge z) \sim (y \wedge z)$,
(12) If $\tau(\mathcal{E}) = \mathcal{E}$, then $\tau = id_\mathcal{E}$,
(13) $\mathrm{Ker}(\tau) = \{1\}$, where $\mathrm{Ker}(\tau) = \{x \in \mathcal{E} | \tau(x) = 1\}$,
(14) $\mathrm{Ker}(\tau)$ is a filter of $\mathcal{E}$,
(15) $\tau(x) = x$ or $\tau(x)$ and x are not comparable,
(16) If $\mathcal{E}$ is linearly order, then $\tau = id_\mathcal{E}$.

Proof. (1) Applying (VE2), we have $\tau(0) \leq 0$ and hence $\tau(0) = 0$.

(2) If $\tau(x) = 1$ for some $x \in \mathcal{E}$ then by (VE2), $1 = \tau(x) \leq x$ giving $x = 1$. The converse follows by (VE1).

(3) Applying (VE2) and (VE3), we have $\tau\tau(x) = \tau(x)$.

(4) By (VE4) and (VE5) we have $\tau(x \rightarrow y) = \tau(x \wedge y \sim x) = \tau(x \wedge y) \sim \tau(x) \leq \tau(x) \wedge \tau(y) \sim \tau(x) = \tau(x) \rightarrow \tau(y)$.

(5) If $x \leq y$, then $x \rightarrow y = 1$. It follows from (VE1) and (4) that $\tau(x) \rightarrow \tau(y) = 1$. Therefore, $\tau(x) \leq \tau(y)$.

(6) Assume that $\tau(x) \leq y$, we have $\tau\tau(x) \leq \tau(y)$. By (3), we get $\tau\tau(x) = \tau(x)$. Thus $\tau(x) \leq \tau(y)$. Conversely, suppose that $\tau(x) \leq \tau(y)$, we have $\tau(x) \leq \tau(y) \leq y$.

(7) Let $y \in \tau(\mathcal{E})$, so there exists $x \in \mathcal{E}$ such that $y = \tau(x)$. Hence $\tau(y) = \tau\tau(x) = \tau(x) = y$. It follows that $y \in Fix_\tau$. Conversely, if $y \in Fix_\tau$, we have $y \in \tau(\mathcal{E})$. Therefore, $\tau(\mathcal{E}) = Fix_\tau$.

(8) By (VE5), we obtain that Fix_τ is closed under $\wedge$.

(9) Since $y \leq x$, we have $\tau(y) \leq \tau(x)$ and $\tau(x) \to \tau(y) = \tau(x) \sim \tau(x) \wedge \tau(y) = \tau(x) \sim \tau(y)$.

(10) By Proposition 2.5(1) and (4),(5), we have $\tau(x \sim y) \leq \tau(x \to y) \leq \tau(x) \to \tau(y)$.

(11) By (E6) and (VE2), one can obtain it very easy and hence we omit the process of this proof.

(12) For any $x \in \mathcal{E}$, we have $x = \tau(x_0)$ for some $x_0 \in \mathcal{E}$. By (3), we have $\tau(x) = \tau(\tau(x_0)) = \tau(x_0) = x$. Therefore, $\tau = id_{\mathcal{E}}$.

(13) Assume that $x \in \mathcal{E}$ but $x \neq 1$ such that $\tau(x) = 1$. Applying (VE2), we have $1 = \tau(x) \leq x$ and hence $x = 1$, which is a contradiction. Therefore, $\mathrm{Ker}(\tau) = \{1\}$.

(14) It is easy to check it and hence we omit the process.

(15) Assume $x \in \mathcal{E}$ such that $\tau(x) \neq x$ and $\tau(x)$ and x are comparable. Then $\tau(x) < x$ or $x < \tau(x)$, from (3),(5), we have $\tau(x) < \tau(x)$, which is a contradiction.

(16) It follows from (15) directly.

From the above Proposition 3.4, one can see that $\tau(\mathcal{E})$ is closed under the operation $\wedge$. However, the following example shows that $\tau(\mathcal{E})$ is not a subalgebra of $\mathcal{E}$ since it is not closed under $\sim$ in general.

Example 3.5. Let $\mathcal{E} = \{0, a, b, c, 1\}$ with $0 \leq a \leq b, c \leq 1$. Consider the operation $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	0	b
c	0	b	0	1	c
1	0	a	b	1	b

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(\mathcal{E}, \sim, \wedge, 1)$ is an equality algebra. Now,we define $\tau(0) = 0$, $\tau(a) = a$, $\tau(b) = a$, $\tau(c) = c$, $\tau(1) = 1$. One can easily check that τ is a very true operator on $\mathcal{E}$. However, $\tau(\mathcal{E})$ is not a subalgebra of $\mathcal{E}$ since $a \sim c = b \notin \tau(\mathcal{E})$.

Although the $\tau(\mathcal{E})$ is not necessary a subalgebra of an equality algebra in general, while it forms an equality algebra after redefined its fuzzy equality, which reveals the essence of the fixed point set.

Theorem 3.6. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is also an equality algebra, where $x \rightsquigarrow y = \tau(x \sim y)$ for all $x, y \in \tau(\mathcal{E})$.

Proof. Now, we will show that $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebra.

For (E1), we show that $(\tau(\mathcal{E}), \wedge, 1)$ is a semilattice with 1 as the greatest element. From Definition 3.1(5), we have that $\tau(\mathcal{E})$ is closed under $\wedge$. Therefore $(\tau(\mathcal{E}), \wedge)$ is a semilattice. For all $x \in \tau(\mathcal{E})$, one can easily check that $x \wedge 1 = x$. Thus, 1 is the greatest element in $\tau(\mathcal{E})$.

For (E2), we will show that $x \rightsquigarrow y = y \rightsquigarrow x$. It is easy to prove.

For (E3), we will show that $x \rightsquigarrow x = 1$. Applying (VE1), we have $\tau(1) = 1$ and hence $x \rightsquigarrow x = \tau(x \sim x) = \tau(1) = 1$.

For (E4), we will show that $x \rightsquigarrow 1 = x$. Since $x \in \tau(\mathcal{E})$, we have $\tau(x) = x$ and hence $x \rightsquigarrow 1 = \tau(x \sim 1) = \tau(x) = x$.

In the similarly way, we can show that (E5)-(E7) hold.

Combine them, we obtain that $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebra.

Note. In fact, it is easily checked that $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebra, where $\tau(x) \rightsquigarrow \tau(y) = \tau(x \sim y)$. Furthermore, we also obtain that if $\tau(x) \rightsquigarrow \tau(y) = 1$, then $\tau(x) = \tau(y)$. Indeed, if $\tau(x), \tau(y) \in Fix_\tau$, by above Theorem 3.6, $\tau(x) \rightsquigarrow \tau(y) = \tau(\tau(x) \sim \tau(y)) = \tau(x) \sim \tau(y)$. Thus $\tau(x) \sim \tau(y) = 1$, and we have $\tau(x) = \tau(y)$ by Proposition 2.5(3).

In what following, we will give an analogy first isomorphism theorem related to very true operator on equality algebras, which will be used in the next section.

Theorem 3.7. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ be an equality algebra. Then the following properties hold:

(1) $\tau : \mathcal{E} \longrightarrow \tau(\mathcal{E})$ is a homomorphism,
(2) The mapping $\tau_0 : \mathcal{E}/\mathrm{Ker}(\tau) \longrightarrow \tau(\mathcal{E})$ defined by $\tau_0([x]) = \tau(x)$ is a isomorphism.

Proof. (1) It follows from (VE1) and (VE5) in Definition 3.1 that $\tau(1) = 1$ and $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$. Moreover, from the above Note, we have $\tau(x) \rightsquigarrow \tau(y) = \tau(x \sim y)$. Therefore $\tau : \mathcal{E} \longrightarrow \tau(\mathcal{E})$ is a homomorphism.

(2) Assume that $[x] = [y]$ and hence $(x, y) \in \mathrm{Ker}(\tau)$. Then $x \sim y \in \mathrm{Ker}(\tau)$, that is, $\tau(x \sim y) = 1$. From the above Note, we have $\tau(x) \rightsquigarrow \tau(y) = 1$ and $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebras and so $\tau(x) = \tau(y)$. Therefore, τ_0 is well defined. Now, we will show that τ_0 is a isomorphism. First, we will show that τ_0 is a homomorphism. From (1), we have $\tau_0([x] \rightharpoonup [y]) = \tau_0([x \sim y]) = \tau(x \sim y) = \tau(x) \rightsquigarrow \tau(y) = \tau_0([x]) \rightsquigarrow \tau_0([y])$. Moreover, we have $\tau_0([x] \bullet [y]) = \tau_0([x \wedge y]) = \tau(x \wedge y) = \tau(x) \wedge \tau(y) = \tau_0([x]) \wedge \tau_0([y])$. Clearly, $\tau_0([1]) = 1$. Hence τ_0 is a homomorphism. Next, we will show that τ_0 is one to one. From the above Note, if $\tau_0([x]) = \tau_0([y])$ then $\tau(x) = \tau(y)$ and hence $\tau(x) \rightsquigarrow \tau(y) = \tau(x \sim y) = 1$, that is means $(x, y) \in \mathrm{Ker}(\tau)$ and hence $[x] = [y]$ and so τ_0 is one to one. Furthermore, since τ is onto, τ_0 is onto. Combine them, we obtain that τ_0 is a isomorphism.

As an application of the properties respect to very true operators, we give a characterization of prelinear equality algebras. For obtaining this important result, we need the following theorem.

Theorem 3.8. The following conditions are equivalent in each very true equality algebra $(\mathcal{E}, \tau)$:

(1) $\tau(x \to y) \leq (x \wedge z) \to (y \wedge z)$,
(2) $\tau(y) \leq z \to (y \wedge z)$,
(3) $\tau(y) \leq u \to (u \wedge (z \to (y \wedge z))$.

Proof. (1) $\Rightarrow$ (2) Assume (1) holds. From Proposition 2.5 (1) and Proposition 3.5 (11), we have $\tau(x \to y) \leq ((1 \wedge z) \sim (y \wedge z)) \leq ((1 \wedge z) \to (y \wedge z)) = z \to (y \wedge z)$.

(2) $\Rightarrow$ (3) Assume (2) holds. By (VE3), we get $\tau(y) \leq \tau\tau(y) \leq \tau(z \to (y \wedge z))$, which implies, by (2) again, $\tau(y) \leq u \to (u \wedge (z \to (z \wedge y)))$.

(3) $\Rightarrow$ (2) Taking $u = 1$, we obtain that (2) holds.

(2) $\Rightarrow$ (1) Assume (2) holds. By (3), we have $\tau(x \to y) \leq (x \wedge z) \to ((x \to y) \wedge (x \wedge z))$. Thus, $\tau(x \to y) \leq (x \wedge z) \to ((x \to y) \wedge x \wedge z)$. Furthermore, from Proposition 2.5(12), we have $\tau(x \to y) \leq (x \wedge z) \to ((x \to y) \wedge x \wedge z) \leq (x \wedge z) \to (y \wedge z)$.

Conditions for an equality algebra to be a prelinear equality algebra is gave via very true operator on equality algebra.

Theorem 3.9. The following conditions are equivalent in each very true equality algebras:

(1) $\mathcal{E}$ is prelinear,
(2) $\tau(x \vee y) = \tau(x) \vee \tau(y)$,
(3) $\tau(x \to y) \vee \tau(y \to x) = 1$,
(4) $\tau((x \to y) \to z) \leq \tau(y \to x) \to z) \to z$.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathcal{E}$ is prelinear equality algebra. By(VE5) and Proposition 2.3, we obtain, for all $x, y \in L$, $\tau(x \vee y) = \tau((x \to y) \to y) \wedge \tau((y \to x) \to x)$. By Proposition 3.4(5) and (VE3), we get $\tau(x \vee y) \leq \tau((x \to y) \to (\tau(x) \vee \tau(y))) \wedge (\tau(y \to x) \to (\tau(x) \vee \tau(y)))$. Hence by Proposition 2.5(13), we obtain $\tau(x \vee y) \leq (\tau(x \to y)) \vee (\tau(y \to x)) \to (\tau(x) \vee \tau(y))$ and hence $\tau(x \vee y) \leq \tau(x) \vee \tau(y)$. The other inequality easily by Proposition 3.4(5).

(2) $\Rightarrow$ (3) straightforward.

(3) $\Rightarrow$ (4) This follows exactly by a proof similar to the proof of the equivalence between prelinearity and $(x \to y) \to z \leq ((y \to x) \to z) \to z)$ in Proposition 2.4.

(4) $\Rightarrow$ (1) Taking very true operator $\tau = id_{\mathcal{E}}$ and direct from Proposition 2.4.

Note. We know a BL-algebras is an prelinear quality algebra satisfies the divisibility. From the above theorem, one can check that the very true prelinear equality algebra essentially generalize BL-algebras with very true operator, which introduced by Hájek [8] in 2001 as a mapping $\tau : \mathcal{E} \to \mathcal{E}$ satisfying conditions (VE1)-(VE3), (4) in Proposition 3.5 and (2) in Theorem 3.9.

In what follows, we will give a relationship between very true equality algebras and sate (morphism) equality algebras, which was introduced by L.C. Ciungu [4] to providing an algebraic foundation for reasoning about probabilities of fuzzy events of a large class of substructural logics based on an fuzzy equality.

Definition 3.10. [4] An equality algebra with internal state or *state equality algebra* is a structure $(\mathcal{E}, \sigma) = (\mathcal{E}, \wedge, \sim, \sigma, 1)$, where $(\mathcal{E}, \wedge, \sim, 1)$ is an equality algebra and $\sigma : \mathcal{E} \to \mathcal{E}$ is a unary operator on $\mathcal{E}$, called internal state or state operator, such that for all $x, y \in \mathcal{E}$ the following conditions are satisfied:

(1) $\sigma(x) \leq \sigma(y)$, whenever $x \leq y$,
(2) $\sigma(x \sim x \wedge y) = \sigma((x \sim x \wedge y) \sim y) \sim \sigma(y)$,
(3) $\sigma(\sigma(x) \sim \sigma(y)) = \sigma(x) \sim \sigma(y)$,
(4) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

Definition 3.11. [4] Let $(\mathcal{E}, \wedge \sim, 1)$ be an equality algebra. A *state morphism operator* on $\mathcal{E}$ is a map $\sigma : \mathcal{E} \to \mathcal{E}$ satisfying the following condition for all $x, y \in \mathcal{E}$:

(1) $\sigma(x \sim y) = \sigma(x) \sim \sigma(y)$,
(2) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$,
(3) $\sigma(\sigma(x)) = \sigma(x)$.

The pair $(\mathcal{E}, \sigma)$ is called a *state morphism equality algebra*.

Theorem 3.12. [4] Any state morphism on a linearly ordered equality algebra $\mathcal{E}$ is an internal state on $\mathcal{E}$.

Theorem 3.13. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then the following conditions are equivalent:

(1) $(\mathcal{E}, \tau)$ is a state morphism equality algebras,
(2) $\tau = id_{\mathcal{E}}$.

Proof. (1) $\Rightarrow$ (2) We note that $(\mathcal{E}, \tau)$ is not only a state morphism equality algebras but also a very true equality algebras. In order to prove this important result, we need some useful result. First, we will prove that $x < y$ implies $\tau(x) < \tau(y)$. Assume that $x < y$, then $\tau(x) \leq \tau(y)$. If $\tau(x) = \tau(y)$, then $\tau(x \to y) = \tau(x) \to \tau(y) = 1$. By Proposition 3.4(13), we have $y \to x = 1$ and hence $y \leq x$, which is a contraction to $x < y$. Now, we will show that $\tau(x) = x$ for all $x \in \mathcal{E}$. By (VE2), we have $\tau(x) \leq x$. Assume that $\tau(x) \neq x$, then $\tau(x) < x$ and hence $\tau\tau(x) < \tau(x)$. Thus $\tau(x) < \tau(x)$, which is a contradiction. Therefore, $\tau = id_{\mathcal{E}}$.

(2) $\Rightarrow$ (1) straightforward.

Theorem 3.14. Let $(\mathcal{E}, \tau)$ be a very true linearly order equality algebra. Then the following conditions are equivalent:
(1) $(\mathcal{E}, \tau)$ is a state equality algebras,
(2) $\tau = id_{\mathcal{E}}$.

Proof. It follows from Proposition 3.4(16), we can easily to obtain this result.

4. Very true filters in very true equality algebras

In this section, we introduce very true filters of very true equality algebras and obtain some important result of them. In particular, using very true filter, we give a characterization of simple very true equality algebras and construct the uniform structures on very true equality algebras.

Definition 4.1. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. A nonempty subset F of $\mathcal{E}$ is called a *very true filter* of $(\mathcal{E}, \tau)$ if F is a filter of $\mathcal{E}$ such that if $x \in F$, then $\tau(x) \in F$ for all $x \in \mathcal{E}$.

Example 4.2. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then $\mathrm{Ker}(\tau) = \{x \in X | \tau(x) = 1\}$ is a very true filter of $(\mathcal{E}, \tau)$.

Example 4.3. Consider the Example 3.3, one can easily check that the very true filter of $(\mathcal{E}, \tau)$ are $\{1\}$, $\{a, b, 1\}$ and $\mathcal{E}$. On the other hand, one can see that $\{b, 1\}$ is not a very true filter of $(\mathcal{E}, \tau)$, but it is a filter of $\mathcal{E}$, that is to say, not every filter is very true filter.

As an application of very true filters, we give a characterizations of simple very true equality algebras. For obtaining this important result, we need the following proposition.

Proposition 4.4. Let $(\mathcal{E}, \tau)$ be a very true equality algebra.

(1) If F is a filter of $\tau(\mathcal{E})$, then $\tau^{-1}(F)$ is a very true filter of $(\mathcal{E}, \tau)$,

(2) If F is a very true filter of $(\mathcal{E},\tau)$, then $\tau(F)$ is a filter of $\tau(\mathcal{E})$.

Proof. (1) Suppose that F is a filter of $\tau(\mathcal{E})$. If $x, x\sim y\in\tau^{-1}(F)$. Then $\tau(x),\tau(x\sim y)\in F$. Since $\tau(x\sim y)\leq\tau(x)\sim\tau(y)$ and $\tau(x)\sim\tau(y)\in F$, thus $\tau(y)\in F$, that is $y\in\tau^{-1}(F)$. Let $x,y\in\mathcal{E}$ such that $x\in\tau^{-1}(F)$ and $x\leq y$. Then $\tau(x)\leq\tau(y)$. Since $\tau(x)\in F$ and $\tau(y)\in\tau(\mathcal{E})$, we can obtain that $\tau(y)\in F$, that is $y\in\tau^{-1}(F)$. Obviously, $1\in\tau^{-}(F)$. Therefore $\tau^{-1}(F)$ is a filter of $\mathcal{E}$.

If $x\in\tau^{-1}(F)$, then $\tau(x)\in F$, so $\tau\tau(x)=\tau(x)\in F$, that is, $\tau(x)\in\tau^{-1}(F)$. Therefore, $\tau^{-1}(F)$ is a very true filter of $(\mathcal{E},\tau)$.

(2) First, we have $\tau(F)=F\cap\tau(\mathcal{E})$. Indeed, if $x\in F\cap\tau(\mathcal{E})$, then $x=\tau(x)$ as $x\in\tau(\mathcal{E})$, and $\tau(x)\in\tau(F)$ as $x\in F$. Thus, we have $x\in\tau(F)$. It follows that $F\cap\tau(\mathcal{E})\subseteq\tau(F)$. Conversely, if $y\in\tau(F)$, then there exists $x\in F$ such that $y=\tau(x)$. Since F is an very true filter of $(\mathcal{E},\tau)$, we obtain $y=\tau(x)\in F$. Therefore, $\tau(F)=F\cap\tau(\mathcal{E})$.

If $x, x\rightsquigarrow y\in\tau(F)=F\cap\tau(\mathcal{E})$, then $\tau(y)\in F\cap\tau(\mathcal{E})=\tau(F)$ and hence $y\in\tau(F)$. On the other hand, if $x\in\tau(F)=F\cap\tau(\mathcal{E})$, $y\in\tau(\mathcal{E})$ such that $x\leq y$, then $y\in F\cap\tau(\mathcal{E})=\tau(F)$. Therefore, $\tau(F)$ is a filter of $\tau(\mathcal{E})$.

Now, we introduce simple very true equality algebras and give a characterizations of it via very true filter.

Definition 4.5. A very true equality algebra $(\mathcal{E},\tau)$ is called *simple very true* if it has exactly two very true filter: $\{1\}$ and $\mathcal{E}$.

Example 4.6. For each equality algebra $\mathcal{E}$, $(\mathcal{E},\tau_1)$ is a simple very true equality algebra, where $\tau_1(1)=1$ and $\tau_1(0)=0$ if $x<1$.

Theorem 4.7. Let $(\mathcal{E},\tau)$ be a very true equality algebra. Then the following are equivalent:

(1) $(\mathcal{E},\tau)$ is a simple very true equality algebra,
(2) $\tau(\mathcal{E})$ is a simple equality algebra.

Proof. (1) $\Rightarrow$ (2) Let F be a filter of $\tau(\mathcal{E})$ and $F\neq\{1\}$. It follows from Proposition 4.4(1) that $\tau^{-1}(F)$ is a very true filter of $(\mathcal{E},\tau)$. Since $(\mathcal{E},\tau)$ is very true simple, we have that $\tau^{-1}(F)=\{1\}$ or $\tau^{-1}(F)=\mathcal{E}$. Notice that $F\subseteq\tau^{-1}(F)$ (if $x\in F$, then $\tau(x)=x$, that is, $x\in\tau^{-1}(F)$, we obtain that $\tau^{-1}(F)\neq\{1\}$. Thus, $\tau^{-1}(F)=\mathcal{E}$. Then $0\in\tau^{-1}(F)$, that is, $0=\tau(0)\in F$. So we obtain that $F=\tau(\mathcal{E})$. Therefore, $\tau(\mathcal{E})$ is simple equality algebra.

(2) $\Rightarrow$ (1) Let F be a very true filter of $(\mathcal{E},\tau)$ and $F\neq\{1\}$. By Proposition 4.4(2), we obtain that $\tau(F)$ is a filter of $\tau(\mathcal{E})$. Since $\tau(\mathcal{E})$ is simple equality algebra, we obtain that $\tau(F)=\{1\}$ or $\tau(F)=\mathcal{E}$. Since $\mathrm{Ker}(\tau)=\{1\}$, we have $F\neq\{1\}$. Thus, $\tau(F)=\mathcal{E}$. Then $0\in\tau(F)$, that is, $0\in F$. It follows that $F=\mathcal{E}$. Therefore $(\mathcal{E},\tau)$ is a simple very true equality algebra.

Note. The above theorem brings a method of how to check a very true equality algebra is simple very true. As an application of above theorem, one can easily check that the very true equality algebra in example 4.6 is simple very true since $\tau(\mathcal{E})=\{0,1\}$ is a simple equality algebra.

For any very true filter F of $(\mathcal{E},\tau)$. Defined by $\tau_F:\mathcal{E}/F\longrightarrow\mathcal{E}/F$ as a mapping $\tau_F([x])=[\tau(x)]$.

Proposition 4.8. Let $(\mathcal{E},\tau)$ be a very true equality algebra and F be a very true filter of very true equality algebra $(\mathcal{E},\tau)$. Then τ_F is a very true operator on $\mathcal{E}/F$.

Proof. First, we will prove that τ_F is well defined. Indeed, assume that $[x]=[y]$ for $x,y\in\mathcal{E}$. Then $(x,y)\in\equiv_F$, i.e., $x\sim y\in F$. Since F is a very true filter and hence $\tau(x\sim y)\in F$. Now, applying (VE4) of Definition 3.1, $\tau(x\sim y)\leq\tau(x)\sim\tau(y)\in F$. Thus $(\tau(x),\tau(y))\in F$ and $[\tau(x)]=[\tau(y)]$. The rest of the proof is easy:

(VE1) $\tau_F([1])=[\tau(1)]=[1]$,
(VE2) $\tau_F([x])=[\tau(x)]\leq[x]$,
(VE3) $\tau_F([x])=[\tau(x)]\leq[\tau\tau(x)]\leq\tau_F\tau_F([x])$,
(VE4) $\tau_F([x]\sim[y])=[\tau(x\sim y)]\leq[\tau(x)\sim\tau(y)]=\tau_F([x])\sim\tau_F([y])$,
(VE5) $\tau_F([x]\wedge[y])=[\tau(x\wedge y)]=[\tau(x)\wedge\tau(y)]=\tau_F([x])\wedge\tau_F([y])$.

Combine them, one can obtain that $(\mathcal{E}/F,\tau_F)$ is a very true equality algebra.

Proposition 4.9. In the very true equality algebra $(\mathcal{E}/Ker(\tau),\tau_{Ker(\tau)})$ we have:

(1) $[x]\leq[y]$ iff $\tau(x\sim x\wedge y)=1$ iff $\tau(x\to y)=1$,
(2) $[x]=[y]$ iff $\tau(x\sim y)=1$.

Proof. (1) Applying the definition of $\mathcal{E}/Ker(\tau)$ we get: $[x]\leq[y]$ iff $[x]\bullet[y]=[x]$ iff $[x]=[x\wedge y]$ iff $[x]\rightharpoonup[x\wedge y]=[1]$ iff $[x\sim x\wedge y]=[1]$ iff $x\sim x\wedge y\in Ker(\tau)$ iff $\tau(x\sim x\wedge y)=1$ iff $\tau(x\to y)=1$.

(2) We have $[x]=[y]$ iff $[x]\rightharpoonup[y]=[1]$ iff $[x\sim y]=[1]$ iff $x\sim y\in Ker(\tau)$ iff $\tau(x\sim y)=1$.

Definition 4.10. Let $(\mathcal{E},\tau)$ be a very true equality algebra and θ be a congruence on $\mathcal{E}$. Then θ is called a *very true congruence* on $(\mathcal{E},\tau)$ if $(x,y)\in\theta$ implies $(\tau(x),\tau(y))\in\theta$ for each $x,y\in\mathcal{E}$.

Example 4.11. Consider the Example 3.4, one can see that $R=\{\{0,0\},\{a,a\},\{b,b\},\{1,1\},\{a,b\},\{b,a\},\{a,1\},\{1,a\},\{1,b\},\{b,1\}\}$ is a very true congruence on a very true equality algebra $(\mathcal{E},\tau)$.

Theorem 4.12. Let $(\mathcal{E},\tau)$ be a very true equality algebra. Then there is a one to one correspondence between its very true filters and its very true congruences.

Proof. Suppose that θ is a very true congruence relation on $(\mathcal{E},\tau)$. Clearly $F_\theta=\{x\in\mathcal{E}|(x,1)\in\theta\}$ is a very true filter of $(\mathcal{E},\tau)$. Now given $x\in F_\theta$, we have $(x,1)\in\eta$ and hence $(\tau(x),1)=(\tau(x),\tau(1))\in\theta$ and therefore $\tau(x)\in F_\theta$. This proves that F_θ is a very true filter on $(\mathcal{E},\tau)$. Conversely, let F be a very true filter. Then θ_F is a very true congruence on very true equality algebra, since for each $(x,y)\in\theta_F$, we have $x\sim y\in F$. Since F is a very true filter and hence $\tau(x\sim y)\in F$. By (VE4), we have $\tau(x)\sim\tau(y)\in\theta_F$, thus θ_F is an very true congruence of $(\mathcal{E},\tau)$. It can be easily shown that $gh(\theta)=\theta$ and $hg(F)=F$, for all very true congruence θ and very true filter F of $(\mathcal{E},\tau)$.

As another applications of very true filter on very true equality algebra, we consider the uniformity structure on a very true equality algebra.

Theorem 4.13. Let $(\mathcal{E},\tau)$ be a very true equality algebra and F be a very true filter of $(\mathcal{E},\tau)$. Define $U_F=\{(x,y)\in\mathcal{E}\times\mathcal{E}|\tau(x)\sim\tau(y)\in F\}$ and $K^*=\{U_F|F$ is a very true filter of $(\mathcal{E},\tau)\}$. Then K^* satisfies the conditions (U_1)-(U_4).

Proof. Now, we will show that K^* satisfies the conditions (U_1)-(U_4).

(U_1) Let $U_F\in K^*$ and $(x,x)\in\triangle$. Since $x\sim x=1\in F$, we have $(x,x)\in U_F$. Therefore (U_1) holds.

(U_2) Note that $(x,y)\in U_F$ if and only if $\tau(x)\sim\tau(y)\in F$ if and only if $\tau(y)\sim\tau(x)\in F$ if and only if $(y,x)\in U_F$ if and only if $(x,y)\in U_F^{-1}$. Thus $U_F^{-1}=U_F\in K^*$. Therefore (U_2) is true.

(U_3) Let $\Sigma(F)=\{F_a|F_a\subseteq F\}$ be the collection of very true filters contained in F. Clearly, $\Sigma(F)$ is not empty. Let G be the very true filter generated by $\cup_a F_a$. Then $U_G\subseteq K^*$. It is sufficient to show that $U_G\circ U_G\subseteq U_F$. If $(x,y)\in U_G\circ U_G$, then there exists $z\in\mathcal{E}$ such that $(x,z)\in U_G$ and $(z,y)\in U_G$. It follows from $(E7)$ in Definition 2.1 that $(x,y)\in U_G$, that is, $\tau(x)\sim\tau(y)\in G$. Since G is the minimal very true filter containing $\cup_a F_a$ and $\cup_a F_a\subseteq F$, it follows that $G\subseteq F$. Thus $\tau(x)\sim\tau(y)\in F$ and hence $(x,y)\in U_F$. Therefore $U_G\circ U_G\subseteq U_F$ is true.

(U_4) This will from the observation that $U_G\cap U_F=U_{G\cap F}$ for all U_G, $U_F\in K^*$. Indeed, if $(x,y)\in U_G\cap U_F$. Then $(x,y)\in U_G$ and $(x,y)\in U_F$, which implies $\tau(x)\sim\tau(y)\in G$ and $\tau(x)\sim\tau(y)\in F$. Thus $\tau(x)\sim\tau(y)\in G\cap F$ and hence $(x,y)\in U_{G\cap F}$. Similary, we can show that $U_{F\cap G}\subseteq U_F\cap U_G$, whence $U_{F\cap G}=U_F\cap U_G$. This completeness the proof.

Theorem 4.14. Let $(\mathcal{E},\tau)$ be a very true equality algebra. Define $K=\{U\in\mathcal{E}\times\mathcal{E}|U_F\subseteq U$, for some $U_F\in K^*\}$. Then K satisfies a uniformity on $(\mathcal{E},\tau)$ and hence the pair $((\mathcal{E},\tau),K)$ is a uniform space.

Proof. Using Theorem 4.13, we can show that K satisfies the conditions $(U_1)-(U_4)$. Now, we will show that it satisfies (U_5). If $U\in K$ and $U\subseteq V\subseteq\mathcal{E}\times\mathcal{E}$. Then there exists a $U_F\in K^*$ such that $U_F\subseteq U\subseteq V$, which implies that $V\in K$. Therefore,$((\mathcal{E},\tau),K)$ is a uniform space.

For $x\in\mathcal{E}$ and $U\in K$, we define

$$U[x]=\{y\in\mathcal{E}|(x,y)\in U\}.$$

Theorem 4.15. Let $(\mathcal{E},\tau)$ be a very true equality algebra. Define $T=\{G\subseteq\mathcal{E}|\tau x\in G,\exists U\in K,U[x]\subseteq G\}$. Then T is a topology on $(\mathcal{E},\tau)$.

Proof. It is clear $\emptyset\in T$ and $\mathcal{E}\in T$. Also from the nature of that definition, it is clear that T is closed under arbitrary union. Finally to show that T is closed under finite intersection, let $G,W\in T$ and suppose $x\in G\cap W$. Then there exist U and $V\in K$ such that $U[x]\subseteq G$ and $V[x]\subseteq W$. Let $N=U\cap V$, then $N\in K$. Also $N[x]\subseteq U[x]\cap V[x]$ and hence $N[x]\subseteq G\cap W$, thus $G\cap W\in T$. Therefore T is a topology on $(\mathcal{E},\tau)$.

Note We denote the uniform topology obtained by an arbitrary family Λ, by T_Λ and If $\Lambda=\{F\}$, we denote it by T_F.

Theorem 4.16. Let $(\mathcal{E},\tau)$ be a very true equality algebra. For each $x\in\mathcal{E}$, the collection $U_x=\{U[x]|U\in K\}$ forms a neighborhood base at x, making $(\mathcal{E},\tau)$ a topological space.

Proof. We note that $x \in U[x]$ for each x. Since $U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x]$, which means that the intersection of neighborhoods is a neighborhood. Finally, if $U[x] \in U_x$ then there exists a $W \in K$ such that $W \circ W \subseteq U$ by (U_3). Then for any $y \in W[x], W[y]$, so this property of neighborhoods is satisfied.

Theorem 4.17. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and $(\mathcal{E}, T_{Ker(\tau)})$ be a compact topological space. Then $\tau(\mathcal{E})$ is finite.

Proof. Since $\mathcal{E} = \cup\{U_F[x] | x \in \mathcal{E}\}$ and $(\mathcal{E}, T_F)$ is a compact topological space. Thus there exist $x_1, x_2 \cdots, x_n \in \mathcal{E}$ such that $\mathcal{E} = \cup U_F[x_i]$. Now, let x be a arbitrary element of $\mathcal{E}$. By Theorem 3.7, $\tau : \mathcal{E} \longrightarrow \tau(\mathcal{E})$ is a homomorphism and by the Note after the Theorem 3.6, $(\tau(\mathcal{E}), \rightsquigarrow, 1)$ is an equality algebra, hence we have $\tau(U_F[x]) = \{\tau(y) | y \in U_F[x]\} = \{\tau(y) | (\tau(x), \tau(y) \in \theta_F\} = \{\tau(y) | \tau(x) \sim \tau(y) \in F\} = \{\tau(y) | \tau(x) = \tau(y)\} = \{\tau(x)\}$. Thus $\tau(U_F[x]) = \{\tau(x)\}$ and so $Fix_\tau = \cup U_F[x_i] = \{\tau(x_1), \cdots, \tau(x_n)\}$, that is, $\tau(\mathcal{E})$ is finite.

Corollary 4.18. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and $(\mathcal{E}, T_{Ker(\tau)})$ be a compact topological space. Then $\tau(\mathcal{E})$ is a compact subset of $\mathcal{E}$.

Proof. It follows from Theorem 4.17 directly and hence we omit it.

5. Conclusion

As we mentioned in the introduction, the study of equality algebras is motivated by the goal to develop appropriate algebraic semantics for fuzzy type theory, so a concept of fuzzy type theory should be introduced based on these algebras. In this paper, motivating by the previous research on very true residuated lattice, very true MV-algebras and very true BL-algebras, we extended the concept of very true operators to the more general fuzzy structures, namely equality algebras. We introduce and study very true equality algebras and prove some new results regarding these structures. The aim of this paper is to provide an algebraic foundation for fuzzy type theory. Since the above topics are of current interest we suggest further directions of research:

1. Characterize very true filter generated by a subset of an very true equality algebra in terms of fuzzy equality operation.

2. Define and characterize subdirectly irreducible very true equality algebras.

3. Establish the logic system corresponding to very true equality algebra and prove the soundness and completeness theorem of this logic.

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A Splitting Iterative Method for a System of Accretive Inclusions in Banach Spaces

Birendra Kumar Sharma[1], Niyati Gurudwan[2], Avantika Awadhiya[3] and Shin Min Kang[4,5,*]

[1]School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur 492010, India
e-mail: sharmabk07@gmail.com

[2]Department of Mathematics, Govt. Rajeev Lochan College, Rajim, Gariaband 493885, India
e-mail: niyati.kuhu@gmail.com

[3]School of studies in Mathematics, Pt. Ravishankar Shukla University, Raipur 492010, India
e-mail: avantika.awadhiya@gmail.com

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

Abstract

In this paper, a system of accretive inclusions is proposed and a splitting iterative method is investigated for solutions of proposed system of operator inclusion problems. Under suitable conditions on the parameters, strong convergence of our splitting iterative method is established in a reflexive Banach space.

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1 Introduction

In the area of nonlinear analysis, the theory of accretive operators is an important and developing field [3, 4]. The class of accretive operators is firmly connected with equations of evolutions found in the heat, wave, Schrödinger and similar other equations [5]. Many problems in operations research and mathematical physics can be written as variational inequalities, equilibrium problems or operator inclusions with accretive operators [2, 10, 17].

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. One popular method for solving the following inclusion problem:

$$\text{find } z \in H \text{ such that } 0 \in Az, \tag{1.1}$$

*Corresponding author

where $A : H \to 2^H$ is an m-accretive operator, is the proximal point algorithm, which was proposed by Martinet [15, 16] and generalized by Rockafellar [20, 21]. Rockafellar [20] proved the weak convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = J^A_{r_n} x_n, \quad n \in \mathbb{N} \tag{1.2}$$

to an element of solution set of problem (1.1). The weak and strong convergence of the sequence $\{x_n\}$ defined by (1.2) have been extensively discussed in Hilbert and Banach spaces (see [7, 28, 29, 30, 31] and the references therein). The proximal point like methods for finding solutions of problem (1.1) have been studied by Lehdili and Moudafi [12] and Tossings [26] in Hilbert spaces and by Sahu and Yao [23] in Banach spaces.

Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning can be mathematically modeled in form of inclusion problem:

to find $z \in C$ such that

$$0 \in (A + B)z, \tag{1.3}$$

where C is a nonempty closed convex subset of H, $A : H \to 2^H$ and $B : C \to H$ are monotone operators. For instance, a stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Fu, \quad u_0 = u(0)$$

can be recast as (1.3) when the governing maximal monotone F is of the form $F = A+B$, see, for example, [13].

The central problem is to iteratively find the solution of the inclusion problem (1.3) when A and B are two monotone operators in a Hilbert space H. One method for finding solutions of problem (1.3) is *splitting method*. A splitting method for (1.3) means an iterative method for which each iteration involves only with the individual operators A and B, but not the sum $A + B$. Splitting methods for linear equations were introduced by Douglas and Rachford [8] and Peaceman and Rachford [18]. Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg [9] and Lions and Mercier [13] (see also [22, 27]).

In this paper, we are interested in the following system of operator inclusion problems:

$$\text{find } z \in C \text{ such that } 0 \in (A_i + B_i)z, \quad i \in \Delta_N := \{1, 2, \cdots, N\}, \tag{1.4}$$

in the framework of a Banach space X, where $N \geq 1$ is a positive integer, C is a nonempty closed convex subset of X, $A_i : X \to 2^X$ is an m-accretive operator such that $\bigcap_{i\in\Delta_N} \overline{D(A_i)} \subseteq C$ and $B_i : C \to X$ an operator. The inclusion problem (1.4) is more general in nature. For instance, if B_i is the operator constantly zero for all $i \in \Delta_N$, the problem (1.4) reduces

$$\text{find } z \in C \text{ such that } 0 \in A_i z, \quad i \in \Delta_N. \tag{1.5}$$

The purpose of this paper is to introduce a forward-backward splitting method to solve the system of operator inclusion problem (1.4) in a Banach space. We prove strong convergence of iterative sequences generated by our algorithm. In Section 2, we give duality mappings, nonexpansive mappings and their properties and accretive operators and their properties. In Section 3, we introduce a forward-backward splitting method and state main theoretical result of the paper. Our iterative method improves and generalizes the corresponding results of inclusion problem (1.5).

2 Preliminaries

2.1 Duality mappings

A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The mapping $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*,\ \|j\|_* = \varphi(\|x\|)\}, \quad x \in X,$$

is called the duality mapping with gauge φ. In the special case where $\varphi(t) = t$, the duality mapping $J_\varphi =: J$ is the classical normalized duality mapping. In the case $\varphi(t) = t^{p-1}$, $p > 1$, the duality mapping $J_\varphi =: J_p$ is called the generalized duality mapping and it is given by

$$J_p(x) := \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \|x\|^{p-1}\}, \quad x \in X.$$

Note that if $p = 2$, then $J_2 = J$ is the normalized duality mapping. By φ we always mean a gauge and by Φ the corresponding function defined by

$$\Phi(t) = \int_0^t \varphi(s)ds.$$

For a smooth Banach space X, we have

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle \quad \text{for all } x, y \in X; \tag{2.1}$$

or considering the normalized duality mapping J, we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle \quad \text{for all } x, y \in X.$$

2.2 Nonexpansive mappings

Let C be a nonempty subset of a Banach space X and $T : C \to X$ a mapping. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

The set of fixed point of T is denoted by $Fix(T)$.

The following result was proved by Kirk [11].

Lemma 2.1. (Kirk [11]) Let X be a reflexive Banach space and let C be a nonempty closed convex bounded subset of X which has normal structure. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point.

A subset C of a Banach space X is called a retract of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a *retraction* of X onto C. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each x in X and $t \ge 0$. If a sunny retraction P is also nonexpansive, then C is said to be a *sunny nonexpansive* retract of X.

The following lemmas will be useful for our main result.

Lemma 2.2. ([1, Corollary 5.6.4]) Let X be a Banach space with a weakly continuous duality mapping $J_\varphi : X \to X^*$ with gauge function φ. Let C be a nonempty closed convex subset of X and $T : C \to C$ an nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $\{x_n - Tx_n\}$ converges strongly to zero, then $x - Tx = 0$.

Lemma 2.3. ([6]) Let X be a strictly convex Banach space. Let C be a nonempty closed convex subset of X and let $N \geq 1$ be a positive integer. For each $i \in \Delta_N$, let $T_i : C \to C$ be a nonexpansive mapping such that $\bigcap_{i\in\Delta_N} Fix(T_i) \neq \emptyset$. Let $\{\delta_i\}_{i\in\Delta_N} \subset (0, 1)$ such that $\sum_{i=1}^N \delta_i = 1$. Then the mapping $\sum_{i=1}^N \delta_i T_i$ is nonexpansive with $Fix(\sum_{i=1}^N \delta_i T_i) = \bigcap_{i\in\Delta_N} Fix(T_i)$.

Lemma 2.4. *Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C and P a retraction from C onto D. Then the following are equivalent:*

(a) P is a sunny and nonexpansive.

(b) $\langle x - Px, J(z - Px)\rangle \leq 0$ for all $x \in C, z \in D$.

(c) $\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.

Lemma 2.5. *Let X be a reflexive Banach space which has a weakly continuous duality map J_φ. Let C be a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $Fix(T)$ is the sunny nonexpansive retract of C.*

The property $(\mathcal{N})$ alludes to the fact that in order to solve the system of operator inclusions (1.4).

Definition 2.6. ([22]) Let C be a nonempty closed convex subset of a Banach space X. An operator $B : C \to X$ is said to satisfy the *property* $(\mathcal{N})$ on $(0, \gamma_{X,B})$ if there exists $\gamma_{X,B} \in (0, \infty]$, depends on X and B, such that $I - \xi B : C \to C$ is nonexpansive for each $\xi \in (0, \gamma_{X,B})$.

Remark 2.7. For a nonexpansive mapping $T : C \to C$ with $B = I - T$, the average mapping $T_w = I - wB$ is always nonexpansive for each $w \in (0, \gamma_{X,B})$, where $\gamma_{X,B} = 1$.

2.3 Accretive operators

Let X be a real Banach space. For an operator $A : X \to 2^X$, we define its domain, range and graph as follows:

$$D(A) = \{x \in X : Ax \neq \emptyset\},$$

$$R(A) = \bigcup\{Az : z \in D(A)\},$$

and

$$G(A) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\},$$

respectively. Thus, we write $A : X \to 2^X$ as follows: $A \subseteq X \times X$. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y \text{ if and only if } y \in Ax.$$

The operator A is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator A is said

to be *maximal accretive* if there is no proper accretive extension of A and *m-accretive* if $R(I+A) = X$, where I stands for the identity operator on X (It follows that $R(I+rA) = X$ for all $r > 0$). If A is m-accretive, then it is maximal accretive, but the converse is not true in general. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A : R(1 + \lambda A) \to D(A)$ by

$$J_\lambda^A = (I + \lambda A)^{-1}.$$

It is called the *resolvent* of A. It is well known that if A is an m-accretive operator on a Banach space X, then for each $\lambda > 0$, the resolvent $J_\lambda^A = (I + \lambda A)^{-1}$ is a single-valued nonexpansive mapping whose domain is entire space X. An accretive operator A defined on a Banach space X is said to satisfy the *range condition* if $\overline{D(A)} \subset R(1+\lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. It is well known that for an accretive operator A which satisfies the range condition, $A^{-1}(0) = Fix(J_\lambda^A)$ for all $\lambda > 0$. We also define the *Yosida approximation* A_r by $A_r = (I - J_r^A)/r$. We know that $A_r x \in AJ_r^A x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq |Ax| = \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know the following [25]: For each $\lambda, \mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, it holds that

$$\left\|J_\lambda^A x - J_\mu^A x\right\| \leq \frac{|\lambda - \mu|}{\lambda}\|x - J_\lambda^A x\|.$$

Lemma 2.8. ([22]) *Let C be a nonempty closed convex subset of a Banach space X, $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$ and $B : C \to X$ an operator such that $Zer(A+B) \neq \emptyset$ and B has the property ($\mathcal{N}$) on $(0, \gamma_{X,B})$, where $\gamma_{X,B}$ is a constant depends on X and B. For $r \in (0, \gamma_{X,B})$, define an operator $J_r^{A,B} : C \to C$ by*

$$J_r^{A,B} x = J_r^A (I - rB)x, \quad x \in C.$$

Then the following statements hold.

(a) $J_r^{A,B}$ is nonexpansive.

(b) $Fix(J_r^{A,B}) = Zer(A + B)$.

Lemma 2.9. ([24]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n y_n \quad \text{for all } n \in \mathbb{N}$$

and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.10. ([14]) *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers and let $\{b_n\}$ be a sequence in $\mathbb{R}$ satisfying the following condition:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following statements hold:

(*a*) *If* $b_n \leq K\alpha_n$ *for all* $n \in \mathbb{N}$ *and for some* $K \geq 0$, *then*

$$a_{n+1} \leq \delta_n a_1 + (1-\delta_n)K + \sum_{j=1}^{n} c_j \quad \text{for all } n \in \mathbb{N},$$

where $\delta_n = \prod_{j=1}^{n}(1-\alpha_j)$ *and hence* $\{a_n\}$ *is bounded.*

(*b*) *If* $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\limsup_{n\to\infty}\frac{b_n}{\alpha_n} \leq 0$, *then* $\{a_n\}_{n=1}^{\infty}$ *converges to zero.*

3 Main results

Now we are ready to prove our main result for solving the system of operator inclusions (1.4) in the framework of Banach space.

Theorem 3.1. *Let* X *be a strictly convex and reflexive Banach space which has a weakly continuous duality map* J_φ. *Let* C *be a nonempty closed convex subset of* X *and let* $N \geq 1$ *be a positive integer. Let* $f : C \to C$ *be a contraction mapping with Lipschitz constant* k_f. *For each* $i \in \Delta_N$, *let* $A_i : X \to 2^X$ *be an* m*-accretive operator such that* $\bigcap_{i\in\Delta_N}\overline{D(A_i)} \subseteq C$ *and* $B_i : C \to X$ *an operator such that* $\mathbb{S} := \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \neq \emptyset$. *For each* $i \in \Delta_N$, *let* B_i *has the property* $(\mathcal{N})$ *on* $(0, \gamma_{X,B_i})$, *where* γ_{X,B_i} *is a constant depends on* X *and* B_i.

Let $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *be real number sequences in* $(0,1)$ *and let* $\{\delta_{n,i}\}$ *be a real number sequence in* $(0,1)$ *for each* $i \in \Delta_N$ *satisfying the following conditions*:

$(C1)$ $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{N}\delta_{n,i} = 1$ *for all* $n \in \mathbb{N}$,

$(C2)$ $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

$(C3)$ $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

$(C4)$ $\lim_{n\to\infty}\delta_{n,i} = \delta_i \in (0,1)$ *for all* $i \in \Delta_N$.

Let $\{x_n\}$ *be a sequence in* C *generated by the following splitting iterative method:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{n=1}^{N}\delta_{n,i} J_{r_i}^{A_i}(I - r_i B)x_n \quad \text{for all } n \in \mathbb{N}, \tag{3.1}$$

where $\{r_i\}_{i\in\Delta_N}$ *is a set of positive real numbers. Then* $\{x_n\}$ *converges strongly to* $x^* \in C$, *which is the unique solution to the following variational inequality:*

$$\text{to find } z \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \text{ such that } \langle (I-f)z, x - z\rangle \geq 0 \tag{3.2}$$

for all $x \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i)$.

Proof. (*a*) Define $T = \sum_{i=1}^{N}\delta_i J_{r_i}^{A_i}(I - r_i B)$. From Lemmas 2.3 and 2.8, we see that T is nonexpansive with $Fix(T) = \bigcap_{i\in\Delta_N} Zer(A_i + B_i)$. From Lemma 2.5 shows that $\mathbb{S}$ is the Sunny nonexpansive retract of C. Let $Q_{\mathbb{S}}$ be the sunny nonexpansive retraction of C onto $\mathbb{S}$. It follows that $Q_{\mathbb{S}}f$ is a contraction. Hence there exists a unique fixed point $x^* \in C$ of $Q_{\mathbb{S}}f$. From Lemma 2.4 that the variational inequality problem (3.2) has a unique solution $x^* \in C$.

(*b*) We proceed with the following steps:

Step I: $\{x_n\}$ *is bounded.*

From (3.1), we have

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\left\|\sum_{i=1}^{N}\delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n-x^*\right\|\\ &\le k_f\alpha_n\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|\\ &\quad+\gamma_n\sum_{i=1}^{N}\delta_{n,i}\|J_{r_i}^{A_i}(I-r_iB)x_n-x^*\|\\ &\le k_f\alpha_n\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\ &= (1-(1-k_f)\alpha_n)\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|\\ &\le \max\left\{\|x_n-x^*\|,\frac{\|f(x^*)-x^*\|}{1-k_f}\right\}\\ &\le \max\left\{\|x_1-x^*\|,\frac{\|f(x^*)-x^*\|}{1-k_f}\right\}.\end{aligned}$$

Thus, $\{x_n\}$ is bounded.

STEP II: $\|x_{n+1}-x_n\|\to 0$ *and* $\|x_n-Tx_n\|\to 0$ *as* $n\to\infty$.

Set $y_n=\sum_{n=1}^{N}\delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n$. Note

$$\begin{aligned}\|y_{n+1}-y_n\| &= \left\|\sum_{i=1}^{N}\delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_{n+1}-\sum_{i=1}^{N}\delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n\right\|\\ &\le \left\|\sum_{i=1}^{N}\delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_{n+1}-\sum_{i=1}^{N}\delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_n\right\|\\ &\quad+\left\|\sum_{i=1}^{N}\delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_n-\sum_{i=1}^{N}\delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n\right\|\\ &\le \sum_{i=1}^{N}\delta_{n+1,i}\|J_{r_i}^{A_i}(I-r_iB)x_{n+1}-J_{r_i}^{A_i}(I-r_iB)x_n\|\\ &\quad+\left\|\sum_{i=1}^{N}(\delta_{n+1,i}-\delta_{n,i})J_{r_i}^{A_i}(I-r_iB)x_n\right\|\\ &\le \|x_{n+1}-x_n\|+\sum_{i=1}^{N}|\delta_{n+1,i}-\delta_{n,i}|\ \|J_{r_i}^{A_i}(I-r_iB)x_n\|.\end{aligned}$$

From (3.1), we have

$$\begin{aligned}x_{n+1} &= \alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n\\ &= \beta_n x_n+(1-\beta_n)z_n,\end{aligned}$$

where

$$z_n=\frac{1}{1-\beta_n}\left[\alpha_n f(x_n)+\gamma_n y_n\right].$$

Hence

$$\begin{aligned} z_{n+1} - z_n &= \frac{1}{1-\beta_{n+1}} \left[\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} y_{n+1}\right] - \frac{1}{1-\beta_n} \left[\alpha_n f(x_n) + \gamma_n y_n\right] \\ &= \frac{1}{1-\beta_{n+1}} \left[\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1}) y_{n+1}\right] \\ &\quad - \frac{1}{1-\beta_n} \left[\alpha_n f(x_n) + (1-\alpha_n-\beta_n) y_n\right] \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left[f(x_{n+1}) - y_{n+1}\right] - \frac{\alpha_n}{1-\beta_n} \left[f(x_n) - y_n\right] + y_{n+1} - y_n. \end{aligned}$$

Note

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n} \|f(x_n) - y_n\| + \|y_{n+1} - \gamma_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n} \|f(x_n) - y_n\| \\ &\quad + \|x_{n+1} - x_n\| + \sum_{i=1}^{N} |\delta_{n+1,i} - \delta_{n,i}| \ \|J_{r_i}^{A_i}(I - r_i B) x_n\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n} \|f(x_n) - y_n\| \\ &\quad + \sum_{i=1}^{N} |\delta_{n+1,i} - \delta_{n,i}| \|J_{r_i}^{A_i}(I - r_i B) x_n\|. \end{aligned}$$

From the conditions $(C1)$-$(C4)$, we get

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.9, we obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $x_{n+1} - x_n = (1-\beta_n)(z_n - x_n)$, we have

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|z_n - x_n\| \leq \|z_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Observe that

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - T x_n\| + \beta_n \|x_n - T x_n\| \\ &\quad + \gamma_n \left\| \sum_{i=1}^{N} (\delta_{n,i} - \delta_i) J_{r_i}^{A_i}(I - r_i B) x_n \right\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - T x_n\| + \beta_n \|x_n - T x_n\| \\ &\quad + \gamma_n \sum_{i=1}^{N} |\delta_{n,i} - \delta_i| \ \|J_{r_i}^{A_i}(I - r_i B) x_n\|, \end{aligned}$$

which implies that

$$(1-\beta_n)\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + \gamma_n \sum_{i=1}^{N} |\delta_{n,i} - \delta_i| \ \|J_{r_i}^{A_i}(I - r_i B)x_n\|.$$

Thus, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

STEP III: $\{x_n\}$ *converges strongly to* x^*.

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J_\varphi(x_n - x^*)\rangle = \lim_{i\to\infty} \langle f(x^*) - x^*, J_\varphi(x_{n_i} - x^*)\rangle.$$

Since X is reflexive, we may further assume that $x_{n_i} \rightharpoonup z$ for some $z \in C$. It follows from Lemma 2.2 that $z \in Fix(T)$. From the weak continuity of the duality mapping J_φ and (3.2) we obtain that

$$\begin{aligned}\limsup_{n\to\infty} \langle f(x^*) - x^*, J_\varphi(x_n - x^*)\rangle &= \lim_{i\to\infty} \langle f(x^*) - x^*, J_\varphi(x_{n_i} - x^*)\rangle \\ &= \langle f(x^*) - x^*, J_\varphi(z - x^*)\rangle \\ &\leq 0.\end{aligned}$$

From (2.1) and (3.1), we have

$$\begin{aligned}&\Phi(\|x_{n+1} - x^*\|) \\ &= \Phi(\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - x^*\|) \\ &\leq \Phi(\|\alpha_n(f(x_n) - f(x^*) + \beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \alpha_n(f(x^*) - x^*)\|) \\ &\leq \Phi(\|\alpha_n(f(x_n) - f(x^*) + \beta_n(x_n - x^*) + \gamma_n(y_n - x^*)\|) \\ &\quad + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\ &\leq \Phi(\alpha_n\|f(x_n) - f(x^*)\| + \beta_n\|x_n - x^*\| + \gamma_n\|y_n - x^*\|) \\ &\quad + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\ &\leq \Phi((1 - (1 - k_f)\alpha_n)\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\ &\leq (1 - (1 - k_f)\alpha_n)\Phi(\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle.\end{aligned}$$

Noticing that $\limsup_{n\to\infty}\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \leq 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Therefore, we conclude from Lemma 2.10 that $\Phi(\|x_n - x^*\|) \to 0$, that is, $\{x_n\}$ converges strongly to x^*. □

Theorem 3.1 is more general in nature due to the property $(\mathcal{N})$ of operators B_i, therefore, we are able to derive the some new and known results from it. To demonstrate the wide range of applicability of our convergence theory, a few examples are detailed below. In particular for $B_i = 0$, we immediately obtain an improvement upon Qing and Lv [19, Theorem 2.1] as follows:

Corollary 3.2. *Let X be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ. Let C be a nonempty closed convex subset of X and let $N \geq 1$ be a positive integer. Let $f : C \to C$ be a contraction mapping with Lipschitz constant k_f. For each $i \in \Delta_N$, let $A_i : X \to 2^X$ be an m-accretive operator such that $\bigcap_{i\in\Delta_N} \overline{D(A_i)} \subseteq C$ such that $\bigcap_{i\in\Delta_N} A_i^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ and let $\{\delta_{n,i}\}$ be a real number sequence in $(0, 1)$ for each $i \in \Delta_N$ satisfying the conditions $(C1)$-$(C4)$.*

Let $\{x_n\}$ be a sequence in C generated in the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{n=1}^{N} \delta_{n,i} J_{r_i}^{A_i} x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{r_i\}_{i\in\Delta_N}$ is a set of positive real numbers. Then $\{x_n\}$ converges strongly to $x^* \in C$, which is the unique solution to the following variational inequality:

$$\text{to find } z \in \bigcap_{i\in\Delta_N} A_i^{-1}(0) \text{ such that } \langle (I - f)z, x - z\rangle \geq 0$$

for all $x \in \bigcap_{i\in\Delta_N} A_i^{-1}(0)$.

Theorem 3.3. *Let X be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ. Let C be a nonempty closed convex subset of X and let $N \geq 1$ be a positive integer. Let $f : C \to C$ be a contraction mapping with Lipschitz constant k_f. For each $i \in \Delta_N$, let $A_i : X \to 2^X$ be an m-accretive operator such that $\bigcap_{i\in\Delta_N} \overline{D(A_i)} \subseteq C$ and $T_i : C \to C$ a nonexpansive with $B_i = I - T_i$ such that $\mathbb{S} := \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \neq \emptyset$.*

Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ and let $\{\delta_{n,i}\}$ be a real number sequence in $(0, 1)$ for each $i \in \Delta_N$ satisfying the conditions $(C1)$-$(C4)$.

Let $\{x_n\}$ be a sequence in C generated in the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{n=1}^{N} \delta_{n,i} J_{r_i}^{A_i}(I - r_i B_i) x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{r_i\}_{i\in\Delta_N}$ is a set of positive real numbers. Then $\{x_n\}$ converges strongly to $x^* \in C$, which is the unique solution to the following variational inequality:

$$\text{to find } z \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \text{ such that } \langle (I - f)z, x - z\rangle \geq 0$$

for all $x \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i)$.

Proof. Note each T_i is nonexpansive with $B_i = I - T_i$. It follows from Remark 2.7 that the average mapping $T_w^{(i)} = I - wB_i$ is always nonexpansive for each $w \in (0, \gamma_{X,B_i})$, where $\gamma_{X,B_i} = 1$. Therefore, Theorem 3.3 follows from Theorem 3.1. □

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Sidi-Israeli Quadrature Method for Steady-State Anisotropic Field Problems by Direct Domain Mapping*

Xin Luo † Jin Huang‡ Tai-Song Xiong§

Abstract

In this paper, the two-dimensional steady-state anisotropic field problems are transformed into the Laplace equation by direct domain mapping, and then the Sidi-Israeli quadrature method is applied to solve the weakly singular boundary integral equation of the Laplace equation. Especially, the kress's variable transformation is used for the polygon case in order to improve the accuracy by smoothing the singularities of the exact solution at the corner points of the boundary. The convergence and error analysis of numerical solutions are given by use of collective compact theory. At last, numerical examples are tested and results verify the theoretical analysis.

Keyword: Boundary integral equation, singularity, variable transformation, convergence

1 Introduction

Consider an anisotropic medium in domain $\Omega \in R^2$ bounded by its boundary $\Gamma = \cup_{j=1}^{m}\Gamma_j$ $(m \geq 1)$ which may consist of m segments each being sufficiently smooth (in the sense of Liapunov). In the absence of heat sources, the equation governing steady-state heat conduction with Dirichlet condition can be described as (see as [1, 2, 5])

$$\begin{cases} \kappa_{ij}\dfrac{\partial^2 u(x)}{\partial x_i \partial x_j} = 0, \quad (i,j=1,2), \\ u(x) = g, \quad x \in \Gamma, \end{cases} \tag{1.1}$$

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†College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, P.R. China, corresponding author: luoxin919@163.com

‡School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, P.R. China

§College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, P.R. China

where u represents the temperature, and $\kappa = \{\kappa_{ij}\}_{1\le i,j\le 2}$ denotes the thermal conductivity matrix which satisfies the symmetry ($\kappa_{12} = \kappa_{21}$) and positive-definite ($|\kappa| = \kappa_{11}\kappa_{22} - \kappa_{12}^2 > 0$) conditions.

Using the following coordinate transformation [2]

$$\hat{x}_1 = \frac{\sqrt{|\kappa|}}{\kappa_{11}} x_1, \quad \hat{x}_2 = x_2 - \frac{\kappa_{12}}{\kappa_{11}} x_1, \tag{1.2}$$

Eq. (1.1) can be written as the 'isotropic' Laplacian form in a mapped plane in the transformed $\hat{x}_i-$system

$$\frac{\partial^2 u(\hat{x})}{\partial \hat{x}_1^2} + \frac{\partial^2 u(\hat{x})}{\partial \hat{x}_2^2} = 0, \quad \hat{x} = (\hat{x}_1, \hat{x}_2) \in \Omega'. \tag{1.3}$$

Then by single-layer potential theory [8], Eq. (1.3) can be converted into the following weakly singular boundary integral equation (BIE):

$$-\frac{1}{2\pi} \int_{\Gamma'} v(\hat{x}) \ln|\hat{x} - \hat{y}| \mathrm{d}s_{\hat{x}} = g(\hat{y}), \quad \hat{y} \in \Gamma', \tag{1.4}$$

where Γ' is the boundary of the transformed domain Ω'. The solution of Eq. (1.4) exists and is unique as long as $C_{\Gamma'} \neq 1$, where $C_{\Gamma'}$ is the logarithmic capacity [11, 12]. As soon as $v(\hat{x})$ is solved from (1.4), the solution $u(\hat{x})$ of the problem (1.3) can be calculated by the following

$$u(\hat{y}) = -\frac{1}{2\pi} \int_{\Gamma'} v(\hat{x}) \ln|\hat{x} - \hat{y}| \mathrm{d}s_{\hat{x}}, \quad \hat{y} \in \Omega'. \tag{1.5}$$

Finally, using the inverse transformation of (1.2)

$$x_1 = \frac{\kappa_{11}}{\sqrt{|\kappa|}} \hat{x}_1, \quad x_2 = \hat{x}_2 + \frac{\kappa_{12}}{\sqrt{|\kappa|}} \hat{x}_1, \tag{1.6}$$

we can obtain the solution $u(x)$ of the problem (1.1).

As far as we know, the most popular numerical methods for engineering problems include, for example, the finite element method (FEM) [17], the finite difference method (FDM) [15], and the boundary element method (BEM) [11, 12]. The former two, used frequently in numerical modeling, are referred to as domain solution techniques and require full discretization of the whole domain and are often computationally costly and mathematically tricky in the volume mesh generation [1]. The BEM has been recognized as an efficient computational method that only the boundary needs to be modeled and owing the high approximation. For the application of BEM for the steady-state anisotropic heat conduction problems (1.1), various types of elements, namely constant, continuous and discontinuous linear elements and continuous and discontinuous quadratic elements has been investigated in the literature [5]. In this article, the Sidi-Israeli quadrature formula [3] is applied to calculate weakly singular integrals. Especially for the case of closed curved polygons Γ', we use the

Kress's variable transformation [4] to overcome the corner singularities and improve the accuracy at the boundary corners.

This paper is organized as follows: in Section 2, the convergence and error analysis are carried out based on the theory of collectively compact operators [6, 7] for closed smooth boundaries. The Kress's variable transformation is introduced to overcome the corner singularities for curved polygons in Section 3. Numerical examples are provided to verify the theoretical results in Section 4, and some useful conclusions are made in Section 5.

2 Sidi-Israeli quadrature method for boundary integral equation on smooth domain

Suppose that the boundary Γ' $(= \partial\Omega')$ to be a smooth closed curve and assume that the curve Γ' can be parameterized by $\hat{y}(t) = \varphi(t) = (\varphi_1(t), \varphi_2(t)) : [0, 2\pi) \to \partial\Omega'$. Then Eq. (1.4) can be written as

$$g(s) = -\frac{1}{2\pi}\int_0^{2\pi} \ln\big|\varphi(s) - \varphi(t)\big| v(t)\mathrm{d}t, \tag{2.1}$$

where $v(t) = |\varphi'(t)|v(\varphi(t))$ and $g(t) = g(\varphi(t))$ are periodic functions with period 2π.

In order to achieve high accuracy for numerical computation of finite-range integrals with weakly singular kernels, the following lemma about Sidi's quadrature formula is introduced.

Lemma 2.1. [3] Assume that the functions $H_1(t,\tau)$ and $H_2(t,\tau)$ are 2ℓ times differentiable on $[0, 2\pi]$. Assume also that the functions $H(t,\tau)$ are periodic with period $T = 2\pi$, and that they are 2ℓ times differentiable on $\widetilde{R} = (-\infty,\infty)\backslash\{\tau + kT\}_{k=-\infty}^{\infty}$. If $a(t,\tau) = H_1(t,\tau)\ln|t-\tau| + H_2(t,\tau)$, then

$$Q_n[a(t,\tau)] = h\sum_{\substack{j=0\\ t_j\neq\tau}}^{n} a(t_j,\tau) + H_2(t_j,\tau)h + \ln(\frac{h}{2\pi})H_1(t_j,\tau)h,\ h = \frac{2\pi}{n},\ t_j = jh,$$

and

$$E_n[a(t,\tau)] = 2\sum_{\mu=1}^{\ell-1}\frac{\zeta'(-2\mu)}{(2\mu)!}\frac{\partial^{2\mu}}{\partial\tau^{2\mu}}\big(H_1(t_j,\tau)\big)h^{2\mu+1} + O\left(h^{2\ell}\right),\quad \text{as}\quad h\to 0,$$

where $\zeta(z)$ is a Riemann function [9, 13] and $E_n[a(t,\tau)] = \int_0^{2\pi} a(t,\tau)dt - Q_n[a(t,\tau)]$.

Define the integral operator

$$(Lv)(s) = \int_0^{2\pi} l(s,t)v(t)\mathrm{d}t, \tag{2.2}$$

with the kernel

$$(l(s,t) = -\frac{1}{2\pi}\ln\big|\varphi(s) - \varphi(t)\big|,$$

and construct the Nyström operator by the Sidi's quadrature formula

$$(L^h v)(s) = -\frac{h}{2\pi}\big[\ln\big(\frac{h}{2\pi}\big) + \ln|\varphi'(s)|\big]v(s) + h\sum_{\substack{j=0\\ t_j\neq s}}^{n-1} l(s,t_j)v(t_j),\ t_j = jh,\ h = \frac{2\pi}{n}.$$

Let $V^h = \text{span}\{e_0(s), e_1(s), \cdots, e_n(s)\} \subset C[0, 2\pi)$ be a piecewise linear function subspace with nodes $\{s_j\}_{j=0}^n$, where $e_i(s)$ is the basis function satisfying $e_i(s_j) = \delta_{ij}$. Define a prolongation operator $P^h : \mathfrak{R}^n \to V^h$ and a restricted operator Q^h satisfying

$$\begin{cases} P^h v = v \cdot e,\ v = (v_0, \cdots, v_n),\ e = (e_0, \cdots, e_n) \in \mathfrak{R}^n, \\ Q^h v = (v(s_0), \cdots, v(s_n)) \in \mathfrak{R}^n,\ \ v \in C[0.2\pi). \end{cases}$$

Then Eq. (2.1) and its approximation equation are

$$\begin{cases} Lv = g \\ L^h v^h = Q^h g \end{cases}$$

where $L^h = [l_{ij}]_{i,j=0}^{n-1}$ and the entries are

$$l_{ij} = \begin{cases} hl(s_i, t_j), & i \neq j, \\ -\frac{h}{2\pi}\ln\big(\frac{h|\varphi'(t_i)|}{2\pi}\big), & i = j. \end{cases}$$

Define the following integral operator

$$(A_0 v)(s) = \int_0^{2\pi} a_0(s,t)v(t)\mathrm{d}t, \tag{2.3}$$

with the kernel

$$a_0(s,t) = -\frac{1}{2\pi}\ln\big|2e^{-1/2}\sin\frac{s-t}{2}\big|.$$

Let $L - A_0 = A_1$, then the integral equation (1.4) can be split into a singularity part and a compact perturbation part

$$A_0 v + A_1 v = g, \tag{2.4}$$

where $(A_1 v)(s) = (v(.), a_1(s,.))_{L^2}$ with the kernel

$$a_1(s,t) = \begin{cases} -\frac{1}{2\pi}\ln\big|e^{1/2}\frac{\varphi(s)-\varphi(t)}{2\sin\frac{s-t}{2}}\big|, & s - t \neq 2\pi Z, \\ \\ -\frac{1}{2\pi}\ln\big(e^{1/2}|\varphi'(s)|\big), & s - t = 2\pi Z. \end{cases}$$

Now we construct the approximations of A_0 and A_1. For the logarithmically singular operators A_0, by the Sidi's quadrature formula [3], we can construct the Fredholm approximation

$$(A_0^h v)(s) = h\sum_{j=0}^{n-1} a_0(s,t_j)v(t_j), \tag{2.5}$$

where

$$a_0(s,t_j)=\begin{cases} -\dfrac{1}{2\pi}\ln\left|2e^{-1/2}\sin\left(\dfrac{s-t_j}{2}\right)\right| & s\neq t_j,\\ -\dfrac{1}{2\pi}\ln\left(\dfrac{e^{-1/2}h}{2\pi}\right) & s=t_j,\end{cases} \tag{2.6}$$

which has the following error bounds:

$$(A_0^h v)(s_i)-(A_0 v)(s_i)=\frac{-2}{\pi}\sum_{\mu=1}^{2l-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[v(s)]^{(2\mu)}\Big|_{s=s_i}h^{2\mu+1}+O(h^{2l}). \tag{2.7}$$

For the integral operators A_1 with periodic kernels, we can construct the Nyström approximation by the trapezoidal rule [8],

$$(A_1^h v)(s)=h\sum_{i=0}^{n-1}a_1(s,t_j)v(t_j)\quad j=0,1,\cdots,n-1,$$

which has the error bounds $O\left(h^{2l}\right)$, $l\in N$.

Consider the discrete approximation of (2.4)

$$(A_0^h+A_1^h)v^h=g^h, \tag{2.8}$$

where $v^h=(v_0^h,v_1^h,\cdots,v_{n-1}^h)^T$, $A_0^h=[a_0(s_i,t_j)]_{i,j=0}^{n-1}$, $A_1^h=[a_1(s_i,t_j]_{i,j=0}^{n-1}$, and $g^h=(g(\varphi(s_0)),\cdots,g(\varphi(s_{n-1})))^T$. Obviously, (2.8) is a linear equation system with n unknowns. Once v^h is solved from (2.8), the solution of (1.5) $u(\hat{y})$ ($\hat{y}\in\Omega'$) can be computed by

$$u^h(\hat{y})=-\frac{h}{2\pi}\sum_{i=0}^{n}v^h(s_i)\left|\hat{x}'(s_i)\right|\ln\left|\hat{x}(s_i)-\hat{y}\right|. \tag{2.9}$$

From (2.6), we have

$$A_0^h=-\frac{h}{2\pi}\begin{bmatrix} \ln\left(e^{-\frac{1}{2}}\frac{h}{2\pi}\right) & \ln\left|2e^{-\frac{1}{2}}\sin\frac{h}{2}\right| & \cdots & \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-1)h}{2}\right| \\ \ln\left|2e^{-\frac{1}{2}}\sin\frac{h}{2}\right| & \ln\left(e^{-\frac{1}{2}}\frac{h}{2\pi}\right) & \cdots & \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-2)h}{2}\right| \\ \vdots & \vdots & \vdots & \vdots \\ \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-1)h}{2}\right| & \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-2)h}{2}\right| & \cdots & \ln\left(e^{-\frac{1}{2}}\frac{h}{2\pi}\right)\end{bmatrix}.$$

By [10], we know that $\|(A_0^h)^{-1}\|\leq cn$. Hence the Eq. (2.8) is equivalent to

$$(E^h+P^h(A_0^h)^{-1}Q^hA_1^h)v^h=P^h(A_0^h)^{-1}Q^hg^h, \tag{2.10}$$

where E^h is the identity operator.

Lemma 2.2. The operator sequence $\left\{P^h(A_0^h)^{-1}Q^hA_0^h:C^3[0,2\pi)\to C[0,2\pi)\right\}$ is uniformly bounded and

$$P^h(A_0^h)^{-1}Q^hA_0\xrightarrow{p}I.$$

where $\xrightarrow{p}$ denotes the pointwisely convergence and I is the embedding operator .

Proof. Let $v \in C^3[0, 2\pi)$ and v^h be the solutions of the auxiliary equations $A_0 v = g$ and $A_0^h v^h = Q^h g$ respectively, where

$$A_0^h Q^h v = A_0^h v(t_i) = h \sum_{\substack{j=0 \\ j \neq i}}^{n-1} a_0(s_i, t_j) v(t_j) - \frac{1}{2\pi} \ln \left(\frac{h e^{-1/2}}{2\pi}\right) v(t_i),$$

and

$$Q^h g = g(s_i) = \int_0^{2\pi} a_0(s_i, t) v(t) \mathrm{d}t.$$

From (2.7), we obtain that

$$Q^h g - A_0^h Q^h v = O\ (h^3),$$

and

$$\|Q^h g - A_0^h Q^h v\|_2 = (n(O\ (h^3))^2)^{\frac{1}{2}} = O\ (h^{\frac{5}{2}}). \tag{2.11}$$

By (2.11), the following holds

$$\begin{aligned} \|Q^h v - (A_0^h)^{-1} Q^h A_0 v\|_2 &= \|Q^h v - (A_0^h)^{-1} Q^h g\|_2 \\ &= \|Q^h v - v^h\|_2 \\ &= \|(A_0^h)^{-1} A_0^h (Q^h v - v^h)\|_2 \\ &\leq \frac{1}{h} \|A_0^h (Q^h v - v^h)\|_2 \\ &= \frac{1}{h} \|A_0^h Q^h v - A_0^h v^h\|_2 \\ &= \frac{1}{h} \|Q^h g - A_0^h Q^h v\|_2 = O\ (h^{\frac{3}{2}}), \end{aligned}$$

the proof of Lemma 2.2 is completed. □

Theorem 2.3. Assume that $\partial\Omega'$ is a simply smooth and closed curve, Q^h is a restricted operator and P^h is a prolongation operator with nodes $\{s_i\}_{i=0}^n$, then the operator sequence $\{P^h (A_0^h)^{-1} Q^h A_1^h\}$ is collectively compactly convergent to $A_0^{-1} A_1$ in $C[0, 2\pi)$, that is,

$$P^h (A_0^h)^{-1} Q^h A_1^h \xrightarrow{c.c} A_0^{-1} A_1.$$

Proof. Since $(P^h (A_0^h)^{-1} Q^h)(P^h A_1^h Q^h) = (P^h (A_0^h)^{-1} Q^h A_0)((A_0)^{-1} P^h A_1^h Q^h)$, we get

$$\|(P^h (A_0^h)^{-1} Q^h)(P^h A_1^h Q^h)\| \leq \|P^h (A_0^h)^{-1} Q^h A_0\|_{0,3} \|(A_0)^{-1} P^h A_1^h Q^h\|_{3,0}.$$

From [10] and by Lemma 2.2, we have $(A_0)^{-1} P^h A_1^h Q^h \xrightarrow{c.c} A_0^{-1} A_1$ and $P^h (A_0^h)^{-1} Q^h A_0 \xrightarrow{p} I$. The proof of Theorem 2.3 is completed. □

Replacing $(A_0^h)^{-1},\ A_0^h,\ A_1^h,\ v^h$ and g^h by $(\hat{A_0}^h)^{-1} = P^h(A_0^h)^{-1}Q^h,\ \hat{A_0}^h = P^hA_0^hQ^h$, $\hat{A_1}^h = P^hA_1^hQ^h$, $\hat{v}^h = P^hv^h$ and $\hat{g}^h = P^h(A_0^h)^{-1}Q^hg^h$, respectively. Then (2.10) can be written as

$$(E^h + (\hat{A_0}^h)^{-1}\hat{A_1}^h)\hat{v}^h = \hat{g}^h. \tag{2.12}$$

Theorem 2.4. Assume that $\partial\Omega'$ is a simply smooth and closed curve, v and $\hat{v}^h$ are the solutions of (2.4) and (2.12), respectively, $x_i \in C^6[0, 2\pi)$ and $g(s) \in C^5[0, 2\pi)$, then the following holds

$$(\hat{v}^h - v)\big|_{s=s_i} = O\ (h^3). \tag{2.13}$$

Proof. By the trapezoidal rule, the asymptotic expansion holds [10]

$$(g - g^h)\big|_{s=s_i} = h^3P^hQ^h\varphi_1\big|_{s=s_i} + O\ (h^5), \tag{2.14}$$

with $\varphi_1(s) = -\zeta^{'}(-2)g''(s)/\pi$. Using (2.7) and (2.14), we can obtain

$$\begin{aligned}
&P^h(A_0^h + A_1^h)Q^h(v^h - v)\big|_{s=s_i}\\
&= g^h - P^h(A_0^h + A_1^h)Q^hv\big|_{s=s_i}\\
&= g^h - [((A_0 + A_1)v - h^3P^hQ^h\varphi_2)\big|_{s=s_i} + O\ (h^5)]\\
&= (g^h - g)\big|_{s=s_i} + h^3P^hQ^h\varphi_2\big|_{s=s_i} + O\ (h^5)\\
&= h^3P^hQ^h\varphi\big|_{s=s_i} + O\ (h^5),
\end{aligned}$$

where $\varphi_2(s) = -\xi^{'}(-2)v''(s)/\pi$, and $\varphi(s) = \varphi_1(s) + \varphi_2(s)$. From Theorem 2.3, we have

$$(E^h + (\hat{A_0}^h)^{-1}A_1^h)(v - \hat{v}^h)\big|_{s=s_i} = h^3(\hat{A_0}^h)^{-1}P^hQ^h\varphi(s)\big|_{s=s_i} + O\ (h^5). \tag{2.15}$$

Since $(E^h + (\hat{A_0}^h)^{-1}A_1^h)^{-1}$ is uniformly bounded, we immediately get

$$(\hat{v}^h - v)\big|_{s=s_i} = O\ (h^3). \tag{2.16}$$

□

3 Corner singularity and convergence analysis

Definition 3.1. [14] A real-valued function γ is said to be a sigmoidal transformation if the following conditions are satisfied:

(i) $\gamma \in C^1[0, 1] \cup C^\infty(0, 1)$ with $\gamma(0) = 0$;

(ii) $\gamma(x) + \gamma(1 - x) = 1,\ 0 \le x \le 1$;

(iii) γ is strictly increasing on $[0, 1]$ and its derivative γ' is strictly increasing on $[0, 1/2]$ with $\gamma'(0) = 0$.

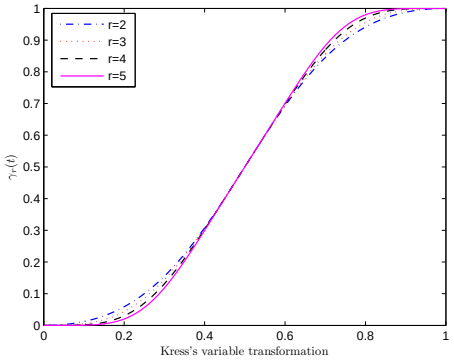


Figure 1: γ_r *transformation*

The Kress's variable transformation is an example of "algebraic" sigmoidal transformations, which was first proposed by Kress [4]. The function γ, defined on $[0,1]$ by

$$\gamma(x) = f(x)/(f(x) + f(1-x)),$$

where

$$f(x) = (x + cB_3(x))^r,$$

here $r > 1$, c is a constant to be determined and B_3 is the Bernoulli polynomial of degree 3 defined by

$$B_3(x) := x(x - 1/2)(x - 1).$$

If we choose $c = -8(1/r - 1/2)$, then we have

$$\gamma_r(t) = \frac{\Theta_1(t)}{\Theta_2(t)} = \frac{(\theta(t))^r}{(\theta(t))^r + (\theta(2\pi - 2\pi t))^r} : [0,1] \to [0,1], \quad r \geq 1, \tag{3.1}$$

where $\theta(t) = (\frac{1}{r} - \frac{1}{2})(1-2t)^3 + \frac{1}{r}(2t-1) + \frac{1}{2}$. The plots for γ_r is shown in Figure 1.

Assume that $\Gamma' = \cup_{q=1}^m \Gamma'_q$ $(q > 1)$ be the boundary of a polygonal domain Ω' in $\mathbb{R}^2$, $\Gamma'_q \in C^{2\ell+1}(q = 1, ..., m,\ \ell \in N)$, and let $g_q = g\big|_{\Gamma'_q}$. Define the boundary integral operators on Γ'_q,

$$(L_{pq}v_q)(\hat{y}) = -\frac{1}{2\pi}\int_{\Gamma'_q} v_q(\hat{x}) \ln\left|\hat{x} - \hat{y}\right| \mathrm{d}s_{\hat{x}},\ \hat{y} = (\hat{y}_1, \hat{y}_2) \in \Gamma'_p \ \ (p, q = 1, ..., m). \tag{3.2}$$

Then Eq. (1.4) can be converted into a matrix operator equation

$$Lv = G, \tag{3.3}$$

where $L = [l_{pq}]_{p,q=1}^m$, $G = (g_1(\hat{y}), ..., g_m(\hat{y}))^T$ and $v = (v_1(\hat{x}), ..., v_m(\hat{x}))^T$. Assume that Γ'_q can be described by the parameter mapping: $\hat{x}_q(s) = \varphi(s) = (\varphi_{q1}(s), \varphi_{q2}(s))$

: $[0,1] \to \Gamma_q'$ with $|\varphi_q^{'}(s)| = [|\varphi_{q1}^{'}(s)|^2 + |\varphi_{q2}^{'}(s)|^2]^{1/2} > 0$. In order to degrade the singularities at corners, we apply the Kress's variable transformation to (3.3) and give the following decomposition of L_{pq},

$$L_{pq} = A_{0(pp)} + A_{1(pq)}.$$

The operators $A_{0(pp)}$ and $A_{1(pq)}$ are singular and compact operators respectively, and their Nyström approximation are

$$\begin{cases} (A_{0(pp)}w_p)(t) = \int_0^1 a_{0(pq)}(t,\tau)w_p(\tau)\mathrm{d}\tau, \quad t \in [0,1], \\ (A_{0(pp)}^{h_p}w_p)(t) = h_p \sum_{\substack{j=1 \\ t\neq\tau_j}}^{n_m} a_{0(pp)}(t,\tau_j)w_p(\tau_j) - h_p \ln\left|e^{-1/2}h_p\right| w_p(t), \ h_p = 1/n_p, \end{cases}$$

and

$$\begin{cases} A_{1(pq)}(w_q)(t) = \int_0^1 a_{1(pq)}(t,\tau)w_q(\tau)\mathrm{d}\tau, \quad t \in [0,1], \\ (A_{1(pq)}^{h_q}w_q)(t) = h_q \sum\limits_{j=1}^{n_q} a_{1(pq)}(t,\tau_j)w_q(\tau_j), \quad t \in [0,1], \ \tau_j = jh \quad p,q = 1,...,m, \end{cases}$$

where

$$a_{0(pp)}(t,\tau) = -\frac{1}{2\pi}\ln\left|2e^{-1/2}\sin\pi(t-\tau)\right|,$$

and

$$a_{1(pq)}(t,\tau) = \begin{cases} -\dfrac{1}{2\pi}\ln\dfrac{|\hat{x}_p(t) - \hat{y}_q(\tau)|}{|2e^{-1/2}\mathrm{sin}\pi(t-\tau)|} & \text{as} \quad p = q, \\ -\dfrac{1}{2\pi}\ln|\hat{x}_p(t) - \hat{y}_q(\tau)| & \text{as} \quad p \neq q, \end{cases}$$

here

$$\begin{aligned} \hat{x}_q(t) &= (\varphi_{q1}(\gamma_r(t)), \varphi_{q2}(\gamma_r(t))), \\ w_q(t) &= v_q(\varphi_q(\gamma_r(t)))|\varphi_q'(\gamma_r(t))|\gamma_r'(t). \end{aligned}$$

Then Eq. (3.3) and its discrete equations are

$$\begin{cases} (A_0 + A_1)W = G, \\ (A_0^h + A_1^h)W^h = G^h. \end{cases} \tag{3.4}$$

where

$$\begin{aligned} A_0 &= \mathrm{diag}(A_{0(11)}, \cdots, A_{0(mm)}), \quad A_1 = [A_{1(pq)}]_{p,q=1}^m, \\ W &= (w_1, \cdots, w_m)^T, \quad G = (g_1, \cdots, g_m)^T, \quad g_q(t) = g_q(\varphi_q(t)), \\ W^h &= (w_1^{h_1}(t_1), ..., w_1^{h_1}(t_{n_1}), ..., w_m^{h_m}(t_1), ..., w_m^{h_m}(t_{n_m}))^T, \\ A_0^h &= \mathrm{diag}(A_{0(11)}^{h_1}, ..., A_{0(mm)}^{h_m}), \quad A_{0(pp)}^h = [a_{0(pp)}(t_j,\tau_i)]_{j,i=1}^{n_p}, \\ A_1^h &= [A_{1(pq)}^{h_q}]_{p,q=1}^m, \quad A_{1(pq)}^h = [a_{1(pq)}(t_j,\tau_i)]_{j,i=1}^{n_p,n_q}, \\ G^h &= (g_1(t_1), ..., g_1(t_{n_1}), ..., g_m(t_1), ..., g_m(t_{n_m}))^T. \end{aligned}$$

The operator $A_{0(pp)}$ is an isometry operator from $(H^r[0,1])^m$ to $(H^{r+1}[0,1])^m$ [11, 12]. In addition, from [10], we know that $A^{h_q}_{0(qq)}$ and A^h_0 are invertible, and $||(A^{h_q}_{0(qq)})^{-1}|| = O(n_q)$ and $||(A^h_0)^{-1}|| = O(h_q^{-1})$, where $\|\cdot\|$ denotes the spectral norm. Hence Eq. (3.4) is equivalent to

$$\begin{cases} (E + A_0^{-1}A_1)W = A_0^{-1}G = \tilde{G} \\ (E^h + (A^h_0)^{-1}A^h_1)W^h = \tilde{G}^h. \end{cases} \tag{3.5}$$

Obviously, the second equation of (3.5) is a system of linear equations with n $(= \Sigma_{j=1}^m n_j)$ unknowns. Once W^h is solved by (3.5), the solution $u(\hat{y})$ $(\hat{y} \in \Omega')$ can be computed by

$$u^h(\hat{y}) = -\frac{1}{2\pi}\sum_{p=1}^{m}\sum_{q=1}^{n_p} h_p \ln|\hat{x}_{pq}(\tau_q) - \hat{y}||\hat{x}'_p(\tau_q)|w^h_q(\tau_q).$$

Let the function $v_q(t) = t^{\alpha_q}\phi_q(t)$ $(0 > \alpha_q \geq -1/2)$, where $\phi_q(t)$ is differentiable enough on $[0,1]$ with $\phi_q(0) \neq 0$. From Taylor's formula we have

$$v_q(t) = \sum_{j=0}^{l} \frac{\phi_q^{(j)}(0)}{j!} t^{j+\alpha_q} + O(t^{l+\alpha_q+1}) \ \text{ as } \ t \to 0^+ \tag{3.6}$$

and

$$\gamma'_r(t) \sim \sum_{j=0}^{\infty} \delta_j t^{r-1+j} \ \text{ as } \ t \to 0^+, \ \text{ and } \ \delta_0 > 0. \tag{3.7}$$

By substituting (3.6) and (3.7) into the expression of $w_q(t)$, then the function $w_q(t)$ can be expressed by

$$w_q(t) = c_1\phi_q(0)t^{r(\alpha_q+1)-1}(1 + O(t)) \ \text{ as } \ t \to 0^+, \tag{3.8}$$

where c_1 is a constant independent of t.

Similarly, let the function $v_q(t) = (1-t)^{\alpha_q}\tilde{\phi}_q(t)$ $(0 > \alpha_q \geq -1/2)$, where $\tilde{\phi}_q(t)$ is differentiable enough on $[0,1]$ with $\tilde{\phi}_q(1) \neq 0$. Then the function $w_q(t)$ can be expressed by

$$w_q(t) = c_2\tilde{\phi}_q(1)(1-t)^{r(\alpha_q+1)-1}(1 + O(1-t))) \ \text{ as } \ t \to 1^-, \tag{3.9}$$

where c_2 is a constant independent of t.

Remark 1 The function $v_q(t)$ has singularities at endpoints $t = 0$ and $t = 1$ [16], but $w_q(t)$ has no singularities by Kress transformation at $t = 0$ and $t = 1$.

Lemma 3.2. Let

$$\tilde{a}_{1(pq)}(t,\tau) = a_{1(pq)}(t,\tau)\gamma'_r(t), \quad \gamma \geq 1, \quad \Gamma_p \cap \Gamma_q \neq \emptyset, \tag{3.10}$$

then $\tilde{a}_{1(pq)}(t,\tau)$ is smooth on $[0,1]^2$.

Proof. By using the continuity of $\tilde{a}_{1(pp)}(t,\tau)$ and the boundness of $\gamma'_r(t)$, we can

immediately complete the proof for the case $p = q$. Let $\Gamma_{p-1} \cap \Gamma_p = P_p = (0, 0)$ and $\beta_p \in (0, 2\pi)$ be the corresponding interior angle. Since $\varphi_{p-1}(1) = \varphi_{p-1}(0)$, the kernel $a_{1(p-1,p)}(t, \tau)$ have singularities at the points $(t, \tau) = (0, 1)$ and $(t, \tau) = (1, 0)$. For convenience of analysis, we only discuss the case in that $(t, \tau) = (1, 0)$. If $(t, \tau) \neq (1, 0)$, we write

$$\tilde{a}_{1(p-1,p)}(t, \tau) = -\frac{1}{4\pi}(S_1(t, \tau) + S_2(t, \tau)), \tag{3.11}$$

where

$$S_1(t, \tau) = \gamma'(t) \ln(|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2)$$

and

$$S_2(t, \tau) = \gamma'(t) \ln[1 - 2|\varphi_{p-1}(t)||\varphi_p(\tau)| \cos \beta_{p-1}/(|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2)].$$

Since

$$\left|2|\varphi_{p-1}(t)||\varphi_p(\tau)| \cos \beta_{p-1}/(|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2)\right| < 1,$$

the function $S_2(t, \tau)$ and its first derivative are bounded. Noting that

$$\gamma_r^{(k)}(0) = \gamma_r^{(k)}(1) = 0, \quad k = 0, \cdots, \gamma,$$

we have

$$|\varphi_{\bar{p}}^{(k)}(0)| = |\varphi_{\bar{p}}^{(k)}(1)| = 0, \quad \bar{p} = p - 1 \text{ or } p, \quad k = 1, \cdots, \gamma.$$

Let $(t, \tau) \in [\varepsilon/2, \varepsilon] \times [1 - \varepsilon, 1 - \varepsilon/2]$ for all $\varepsilon > 0$, we have

$$|S_1(t, \tau)| = O(\varepsilon^{r-1}| \ln \varepsilon|),$$

so $S_1(t, \tau)$ is also bounded. In addition, from

$$\begin{aligned}\left|\frac{\partial}{\partial \tau} S_1(t, \tau)\right| &\leq O(t^{r-1}) \frac{2|\varphi_p(\tau)||\hat{x}'_{p-1}(\gamma_r(\tau))||\gamma'_r(\tau)|}{|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2} \\ &= O(\varepsilon^{r-1}) O(\varepsilon^{2r-2})/O(\varepsilon^{2r-2}) = O(\varepsilon^{r-1}),\end{aligned}$$

we know $\frac{\partial \tilde{a}_{1(pq)}(t,\tau)}{\partial \tau}$ is also continuous in $(C[0, 1])^2$. The proof of Lemma 3.2 is completed. □

Suppose that

$$t_\nu = (\nu + 1)/2 \quad \text{for } -1 < \nu \leq 1,$$

so that $-1 < \nu \leq 1$ with $t_0 = 1/2$ and $t_1 = 1$. The offset trapezoidal rule $Q_n^{[\nu,r]} f$ with

transformation γ_r, $n \in N$, $-1 < \nu \leq 1$ is defined by

$$Q_n^{[\nu,r]} f = \begin{cases} \frac{1}{n} \sum_{j=0}^{n-1} f(\gamma_r((j+t_\nu)/n)) \gamma_r'((j+t_\nu)/n), & -1 < \nu < 1, \\ \frac{1}{n} \sum_{j=1}^{n-1} f(\gamma_r(j/n)) \gamma_r'(j/n), & \nu = 1. \end{cases}$$

At the same time, we define the truncation error $E_n^{[\nu,r]} f$ by

$$E_n^{[\nu,r]} f := \mathrm{I} f - Q_n^{[\nu,r]} f. \tag{3.12}$$

Let us assume that near $x = 0$ we can write

$$\gamma_r(x) = c_0(r) x^r (1 + \sum_{k=1}^{\infty} d_k(r) x^k). \tag{3.13}$$

Theorem 3.3. [14] Assume that f is holomorphic at both 0 and 1, and $f_r(\tau) := f(\gamma_r(\tau)) \gamma_r'(\tau)$, $0 \leq \tau \leq 1$ can be continuous into the strip S such that

(i) $f_r(\tau)$ is continuous in S and holomorphic in $\mathrm{int}(S)$;

(ii) $f_r(\tau) = o(exp(2\pi n |Rz|))$ as $Rz \to \infty$ in S, uniformly with respect to Rz.

For any sigmoidal transformation γ_r of order $r > 1$ and for $-1 < \nu \leq 1$ then for $n >> 1$

$$\begin{aligned} (2\pi n)^r E_n^{[0,r]} f \sim 2 c_0(r) \Gamma(r+1)(f(0) + f(1)) \times \{ & \cos(r\pi/2)(1 - 2^{1-r}) \zeta(r) \\ & - \sin(r\pi/2)(r+1) d_1(r)(1 - 2^{-r}) \zeta(r+1)/(2\pi n)\} + O(1/n^{\min(2,r)}), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} (2\pi n)^r E_n^{[1,r]} f \sim 2 c_0(r) \Gamma(r+1)(f(0) + f(1)) \times \{ & \cos(r\pi/2) \zeta(r) \\ & - \sin(r\pi/2)(r+1) d_1(r) \zeta(r+1)/(2\pi n)\} + O(1/n^{\min(2,r)}). \end{aligned} \tag{3.15}$$

Theorem 3.4. [14] Suppose f is defined on S by

$$f(z) = z^\alpha (1-z)^\beta g(z) \quad \text{for } \alpha,\ \beta > 1, \tag{3.16}$$

where g is holomorphic on S, real on $[0,1]$ and such that $g(0) \neq 0$, $g(1) \neq 0$. Let γ_r be a sigmoidal transformation of order r, $r > 1$. Then for $n >> 1$

$$n^r E_n^{[\nu,r]} f \sim J_\nu(\alpha, r, n) g(0) + J_{-\nu}(\beta, r, n) g(1),$$

where the strip S of the complex $z-$ plane defined by

$$S := \{z : 0 \leq x = \mathcal{R}z \leq 1,\ -\infty < y = \mathcal{I}z < \infty\},$$

and

$$J_\nu(\alpha,r,n) := -r(c_0(r))^{1+\alpha}\{\zeta(1-r(\alpha+1),t_\nu) + (\alpha+1+1/r)d_1(r)\zeta(-r(\alpha+1),t_\nu)/n\}/n^{r\alpha}. \tag{3.17}$$

and

$$t_{-\nu} = 1 - t_\nu, \quad \text{for } -1 < \nu \le 1.$$

Remark 2 If we choose $r = 3$ for γ_r, then $\gamma_3 \sim O\,(t^3)$ and $d_1(r) = 0$. In addition, if $\beta = 0$, we have

$$E_n^{[\nu,r]} f \sim n^{-\omega}, \quad \omega = \min\{r, (\alpha+1)r\}. \tag{3.18}$$

For the Nyström approximation operator $A_{1(pq)}^{h_q}$ of the integral operator $A_{1(pq)}$, we have the error bounds [8]

$$(A_{1(pq)}w_q)(t) - (A_{1(pq)}^{h_q}w_q)(t) = O\,(h^r), \text{ for } \Gamma_p = \Gamma_q \text{ or } \Gamma_p \cap \Gamma_q = \emptyset, \ r \in N, \tag{3.19}$$

and

$$A_{1(pq)}(w_q)(t) - (A_{1(pq)}^{h_q}w_q)(t) = O\,(h^\omega), \text{ for } \Gamma_p \cap \Gamma_q \in \{P_q\}, \tag{3.20}$$

where $\omega = \min\{r, (\alpha+1)r\}$.

For the approximate operator $A_{0(pp)}^{h_p}$ of the logarithmically singular operator $A_{0(pp)}$,

$$\begin{aligned}(A_{0(pp)}^{h_p}w_p)(t) = -\frac{h_l}{2\pi}&\left\{\sum_{\substack{j=1\\ t\neq\tau_j}}^{n_q} \ln\left|2e^{-1/2}\sin\pi(t-\tau_j)\right| w_p(\tau_j)\right\} \\ &- \frac{h_p}{2\pi}\left\{\ln\left|2e^{-1/2}h_p\right| w_p(t)\right\} \quad (i = 1, ..., n_p),\end{aligned} \tag{3.21}$$

which have the error bounds [3]

$$(A_{0(pq)}^{h_p}w_p)(t) - (A_{0(pp)}w_p)(t) = -\frac{2}{\pi}\sum_{\mu=1}^{2\ell-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[w_p(t)]^{(2\mu)}h_p^{2\mu+1} + O(h_p^{2\ell}), \quad t \in \{t_i\},$$

where $\zeta'(t)$ is the derivative of the Riemann zeta function.

From (3.21), we can obtain

$$A_{0(pp)}^{h_p} = -\frac{h_p}{2\pi}\begin{bmatrix} \ln(\frac{h_p}{e^{1/2}}) & \ln(\frac{\sin(\pi h_p)}{e^{1/2}/2}) & \cdots & \ln(\frac{\sin((n_p-1)\pi h_p)}{e^{1/2}/2}) \\ \ln(\frac{\sin(\pi h_m)}{e^{-1/2}/2}) & \ln(\frac{h_p}{e^{1/2}}) & \cdots & \ln(\frac{\sin((n_p-2)\pi h_p)}{e^{1/2}/2}) \\ \vdots & \vdots & \vdots & \vdots \\ \ln(\frac{\sin((n_p-1)\pi h_p)}{e^{1/2}/2}) & \ln(\frac{\sin((n_p-2)\pi h_p)}{e^{1/2}/2}) & \cdots & \ln(\frac{h_p}{e^{1/2}}) \end{bmatrix},$$

Define the subspace $C_0[0,1] = \{v(t) \in C[0,1] : v(t)/\gamma_3(\pi t) \in C[0,1]\}$ of the space $C[0,1]$ with the norm $||v||^* = \max_{0\le t\le 1}|v(t)/\gamma_3(\pi t)|$. Let $S^{h_p} = \text{span}\{e_j(t), j = 1, ..., n_p\} \subset C_0[0,1]$ be a piecewise linear function subspace with the basis nodes

$\{t_i\}_{i=1}^{n_p}$, where $e_j(t)$ are the basis functions satisfying $e_{pj}(t_{pi}) = \delta_{ji}$. Also define a prolongation operator $I^{h_p} : \Re^{n_p} \to S^{h_p}$ and a restricted operator $R^{h_p} : C_0[0,1] \to \Re^{n_p}$ satisfying

$$\begin{cases} P^{h_p} v = \sum_{j=1}^{n_p} v_{pj} e_{pj}(\tau), \quad v = (v_{p1}, \cdots, v_{pn_p}) \in \mathfrak{R}^n, \\ Q^{h_p} v = (v(\tau_{p1}), \cdots, v(\tau_{pn_p})) \in \mathfrak{R}^n, \quad v \in C[0, 2\pi). \end{cases}$$

For the properties of $A_{0(pp)}^{h_p}$, we have the following lemma from [10].

Lemma 3.5. The operator sequence $\{P^{h_p}(A_{0(pp)}^{h_p})^{-1} Q^{h_p} A_{0(pp)} : C^2[0,1] \to C[0,1]\}$ is uniformly bounded and

$$P^{h_p}(A_{0(pp)}^{h_p})^{-1} Q^{h_p} A_{0(pp)} \xrightarrow{p} I. \tag{3.22}$$

If $|p-q| \neq 1$ or $m-1$, by the definition of $A_{1(pq)}$ we know the kernel $a_{1(pq)}(t,\tau)$ of the operator $A_{1(pq)}$ and its derivatives of higher order are continuous .

Lemma 3.6. Let $\Gamma' = \cup_{q=1}^{m} \Gamma'_q$ satisfy $C_{\Gamma'} \neq 1$, and also let

$$\bar{A}_{1(pq)}^{h_q} = \begin{cases} A_{1(pq)}^{h_q}, & \Gamma'_p = \Gamma'_q \text{ or } \Gamma'_p \cap \Gamma'_q = \emptyset, \\ \tilde{A}_{1(pq)}^{h_q}, & \Gamma'_p \cap \Gamma'_q \in \{P_q\}, \end{cases}$$

where the kernel $\tilde{a}_{1(pq)}(t,\tau)$ of $\tilde{A}_{1(pq)}$ is defined by (3.10). Then under the transformation (3.1), we have

$$\|(A_{0(pp)})^{-1} \bar{A}_{1(pq)}^{h_q}\|_{2,0} \leq M \tag{3.23a}$$

and

$$P^{h_p}(A_{0(pp)}^{h_p})^{-1} Q^{h_p} \bar{A}_{1(pq)}^{h_q} \xrightarrow{c.c} (A_{0(pp)})^{-1} A_{1(pq)}, \quad \text{in} \quad C[0,1] \to C[0,1], \tag{3.23b}$$

where M is a constant.

proof. From [10] and by Lemma 3.2, $a_{1(pq)}(t,\tau)$ and $\tilde{a}_{1(pq)}(t,\tau)$ are continuous on $(C^2[0,1])^2$, and then we have (3.23a). Using the results of Lemma 3.5, and by

$$\begin{aligned} \|P^{h_p}(A_{0(pp)}^{h_p})^{-1} Q^{h_p} \bar{A}_{1(pq)}^{h_q}\|_{0,0} &= \|(P^{h_p}(A_{0(pp)}^{h_p})^{-1} Q^{h_p} A_{0(pp)})((A_{0(pp)})^{-1} \bar{A}_{1(pq)}^{h_q})\|_{0,0} \\ &\leq \|P^{h_p}(A_{0(pp)}^{h_p})^{-1} Q^{h_p} A_{0(pp)}\|_{0,2} \|(A_{0(pp)})^{-1} \bar{A}_{1(pq)}^{h_q}\|_{2,0} \\ &\leq C, \end{aligned}$$

where C is a constant. Thus, we complete the proof of Lemma 3.6. □

Consider the following discrete equation

$$(E^h + P^h(A_0^h)^{-1} Q^h A_1^h) W^h = P^h(A_0^h)^{-1} Q^h G^h, \tag{3.24}$$

where P^h =diag$(P^{h_1}, ..., P^{h_m})$ and Q^h =diag$(Q^{h_1}, ..., Q^{h_m})$.

Theorem 3.7. Assume $\Gamma' = \cup_{q=1}^{m} \Gamma'_q$ satisfy $C_{\Gamma'} \neq 1$, and Γ'_q $(q = 1, ..., m)$ are smooth curves. Then we have

$$P^h(A_0^h)^{-1} Q^h A_1^h \xrightarrow{c.c} A_0^{-1} A_1. \tag{3.25}$$

Proof. Let $\mathcal{B} = \{z : ||z|| \leq 1, z \in (C[0,1])^m\}$ be the unit ball in the space $V = (C[0,1])^m$, and

$$H = \{H^{(1)}, H^{(2)}, \cdots\}, \quad H^{(n)} = \{h_1^{(n)}, \cdots, h_m^{(n)}\}$$

are multi-parameter sequences. Also let $\max_{1\leq q\leq m} h_q^{(n)} \to 0$ as $n_q \to \infty$. Choosing the sequence $\{Z_h, h \in H\} \subset \Theta$ and

$$Z_h = (Z_{1h}, \cdots, Z_{mh}), \quad Z_{qh} = (z_{q1}, \cdots, z_{qn_q}),\ q = 1, \cdots, m,$$

satisfying

$$\max_{1\leq p\leq m} \max_{0\leq q\leq n_p} \max_{0\leq t\leq 1} |z_{pq}^h(t)/\gamma_3(\pi t)| \leq 1. \tag{3.26}$$

From

$$P^h(A_0^h)^{-1}Q^h = \begin{bmatrix} P^{h_1}(A_{0(11)}^{h_1})^{-1}Q^{h_1} & & & \\ & P^{h_2}(A_{0(22)}^{h_2})^{-1}Q^{h_2} & & \\ & & \ddots & \\ & & & P^{h_m}(A_{0(mm)}^{h_m})^{-1}Q^{h_m} \end{bmatrix}$$

and

$$P^hA_1^hQ^h = \begin{bmatrix} P^{h_1}A_{1(11)}^{h_1}Q^{h_1} & P^{h_2}A_{1(12)}^{h_2}Q^{h_2} & \cdots & P^{h_m}A_{1(1m)}^{h_m}Q^{h_m} \\ P^{h_1}A_{1(21)}^{h_1}Q^{h_1} & P^{h_2}A_{1(22)}^{h_2}Q^{h_2} & \cdots & P^{h_m}A_{1(2m)}^{h_m}Q^{h_m} \\ \vdots & \vdots & & \vdots \\ P^{h_1}A_{1(m1)}^{h_1}Q^{h_1} & P^{h_2}A_{1(m2)}^{h_2}Q^{h_2} & \cdots & P^{h_m}A_{1(mm)}^{h_m}Q^{h_m} \end{bmatrix},$$

we have

$$P^h(A_0^h)^{-1}Q^hA_1^hQ^hZ_h = \begin{bmatrix} \sum_{q=1}^m P^{h_1}(A_{0(11)}^{h_1})^{-1}Q^{h_1}A_{1(1q)}^{h_q}Q^{h_q}Z_{qh} \\ \sum_{q=1}^m P^{h_2}(A_{0(22)}^{h_2})^{-1}Q^{h_2}A_{1(2q)}^{h_q}Q^{h_q}Z_{qh} \\ \vdots \\ \sum_{q=1}^m P^{h_m}(A_{0(mm)}^{h_m})^{-1}Q^{h_m}A_{1(mq)}^{h_q}Q^{h_q}Z_{qh} \end{bmatrix}.$$

If $\Gamma_p' \cap \Gamma_q' = \emptyset$, from Lemma 3.6 we obtain

$$P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}A_{1(pq)}^{h_q} \xrightarrow{cc} A_{0(pp)}^{-1}A_{1(pq)}. \tag{3.27}$$

If $\Gamma_p' \cap \Gamma_q' \neq \emptyset$, using Lemma 3.5 and Lemma 3.6, and by

$$\begin{aligned} ||P^{h_p}(A_{0(pp)}^h)^{-1}Q^{h_p}A_{1(pq)}^{h_q}Q^{h_q}Z_{qh}||_0 &= ||P^{h_p}(A_{0(pp)}^h)^{-1}Q^{h_p}\tilde{A}_{1(pq)}^{h_q}Q^{h_q}Z_{qh}/\gamma_3(\pi t)||_0 \\ &\leq \left\|P^{h_{pp}}(A_{0(pp)}^h)^{-1}Q^{h_p}A_{0(pp)}\right\|_{0,3} ||A_{0(pp)}^{-1}\tilde{A}_{1(pq)}^{h_q}||_{3,0} \left\|||Q^{h_q}Z_{qh}\right\|^*, \end{aligned} \tag{3.28}$$

we can find an infinite subsequence in $\{P^{h_p}(A^{h_p}_{0(pp)})^{-1}Q^{h_p}A^{h_q}_{1(pq)}Q^{h_q}Z_{qh}\}$ which converges as $h \to 0$. Hence, there exists an infinite subsequence $H_l \subset H$ such that (3.28) converges, As above, there exists an infinite subsequence $\{H_l,\ l = 1, \cdots, m\}$ such that $\{P^h(A^h_0)^{-1}Q^hA^h_1,\ h \in H_m\}$ is a convergent sequence in the space $V = (C_0[0,1])^m$. This shows that $\{P^h(A^h_0)^{-1}Q^hA^h_1\}$ is a collectively compact sequence, and $P^h(A^h_0)^{-1}Q^hA^h_1$ is pointwisely convergent to $A_0^{-1}A_1$. The proof of Theorem 3.7 is completed. □

Similar to Theorem 2.4, we have the following theorem.

Theorem 3.8. Assume $\Gamma' = \cup_{q=1}^m \Gamma'_q$ satisfy $C_{\Gamma'} \neq 1$, $g_q = g|_{\Gamma'_q} \in C^6(\Gamma'_q)$, then when we choose an appropriate number r in (3.18) such that $\omega > 3$, there exists a vector function $\Phi = (\Phi_1, ..., \Phi_m)^T \in (C_0[0,1])^m$ independent of $h = (h_1, ..., h_m)^T$ such that the following multi-parameter asymptotic expansions hold at nodes

$$w - \hat{w}^h = \mathrm{diag}(h_1^3, ..., h_m^3)\Phi + O(h^5_{\max})e, \tag{3.29}$$

where $h_{\max} = \max_{1\le q\le m} h_q$, and $e = (1, 1, \cdots, 1)^T$ is a m dimensional vector.

4 Numerical experiments

In this section, two numerical examples are presented to verify the efficiency of the Sidi-Israeli quadrature method for anisotropic heat conduction problems.

Suppose that $e_n = \left|u - u_n\right|$ be the errors by Sidi-Israeli quadrature method using n boundary nodes, and let $r_n = \log_2(e_n/e_{n/2})$ be the error ratio.

Example 1. [5] Consider the steady state heat conduction in an anisotropic material in the two dimensional disc Ω of radius unity. The thermal conductivity tensor is chosen to be $\kappa_{11} = 5.0$, $\kappa_{12} = \kappa_{21} = 2.0$, and $\kappa_{22} = 1.0$. Dirichlet boundary conditions corresponding to the analytical solution $u(x_1, x_2) = x_1^3/5 - x_1^2x_2 + x_1x_2^2 + x_2^3/3$ are applied to the whole boundary $\Gamma = \{(x_1, x_2)|x_1^2 + x_2^2 = 1\}$. Under the transformation (1.2), the physical domain is distorted into an oblique ellipse Ω' with the boundary $\Gamma' = \{(\hat{x}_1, \hat{x}_2)|(5\hat{x}_1)^2 + (\hat{x}_2 + 2\hat{x}_1)^2 = 1\}$ on the mapped plane, as shown in Fig. 2. The computed values at the interior points $P_1 = (0.2, 0.2)$, $P_2 = (0.4, 0.4)$ and $P_3 = (0.6, 0.6)$ using different boundary nodes are listed in Table 1, from the numerical results we can see that $r_n \approx 3$.

In addition, the numerical solution u of the interior points along the line $x_2 = x_1$ are computed, where $x_1 = \sqrt{2}/2\rho$ and $\rho = -0.9 : 0.05 : 0.9$. The plots of computed errors are shown in Figure 3 to Figure 5.

Example 2. Consider a square domain Ω with the boundary $\Gamma = \sum_{q=1}^4 \Gamma_q$, where $\Gamma_1 = \{(x_1, 0) :\ 0 \le x_1 \le 1\}$, $\Gamma_2 = \{(1, x_2) :\ 0 \le x_2 \le 1\}$, $\Gamma_3 = \{(x_1, 1) :\ 0 \le x_1 \le 1\}$, and $\Gamma_4 = \{(0, x_2) :\ 0 \le x_2 \le 1\}$. The invariant coefficients are chosen to be $\kappa_{11} = 1$, $\kappa_{12} = 0.5$, and $\kappa_{22} = 1$. The Dirichlet condition are applied to the Γ is $u(x_1, x_2) = \frac{1}{2}x_1^2 + x_1x_2 - x_2^2 + 2$. Let each boundary Γ_q $(q = 1, \cdots, 4)$ be divided into 2^k $(k = 4, \cdots, 10)$ segments. The physical domain and the 'isotropic' mapped domain (parallelogram) are shown in Fig. 6. In order to overcome the singularities at the corners, we use Sidi-Israeli quadrature method with the Kress's

Table 1: The errors for u at the points $P_1 = (0.2, 0.2)$, $P_2 = (0.4, 0.4)$ and $P_3 = (0.6, 0.6)$

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}
$e_n(P_1)$	8.633E-4	2.420E-5	3.672E-6	4.589E-7	5.735E-8	7.169E-9	8.961E-10
$r_n(P_1)$	-	5.517	2.720	3.000	3.000	3.000	3.000
$e_n(P_2)$	1.533E-3	9.791E-6	6.180E-6	7.621E-7	9.525E-8	1.191E-8	1.488E-9
$r_n(P_2)$	-	7.291	0.644	3.020	3.000	3.000	3.000
$e_n(P_3)$	2.377E-3	1.532E-3	7.495E-5	3.283E-7	9.447E-8	1.178E-8	1.472E-9
$r_n(P_3)$	-	0.6324	4.355	7.835	1.797	3.004	3.000

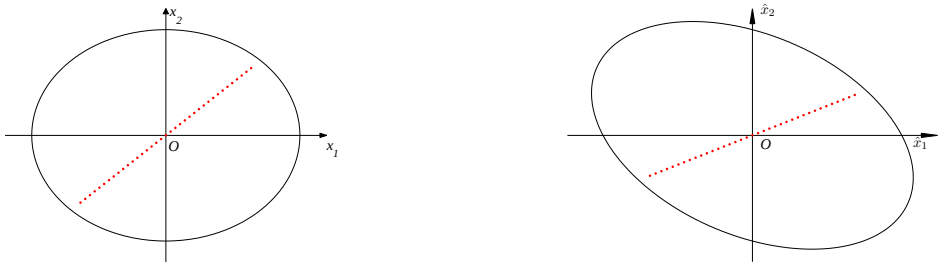


Figure 2: *Left: The physical domain Ω; Right: The mapped domain Ω'.*

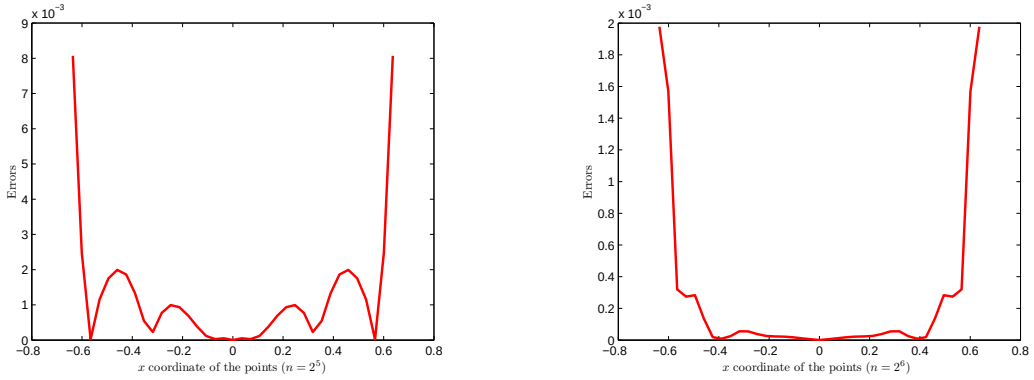


Figure 3: *Left: Errors for u by 2^5 boundary nodes; Right: Errors for u by 2^6 boundary nodes.*

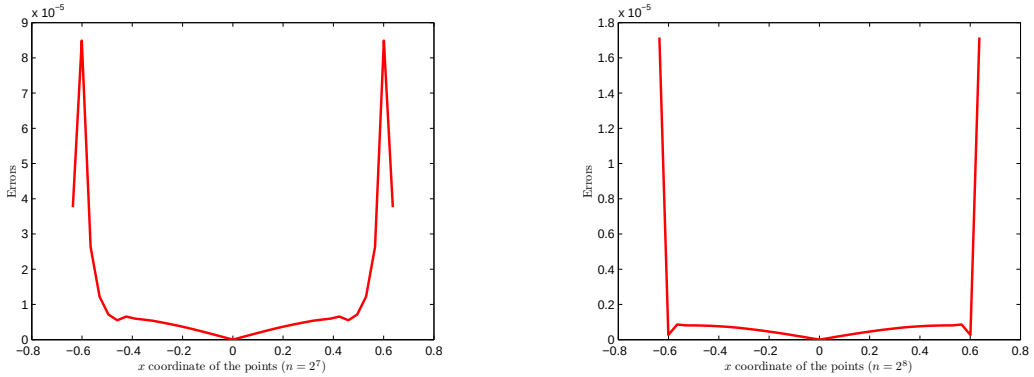


Figure 4: *Left: Errors for u by 2^7 boundary nodes; Right: Errors for u by 2^8 boundary nodes.*

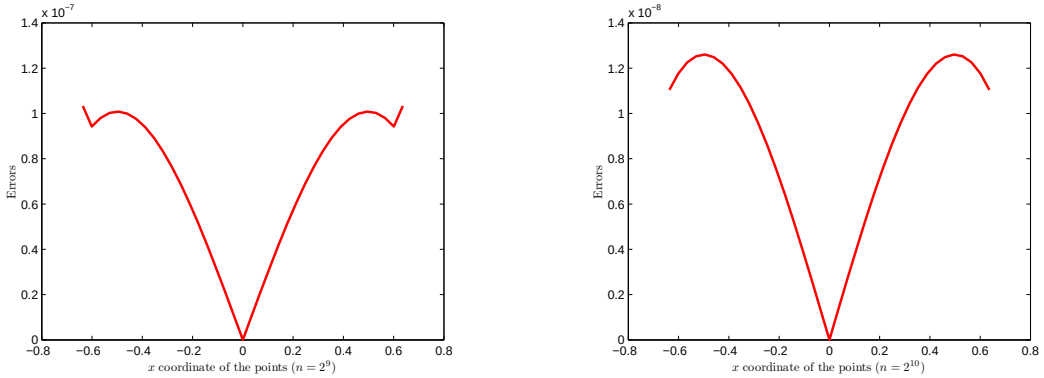


Figure 5: *Left: Errors for u by 2^9 boundary nodes; Right: Errors for u by 2^{10} boundary nodes.*

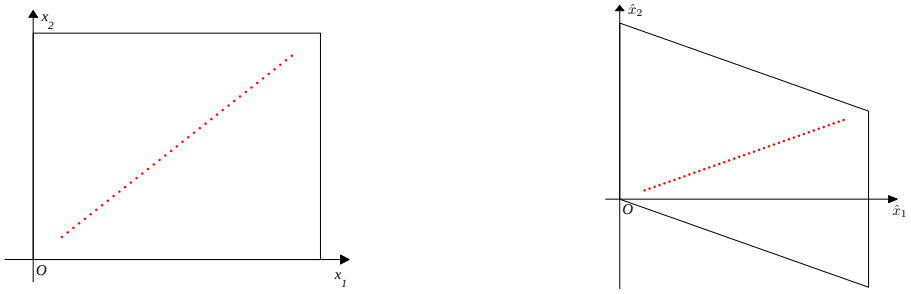


Figure 6: *Left: The physical domain* Ω*; Right: The mapped domain* Ω'.

variable transformation γ_3 for this problem. The computed values at the interior points $P_1 = (0.1, 0.1)$, $P_2 = (0.3, 0.3)$ and $P_3 = (0.5, 0.5)$ using n $(= 4 \times 2^k, k = 4, \cdots, 10$ nodes are listed in Table 2, from the numerical results we can also see that $r_n \approx 3$.

In addition, the numerical solution u of the interior points along the line $x_2 = x_1$ are computed, where $x_1 = \rho$ and $\rho = 0.1 : 0.02 : 0.9$. The plots of computed errors are shown in Figure 7 to Figure 9.

Table 2: The errors for u at the points $P_1 = (0.1, 0.1)$, $P_2 = (0.3, 0.3)$ and $P_3 = (0.5, 0.5)$

n	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
$e_n(P_1)$	1.154E-4	1.254E-5	1.565E-6	1.956E-7	2.444E-8	3.055E-9	3.819E-10
$r_n(P_1)$	-	3.202	3.002	3.000	3.000	3.000	3.000
$e_n(P_2)$	1.158E-4	1.445E-5	1.805E-6	2.256E-7	2.820E-8	3.525E-9	4.406E-10
$r_n(P_2)$	-	3.003	3.001	3.000	3.000	3.000	3.000
$e_n(P_3)$	1.198E-4	1.495E-5	1.869E-6	2.336E-7	2.919E-8	3.649E-9	4.561E-10
$r_n(P_3)$	-	3.002	3.001	3.000	3.000	3.000	3.000

5 Conclusions

In this paper, the Sidi-Israeli quadrature method is used to solve the boundary integral equations of steady state anisotropic heat conduction problems on the two-dimensional domain with smooth boundaries and polygons respectively. Especially, in order to provide a good accuracy in the solution near the singular points, the Kress's variable transformation is used for the weakly singular integral equations of

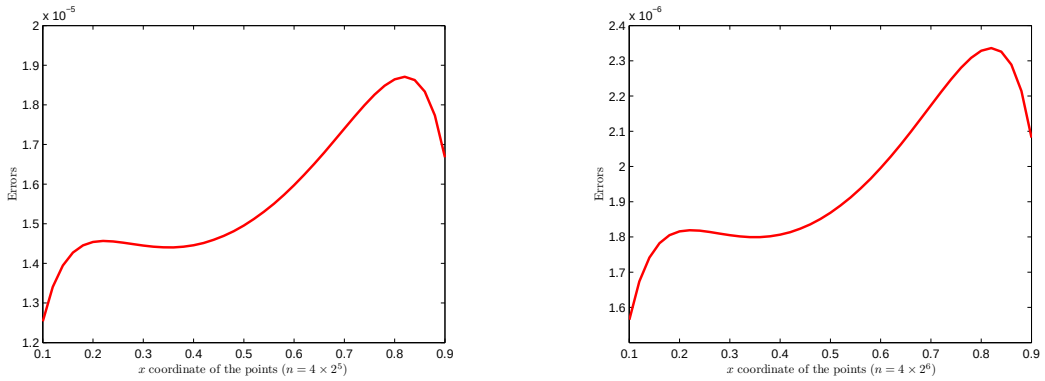


Figure 7: *Left: Errors for u by 4×2^5 boundary nodes; Right: Errors for u by 4×2^6 boundary nodes.*

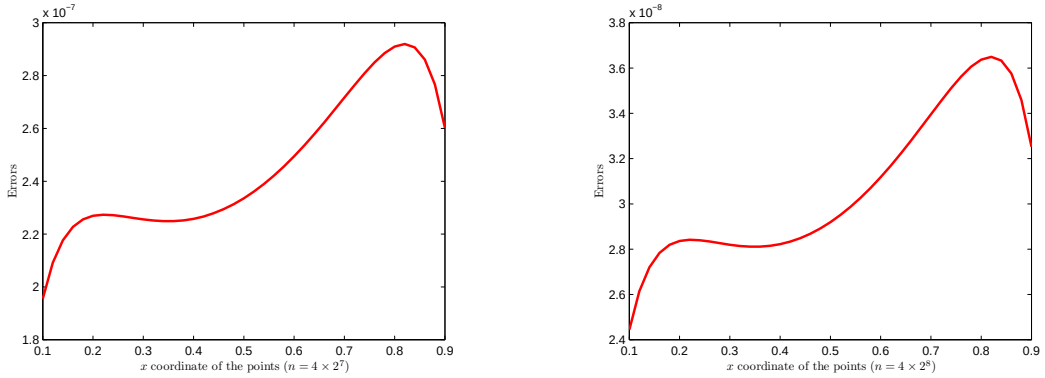


Figure 8: *Left: Errors for u by 4×2^7 boundary nodes; Right: Errors for u by 4×2^8 boundary nodes.*

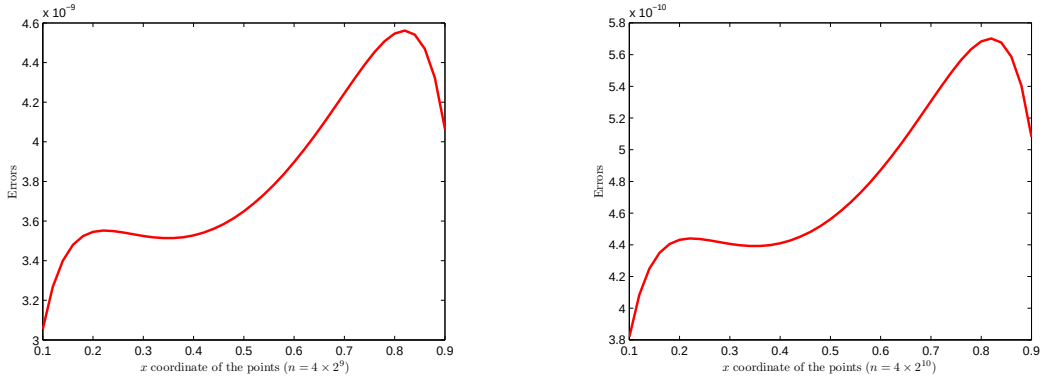


Figure 9: *Left: Errors for u by 4×2^9 boundary nodes; Right: Errors for u by 4×2^{10} boundary nodes.*

problems (1.1). The numerical results show that the presented algorithm has a high accuracy of $O\left(n^{-3}\right)$, which coincides with our theoretical analysis.

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Hyers-Ulam stability of an additive set-valued functional equation

Gang Lu, Jun Xie, Choonkil Park* and Yuanfeng Jin*

Abstract. In this paper, we define the following additive set-valued functional equation

$$\begin{aligned} &f(2x+3y-z)+f(2y+3z-x)+f(3x+2z-y)\\ &=f(x+y)+f(y+z)+f(x+z)+f(2x)+f(2y)+f(2z) \end{aligned} \tag{1}$$

and prove the Hyers-Ulam stability of the above additive set-valued functional equation.

1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [19] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach (see [1, 4, 5, 7, 8, 14, 15, 20, 21, 22, 23, 24, 25, 26, 27]).

It is easy to show that if $f:\mathbb{R}\to\mathbb{R}$ is a solution of the inequality

$$|f(x+y)-f(x)-f(y)|<\varepsilon \tag{1.1}$$

for some $\varepsilon>0$ then there exists a linear function $g(x)=mx, m\in\mathbb{R}$, such that $|f(x)-g(x)|<\varepsilon$ for all $x\in\mathbb{R}$.

The inequality (1.1) can be written as the form

$$f(x+y)-f(x)-f(y)\in B(0,\varepsilon),$$

where $B(0,\varepsilon):=(-\varepsilon,\varepsilon)$. Hence we have

$$f(x+y)+B(0,\varepsilon)\subseteq f(x)+B(0,\varepsilon)+f(y)+B(0,\varepsilon)$$

and denoting by $F(x)=f(x)+B(0,\varepsilon), x\in\mathbb{R}$, we get

$$F(x+y)\subseteq F(x)+F(y), x,y\in\mathbb{R}$$

and

$$g(x)\in F(x).$$

Let Y be a real normed space. The family of all closed and convex subsets, containing 0, of Y will be denoted by $ccz(Y)$.

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*Corresponding authors.

Hyers-Ulam stability of an additive set-valued functional equation

Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$\begin{aligned} A+B &= \{x \in X : x = a+b, \quad a \in A, b \in B\}, \\ \lambda A &= \{x \in X : x = \lambda a, \quad a \in A\}. \end{aligned}$$

Lemma 1.1. ([13]) *Let λ and μ be real numbers. If A and B are nonempty subsets of a real vector space X, then*

$$\lambda(A+B) = \lambda A + \lambda B,$$
$$(\lambda+\mu)A \subseteq \lambda A + \mu B.$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda+\mu)A = \lambda A + \mu A.$$

A subset $A \subseteq X$ is said to be a *cone* if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero*.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 10, 11, 12]).

2. Stability of the set-valued functional equation (1)

In this section, let X be a real vector space, $A \subseteq X$ a cone with zero and Y a Banach space.

The following theorem is similar to the results of [16] and [18]

Theorem 2.1. *If $F : A \to ccz(Y)$ is a set-valued map satisfying*

$$\begin{aligned} &F(2x+3y-z) + F(2y+3z-x) + F(3x+2z-y) \\ &\subseteq F(x+y) + F(y+z) + F(x+z) + F(2x) + F(2y) + F(2z) \end{aligned} \tag{2.1}$$

and

$$\sup\{diam(F(x)) : x \in A\} < +\infty$$

for all $x, y, z \in A$, then there exists a unique additive mapping $g : A \to Y$ such that $g(x) \in F(x)$.

Proof. Take an element $x \in A$. Letting $y = z = x$ in (2.1) and using Lemma 1.1, we get

$$3F(4x) \quad \subseteq \quad 6F(2x). \tag{2.2}$$

Replacing $2x$ by $2^n x$ in (2.2), we obtain

$$F\left(2^{n+1}x\right) \subseteq 2F\left(2^n x\right)$$

and

$$\frac{F(2^{n+1}x)}{2^{n+1}} \subseteq \frac{F(2^n x)}{2^n}.$$

Denoting by $F_n(x) = \frac{F(2^n x)}{2^n}, x \in A, n \in \mathbb{N}$, we obtain that $(F_n(x))_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_n(x)) = \frac{1}{2^n} diam\left(F\left(2^n x\right)\right).$$

Taking account of $\sup\{diam(F(x)) : x \in A\} < +\infty$, we get

$$\lim_{n\to\infty} diam(F_n(x)) = 0.$$

Using the Cantor theorem for the sequence $(F_n(x))_{n\geq 0}$, we obtain that the intersection $\cap_{n\geq 0} F_n(x)$ is a singleton set and we denote this intersection by $g(x)$ for all $x \in A$. Thus we get a mapping $g : A \to Y$ and $g(x) \in F_0(x) = F(x)$ for all $x \in A$.

We now show that g is additive. For all $x_i \in A, i = 1, 2, \cdots, N$ and $n \in \mathbb{N}$,

$$\begin{aligned} & F_n(2x+3y-z) + F_n(2y+3z-x) + F_n(3x+2z-y) \\ = \; & \frac{F(2^n(2x+3y-z))}{2^n} + \frac{F(2^n(2y+3z-x))}{2^n} + \frac{F(2^n(3x+2z-y))}{2^n} \\ \subseteq \; & \frac{F(2^n(x+y)) + F(2^n(y+z)) + F(2^n(x+z)) + F(2^n(2x)) + F(2^n(2y)) + F(2^n(2z))}{2^n}. \\ = \; & F_n(x+y) + F_n(y+z) + F_n(x+z) + F_n(2x) + F_n(2y) + F_n(2z). \end{aligned}$$

By the definition of g, we obtain

$$\begin{aligned} & g(2x+3y-z) + g(2y+3z-x) + g(3x+2z-y) \\ & = \bigcap_{n=0}^{\infty} F_n(2x+3y-z) + \bigcap_{n=0}^{\infty} F_n(2y+3z-x) + \bigcap_{n=0}^{\infty} F_n(3x+2z-y) \\ & \subseteq \bigcap_{n=0}^{\infty} \{F_n(2x+3y-z) + F_n(2y+3z-x) + F_n(3x+2z-y)\} \\ & \subseteq \bigcap_{n=0}^{\infty} \{F_n(x+y) + F_n(y+z) + F_n(x+z) + F_n(2x) + F_n(2y) + F_n(2z)\} \end{aligned}$$

and $g(x_i) \in F_n(x_i)$. Thus we get

$$\begin{aligned} & \|g(2x+3y-z) + g(2y+3z-x) + g(3x+2z-y) \\ & -g(x+y) - g(y+z) - g(z+x) - g(2x) - g(2y) - g(2z)\| \\ & \leq diam\,(F_n\,(x+y)) + diam\,(F_n\,(y+z)) + diam\,(F_n\,(x+z)) + diam\,(F_n\,(2x)) \\ & + diam\,(F_n\,(2y)) + diam\,(F_n\,(2z)) \end{aligned}$$

which tends to zero as n tends to ∞. Thus

$$\begin{aligned} & g(2x+3y-z) + g(2y+3z-x) + g(3x+2z-y) \\ & = g(x+y) + g(y+z) + g(x+z) + g(2x) + g(2y) + g(2z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in A$.

Letting $x = y = z = 0$ in (2.3), we have $3g(0) = 6g(0)$. Thus $g(0) = 0$. Letting $x = y = z$ in (2.3), we get $g(2x) = 2g(x)$ for all $x \in A$. And letting $y = z = 0$ in (2.3), we have

$$g(-x) + g(3x) = 2g(x) = g(2x) \tag{2.4}$$

for all $x \in A$. Letting $z = -x, y = 2x$ in (2.4), we get

$$g(z) + g(y-z) = g(y) \tag{2.5}$$

for all $y, z \in A$. Letting $y = 0$ in (2.5), we have $g(-z) = -g(z)$ for all $z \in A$. Hence

$$g(y-z) = g(y) + g(-z)$$

for all $y, z \in A$. That is, g is an additive mapping. □

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Gang Lu
Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110178, P. R. China; Department of Mathematics, Zhejiang University, Hangzhou 310027, P. R. China
E-mail address: lvgang1234@hanmail.net

Jun Xie
Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110178, P. R. China
E-mail address: 583193617@qq.com

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences,, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

Yuanfeng Jin
Department of Mathematics, Yanbian University, Yanji 133001, P. R. China
E-mail address: yfkim@ybu.edu.cn
E-mail address: 1121646592@qq.com

Topological spaces induced by fuzzy prime ideals in BCC-algebras

Sun Shin Ahn[1] and Keum Sook So[2,*]

[1]**Department of Mathematics Education, Dongguk University, Seoul 04620, Korea**

[2]**Department of Mathematics, Hallym University, Chuncheon, 24252, Korea**

Abstract. In this paper, we construct a topological space on the set of all fuzzy prime ideals of a commutative BCC-algebra X, and we discuss fuzzy prime ideals in commutative BCC-algebras.

1. Introduction

In 1966, Imai and Iséki ([7]) defined a class of algebras of type (2,0), called *BCK-algebras* which generalized the notion of an algebra of sets with the set subtraction as the only fundamental non-nullary operation, and also the notion of implication algebras ([8]). The class of all BCK-algebras is a quasi-variety. Iséki posed an interesting problem whether the class of BCK-algebras is a variety. That problem was solved by Wroński ([11]), who proved that BCK-algebra do not form a variety. In connection with this problem, Komori ([9]) introduced the notion of BCC-algebras, and Dudeck ([1, 2]) redefined the notion of BCC-algebras by using a dual form of the original definition in the sense of Komori. In [5], Dudek and Zhang introduced a new notion of ideals in BCC-algebras and described some connections between such ideals and congruences. Dudek and Jun ([3]) considered the fuzzification of ideals in BCC-algebras. Dudek, Jun and Stojakovic ([4]) described fuzzy BCC-ideals and its image. In this paper, we define a topology on the set of all fuzzy prime ideals of a commutative BCC-algebra X and the resulting space, denoted by F-spec(X), and obtain some related properties.

2. Preliminaries

By a *BCC-algebra* ([6]) we mean a nonempty set X with a constant 0 and a binary operation " $*$ " satisfying axioms: for all $x, y, z \in X$,

(I) $((x * y) * (z * y)) * (x * z) = 0$,

(II) $0 * x = 0$,

(III) $x * 0 = x$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call X a BCC-algebra. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. Then $\leq$ is a partial ordering on X. The relation " $\leq$ " is called a *BCC-order* on X. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

In a BCC-algebra X, the following hold: for any $x, y, z \in X$,

(2.1) $x * x = 0$,

(2.2) $(x * y) * x = 0$,

(2.3) $x \leq y \Rightarrow x * z \leq y * z$,

[0]*Correspondence: +82 33 248 2011(phone), +82 33 256 2011(fax) (K. S. So).

(2.4) $x \leq y \Rightarrow z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra, but there exist BCC-algebras which are not BCK-algebras (cf. [2]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(2.5) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Definition 2.1 ([5]). Let X be a BCC-algebra and $\emptyset \neq I \subseteq X$. I is called an *ideal* (or a *BCK-ideal*) of X if it satisfies the following conditions:

(i) $0 \in I$,

(ii) $x * y, y \in I$ imply $x \in I$ for all $x, y \in X$.

Theorem 2.2 ([5]). *In a BCC-algebra X, every ideal of X is a subalgebra of X.*

Definition 2.3 ([5]). Let X be a BCC-algebra and $\emptyset \neq I \subseteq X$. I is called a *BCC-ideal* of X if it satisfies the following conditions:

(i) $0 \in I$,

(ii) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.

Lemma 2.4 ([5]). *In a BCC-algebra X, any BCC-ideal of X is an ideal of X.*

Corollary 2.5 ([5]). *Any BCC-ideal X of a BCC-algebra X is a subalgebra of X.*

Remark. *In a BCC-algebra, a subalgebra need not be an ideal, and an ideal need not be a BCC-ideal in general* (*see* [2, 4]).

We now review some fuzzy logic concept. Let X be a BCC-algebra. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$. The set $\mu_t := \{x \in X | \mu(x) \geq t\}$, where $t \in [0, 1]$ is fixed, is called a *level set* of X. By $Im(\mu)$ we denote the image set of μ. A fuzzy set $\mu : X \to [0, 1]$ is called a *fuzzy subalgebra* ([3]) of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.6. For $t \in [0, 1]$, *fuzzy point* x_t is a fuzzy subset of X such that

$$x_t(y) := \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Definition 2.7 ([3]). A fuzzy set μ in a BCC-algebra X is called a *fuzzy BCK-ideal* if

(i) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Lemma 2.8 ([3]). *Let X be a BCC-algebra and μ be a fuzzy BCK-ideal of X.*

(i) *if $x * y = 0$, then $\mu(x) \geq \mu(y)$ for any $x, y \in X$,*

(ii) *$\mu(x * y) \geq \min\{\mu(x * z), \mu(z * y)\}$ for all $x, y, z \in X$.*

Definition 2.9 ([3]). A fuzzy set μ in a BCC-algebra X is called a *fuzzy BCC-ideal* if

(i) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(ii) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}$ for all $x, y, z \in X$.

Any fuzzy BCC-ideal is a fuzzy BCK-ideal in BCC-algebras.

Lemma 2.10 ([4]). *If μ is a fuzzy BCC-ideal of a BCC-algebra X, then, for any $x, y, z \in X$,*

(i) $x \leq y$ *implies* $\mu(y) \leq \mu(x)$,
(ii) $\mu(x * y) = \mu(0)$, *then* $\mu(x) \geq \mu(y)$,
(iii) $\mu(x * y) \geq \mu(x)$,
(iv) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$,
(v) $\mu(x * (y * z)) \geq \min\{\mu(x), \mu(y), \mu(z)\}$,
(i) $\mu((x * y) * (x * z)) \geq \mu(z * y)$.

Proposition 2.11 ([3]). *A fuzzy set μ in a BCC-algebra X is a fuzzy $BCK(BCC$, resp.$)$-ideal (subalgebra, resp.) if and only if for every $t \in [0, 1]$, the level subset μ_t is either empty or a $BCK(BCC$, resp.$)$-ideal (subalgebra, resp.) of X.*

Theorem 2.12. If I is an ideal of a BCC-algebra, then the characteristic function $\chi_I : X \to [0, 1]$ of I, is a fuzzy ideal of X with $I = X_{\chi_I}$, where $X_{\chi_I} =: \{x \in X | \chi_I(x) = \chi_I(0)$.

Proof. It is easily checked that χ_I is a fuzzy ideal of X. Given an ideal I, we have

$$\begin{aligned} X_{\chi_I} =& \{x \in X | \chi_I(x) = \chi_I(0)\} \\ =& \{x \in X | \chi_I(x) = 1\} \\ =& I. \end{aligned}$$

□

3. Toplogical Spaces by fuzzy prime ideals

Theorem 3.1. *Let X be a BCC-algebra and let $\{\eta_i\}_{\{i \in \Lambda\}}$ be a family of fuzzy $BCK(BCC$, resp.$)$-ideals of X. Then $\cap_{i \in \Lambda} \eta_i$ is a fuzzy $BCK(BCC$, resp.$)$-ideal of X.*

Proof. Straighforward. □

If μ is a fuzzy subset of a BCC-algebra X, then the ideal generated by μ which is denoted by $\langle \mu \rangle$ is defined as follows:

$$\langle \mu \rangle = \cap\{\eta | \mu \subseteq \eta, \eta \text{ is a fuzzy } BCC(BCK, \text{ resp.})\text{-ideal of } X\}.$$

For all x, y in a BCC-algebra X, $y * (y * x)$ is denoted by $x \wedge y$. A BCC-algebra X is said to be *commutative* ([2]) if $x * (x * y) = y * (y * x)$, for all $x, y \in X$, i.e., $x \wedge y = y \wedge x$. If X is a commutative BCC-algebra, then it is easy to check that

$$x \wedge y \leq x \text{ and } x \wedge y \leq y. \qquad (*)$$

A proper ideal P of a BCC-algebra X is said to be *prime* if for all ideals A, B of X such that $AB \subseteq P$, either $A \subseteq P$ or $B \subseteq P$, where

$$AB = \{a \wedge b | a \in A, b \in B\}.$$

In what follows, let X be a BCC-algebra, unless otherwise specified.

Definition 3.2. Let μ and η be two fuzzy subsets of X. Then the fuzzy set $\mu\eta$ is defined by

$$\mu\eta(x) = \sup\{\min\{\mu(y), \eta(z)\} | x = y \wedge z\}.$$

Clearly, since $x \wedge x = x$, each $x \in X$ is expressible as $x = y \wedge z$ for some $y, z \in X$.

Definition 3.3. A non-constant fuzzy ideal μ of X is said to be *fuzzy prime* if for all fuzzy ideals θ, σ such that $\theta\sigma \subseteq \mu$, either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.

Lemma 3.4. *Let μ and η be two fuzzy $BCK(BCC$, resp.$)$-ideals of X. Then $\mu\eta \subseteq \mu \cap \eta$.*

Proof. Let $x \in X$ such that $x = a \wedge b$ for some $a, b \in X$ and let μ, η be fuzzy $BCK(BCC$, resp.$)$-ideals of X. Then $\mu(a) \leq \mu(a \wedge b) = \mu(x)$ and $\eta(b) \leq \eta(a \wedge b) = \eta(x)$ from Lemma 2.10-(i). Hence $\min\{\mu(a), \eta(b)\} \leq (\mu \cap \eta)(x)$. Thus $\mu\eta(x) \leq (\mu \cap \eta)(x)$. □

Let Y be the set of all fuzzy prime ideals of X. Let $V(\theta) := \{\mu \in Y | \theta \subseteq \mu\}$, where θ is any fuzzy subset of X. Put $Y(\theta) = Y \setminus V(\theta)$, the complement of $V(\theta)$ in Y.

Lemma 3.5. *If θ is a fuzzy subset of X, then $V(\langle\sigma\rangle) = V(\sigma)$. In particular, $V(\langle x_\beta\rangle) = V(x_\beta)$ for any fuzzy point x_β of X.*

Proof. Clearly, $V(\sigma) \subseteq V(\langle\sigma\rangle)$. Now, let $\mu \in V(\langle\sigma\rangle) = \{\mu \in Y | \langle\sigma\rangle \subseteq \mu\}$. Then we have $\langle\sigma\rangle \subseteq \mu$. Since $\sigma \subseteq \langle\sigma\rangle$, we have $\sigma \subseteq \mu$ which implies that $\mu \in V(\sigma)$. Thus $V(\langle\sigma\rangle) \subseteq V(\sigma)$. □

Theorem 3.6. *Let $\tau = \{Y(\theta) | \theta$ is a fuzzy $BCK(BCC)$-ideal of $X\}$. Then the pair (Y, τ) is a topological space.*

Proof. Consider the fuzzy ideals θ and σ of Y defined by $\theta(x) := 0$ and $\sigma(x) := 1$ for all $x \in X$. Then $V(\theta) = Y$ and $V(\sigma) = \emptyset$ so that $\emptyset, Y \in \tau$. Now let θ_1 and θ_2 be two fuzzy $BCK(BCC)$-ideals of X. We show that $V(\theta_1) \cup V(\theta_2) = V(\theta_1 \cap \theta_2)$. To do this, if $\mu \in V(\theta_1) \cup V(\theta_2)$, then $\mu \in V(\theta_1)$ or $\mu \in V(\theta_2)$, i.e., $\theta_1 \subseteq \mu$ or $\theta_2 \subseteq \mu$ and hence $\theta_1 \cap \theta_2 \subseteq \mu$. Therefore $\mu \in V(\theta_1 \cap \theta_2)$. Thus $V(\theta_1) \cup V(\theta_2) \subseteq V(\theta_1 \cap \theta_2)$. On the other hand, if $\mu \in V(\theta_1 \cap \theta_2)$, then $\theta_1 \cap \theta_2 \subseteq \mu$. By Lemma 3.4, $\theta_1\theta_2 \subseteq \theta_1 \cap \theta_2 \subseteq \mu$ and $\theta_1\theta_2 \subseteq \mu$. Since μ is a fuzzy prime ideal of X, $\theta_1 \subseteq \mu$ or $\theta_2 \subseteq \mu$. Hence $\mu \in V(\theta_1) \cup V(\theta_2)$. Therefore $V(\theta_1 \cap \theta_2) \subseteq V(\theta_1) \cup V(\theta_2)$ and hence $V(\theta_1) \cup V(\theta_2) = V(\theta_1 \cap \theta_2)$. Thus $Y(\theta_1) \cap Y(\theta_2) = Y(\theta_1 \cap \theta_2)$, i.e., τ is closed under finite intersection.

Now we will prove that if $\{\theta_i\}_{i\in\Lambda}$ is a family of fuzzy $BCK(BCC)$-ideal of X, then

$$\cap_{i\in\Lambda} V(\theta_i) = V(\langle\cup_{i\in\Lambda}\theta_i\rangle). \qquad (**)$$

Let $\mu \in Y$. Then we have

$$\begin{aligned} \mu \in V(\theta_i), \forall i \in \Lambda &\Leftrightarrow \theta_i \subseteq \mu, \forall i \in \Lambda \\ &\Leftrightarrow \cup_{i\in\Lambda}\,\theta_i \subseteq \mu \\ &\Leftrightarrow \langle\cup_{i\in\Lambda}\theta_i\rangle \subseteq \mu \\ &\Leftrightarrow \mu \in V(\langle\cup_{i\in\Lambda}\theta_i\rangle). \end{aligned}$$

Therefore $(**)$ holds. Thus $\cap_{i\in\Lambda} Y(\theta_i) = Y(\langle\cup_{i\in\Lambda}\theta_i\rangle)$. This proves that τ is closed under arbitrary union. Thus (Y, τ) is a topological space. □

The topological space (Y,τ) described in Theorem 3.6 is called a *fuzzy spectrum* of Y or F*-spectrum* of Y and is denoted by F-spec (Y).

Theorem 3.7. *Let (Y,τ) be a topological space. Then the subfamily $\mathcal{B} = \{Y(x_\beta)|x \in X, \beta \in (0,1]\}$ of τ is a base for τ.*

Proof. It is enough to show that for all $Y(\theta) \in \tau$ and $\mu \in Y(\theta)$ there exists $Y(x_\beta) \in \mathcal{B}$ such that $\mu \in Y(x_\beta)$ and $Y(x_\beta) \subseteq Y(\theta)$. To do this, if $Y(\theta) \in \tau$ and $\mu \in Y(\theta)$, then $\theta \not\subseteq \mu$. Hence there exists $x \in X$ such that $\theta(x) > \mu(x)$. If $\theta(x) = \beta$, then

$$\mu \in Y(x_\beta). \tag{1}$$

If $\sigma \in V(\theta)$ is an arbitrary element, then $\sigma(x) \geq \theta(x) = \beta = x_\beta(x)$, which implies that $x_\beta \subseteq \sigma$. Therefore $\sigma \in V(x_\beta)$ and hence $V(\theta) \subseteq V(x_\beta)$. Thus we have

$$Y(x_\beta) \subseteq Y(\theta). \tag{2}$$

By (1) and (2), the proof is complete. □

4. **Fuzzy prime ideals of commutative BCC-algebras**

Proposition 4.1. *Let μ be a fuzzy $BCK(BCC)$-ideal of a BCC-algebra X. Then $X_\mu := \{x \in X|\mu(x) = \mu(0)\}$ is a $BCK(BCC)$-ideal of X.*

Proof. Straightforward. □

If μ is not a fuzzy $BCK(BCC)$-ideal of a BCC-algebra X, then Proposition 4.1 need not be true as shown in the following example.

Example 4.2. Let X be a BCC-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then $(X;*,0)$ is not a BCK-algebra, since $(2*1)*4 = 1 \neq 0 = (2*4)*1$. Let $S := \{0,1,2,3,4\}$ and $T := \{0,1,2\}$. Then S is a BCC-ideal of X and T is a BCC-subalgebra of X, but not a BCK-ideal of X, since $3*2 = 0 \in T$ and $3 \notin T$ ([5]). Let $\mu : X \to [0,1]$ be a map defined by $\mu(0) = \mu(1) = \mu(2) = 1$ and $\mu(3) = \mu(4) = \mu(5) = \frac{1}{2}$. Then μ is not a BCK-ideal, since $\frac{1}{2} = \mu(3) < \min\{\mu(3*2), \mu(2)\} = 1$. $X_\mu = \{0,1,2\}$ is not a BCK-ideal of X, since $1 = 3*2 \in X_\mu$ and $2 \in X_\mu$, but $3 \notin X_\mu$. Define a fuzzy subset ν in X by $\nu(0) = \nu(1) = \nu(5) = 0.9$ and $\nu(2) = \nu(3) = \nu(4) = 0.3$. Then ν is not a fuzzy BCC-ideal of X, since $\nu(4*2) = \nu(4) = 0.3 < \min\{\nu((4*5)*2) = \nu(1*2) = \nu(0) = 0.9, \nu(5) = 0.9\} = 0.9$. But $X_\nu = \{0,1,5\}$ is not a BCC-ideal of X, since $(4*5)*0 = 1,5 \in X_\nu$ but $4*0 = 4 \notin X_\nu$.

A proper ideal P of a BCC-algebra X is said to be *s-prime* if

$$x \wedge y \in P \Leftrightarrow x \in P \text{ or } y \in P, \text{ for all } x,y \in X.$$

Definition 4.3. Let X be a commutative BCC-algebra. A non-constant fuzzy BCK-ideal (or fuzzy ideal) μ of X is said to be *s-prime* if for all $x, y \in X$, either $\mu(x \wedge y) = \mu(x)$ or $\mu(x \wedge y) = \mu(y)$.

Lemma 4.4. *A non-constant fuzzy set μ of a commutative X is a fuzzy s-prime ideal of X if and only if for each $t \in [0,1]$, μ_t is either empty or an s-prime ideal of X if it is proper.*

Proof. Suppose that μ is a fuzzy s-prime ideal of X. For each $t \in [0,1]$, assume that $\mu_t \neq \emptyset$ and $x \wedge y \in \mu_t$, where $x, y \in X$. Then $\mu(x \wedge y) \geq t$. Since μ is a fuzzy s-prime ideal of X, we obtain $\mu(x \wedge y) = \mu(x) \geq t$ or $\mu(x \wedge y) = \mu(y) \geq t$. Hence either $x \in \mu_t$ or $y \in \mu_t$. Thus μ_t is an s-prime ideal of X.

Conversely, if μ is not an s-prime ideal of X, then $\mu(x \wedge y) \neq \mu(x)$ and $\mu(x \wedge y) \neq \mu(y)$ for all $x, y \in X$. Let $x \wedge y \in \mu_t$ for all $x, y \in X$. Then $\mu(x \wedge y) \geq t$. Since μ is not an s-prime ideal of X, we obtain $\mu(x) < t$ and $\mu(y) < t$. Hence $x \notin \mu_t$ and $y \notin \mu_t$, which is a contradiction. □

Lemma 4.5. *Let μ be a fuzzy prime ideal of a BCC-algebra X. Then for any $t \in [0,1]$, μ_t is either empty or a prime ideal of X if it is proper.*

Proof. Let $t \in [0,1]$ and $\mu_t \neq \emptyset$. By Proposition 2.11, μ_t is a BCK-ideal of X. Now let A, B be two ideals of X such that

$$AB \subseteq \mu_t = \{x \in X | \mu(x) \geq t\}.$$

If we define the fuzzy subsets $\theta := \chi_A$ and $\sigma := \chi_B$, then it is easy to show that $\theta\sigma \subseteq \mu$, which implies $\theta \subseteq \mu$ or $\sigma \subseteq \mu$, since μ is a fuzzy prime ideal of X. It follows that $A \subseteq \mu_t$ or $B \subseteq \mu_t$. □

Lemma 4.6. *Let X be a commutative BCC-algebra X. If $z \leq x$ and $z \leq y$ for all $x, y, z \in X$, then $z \leq x \wedge y$.*

Proof. Since $z \leq x$ and $z \leq y$, we have $z * x = 0$ and $z * y = 0$. Then $z = z * 0 = z * (z * x) = x * (x * z)$ and $z = z * 0 = z * (z * y) = y * (y * z)$, since X is commutative. Hence $z = x * (x * z) = x * (x * (y * (y * z))) \leq x * (x * y) = y \wedge x = x \wedge y$. This competes the proof. □

A BCC-algebra X is said to be *positive implicative* ([2]) if for any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Lemma 4.7. *Let X be a positive implicative BCC-algebra which is commutative. Then a proper ideal P of X is an s-prime ideal of X if and only if P is a prime ideal of X.*

Proof. Suppose that P is an s-prime ideal such that $AB \subseteq P$ for some ideals A, B of X. In order to prove that $A \subseteq P$ or $B \subseteq P$, let us assume that neither $A \subseteq P$ nor $B \subseteq P$. Then there exist $a \in A$, $b \in B$ such that $a \notin P$ and $b \notin P$. Since $a \wedge b \in AB$ and $AB \subseteq P$, we have $a \wedge b \in P$. Since P is an s-prime ideal of X, $a \in P$ or $b \in P$, which is a contradiction. Thus $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Conversely, suppose that for any ideals A, B of X, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. We prove that P is an s-prime ideal of X. Let $a \wedge b \in P$, where $a, b \in X$. Put $A(a) := \{x \in X | x \leq a\}$ and $A(b) := \{y \in X | y \leq b\}$. Clearly, $0, a \in A(a)$. Let $x * y, y \in A(a)$. Then $x * y \leq a$ and $y \leq a$. Since X is positive implicative, we have $(x * y) * a = (x * a) * (y * a) = (x * a) * 0 = x * a = 0$. Hence $x \in A(a)$. Therefore $A(a)$ is a BCK-ideal of X. Similarly, $A(b)$ is a BCK-ideal of X. We claim that $A(a)A(b) \subseteq P$. Let $x \in A(a)$ and $y \in A(b)$. Then $x \leq a$ and $y \leq b$. Since X is commutative, we obtain $x \wedge y \leq x$. Since $(X, \leq)$ is a partially ordered set, we have $x \wedge y \leq a$. Similarly, $x \wedge y \leq b$. By Lemma 4.6, we obtain $x \wedge y \leq a \wedge b$ and $a \wedge b \in P$. Since P is a BCK-ideal, $x \wedge y \in P$. Hence $A(a)B(b) \subseteq P$. By hypothesis, $A(a) \subseteq P$ or $A(b) \subseteq P$. Since $a \in A(a), b \in A(b)$, we have $a \in P$ or $b \in P$. Thus P is an s-prime ideal of X. □

Theorem 4.8. *Let μ be a fuzzy prime ideal of a BCC-algebra X. Then X_μ is a prime ideal of X.*

Proof. Clearly, $X_\mu = \mu_{\mu(0)}$. By Lemma 4.5, X_μ is a prime ideal of X. □

Theorem 4.9. *Let μ be a fuzzy prime ideal of a positive implicative BCC-algebra X which is commutative. Then μ is a fuzzy s-prime ideal of X.*

Proof. Let μ be a fuzzy prime ideal of X. By Lemma 4.5, μ_t is a prime ideal of X. Using Lemma 4.7, μ_t is an s-prime ideal of X. It follows from Lemma 4.4 that μ is a fuzzy s-prime ideal of X. □

The following example shows that the converse of Theorem 4.9 does not hold.

Example 4.10. Let $X := \{0, e\}$ be a set with the following table:

$*$	0	e
0	0	0
e	e	0

Then $(X; *, 0)$ is a positive implicative BCC-algebra which is commutative. Define the fuzzy subset μ of X by $\mu(0) = 0.7, \mu(e) = 0$. Clearly μ is a fuzzy s-prime ideal of X. Now consider the fuzzy ideals σ and θ of X which are defined by $\sigma(x) = \frac{1}{2}$ for all $x \in X$ and $\theta(e) = 0, \theta(0) = 1$. Then we have $\sigma\theta \subseteq \mu$ but $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Thus μ is not a fuzzy prime ideal of X.

Theorem 4.11. *Let X be a commutative BCC-algebra. Then I is an s-prime ideal of X if and only if χ_I is a fuzzy s-prime ideal of X.*

Proof. Suppose that I is an s-prime ideal of X. By Theorem 2.12, χ_I is a fuzzy ideal of X. Since I is proper, χ_I is a non-constant function. Let $x, y \in X$. If $x \in I$ or $y \in I$, then $x \wedge y \in I$. Hence $\chi_I(x \wedge y) = 1 = \chi_I(x) \vee \chi_I(y)$. If $x \notin I$ and $y \notin I$, then $x \wedge y \notin I$. Hence $\chi_I(x \wedge y) = 0 = \chi(x) \vee \chi_I(y)$. Thus χ_I is a fuzzy s-prime ideal of X.

Conversely, since $I = X_{\chi_I}$, if χ_I is a fuzzy s-prime ideal of X, it follows by Lemma 4.4 that I is an s-prime ideal of X. □

Corollary 4.12. *Let X be a positive implicative BCC-algebra which is commutative. Then P is a prime ideal of X if and only if χ_P is a fuzzy prime ideal of X.*

Proof. Let P be a prime ideal of X. Then χ_P is a fuzzy ideal of X. Now let θ, σ be two fuzzy ideals such that $\theta\sigma \subseteq \chi_P$. We shall show that

$$\theta \subseteq \chi_P \text{ or } \sigma \subseteq \chi_P. \qquad (***)$$

If $(***)$ does not hold, then there exist $x, y \in X \setminus P$ such that $\theta(x) > 0$ and $\sigma(y) > 0$. By Lemma 4.7, we have $x \wedge y \notin P$. Since $\theta\sigma \subseteq \chi_P$, we have

$$0 < \min\{\theta(x), \sigma(y)\} \leq \theta\sigma(x \wedge y) \leq \chi_P(x \wedge y).$$

In other words, $x \wedge y \in P$, which is a contradiction. Hence $(***)$ holds.

Conversely, let χ_P be a fuzzy prime ideal. By Theorem 4.9, χ_P is a fuzzy s-prime ideal. By Theorem 4.11, P is an s-prime ideal of X. By Lemma 4.7, P is a prime ideal of X. □

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RIESZ FUZZY NORMED SPACES AND STABILITY OF A LATTICE PRESERVING FUNCTIONAL EQUATION

CHOONKIL PARK, EHSAN MOVAHEDNIA*, SEYED MOHAMMAD SADEGH MODARRES MOSADEGH, AND MOHAMMAD MURSALEEN

Abstract. The main objective of this paper is to introduce and to study fuzzy normed Riesz spaces. By the direct method, we prove the Hyers-Ulam stability of the following lattice preserving functional equation in fuzzy Banach Riesz space

$$\mathcal{N}_2\left(f(\tau x \vee \eta y) - \tau f(x) \vee \eta f(y), t\right) \geq \mathcal{N}_1(\varphi(\tau x \vee \eta y, \tau x \wedge \eta y), t)$$

where $(\mathcal{X}, \mathcal{N}_1), (\mathcal{Y}, \mathcal{N}_2)$ are fuzzy normed Riesz space and fuzzy Banach Riesz space, respectively; and $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $\tau, \eta \geq 1$ and $\alpha \in [0, 1)$.

1. Introduction

Riesz spaces are named after Frigyes Riesz who first defined them in [1] . Riesz spaces are real vector spaces equipped with a partial order. Under this partial order the Riesz space must satisfy some axioms, including the axiom that it is a lattice.

For the basic theory of vector lattices (Riesz spaces) and Banach lattices and for unexplained terminology we refer to [2, 3, 4].

In 1984, Katrasas [5] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [6, 7]. In particular, Bag and Samanta [8], following Cheng and Mordeson [9], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [10]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces.

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation $\mathcal{D}$ must be close to an exact solution of $\mathcal{D}$? If the problem accepts a solution, we say that the equation $\mathcal{D}$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [11] in 1940. In 1941, Hyers [12] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The result of Hyers was generalized by Rassias [13] for linear mapping by considering an unbounded Cauchy difference. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]). Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [26]–[31].

In this paper, Riesz fuzzy normed spaces are defined and the stability condition are verified.

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*Corresponding author.

2. Preliminary estimates

A non empty set $\mathcal{M}$ with a relation "$\leq$" is said to be an ordered set whenever the following conditions are satisfied :

1. $x \leq x$ for every $x \in \mathcal{M}$.
2. $x \leq y$ and $y \leq x$ implies that $x = y$.
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for all $x, y \in \mathcal{M}$ either $x \leq y$ or $y \leq x$, then $\mathcal{M}$ is called a totally ordered set. Let $\mathcal{A}$ be subset of an ordered set $\mathcal{M}$. $x \in \mathcal{M}$ is called an upper bound of $\mathcal{A}$ if $y \leq x$ for every $y \in \mathcal{A}$. $z \in \mathcal{M}$ is called a lower bound of $\mathcal{A}$ if $y \geq z$ for all $y \in \mathcal{A}$. Moreover, if there is an upper bound of $\mathcal{A}$, then $\mathcal{A}$ is said to be bounded from above. If there is a lower bound of $\mathcal{A}$, then $\mathcal{A}$ is said to be bounded from below. If $\mathcal{A}$ is bounded from above and from below, then we will briefly say that $\mathcal{A}$ is order bounded.

An order set $(\mathcal{M}, \leq)$ is called a *lattice* if any two elements $x, y \in \mathcal{M}$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

A real vector space $\mathcal{E}$ which is also an ordered set is an ordered vector space if the order and the vector space structure are compatible in the following sense:

1. If $x, y \in \mathcal{E}$ such that $x \leq y$, then $x + z \leq y + z$ for all $z \in \mathcal{E}$.
2. If $x, y \in \mathcal{E}$ such that $x \leq y$, then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

$(\mathcal{E}, \leq)$ is called a Riesz space if $(\mathcal{E}, \leq)$ is a lattice and ordered vector space.

A norm $\|\cdot\|$ on Riesz space $\mathcal{E}$, is called a lattice norm if $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. In the latter case $(\mathcal{E}, \|\cdot\|)$ is called a normed Riesz space.

$(\mathcal{E}, \|\cdot\|)$ is called a Banach lattice if for all $x, y \in \mathcal{E}$

1. $(\mathcal{E}, \|\cdot\|)$ is a Banach space.
2. $\mathcal{E}$ is a Riesz space.
3. $\|\cdot\|$ is a lattice norm.

Example 2.1. *Suppose that $\mathcal{X}$ is compact Hausdorff space. We denote by $\mathcal{C}(\mathcal{K})$ the Banach space of all real continuous functions on $\mathcal{X}$. Let "$\leq$" be a point-wise order on $\mathcal{C}(\mathcal{K})$, $f \leq g$ if and only if $f(t) \leq g(t)$ for all $t \in K$. It is easy to see that $(\mathcal{C}(\mathcal{K}), \leq)$ is a Banach lattice.*

Let $\mathcal{E}$ be a Riesz space and let the positive cone $\mathcal{E}^+$ of $\mathcal{E}$ consist of all $x \in \mathcal{E}$ such that $x \geq 0$. For every $x \in \mathcal{E}$ let

$$x^+ = x \vee 0 \quad x^- = -x \vee 0 \quad |x| = x \vee -x.$$

Let $\mathcal{E}$ be a Riesz space. For all $x, y, z \in \mathcal{E}$ and $a \in \mathcal{R}$ the following assertions hold

1. $x + y = x \vee y + x \wedge y$, $-(x \vee y) = -x \wedge y$.
2. $x + (y \vee z) = (x + y) \vee (x + z)$, $x + (y \wedge z) = (x + y) \wedge (x + z)$.
3. $|x| = x^+ + x^-$, $|x + y| \leq |x| + |y|$.
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$.
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$, $(x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$.

A Riesz space $\mathcal{E}$ is **Archimedean** if $x \leq 0$ holds whenever the set $\{nx : n \in \mathbf{N}\}$ is bounded from above.

Definition 2.1. [2] *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach lattices. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ is called positive if $\mathcal{T}(\mathcal{X}^+) = \{\mathcal{T}(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$.*

Theorem 2.1. [3] *For an operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ between two Riesz spaces the following statements are equivalent:*

1. *$\mathcal{T}$ is a lattice homomorphism.*
2. *$\mathcal{T}(x^+) = \mathcal{T}(x)^+$ for all $x \in \mathcal{X}$.*
3. *$\mathcal{T}(x \wedge y) = \mathcal{T}(x) \wedge \mathcal{T}(y)$.*
4. *If $x \wedge y = 0$ in $\mathcal{X}$, then $\mathcal{T}(x) \wedge \mathcal{T}(y) = 0$ holds in $\mathcal{Y}$.*

5. $\mathcal{T}(|x|) = |\mathcal{T}(x)|$.

Definition 2.2. [4] *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach lattices and let $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ be a positive mapping. We define*

($\mathcal{P}_1$) *lattice homomorphism functional equation:*

$$\mathcal{T}(|x| \vee |y|) = \mathcal{T}(|x|) \vee \mathcal{T}(|y|);$$

($\mathcal{P}_2$) *semi-homogeneity: for all $x \in \mathcal{X}$ and every number $\alpha \in \mathcal{R}^+$*

$$\mathcal{T}(\alpha|x|) = \alpha\mathcal{T}(|x|).$$

Remark 2.1. [4] *Given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$, let a positive mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfy the property ($\mathcal{P}_1$). Then the following statements are valid.*

1. *$f(|x \vee y|) \leq f(|x|) \vee f(|y|)$ for all $x, y \in \mathcal{X}$.*
2. *The semi-homogeneity implies that $f(0) = 0$.*
3. *f is an increasing operator, in the sense that if $x, y \in \mathcal{X}$ are such that $|x| \leq |y|$, then $f(|x|) \leq f(|y|)$.*

3. Main results

Definition 3.1. *Let $(\mathcal{X}, \leq)$ be a Riesz space. A function $\mathcal{N} : \mathcal{X} \times \mathcal{R} \to [0, 1]$ is called a Riesz fuzzy norm on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathcal{R}$*

($\mathcal{N}_1$) $\mathcal{N}(x, t) = 0$ for $t \leq 0$;
($\mathcal{N}_2$) $x = 0$ if and only if $\mathcal{N}(x, t) = 1$ for all t> 0 ;
($\mathcal{N}_3$) $\mathcal{N}(cx, t) = \mathcal{N}(x, t/|c|)$ if c$\neq 0$;
($\mathcal{N}_4$) $\mathcal{N}(x + y, t + s) \geq \min\{\mathcal{N}(x, t), \mathcal{N}(y, s)\}$;
($\mathcal{N}_5$) $\mathcal{N}(x, .)$ is a non-decreasing function of R and

$$lim_{t\to\infty}\, \mathcal{N}(x, t) = 1 \ ;$$

($\mathcal{N}_6$) for $x \neq 0, \mathcal{N}(x, .)$ is continuous on R ;
($\mathcal{N}_7$) $\mathcal{N}(x, t) \geq \mathcal{N}(y, t)$ whenever $|x| \leq |y|$.
Then $(\mathcal{X}, \leq, \mathcal{N})$ is called a Riesz fuzzy normed space.

Example 3.1. *Let $(\mathcal{X}, \leq, \|.\|)$ be a normed Riesz space. One can easily verify that for each $k > 0$,*

$$\mathcal{N}_k(x, t) = \begin{cases} \dfrac{t}{t + k\|x\|} & \text{if } t > 0 \\ \\ 0 & \text{if } t \leq 0 \end{cases}$$

defines a Riesz fuzzy norm on $\mathcal{X}$.

Note that $(\mathcal{N}_1) - (\mathcal{N}_6)$ have been checked in [8]. We show that $(\mathcal{N}_7)$ is satisfied. Suppose that $|x| \leq |y|$ for all $x, y \in \mathcal{X}$. Then $||x|| \leq ||y||$ since $(\mathcal{X}, \leq, \|\cdot\|)$ is a normed Riesz space. So

$$\frac{t}{t + k\|x\|} \geq \frac{t}{t + k\|y\|}$$

and so

$$\mathcal{N}(x, t) \geq \mathcal{N}(y, t)$$

for all $t > 0$ and $k > 0$. Therefore, $(\mathcal{X}, \leq, \mathcal{N})$ is a Riesz fuzzy normed space.

Example 3.2. *Let $(\mathcal{X}, \leq, \|.\|$) be a normed Riesz space. We define*

$$\mathcal{N}(x, t) = \begin{cases} 0 & \text{if } t \leq \|x\| \\ \\ 1 & \text{if } t > \|x\|. \end{cases}$$

It is very easy to show that $(\mathcal{X}, \leq, \mathcal{N})$ is a Riesz fuzzy normed space.

Remark 3.1. *Convergent and Cauchy sequences in Riesz fuzzy normed space are same as in fuzzy normed space.*

Definition 3.2. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be convergent if there exists an $x \in \mathcal{X}$ such that $lim_{n\to\infty}\mathcal{N}(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by*

$$\mathcal{N} - lim_{n\to\infty}\mathcal{N}(x_n - x, t) = x.$$

Definition 3.3. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be Cauchy if for each $\epsilon > 0$ and each $\delta > 0$ there exists an $n_0 \in \mathbf{N}$ such that*

$$\mathcal{N}(x_m - x_n, \delta) > 1 - \epsilon \ (m, n \geq n_0).$$

Definition 3.4. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. A sequence y_n in $\mathcal{X}$ is called order fuzzy convergent to y as $n \to \infty$ if there exists a sequence $x_n \downarrow 0$ in $\mathcal{X}$ as $n \to \infty$ and*

$$\mathcal{N}(y_n - y, t) \geq \mathcal{N}(x_n, t)$$

for all $n \in \mathbf{N}$. We write $y = of - \lim_{n\to\infty} y_n$.

It is well-known that every convergent sequence in a fuzzy normed Riesz space is Cauchy. If each Cauchy sequence is convergent, then the Riesz fuzzy norm is said to be complete and the fuzzy normed Riesz space is called a fuzzy Banach Riesz space.

Theorem 3.1. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space and let $\{x_n\}, \{y_n\}$ be sequences in $\mathcal{X}$ such that*

$$x = of - \lim_{n\to\infty} x_n \quad and \quad y = of - \lim_{n\to\infty} y_n.$$

Then

$$\begin{aligned} x_n + y_n &= of - \lim_{n\to\infty} x + y, \\ x_n \vee y_n &= of - \lim_{n\to\infty} x \vee y, \\ x_n \wedge y_n &= of - \lim_{n\to\infty} x \wedge y. \end{aligned}$$

Theorem 3.2. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then lattice operators are continuous.*

Proof. Assume that

$$\lim_{n\to\infty} \mathcal{N}(x_n - x, s) = 1 \quad \lim_{n\to\infty} \mathcal{N}(y_n - y, t) = 1 \tag{3.1}$$

for all $t, s > 0$. Therefore,

$$\begin{aligned} \mathcal{N}(x_n \wedge y_n - x \wedge y, t + s) &= \mathcal{N}(x_n \wedge y_n - x_n \wedge y + x_n \wedge y - x \wedge y, t + s) \\ &\geq \min\{\mathcal{N}(x_n \wedge y_n - x_n \wedge y, t), \mathcal{N}(x_n \wedge y - x \wedge y, s)\} \\ &\geq \min\{\mathcal{N}(y_n - y, t), \mathcal{N}(x_n - x, s)\}. \end{aligned}$$

So

$$\lim_{n\to\infty} \mathcal{N}(x_n \wedge y_n - x \wedge y, t + s) = 1.$$

It is easy to see that the other lattice operations are continuous. □

Theorem 3.3. *Every fuzzy normed Riesz space is Archimedean.*

Proof. Let $(\mathcal{X}, \leq, \mathcal{N})$ be fuzzy normed Riesz space. We show that $\mathcal{X}$ has Archimedean properties. Suppose that $x, y \in \mathcal{X}_+$ and $nx \leq y$ for all $n \in \mathbf{N}$.

$$\mathcal{N}(nx, t) \geq \mathcal{N}(y, t) \qquad t > 0$$

and so

$$\mathcal{N}\left(x, \frac{t}{n}\right) \geq \mathcal{N}(y, t) \qquad t > 0.$$

Therefore,

$$\mathcal{N}(x, t) \geq \mathcal{N}(y, nt) \qquad t > 0$$

and

$$\mathcal{N}(x, t) \geq \mathcal{N}(n^{-1}y, t) \qquad t > 0.$$

Since $n \in \mathbf{N}$ is arbitrary as $n \to \infty$, we have $x = 0$. Hence $\mathcal{X}$ has Archimedean properties. □

Theorem 3.4. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then the positive cone $\mathcal{E}_+$ is closed.*

Proof. Assume that $x_n \in \mathcal{E}_+$

$$\lim_{n\to\infty} \mathcal{N}\left(x_n - x, \frac{t}{2}\right) = 1 \quad \text{for all } t > 0, x \in \mathcal{E}.$$

By Theorem 3.2, we have

$$\lim_{n\to\infty} \mathcal{N}\left(x_n \vee 0 - x \vee 0, \frac{t}{2}\right) = 1 \quad \text{for all } t > 0, x \in \mathcal{E}.$$

So $x_n \vee 0 = x_n$ since $x_n \in \mathcal{E}_+$. Therefore,

$$\lim_{n\to\infty} \mathcal{N}\left(x_n - x \vee 0, \frac{t}{2}\right) = 1 \quad \text{for all } t > 0, x \in \mathcal{E}$$

and hence by $(\mathcal{N}_4)$, we get

$$\mathcal{N}\left(x - x \vee 0, t\right) \geq \min\left\{\mathcal{N}\left(x_n - x \vee 0, \frac{t}{2}\right), \mathcal{N}\left(x_n - x, \frac{t}{2}\right)\right\}$$

for all $t > 0$ and $x \in \mathcal{E}$. Two terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, and so $x = x \vee 0$. Hence $x \in \mathcal{E}_+$. Thus the proof is complete. □

Theorem 3.5. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. For every increasing convergent sequence $\{x_n\} \subset \mathcal{E}$*

$$\lim_{n\to\infty} \mathcal{N}(x_n - u, t) = 1 \quad \textit{for all} \quad t > 0,$$

where $u = \sup\{x_n \ : \ n \in \mathbf{N}\}$.

Proof. Suppose that $\{x_n\}$ is an increasing convergent sequence and

$$\lim_{n\to\infty} \mathcal{N}(x_n - x, t) = 1 \quad \text{for all} \quad t > 0 \text{ and all } n \in \mathbf{N}. \tag{3.2}$$

For every $m \geq n$, we have

$$x_m - x_n \in \mathcal{E}_+.$$

It follows from Theorem 3.4 that $x - x_n \geq 0$ and $x_n \leq u \leq x$ for all $n \in \mathbf{N}$. So by $(\mathbf{N}_7)$

$$\mathcal{N}(x_n - x, t) \leq \mathcal{N}(x_n - u, t) \text{ for all } t > 0.$$

Therefore, as $n \to \infty$, we have $\lim_{n\to\infty} \mathcal{N}(x_n - u, t) = 1$ and hence $u = x$. This completes the proof. □

Definition 3.5. *The sequence $\{x_n\}$ is called uniformly bounded if there exist $e \in \mathcal{E}_+$ and $a_n \in l^1$ such that $x_n \leq a_n \cdot e$.*

Definition 3.6. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then $\mathcal{E}$ is called uniformly complete if $\sup\{\sum_{i=1}^{n} x_i \ : \ n \in \mathbf{N}\}$ exists for every uniformly bounded sequence $x_n \subset \mathcal{E}_+$.*

Theorem 3.6. *Every fuzzy Banach Riesz space is uniformly complete.*

Proof. Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy Banach Riesz space and $\{x_n\} \subset \mathcal{E}_+$ be a sequence such that $x_n \leq a_n e$ for a suitable sequence $\{a_n\} \in l^1$ and some $e \in \mathcal{E}_+$. We show that $\sup\{\sum_{i=1}^{n} x_i \ : \ n \in \mathbf{N}\}$ exists. We set

$$y_n = x_1 + x_2 + ... + x_n \quad \text{and} \quad b_n = \sum_{j=n+1}^{\infty} a_j. \tag{3.3}$$

By (3.3) and $(\mathcal{N}_7)$ we have

$$\begin{aligned} \mathcal{N}(y_{n+p} - y_n, t) &= \mathcal{N}(x_{n+1} + \cdots + x_{n+p}, t) \\ &\geq \mathcal{N}(\sum_{j=1}^{\infty} a_{n+j} \cdot e, t) \\ &= \mathcal{N}(b_n \cdot e, t) \end{aligned}$$

for all $t > 0$. As $n \to \infty$, we get

$$\lim_{n\to\infty} \mathcal{N}(y_{n+p} - y_n, t) = 1.$$

So (y_n) is a Cauchy sequence in fuzzy Banach Riesz space and therefore there exists $y \in \mathcal{E}$ such that $y_n \longrightarrow^{\mathcal{N}} y$. Since y_n is increasing and convergence sequence, by Theorem 3.5, we have

$$\lim_{n\to\infty} \mathcal{N}(y_n - \vee y_n, t) = 1$$

that is, $y_n \longrightarrow^{\mathcal{N}} \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in \mathbf{N}\}$. Using a unique limit we have

$$y = \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in \mathbf{N}\}.$$

Thus the proof is complete. □

Definition 3.7. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. $\mathcal{A} \subset \mathcal{E}$ is solid if*
(1) *$f \in \mathcal{A}$ if and only if $|f| \in \mathcal{A}$;*
(2) *if $0 \leq f \in \mathcal{A}$ and $g \in \mathcal{E}_+$ then $f \wedge g \in \mathcal{A}$.*

Definition 3.8. *Every solid subset I of $\mathcal{E}$ is called an ideal in $\mathcal{E}$.*

Definition 3.9. *An ordered closed ideal is referred to as a band.*

Theorem 3.7. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then the closure of every solid subset of $\mathcal{E}$ is solid.*

Proof. Let $\mathcal{A} \subset \mathcal{E}$ be solid and $f \in \overline{\mathcal{A}}$. We show that $|f| \in \overline{\mathcal{A}}$. There exists $\{f_n\} \in \mathcal{A}$ such that $f_n \longrightarrow^{\mathcal{N}} f$. It follows from $||f_n| - |f|| \leq |f_n - f|$ and $(\mathcal{N}_7)$ that

$$\begin{aligned} \mathcal{N}(|f_n| - |f|, t) &\geq \mathcal{N}(|f_n - f|, t) \\ &= \mathcal{N}(f_n - f, t) \end{aligned}$$

for all $n \in \mathbf{N}$ and $t > 0$. Therefore, $|f_n| \to^{\mathcal{N}} |f|$ as n tends to infinity. Hence $|f| \in \overline{\mathcal{A}}$, since $\mathcal{A}$ is solid and $f_n \in \mathcal{A}$. Conversely, assume that $|f| \in \overline{\mathcal{A}}$. Then there exists $f_n \in \mathcal{A}_+$ such that $f_n \longrightarrow^{\mathcal{N}} |f|$. By (3.2) we have

$$f_n \wedge f \longrightarrow^{\mathcal{N}} f \wedge |f| = f.$$

Therefore, $f \in \overline{\mathcal{A}}$, since $f_n \wedge f \in \mathcal{A}$.
Now, assume that $0 \leq f \in \overline{\mathcal{A}}$ and $g \in \mathcal{E}_+$. There exists a sequence $\{f_n\} \in \mathcal{A}_+$ such that $f_n \longrightarrow^{\mathcal{N}} f$. Hence by (3.2)

$$f_n \wedge g \rightarrow^{\mathcal{N}} f \wedge g.$$

So $f_n \wedge g \in \mathcal{A}$. Since $f_n \in \mathcal{A}$ and $\mathcal{A}$ is solid, $f \wedge g \in \overline{\mathcal{A}}$. □

Theorem 3.8. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then every band in $\mathcal{E}$ is closed.*

Proof. Suppose that $\mathcal{B}$ is a band and assume that $\{f_n\} \subset \mathcal{B}$ is a sequence such that $f_n \longrightarrow^{\mathcal{N}} f$ for some $f \in \mathcal{E}$. It follows from (3.2) that

$$|f_n| \wedge |f| \rightarrow^{\mathcal{N}} |f|$$

as $n \to \infty$. For every $n \in \mathbf{N}$, let

$$g_n = (|f_n| \vee \cdots \vee |f_1|) \wedge |f|.$$

Then $\{g_n\}$ is an increasing sequence and

$$g_n = (|f_n| \wedge |f|) \vee ... \vee (|f_1| \wedge |f|).$$

So $|f_n| \wedge |f| \leq g_n \leq |f|$. Therefore, by $(\mathcal{N}_7)$ we have

$$\mathcal{N}(|f| - g_n, t) \geq \mathcal{N}(|f| - |f_n| \wedge |f|, t)$$

for all $t > 0$. Hence $g_n \longrightarrow^{\mathcal{N}} |f|$ as $n \to \infty$. □

4. Hyers-Ulam stability of lattice homomorphisms in fuzzy normed Riesz spaces

Using the direct method, we prove the Hyers-Ulam stability of lattice homomorphisms in fuzzy Banach Riesz space as below.

Theorem 4.1. *Let f be a positive operator from a fuzzy normed Riesz space $(\mathcal{X}, \mathcal{N}_1)$ to a fuzzy Banach Riesz space $(\mathcal{Y}, \mathcal{N}_2)$ such that*

$$\mathcal{N}_2\left(f(\tau x \vee \eta y) - \tau f(x) \vee \eta f(y), t\right) \geq \mathcal{N}_1(\varphi(\tau x \vee \eta y, \tau x \wedge \eta y), t) \tag{4.4}$$

for all $x, y \in \mathbf{X}$ and $t > 0$. Here $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $\tau, \eta \geq 1$ and for which there are a number $\alpha \in [0, 1)$ and a unique positive operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ satisfying the properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ for $x \in \mathcal{X}_+$ and the inequality

$$\mathcal{N}_2(\mathcal{T}(x) - f(x), t) \geq \mathcal{N}_1(\varphi(x, x), \frac{\tau - \tau^\alpha}{\tau^\alpha}, t).$$

Proof. Putting $y = x$ and $\tau = \eta$ in (**??**), we have

$$\begin{aligned} \mathcal{N}_2(f(\tau x) - \tau f(x), t) &\geq \mathcal{N}_1(\varphi(\tau x, \tau x), t) \\ &\geq \mathcal{N}_1\left(\tau^\alpha \varphi(x, x), t\right) \\ &= \mathcal{N}_1\left(\varphi(x, x), \frac{t}{\tau^\alpha}\right). \end{aligned}$$

Therefore,

$$\mathcal{N}_2\left(\frac{1}{\tau} f(\tau x) - f(x), \tau^{\alpha-1} t\right) \geq \mathcal{N}_1(\varphi(x, x), t). \tag{4.5}$$

Now, replacing x by τx in (4.5) and using the assumption that $\varphi(\tau x, \tau x) \leq \tau^{\alpha}\varphi(x,x)$ and the property $(\mathcal{N}_3)$ and $(\mathcal{N}_7)$ of Definition 3.1, we obtain

$$\begin{aligned}\mathcal{N}_2\left(\frac{1}{\tau}f(\tau^2x)-f(\tau x),\tau^{\alpha-1}t\right) &\geq \mathcal{N}_1(\varphi(\tau x,\tau x),t)\\ &\geq \mathcal{N}_1\left(\tau^{\alpha}\varphi(x,x),t\right)\\ &= \mathcal{N}_1\left(\varphi(x,x),\frac{t}{\tau^{\alpha}}\right).\end{aligned}$$

Hence

$$\mathcal{N}_2\left(\frac{1}{\tau^2}f(\tau^2x)-\frac{1}{\tau}f(\tau x),\tau^{2\alpha-2}t\right) \geq \mathcal{N}_1(\varphi(x,x),t). \tag{4.6}$$

By comparing (4.5) and (4.6) and using property $(\mathcal{N}_4)$, we obtain

$$\mathcal{N}_2\left(\frac{1}{\tau^2}f(\tau^2x)-f(x),\left(\tau^{\alpha-1}+\tau^{2(\alpha-1)}\right)t\right) \geq \mathcal{N}_1(\varphi(x,x),t). \tag{4.7}$$

Again, replacing x by τx in (4.7), we have

$$\mathcal{N}_2\left(\frac{1}{\tau^3}f(\tau^3x)-\frac{1}{\tau}f(\tau x),\left(\tau^{2(\alpha-1)}+\tau^{3(\alpha-1)}\right)t\right) \geq \mathcal{N}_1(\varphi(x,x),t). \tag{4.8}$$

By comparing (4.5) and (4.8) and the property $(\mathcal{N}_4)$, we obtain

$$\mathcal{N}_2\left(\frac{1}{\tau^3}f(\tau^3x)-f(x),\left(\tau^{(\alpha-1)}+\tau^{2(\alpha-1)}+\tau^{3(\alpha-1)}\right)t\right) \geq \mathcal{N}_1(\varphi(x,x),t).$$

With this process, we obtain

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^nx)-f(x),\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1(\varphi(x,x),t) \tag{4.9}$$

for all $n \in \mathbf{N}$. If $m \in \mathbf{N}$ and $n > m > 0$, then $n-m \in \mathbf{N}$. Replacing n by $n-m$ in (4.9), we get

$$\mathcal{N}_2\left(\frac{1}{\tau^{n-m}}f(\tau^{n-m}x)-f(x),\sum_{k=1}^{n-m}\tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1(\varphi(x,x),t). \tag{4.10}$$

By replacing x by $\tau^m x$ and using $(\mathcal{N}_7)$, we obtain

$$\begin{aligned}\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^nx)-\frac{1}{\tau^m}f(\tau^mx),\frac{1}{\tau^m}\sum_{k=1}^{n-m}\tau^{k(\alpha-1)}t\right) &\geq \mathcal{N}_1(\varphi(\tau^mx,\tau^mx),t)\\ &\geq \mathcal{N}_1\left(\tau^{m\alpha}\varphi(x,x),t\right)\\ &= \mathcal{N}_1\left(\varphi(x,x),\frac{t}{\tau^{m\alpha}}\right).\end{aligned}$$

It follows that

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^nx)-\frac{1}{\tau^m}f(\tau^mx),\frac{1}{\tau^m}\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1\left(\varphi(x,x),t\right). \tag{4.11}$$

Let $c > 0$, and let ϵ be given. Since $\lim_{t\to\infty}\mathcal{N}_1(\varphi(x,x),t) = 1$, there is some $t_0 > 0$ such that

$$\mathcal{N}_1(\varphi(x,x),t) \geq 1-\epsilon.$$

Fix $t > t_0$. The convergence of series $\sum_{k=1}^{\infty}\tau^{k(\alpha-1)}$ guarantees that there exists some $n_0 \geq 0$ such that, for each $n > m > n_0$, the inequality

$$\sum_{k=m+1}^{n}\tau^{k(\alpha-1)} < c$$

holds. It follows that

$$\begin{aligned}\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x),c\right) &\geq \mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x),\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t\right)\\ &\geq \mathcal{N}_1(\varphi(x,x),t)\geq 1-\epsilon.\end{aligned}$$

Hence $\left\{\frac{f(\tau^n x)}{\tau^n}\right\}$ is a Cauchy sequence in fuzzy Banach Riesz space $(\mathcal{Y},\mathcal{N}_2)$ and thus this sequence converges to some $\mathcal{T}(x)\in\mathcal{Y}$. It means that

$$\mathcal{T}(x)=\mathcal{N}_2-\lim_{n\to\infty}\frac{f(\tau^n x)}{\tau^n}$$

Furthermore, by putting $m=0$ in (4.11), we have

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-f(x),\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1(\varphi(x,x),t).$$

So

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-f(x),t\right) \geq \mathcal{N}_1\left(\varphi(x,x),\frac{t}{\sum_{k=1}^{n}\tau^{k(\alpha-1)}}\right).$$

As $n\to\infty$, we have

$$\mathcal{N}_2\left(\mathcal{T}(x)-f(x),t\right) \geq \mathcal{N}_1\left(\varphi(x,x),\frac{\tau-\tau^\alpha}{\tau^\alpha}t\right).$$

Next, we show that $\mathcal{T}$ satisfies $(\mathcal{P}_1)$. Putting $\tau=\eta=\tau^n$ in (4.4), we get

$$\begin{aligned}\mathcal{N}_2\left(f\left(\tau^n x\vee\tau^n y\right)-\tau^n f(x)\vee\tau^n f(y),t\right) &\geq \mathcal{N}_1\left(\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y,t\right)\\ &\geq \mathcal{N}_1\left(\varphi(x\vee y,x\wedge y),\frac{t}{\tau^{n\alpha}}\right).\end{aligned}$$

Substituting x with $\tau^n x$ and y with $\tau^n y$ in this last inequality, one can get

$$\begin{aligned}&\mathcal{N}_2\left(f(\tau^n\left(\tau^n x\vee\tau^n y\right))-\tau^n f(\tau^n x)\vee\tau^n f(\tau^n y),t\right)\\ &\geq \mathcal{N}_1\left(\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y,\frac{t}{\tau^{n\alpha}}\right)\\ &\geq \mathcal{N}_1\left(\varphi(x\vee y,x\wedge y),\frac{t}{\tau^{2n\alpha}}\right),\end{aligned}$$

which yields

$$\mathcal{N}_2\left(\frac{f\left(\tau^{2n}(x\vee y)\right)}{\tau^{2n}}-\frac{f(\tau^n x)}{\tau^n}\vee\frac{f(\tau^n y)}{\tau^n},\frac{t}{\tau^{2n}}\right) \geq \mathcal{N}_1\left(\varphi(x\vee y,x\wedge y),\tau^{2(1-\alpha)}t\right).$$

The term on the right-hand side of the above inequality tends to 1 as $n\to\infty$. By Theorem 3.2, we obtain

$$\mathcal{N}_2(\mathcal{T}(x\vee y)-\mathcal{T}x\vee\mathcal{T}y,t) \geq 1.$$

This means that

$$\mathcal{T}(x\vee y) = \mathcal{T}x\vee\mathcal{T}y,$$

consequently, the property $(\mathcal{P}_1)$ holds.

Next, we show that $\mathcal{T}(\tau x)=\tau\mathcal{T}(x)$ for all $x\in\mathbf{X}_+$ and $\tau\geq 1$. In fact, in the inequality (4.4), choose $\eta=\tau$, $y=0$ and substitute $2^n\tau$ for τ and consider Remark 2.1.

$$\mathcal{N}_2\left(f\left(2^n\tau x\vee 0\right)-2^n\tau f(x)\vee 0,t\right) \geq \mathcal{N}_1\left(\varphi\left(2^n\tau x,0\right),t\right)$$

for all $x \in \mathbf{X}$. Now we replace x with $2^n x$. Consequently, by (2.1)

$$\begin{aligned}\mathcal{N}_2\left(\frac{f\left(4^n\tau x\right)}{4^n}-\frac{\tau f(2^n x)}{2^n},\frac{t}{4^n}\right) &\geq \mathcal{N}_1\left(\varphi\left(4^n\tau x,0\right),t\right)\\ &\geq \mathcal{N}_1\left(4^{n\alpha}\tau^{\alpha}\varphi\left(x,0\right),t\right).\end{aligned}$$

Therefore,

$$\mathcal{N}_2\left(\frac{f\left(4^n\tau x\right)}{4^n}-\frac{\tau f(2^n x)}{2^n},t\right) \geq \mathcal{N}_1\left(4^{n(\alpha-1)}\tau^{\alpha}\varphi\left(x,0\right),t\right).$$

The term on the right-hand side of the above inequality tends to 1 as $n \to \infty$. Thus

$$\mathcal{T}(\tau x) = \tau(\mathcal{T}x)$$

for all $x \in \mathbf{X}_+$. □

Theorem 4.2. *Let $\mathcal{X}, \mathcal{Y}$ be Banach lattices and $p : [0,\infty) \to [0,\infty)$ be a continuous function. Consider a positive map $f : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\nu \in \mathbf{R}$ and $0 \leq r < 1$ such that*

$$\mathcal{N}_2\left(f(\alpha|x| \vee \beta|y|) - \frac{\alpha p(\alpha) f(|x|) \vee \beta p(\beta) f(|y|)}{p(\alpha) \vee p(\beta)}, t\right) \geq \mathcal{N}_1\left(\nu\left(x^r \vee y^r\right), t\right) \tag{4.12}$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbf{R}^+$. Then there exists a unique positive mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies the properties $(\mathcal{P}_1), (\mathcal{P}_2)$ and the inequality

$$\mathcal{N}_2\left(F(|x|) - T(|x|), t\right) \geq \mathcal{N}_1\left(\frac{2\nu x}{2-2^r}, t\right).$$

Proof. Putting $\alpha = \beta = 2$ and $x = y$ in (4.12), we get

$$\mathcal{N}_2\left(f(2|x|) - \frac{2p(2)f(|x|) \vee 2p(2)f(|x|)}{p(2) \vee p(2)}, t\right) \geq \mathcal{N}_1\left(\nu x^r, t\right)$$

for all $x \in \mathcal{X}$ and $r \in [0,1)$. Therefore,

$$\begin{aligned}\mathcal{N}_2\left(f(2|x|) - 2f(|x|), t\right) &\geq \mathcal{N}_1\left(\nu x^r, t\right),\\ \mathcal{N}_2\left(\frac{1}{2}f(2|x|) - f(|x|), t\right) &\geq \mathcal{N}_1\left(\nu x^r, 2t\right).\end{aligned}$$

The rest of the proof is similar to the previous one. □

5. Conclusion

In the classical Riesz space theory, Banach lattice requires more attention. In the present research work, we briefly introduced and studied the fuzzy normed Riesz spaces. Thus we think that there are many open problems and applications in this new research area. For example we will introduce M-space, L-space and order unit in fuzzy normed Riesz spaces in our future research work.

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Choonkil Park

Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea

E-mail address: baak@hanyang.ac.kr

Ehsan Movahednia

Department of Mathematics, Behbahan Khatam Al-Anbia University of Technology, Behbahan, Iran

E-mail address: movahednia@bkatu.ac.ir

Seyed Mohammad Sadegh Modarres Mosadegh

Department of mathematics, University of Yazd, P. O. Box 89195-741, Yazd, Iran

E-mail address: smodarres@yazd.ac.ir

Mohammad Mursaleen

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

E-mail address: mursaleenm@gmail.com

BIPOLAR FUZZY SETS OF BCK-MODULES

M. A. ALGHAMDI[1], N. O. ALSHEHRI[2], AND N. M. MUTHANA[3]

Abstract. The notion of bipolar fuzzy BCK-submodules are introduced, and some characterizations of bipolar fuzzy BCK-submodules are given. The conceopt of homomorphic images and preimages of bipolar fuzzy BCK-submodules are investigated. Normality and completely normality of bipolar fuzzy BCK-submodules are discussed.

keywords: bipolar fuzzy BCK-submodule, normal (completely normal) bipolar fuzzy BCK-submodule, maximal bipolar fuzzy BCK-submodule.

1. Introduction

In the traditional fuzzy sets, which presented by Zadeh [8] in 1965, the membership of elements are expressed in degrees ranging from 0 to 1. The membership degree 0 is assigned to elements which do not satisfy a corresponding property to the concerned fuzzy set. It is of interest to know whether these elements are satisfying a counter-property of our fuzzy set, but the restriction of the membership degrees to the interval [0,1] led to a great difficulty in doing this. For this reason, Lee [7] introduced the concept of the bipolar-valued fuzzy sets as an extension of the fuzzy sets. In the case of bipolar-valued fuzzy sets, the membership degrees range is increased from the interval [0,1] to the interval [-1,1]. The representation of bipolar-valued fuzzy sets express the difference of contrary elements from irrelevant elements.

The notion of bipolar-valued fuzzy subalgebra and bipolar-valued fuzzy ideal was introduced by Lee [6].

H. A. S. Abujabal, M. Aslam and A. B. Thaheem [1], introduced the notion of BCK-modules as an action of BCK-algebra over a commutative group. The concept of fuzzy BCK-submodules was introduced by M. Bakhshi [2], where he characterized the fuzzy BCK-submodules and provided some operations of it.

In this paper, we apply the notion of bipolar-valued fuzzy set on BCK-modules and introduce the notion of bipolar-valued fuzzy BCK-submodules. Then we present some characterization of bipolar-valued fuzzy BCK-submodules by means of positive t-level cut and negative s-level cut. Moreover, a certain form of bipolar fuzzy BCK-submodules is derived from a given BCK-submodule. We investigate the homomorphic image and preimage of the bipolar-valued fuzzy BCK-submodules under some conditions. The later work is devoted to discuss the normality and completely normality of bipolar fuzzy BCK-submodules. A maximal bipolar fuzzy BCK-submodule is defined and its range is specified quit so. Many examples are given to illustrate our concepts and results.

2. Preliminaries

In this section we review some definitions and results regarding BCK-algebras, BCK-modules and bipolar fuzzy sets.

By a BCK-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(I) $((x * y) * (x * z))(z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ implies $x = y$, for all $x, y, z \in X$.

Let $(X, *, 0)$ be a BCK-algebra. Then X is a partially ordered set with the partial ordering $\leq$ defined on X by: $x \leq y$ if and only if $x * y = 0$. X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$. X is said to be commutative (implicative) if $x \wedge y = y \wedge x$ $(x * (y * x) = x)$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$.

Definition 2.1[1] Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), we mean an abelian group M with an operation $X \times M \longrightarrow M$ with $(x, m) \mapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$,

(1) $(x \wedge y)m = x(ym)$,
(2) $x(m + n) = xm + xn$,
(3) $0m = 0$.

Moreover, if X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary.

Example 2.2 Any bounded implicative BCK-algebra X forms an X-module, where "+ "is defined as $x + y = (x * y) \vee (y * x)$ and $xy = x \wedge y$.

A subgroup N of an X-module M is called submodule of M if N is also an X-module.

Theorem 2.3 [2] A subset N of a BCK-module M is a BCK-submodule of M if and only if $n_1 - n_2, xn \in N$ for all $n_1, n_2, n \in N$ and $x \in X$.

Definition 2.4 [1] Let M, N be modules over a BCK-algebra X. A mapping $f : M \longrightarrow N$ is called an X-homomorphism if

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$
(2) $f(xm) = xf(m)$ for all $m_1, m_2, m \in M$, $x \in X$.

A BCK-module homomorphism is said to be monomorphism (epimorphism) if it is one to one (onto). If it is both one to one and onto, then we say that it is an isomorphism.

Let X be the universe of discourse. A bipolar valued fuzzy set Φ of X is an object having the form

$$\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$$

where $\Phi^+ : X \longrightarrow [0, 1]$ and $\Phi^- : X \longrightarrow [-1, 0]$ are mappings. The positive membership degree $\Phi^+(x)$ denotes the satisfaction degree of an element x to the

property corresponding to a bipolar -valued fuzzy set $\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$, and the negative membership degree $\Phi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$.

For the sake of simplicity, we shall use the symbol $\Phi = (X; \Phi^+, \Phi^-)$ for the bipolar-valued fuzzy set $\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

For a bipolar fuzzy set $\Phi = (X; \Phi^+, \Phi^-)$ and $(t, s) \in [0, 1] \times [-1, 0]$, we define

$$\begin{aligned} P(\Phi; t) &= \{x \in X \mid \Phi^+(x) \geq t\}, \\ N(\Phi; s) &= \{x \in X \mid \Phi^-(x) \leq s\} \end{aligned}$$

which are called the positive t-level cut of $\Phi = (X; \Phi^+, \Phi^-)$ and the negative s-level cut of $\Phi = (X, \Phi^+, \Phi^-)$, respectively. For $k \in [0, 1]$, the set

$$L(\Phi; k) = P(\Phi; k) \cap N(\Phi; -k)$$

is called the k-level cut of $\Phi = (X; \Phi^+, \Phi^-)$ (see [6]).

If $\Phi = (X; \Phi^+, \Phi^-)$ and $\Psi = (X; \Psi^+, \Psi^-)$ are bipolar fuzzy sets defined on X, then the union and the intersection of Φ and Ψ are bipolar fuzzy sets of X defined as follows:

$\Phi \cup \Psi = (X; (\Phi \cup \Psi)^+, (\Phi \cup \Psi)^-)$ and $\Phi \cap \Psi = (X; (\Phi \cap \Psi)^+, (\Phi \cap \Psi)^-)$, respectively, where

$$(\Phi \cup \Psi)^+(x) = \max\left\{\Phi^+(x), \Psi^+(x)\right\}, \ (\Phi \cup \Psi)^-(x) = \min\left\{\Phi^-(x), \Psi^-(x)\right\},$$

and

$$(\Phi \cap \Psi)^+(x) = \min\left\{\Phi^+(x), \Psi^+(x)\right\}, \ (\Phi \cap \Psi)^-(x) = \max\left\{\Phi^-(x), \Psi^-(x)\right\},$$

for all $x \in X$.

Definition 2.5 [3] Let $\Phi = (X; \Phi^+, \Phi^-)$ and $\Psi = (X; \Psi^+, \Psi^-)$ be bipolar fuzzy sets of X. If $\Psi^+(x) \geq \Phi^+(x)$ and $\Psi^-(x) \leq \Phi^-(x)$ for all $x \in X$, then we say that $\Psi = (X; \Psi^+, \Psi^-)$ is a bipolar fuzzy extension of $\Phi = (X; \Phi^+, \Phi^-)$ (simply Φ is subset of Ψ) and we write $\Phi \subseteq \Psi$.

In what follows, X will denote a bounded BCK-algebra and M, N are X-modules unless otherwise specified.

3. **Bipolar Fuzzy BCK-Submodules**

In this section applying bipolar fuzzy sets theory to BCK-modules, we introduce the notion of bipolar fuzzy BCK-submodules and discuss their properties.

Definition 3.1 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-)$ of a BCK-module M is said to be a bipolar fuzzy BCK-submodule if it satisfies:

(BFS1) $\Phi^+(m_1 + m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and
$\Phi^-(m_1 + m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$,

(BFS2) $\Phi^+(-m) = \Phi^+(m)$ and $\Phi^-(-m) = \Phi^-(m)$,

(BFS3) $\Phi^+(xm) \geq \Phi^+(m)$ and $\Phi^-(xm) \leq \Phi^-(m)$.

For the sake of simplicity, we shall use the symbols $\mathcal{BF}(M)$ and $\mathcal{BFS}(M)$ for the set of all bipolar fuzzy sets of M, and the set of all bipolar fuzzy BCK-submodules of M, respectively.

Example 3.2 Let $X = P(\mathbb{Z})$ with $*$ defined by $A * B = A \cap B^c$ and 0 is the empty set $\varnothing$, then X is a BCK-algebra. $M = \mathbb{Z}^{\mathbb{Z}} = \{f \mid f : \mathbb{Z} \longrightarrow \mathbb{Z}\}$, considered with the traditional addition of maps and 0 is the zero map, is an abelian group. If we define an action of X on M by $Af = \chi_A f$, then M forms an X-module. Define a bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-)$ on M by

$$\Phi^+(f) = \begin{cases} 1 & \text{if } f = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Phi^-(f) = \begin{cases} -1 & \text{if } f = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then Φ is a bipolar fuzzy BCK-submodule of M.

Example 3.3 Let $X = \{0, a, b, c, d\}$ and consider the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	d
d	d	d	d	d	0

Tab. 3.1

Then $(X, *, 0)$ is a commutative BCK-algebra which is not bounded. The subset $M = \{0, a, b, c\}$ of X along with the operation $+$ defined by Table 3.2 is an abelian group. Table 3.3 shows the action of X on M ($xm = x \wedge m$). Consequently, M forms an X-module.

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Tab. 3.2

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	b	b
c	0	a	b	c
d	0	0	0	0

Tab. 3.3

Now let $\Phi = (M; \Phi^+, \Phi^-)$ be a bipolar fuzzy set on M defined as follows:

M	0	a	b	c
Φ^+	1	0.7	0.7	0.7
Φ^-	-0.8	-0.6	-0.6	-0.6

Tab. 3.4

Then $\Phi = (M; \Phi^+, \Phi^-)$ is a bipolar fuzzy BCK-submodule of M.

Theorem 3.4 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ if and only if
(i) $\Phi^+(xm) \geq \Phi^+(m)$ and $\Phi^-(xm) \leq \Phi^-(m)$,
(ii) $\Phi^+(m_1 - m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and
$\Phi^-(m_1 - m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$ for all $m, m_1, m_2 \in M$ and $x \in X$.
Proof Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then

$$\begin{aligned} \Phi^+(m_1 - m_2) &\geq \min\{\Phi^+(m_1), \Phi^+(-m_2)\} \\ &= \min\{\Phi^+(m_1), \Phi^+(m_2)\}, \\ \Phi^-(m_1 - m_2) &\leq \max\{\Phi^-(m_1), \Phi^-(-m_2)\} \\ &= \max\{\Phi^-(m_1), \Phi^-(m_2)\}. \end{aligned}$$

Conversely, assume that (i) and (ii) are satisfied. Put $x = 0$ in (i), then $\Phi^+(0) \geq \Phi^+(m)$, and $\Phi^-(0) \leq \Phi^-(m)$ for all $m \in M$. So using (ii), we have

$$\begin{aligned} \Phi^+(-m) &= \Phi^+(0 - m) \geq \min\{\Phi^+(0), \Phi^+(m)\} \geq \Phi^+(m), \\ \Phi^+(m) &= \Phi^+(0 - (-m)) \geq \min\{\Phi^+(0), \Phi^+(-m)\} \geq \Phi^+(-m), \end{aligned}$$

which implies that

$$\Phi^+(-m) = \Phi^+(m).$$

Moreover,

$$\begin{aligned} \Phi^+(m_1 + m_2) &= \Phi^+(m_1 - (-m_2)) \\ &\geq \min\{\Phi^+(m_1), \Phi^+(-m_2)\} \\ &= \min\{\Phi^+(m_1), \Phi^+(m_2)\}. \end{aligned}$$

We can verify that $\Phi^-(-m) = \Phi^-(m)$ and $\Phi^-(m_1 + m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$ by similar argument.

Theorem 3.5 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ if and only if
(i) $\Phi^+(0) \geq \Phi^+(m)$ and $\Phi^-(0) \leq \Phi^-(m)$,
(ii) $\Phi^+(x_1 m_1 - x_2 m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and $\Phi^-(x_1 m_1 - x_2 m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$
for all $m, m_1, m_2 \in M$ and $x_1, x_2 \in X$.
Proof Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ and let $m, m_1, m_2 \in M$ and $x_1, x_2 \in X$. (i) is already shown in the proof of Theorem 3.4. Moreover,

$$\begin{aligned} \Phi^+(x_1 m_1 - x_2 m_2) &\geq \min\{\Phi^+(x_1 m_1), \Phi^+(x_2 m_2)\} \\ &\geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}, \end{aligned}$$

and

$$\begin{aligned} \Phi^-(x_1 m_1 - x_2 m_2) &\leq \max\{\Phi^-(x_1 m_1), \Phi^-(x_2 m_2)\} \\ &\leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}. \end{aligned}$$

Now, let $\Phi = (M; \Phi^+, \Phi^-)$ be a bipolar fuzzy set of M. Assume that (i) and (ii) hold. Let $m, m_1, m_2 \in M$ and $x \in X$, then $\Phi^+(xm) = \Phi^+(xm - 0) = \Phi^+(xm - 0m) \geq \min\{\Phi^+(m)), \Phi^+(m)\} = \Phi^+(m)$, and $\Phi^-(xm) = \Phi^-(xm - 0) = \Phi^-(xm - 0m) \leq \max\{\Phi^-(m)), \Phi^-(m)\} = \Phi^-(m)$. Now

$\Phi^+(m_1-m_2)=\Phi^+(1m_1-1m_2)\geq\min\{\Phi^+(m_1),\Phi^+(m_2)\}$, and $\Phi^-(m_1-m_2)=\Phi^-(1m_1-1m_2)\leq\max\{\Phi^-(m_1),\Phi^-(m_2)\}$. Using Theorem 3.4, $\Phi=(M;\Phi^+,\Phi^-)$ is a bipolar fuzzy BCK-submodule of M.

Theorem 3.6 Let $\Phi=(M;\Phi^+,\Phi^-)\in\mathcal{BF}(M)$. Then $\Phi\in BFS(M)$ if and only if $\varnothing\neq P(\Phi;t)$ and $\varnothing\neq N(\Phi;s)$ are submodules of M for all $(t,s)\in[0,1]\times[-1,0]$.

Proof Assume that $\Phi=(M;\Phi^+,\Phi^-)\in\mathcal{BFS}(M)$ and $(t,s)\in[0,1]\times[-1,0]$ be such that $\varnothing\neq P(\Phi;t),\varnothing\neq N(\Phi;s)$. Let m, $m_1,m_2\in P(\Phi;t)$ and $m',m'_1,m'_2\in N(\Phi;s)$ and $x\in X$. Then

$$\begin{aligned}\Phi^+(m_1-m_2) &\geq \min\{\Phi^+(m_1),\Phi^+(m_2)\}\geq t,\\ \Phi^-(m'_1-m'_2) &\leq \max\{\Phi^-(m'_1),\Phi^-(m'_2)\}\leq s.\end{aligned}$$

i.e. $m_1-m_2\in P(\Phi;t)$ and $m'_1-m'_2\in N(\Phi;s)$.

Further,

$$\begin{aligned}\Phi^+(xm) &\geq \Phi^+(m)\geq t,\\ \Phi^-(xm') &\leq \Phi^-(m')\leq s.\end{aligned}$$

Thus we have $xm\in P(\Phi;t)$ and $xm'\in N(\Phi;s)$. Hence $P(\Phi;t)$ and $N(\Phi;s)$ are submodules of M.

Conversely, assume that $\varnothing\neq P(\Phi;t)$ and $\varnothing\neq N(\Phi;s)$ are submodules of M for all $(t,s)\in[0,1]\times[-1,0]$. For $m,m'\in M$, let $t_0=\min\{\Phi^+(m),\Phi^+(m')\}$ and $s_0=\max\{\Phi^-(m),\Phi^-(m')\}$. Then $m,m\prime\in P(\Phi;t_0)$ and $m,m\prime\in N(\Phi;s_0)$. Since $P(\Phi;t_0)$ and $N(\Phi;s_0)$ are submodules of M, then $m-m\prime\in P(\Phi;t_0)$ and $m-m\prime\in N(\Phi;s_0)$, which means that

$$\Phi^+(m-m')\geq t_0=\min\{\Phi^+(m),\Phi^+(m')\},$$

and

$$\Phi^-(m-m')\leq s_0=\max\{\Phi^-(m),\Phi^-(m')\}.$$

Now, let $\Phi^+(m)=t_1$, $\Phi^-(m')=s_1$ and $x\in X$. Then $m\in P(\Phi,t_1)$ and $m'\in N(\Phi;s_1)$ which implies that $xm\in P(\Phi;t_1)$ and $xm'\in N(\Phi;s_1)$. i.e. $\Phi^+(xm)\geq t_1=\Phi^+(m)$, and $\Phi^-(xm')\leq s_1=\Phi^-(m')$. Thus by Theorem 3.4, $\Phi\in\mathcal{BFS}(M)$.

Corollary 3.7 If $\Phi=(M;\Phi^+,\Phi^-)\in\mathcal{BFS}(M)$, then the intersection of a non-empty positive t-level cut and a non-empty negative s-level cut of $\Phi=(M;\Phi^+,\Phi^-)$ is a submodule of M for all $(t,s)\in[0,1]\times[-1,0]$. In particular, the non-empty k-level cut of $\Phi=(M;\Phi^+,\Phi^-)$ is a submodule of M for all $k\in[0,1]$.

The union of a non-empty positive t-level cut and a non-empty negative s-level cut of $\Phi=(M;\Phi^+,\Phi^-)\in\mathcal{BFS}(M)$ is not a submodule of M in general as seen in the following example.

Example 3.8 Let $X=\{0,a,b,c,d,1\}$ with a binary operation $*$ defined on Table 3.5. For the subset $M=\{0,a,c,d\}$ of X, define an operation $+$ as $x+y=(x*y)\vee(y*x)$. It follows by Table 3.6 that $(M,+)$ is an abelian group. M is an X-module according to Table 3.7.

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
1	1	d	c	b	a	0

Tab. 3.5

$+$	0	a	c	d
0	0	a	c	d
a	a	0	d	c
c	c	d	0	a
d	d	c	a	0

Tab. 3.6

$\wedge$	0	a	c	d
0	0	0	0	0
a	0	a	0	a
b	0	a	0	a
c	0	0	c	c
d	0	a	c	d
1	0	a	c	d

Tab. 3.7

Define a bipolar fuzzy set on M by the following table:

M	0	a	c	d
Φ^+	0.9	0.6	0.4	0.4
Φ^-	-1	-0.5	-0.7	-0.5

Tab. 3.8

We can easily check that $\Phi = (X; \Phi^+, \Phi^-)$ is a bipolar fuzzy BCK-submodule of M. The positive 0.5-level cut is $P(\Phi; 0.5) = \{0, a\}$, and the negative -0.7-level cut is $N(\Phi; -0.7) = \{0, c\}$. It is clear that $P(\Phi; 0.5) \cup N(\Phi; -0.7) = \{0, a, c\}$ is not a submodule of M. Furthermore, $P(\Phi; 0.6) \cup N(\Phi; -0.6) = \{0, a, c\}$ is not a submodule of M.

A sufficient condition for $P(\Phi; k) \cup N(\Phi; -k)$ to be a submodule of M is given in the next theorem. without this condition, $P(\Phi; k) \cup N(\Phi; -k)$ need not be a submodule of M as we have already seen.

Theorem 3.9 If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ such that

$$(*) \qquad \Phi^+(m) + \Phi^-(m) \geq 0,$$

or

$$(**) \qquad \Phi^+(m) + \Phi^-(m) \leq 0,$$

for all $m \in M$, then the union of a non-empty positive k-level cut and a non-empty negative $-k$-level cut of $\Phi = (M; \Phi^+, \Phi^-)$ is a submodule of M for all $k \in [0, 1]$.

Proof Let $k \in [0,1]$ such that $P(\Phi;k) \neq \varnothing$ and $N(\Phi;-k) \neq \varnothing$. Then they are submodules of M by Theorem 3.6. Let $x \in X$ and $m, m_1, m_2 \in P(\Phi;k) \cup N(\Phi;-k)$. If $m \in P(\Phi;k)$, then

$$xm \in P(\Phi;k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

If $m \in N(\Phi^-,-k)$, then

$$xm \in N(\Phi;-k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

Now we shall prove that $m_1 - m_2 \in P(\Phi^+;k) \cup N(\Phi^-;-k)$. We have the following three cases:

i. $m_1, m_2 \in P(\Phi;k)$,
ii. $m_1, m_2 \in N(\Phi;-k)$,
iii. $m_1 \in P(\Phi;k)$, $m_2 \in N(\Phi;-k)$.

Case i. implies that

$$m_1 - m_2 \in P(\Phi;k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

Case ii. gives

$$m_1 - m_2 \in N(\Phi;-k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

In case iii. $\Phi^+(m_1) \geq k$ and $\Phi^-(m_2) \leq -k$. If we consider (*), then $\Phi^+(m_2) + \Phi^-(m_2) \geq 0$ which means that $\Phi^+(m_2) \geq k$ and so

$$m_1 - m_2 \in P(\Phi;k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

If we consider (**), then $\Phi^+(m_1) + \Phi^-(m_1) \leq 0$ implies that $\Phi^-(m_1) \leq -k$. Hence

$$m_1 - m_2 \in N(\Phi;-k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

Therefore $P(\Phi;k) \cup N(\Phi;-k)$ is a submodule of M.

For a bipolar fuzzy set $\Phi = (M;\Phi^+,\Phi^-)$ and an element $m \in M$, we shall write $\Phi(m) = (\alpha,\beta)$ in the meaning of $\Phi^+(m) = \alpha$ and $\Phi^-(m) = \beta$.

Theorem 3.10 Let M be a module over a BCK-algebra X and $\varnothing \neq N \subseteq M$. Suppose that $\Phi = (M;\Phi^+,\Phi^-)$ is a bipolar fuzzy set on M defined as follows:

$$\Phi(m) = \begin{cases} (\alpha,\gamma) & \text{if } m \in N, \\ (\beta,\delta) & \text{otherwise,} \end{cases}$$

where $(\alpha,\gamma),(\beta,\delta) \in [0,1]\times[-1,0]$ with $\alpha > \beta$ and $\gamma < \delta$. Then $\Phi = (M;\Phi^+,\Phi^-)$ is a bipolar fuzzy BCK-submodule of M if and only if N is a submodule of M.

Proof Assume that $\Phi = (M;\Phi^+,\Phi^-) \in \mathcal{BFS}(M)$ and we shall prove that N is a submodule of M. Let $n \in N$ and $x \in X$. Then $\Phi^+(xn) \geq \Phi^+(n) = \alpha > \beta$ which implies that $\Phi^+(xn) = \alpha$ i.e. $xn \in N$.

Now let $n_1, n_2 \in N$, then $\Phi^+(n_1) = \Phi^+(n_2) = \alpha$ and $\Phi^+(n_1 - n_2) \geq \min\{\Phi^+(n_1),$ $\Phi^+(n_2)\} = \alpha > \beta$ and this gives $\Phi^+(n_1 - n_2) = \alpha$ i.e. $n_1 - n_2 \in N$. Hence N is a submodule of M.

Conversely, let N be a submodule of M and let $m, m_1, m_2 \in M$ and $x \in X$. We shall prove that $\Phi^+(xm) \geq \Phi^+(m)$ and $\Phi^-(xm) \leq \Phi^-(m)$. If $m \in N$,then $xm \in N$ and we obtain

$$\Phi^+(xm) = \alpha = \Phi^+(m),$$

and

$$\Phi^-(xm) = \gamma = \Phi^-(m).$$

If $m \notin N$, then $\Phi^+(m) = \beta$ and $\Phi^-(m) = \delta$. So we have

$$\Phi^+(xm) \geq \beta = \Phi^+(m)$$

and

$$\Phi^-(xm) \leq \delta = \Phi^-(m).$$

To show that

$\Phi^+(m_1 - m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and $\Phi^-(m_1 - m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$,

we consider the following cases:

i. $m_1, m_2 \in N$,
ii. $m_1, m_2 \notin N$,
iii. $m_1 \in N, m_2 \notin N$.

For case i. we have $m_1 - m_2 \in N$ and so

$$\Phi^+(m_1 - m_2) = \alpha = \min\{\Phi^+(m_1), \Phi^+(m_2)\}$$

and

$$\Phi^-(m_1 - m_2) = \gamma = \max\{\Phi^-(m_1), \Phi^-(m_2)\}.$$

For case ii. and iii. we have

$$\Phi^+(m_1 - m_2) \geq \beta = \min\{\Phi^+(m_1), \Phi^+(m_2)\}$$

and

$$\Phi^-(m_1 - m_2) \leq \delta = \max\{\Phi^-(m_1), \Phi^-(m_2)\}.$$

Therefore $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$.

For a submodule N of M, denote by $\Phi_N = (M, \Phi_N^+, \Phi_N^-)$ the bipolar fuzzy set defined in the theorem above with $(\alpha, \gamma) = (1, -1)$ and $(\beta, \delta) = (0, 0)$.

Example 3.11 Let $f : M \longrightarrow N$ be a homomorphism of BCK-modules. We know that $\ker f$ and $\operatorname{Im} f$ are submodules of M and N respectively. So $\Phi_{\ker f} = (M; \Phi^+_{\ker f}, \Phi^-_{\ker f})$ and $\Psi_{\operatorname{Im} f} = (N; \Psi^+_{\operatorname{Im} f}, \Psi^-_{\operatorname{Im} f})$ are bipolar fuzzy BCK-submodules.

Definition 3.12 Let $f : M \longrightarrow N$ be a BCK-module homomorphism and let $\Phi \in \mathcal{BFS}(M)$. Then the homomorphic image $f(\Phi) = (N; f(\Phi^+), f(\Phi^-))$ of Φ under f defined as follows:

$$f(\Phi^+)(n) = \begin{cases} \sup\limits_{m \in f^{-1}(n)} \Phi^+(m) & \text{if } f^{-1}(n) \neq \varnothing, \\ 0 & \text{if } f^{-1}(n) = \varnothing, \end{cases}$$

and

$$f(\Phi^-)(n) = \begin{cases} \inf\limits_{m \in f^{-1}(n)} \Phi^-(m) & \text{if } f^{-1}(n) \neq \varnothing, \\ 0 & \text{if } f^{-1}(n) = \varnothing. \end{cases}$$

Theorem 3.13 Let $f : M \longrightarrow N$ be a BCK-module epimorphism. If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then the homomorphic image $f(\Phi) \in \mathcal{BFS}(N)$.

Proof According to Theorem 3.6, it is sufficient to prove that $P(f(\Phi); t)$ and $N(f(\Phi); s)$ are submodules of N for all $(t, s) \in [0, 1] \times [-1, 0]$ satisfying $P(f(\Phi); t) \neq$

$\varnothing$, $N(f(\Phi);s) \neq \varnothing$. Let $n_0, n_1, n_2 \in P(f(\Phi);t)$ and $x \in X$. Since f is an epimorphism, then there exist $m_0 \in f^{-1}(n_0), m_1 \in f^{-1}(n_1), m_2 \in f^{-1}(n_2)$ such that $\Phi^+(m_i) \geq t$, $i = 0, 1, 2$. Now

$$f(\Phi^+)(xn_0) = \sup_{m \in f^{-1}(xn)} \Phi^+(m) \geq \Phi^+(xm_0) \geq \Phi^+(m_0) \geq t,$$

and

$$f(\Phi^+)(n_1 - n_2) = \sup_{m \in f^{-1}(n_1 - n_2)} \Phi^+(m) \geq \Phi^+(m_1 - m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\} \geq t.$$

Which implies that $xn_0, n_1 - n_2 \in P(f(\Phi);t)$. Therefore $P(f(\Phi);t)$ is a submodule of N for all $t \in [0,1]$. Analogously, we can verify that $N(f(\Phi);s)$ is a submodules of N for all $s \in [-1,0]$. This completes the proof.

Definition 3.14 Let $f : M \longrightarrow N$ be a homomorphism of BCK-modules, and $\Psi = (N; \Psi^+, \Psi^-)$ be a bipolar fuzzy set of N. Then the preimage of Ψ, $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-))$, is the bipolar fuzzy set on M given by $f^{-1}(\Psi^+)(m) = \Psi^+(f(m))$, $f^{-1}(\Psi^-)(m) = \Psi^-(f(m))$ for all $m \in M$.

Theorem 3.15 Let $f : M \longrightarrow N$ be a homomorphism of BCK-modules, and $\Psi = (N; \Psi^+, \Psi^-) \in \mathcal{BFS}(N)$, then the preimage $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-)) \in \mathcal{BFS}(M)$.

Proof Suppose that $\Psi = (N; \Psi^+, \Psi^-) \in \mathcal{BFS}(N)$ and f is a homomorphism of BCK-modules from M to N. Then for all $m_1, m_2 \in M$, we have

$$\begin{aligned} f^{-1}(\Psi^+)(m_1 - m_2) &= \Psi^+(f(m_1 - m_2)) = \Psi^+(f(m_1) - f(m_2)) \\ &\geq \min\{\Psi^+(f(m_1)), \Psi^+(f(m_2))\} \\ &= \min\{f^{-1}(\Psi^+)(m_1), f^{-1}(\Psi^+)(m_2)\} \end{aligned}$$

Moreover, let $x \in X$ and $m \in M$. then

$$\begin{aligned} f^{-1}(\Psi^+)(xm) &= \Psi^+(f(xm)) = \Psi^+(xf(m)) \\ &\geq \Psi^+(f(m)) = f^{-1}(\Psi^+)(m). \end{aligned}$$

Analogously, we have

$$f^{-1}(\Psi^-)(m_1 - m_2) \leq \max\{f^{-1}(\Psi^-)(m_1), f^{-1}(\Psi^-)(m_2)\}$$

and

$$f^{-1}(\Psi^-)(xm) \leq \max f^{-1}(\Psi^-)(m).$$

Hence, $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-))$ is a bipolar fuzzy BCK-submodule of M.

Theorem 3.16 Let $f : M \longrightarrow N$ be an epimorphism of BCK-modules. If $\Psi = (N; \Psi^+, \Psi^-)$ is a bipolar fuzzy set on N such that the preimage $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-)) \in \mathcal{BFS}(M)$, then $\Psi \in \mathcal{BFS}(N)$.

Proof Let $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-))$ be a bipolar fuzzy BCK-submodule on M and let $f : M \longrightarrow N$ be an epimorphism. For $n_0, n_1, n_2 \in N$, there exist $m_0, m_1, m_2 \in M$ such that $f(m_0) = n_0$, $f(m_1) = n_1$, and $f(m_2) = n_2$.

Now

$$\begin{aligned}\Psi^+(n_1-n_2) &= \Psi^+(f(m_1-m_2)) = f^{-1}(\Psi^+)(m_1-m_2)\\ &\geq \min\{f^{-1}(\Psi^+)(m_1), f^{-1}(\Psi^+)(m_2)\}\\ &= \min\{\Psi^+(f(m_1), \Psi^+(f(m_2))\}\\ &\quad \min\{\Psi^+(n_1), \Psi^+(n_2)\}\end{aligned}$$

and

$$\Psi^+(xn_0) = \Psi^+(f(xm_0)) = f^{-1}(\Psi^+)(xm_0) \geq f^{-1}(\Psi^+)(m_0) = \Psi^+(f(m_0)) = \Psi^+(n_0),$$

for all $x \in X$. By similar argument, we have

$$\Psi^-(n_1-n_2) \leq \max\{\Psi^-(n_1), \Psi^-(n_2)\},$$

and

$$\Psi^-(xn_0) \leq \Psi^-(n_0).$$

This finishes the proof.

For a bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-)$, we define M_Φ to be the set of all elements $m \in M$ with $\Phi(m) = \Phi(0)$.

Proposition 3.17 If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then M_Φ is a submodule of M.

Proof Clearly $M_\Phi \neq \varnothing$, since $0 \in M_\Phi$. Let $m, m_1, m_2 \in M_\Phi$ and $x \in X$. Then

$$\Phi^+(0) \geq \Phi^+(xm) \geq \Phi^+(m) = \Phi^+(0)$$

i.e. $\Phi^+(xm) = \Phi^+(0)$. Similarly, $\Phi^-(xm) = \Phi^-(0)$ and so $xm \in M_\Phi$. Furthermore,

$$\Phi^+(0) \geq \Phi^+(m_1-m_2) \geq \min\left\{\Phi^+(m_1), \Phi^+(m_2)\right\} = \Phi^+(0),$$

which means $\Phi^+(m_1-m_2) = \Phi^+(0)$. Analogously, $\Phi^-(m_1-m_2) = \Phi^-(0)$. Hence $m_1-m_2 \in M_\Phi$. Therefore M_Φ is a submodule of M.

Definition 3.18 If $\Phi(0) = (1,-1)$ for $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then Φ is said to be normal.

Theorem 3.19 Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$. The normalization $\bar{\Phi} = (M; \bar{\Phi}^+, \bar{\Phi}^-)$ of Φ defined by $\bar{\Phi}^+(m) = \Phi^+(m) + 1 - \Phi^+(0)$ and $\bar{\Phi}^-(m) = \Phi^-(m) - 1 - \Phi^-(0)$, for all $m \in M$, is normal bipolar fuzzy BCK-submodule of M containing Φ.

Proof Clearly, $\bar{\Phi}^+(m) \geq \Phi^+(m)$ and $\bar{\Phi}^-(m) \leq \Phi^-(m)$ for all $m \in M$. i.e.$\Phi \subseteq \bar{\Phi}$. Now let $m, m_1, m_2 \in M$ and $x \in X$. Then

$$\bar{\Phi}^+(xm) = \Phi^+(xm) + 1 - \Phi^+(0) \geq \Phi^+(m) + 1 - \Phi^+(0) = \bar{\Phi}^+(m),$$

and

$$\begin{aligned}\bar{\Phi}^+(m_1-m_2) &= \Phi^+(m_1-m_2) + 1 - \Phi^+(0)\\ &\geq \min\left\{\Phi^+(m_1), \Phi^+(m_2)\right\} + 1 - \Phi^+(0)\\ &= \min\left\{\Phi^+(m_1) + 1 - \Phi^+(0), \Phi^+(m_2) + 1 - \Phi^+(0)\right\}\\ &= \min\left\{\bar{\Phi}^+(m_1), \bar{\Phi}^+(m_2)\right\}.\end{aligned}$$

Analogously, $\bar{\Phi}^{-}(xm) \leq \bar{\Phi}^{-}(m)$ and $\bar{\Phi}^{-}(m_1 - m_2) \leq \max\left\{\bar{\Phi}^{-}(m_1), \bar{\Phi}^{-}(m_2)\right\}$. Hence $\bar{\Phi} \in \mathcal{BFS}(M)$. Moreover,

$$\bar{\Phi}^{+}(0) = \Phi^{+}(0) + 1 - \Phi^{+}(0) = 1,$$

and

$$\bar{\Phi}^{-}(0) = \Phi^{-}(0) - 1 - \Phi^{-}(0) = -1.$$

Which means that $\bar{\Phi}$ is normal.

Let $\mathcal{S}(M)$ (respectively $\mathcal{N}(M)$) denote the set of all submodules (respectively, normal bipolar fuzzy BCK-submodules) of M. We define functions $F : \mathcal{S}(M) \longrightarrow \mathcal{N}(M)$ and $G : \mathcal{N}(M) \longrightarrow \mathcal{S}(M)$ by $F(N) = \Phi_N$ and $G(\Phi) = M_\Phi$. Then $GF = 1_{\mathcal{S}(M)}$ and $FG(\Phi) = F(M_\Phi) = \Phi_{M_\Phi} \subseteq \Phi$.

Note that $\mathcal{S}(M)$ (respectively $\mathcal{N}(M)$) is a poset under the set inclusion (respectively, bipolar fuzzy set inclusion).

Theorem 3.20 If $N, K \in \mathcal{S}(M)$, then $\Phi_{N\cap K} = \Phi_N \cap \Phi_K$, that is, $F(N \cap K) = F(N) \cap F(K)$. If $\Phi, \Psi \in \mathcal{N}(M)$, then $M_{\Phi\cap\Psi} = M_\Phi \cap M_\Psi$, that is, $G(\Phi \cap \Psi) = G(\Phi) \cap G(\Psi)$.

Proof Let $m \in M$. If $m \in N \cap K$, then $\Phi_{N\cap K}(m) = (1, -1)$. From $m \in N$ and $m \in K$ it follows that $\Phi_N(m) = \Phi_K(m) = (1, -1)$. Hence

$$(\Phi_N \cap \Phi_K)^{+}(m) = \min\{\Phi_N^{+}(m), \Phi_K^{+}(m)\} = 1 = \Phi_{N\cap K}^{+}(m),$$

and

$$(\Phi_N \cap \Phi_K)^{-}(m) = \max\{\Phi_N^{-}(m), \Phi_K^{-}(m)\} = -1 = \Phi_{N\cap K}^{-}(m).$$

If $m \notin N \cap K$, then $m \notin N$ or $m \notin K$. Thus

$$(\Phi_N \cap \Phi_K)^{+}(m) = \min\{\Phi_N^{+}(m), \Phi_K^{+}(m)\} = 0 = \Phi_{N\cap K}^{+}(m),$$

and

$$(\Phi_N \cap \Phi_K)^{-}(m) = \max\{\Phi_N^{-}(m), \Phi_K^{-}(m)\} = 0 = \Phi_{N\cap K}^{-}(m).$$

Therefore $\Phi_{N\cap K} = \Phi_N \cap \Phi_K$, and so $F(N \cap K) = F(N) \cap F(K)$. Now let $\Phi, \Psi \in \mathcal{N}(M)$. Then

$$\begin{aligned}
M_{\Phi\cap\Psi} &= \left\{m \in M \mid (\Phi \cap \Psi)^{+}(m) = 1, (\Phi \cap \Psi)^{-}(m) = -1\right\} \\
&= \left\{m \in M \mid \min\{\Phi^{+}(m), \Psi^{+}(m)\} = 1, \max\left\{\Phi^{-}(m), \Psi^{-}(m)\right\} = -1\right\} \\
&= \left\{m \in M \mid \Phi^{+}(m) = \Psi^{+}(m) = 1, \Phi^{-}(m) = \Psi^{-}(m) = -1\right\} \\
&= \left\{m \in M \mid \Phi^{+}(m) = 1, \Phi^{-}(m) = -1\right\} \cap \\
&\quad \left\{m \in M \mid \Psi^{+}(m) = 1, \Psi^{-}(m) = -1\right\} \\
&= \{m \in M \mid \Phi(m) = \Phi(0)\} \cap \{m \in M \mid \Psi(m) = \Psi(0)\} \\
&= M_\Phi \cap M_\Psi,
\end{aligned}$$

that is, $G(\Phi \cap \Psi) = G(\Phi) \cap G(\Psi)$. This completes the proof.

Proposition 3.21 Let $\Phi = (M; \Phi^{+}, \Phi^{-})$ be a non-constant normal bipolar fuzzy BCK-submodule which is maximal in $(\mathcal{N}(M), \subseteq)$, then Φ takes only a value among $(0, 0), (1, 0), (0, -1)$ and $(1, -1)$.

Proof Let $\Phi = (M; \Phi^{+}, \Phi^{-})$ be a non-constant maximal element in $(\mathcal{N}(M), \subseteq)$. Since Φ is normal, then $\Phi(0) = (1, -1)$. Let $m_0 \in M$ be such that $\Phi^{+}(m_0) \neq 1$.

Then $\Phi^{+}(m_0) = 0$. Otherwise, $0 < \Phi^{+}(m_0) < 1$. Consider $\Psi = (M, \Psi^{+}, \Psi^{-})$ defined by

$$\Psi^{+}(m) = \frac{1}{2}(\Phi^{+}(m) + \Phi^{+}(m_0)),\ \Psi^{-}(m) = \frac{1}{2}(\Phi^{-}(m) + \Phi^{-}(m_0)),$$

for all $m \in M$. Clearly Ψ is well defined. Let $m, m_1, m_2 \in M$, then

$$\begin{aligned}
\Psi^{+}(m_1 - m_2) &= \frac{1}{2}(\Phi^{+}(m_1 - m_2) + \Phi^{+}(m_0)) \\
&\geq \frac{1}{2}(\min\{\Phi^{+}(m_1), \Phi^{+}(m_2)\} + \Phi^{+}(m_0)) \\
&= \min\{\frac{1}{2}(\Phi^{+}(m_1) + \Phi^{+}(m_0)), \frac{1}{2}(\Phi^{+}(m_2) + \Phi^{+}(m_0))\} \\
&= \min\left\{\Psi^{+}(m_1), \Psi^{+}(m_2)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\Psi^{+}(xm) &= \frac{1}{2}(\Phi^{+}(xm) + \Phi^{+}(m_0)) \\
&\geq \frac{1}{2}(\Phi^{+}(m) + \Phi^{+}(m_0)) \\
&= \Psi^{+}(m).
\end{aligned}$$

By similar argument, we can show

$$\Psi^{-}(m_1 - m_2) \leq \max\{\Psi^{-}(m_1), \Psi^{-}(m_2)\},$$

and

$$\Psi^{-}(xm) \leq \Psi^{-}(m).$$

Hence, $\Psi \in \mathcal{BFS}(M)$. Clearly, $\bar{\Psi}$ is non-constant and by Theorem 3.19, $\bar{\Psi} = \left(M, \bar{\Psi}^{+}, \bar{\Psi}^{-}\right) \in \mathcal{N}(M)$. Now for all $m \in M$, we have

$$\begin{aligned}
\bar{\Psi}^{+}(m) &= \Psi^{+}(m) + 1 - \Psi^{+}(0) \\
&= \frac{1}{2}(\Phi^{+}(m) + \Phi^{+}(m_0)) + 1 - \frac{1}{2}(\Phi^{+}(0) + \Phi^{+}(m_0)) \\
&= \frac{1}{2}(1 + \Phi^{+}(m)) \geq \Phi^{+}(m),
\end{aligned}$$

and

$$\begin{aligned}
\bar{\Psi}^{-}(m) &= \Psi^{-}(m) - 1 - \Psi^{-}(0) \\
&= \frac{1}{2}(\Phi^{-}(m) + \Phi^{-}(m_0)) - 1 - \frac{1}{2}(\Phi^{-}(0) + \Phi^{-}(m_0)) \\
&= \frac{1}{2}(\Phi^{-}(m) - 1) \\
&\leq \Phi^{-}(m).
\end{aligned}$$

Furthermore,

$$\bar{\Psi}^{+}(m_0) = \frac{1}{2}(1 + \Phi^{+}(m_0)) > \Phi^{+}(m_0).$$

This means that Φ is a proper subset of $\bar{\Psi}$, which contradicts the maximality of Φ in $\mathcal{N}(M)$. Thus the possible values of Φ^{+} are only 0 and 1. Likewise, we can show that 0 and -1 are the only possible values of Φ^{-}. Therefore, Φ takes only a value among $(0,0), (1,0), (0,-1)$ and $(1,-1)$. This finishes the proof.

For $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, consider the following sets:

$$M_\Phi^{(0,0)} = \{m \in M \mid \Phi^+(m) \geq 0, \Phi^-(m) \leq 0\} = M,$$
$$M_\Phi^{(1,0)} = \{m \in M \mid \Phi^+(m) \geq 1, \Phi^-(m) \leq 0\} = \{m \in M \mid \Phi^+(m) = 1\},$$
$$M_\Phi^{(0,-1)} = \{m \in M \mid \Phi^+(m) \geq 0, \Phi^-(m) \leq -1\} = \{m \in M \mid \Phi^-(m) = -1\},$$
$$\begin{aligned} M_\Phi^{(1,-1)} &= \{m \in M \mid \Phi^+(m) \geq 1, \Phi^-(m) \leq -1\} \\ &= \{m \in M \mid \Phi^+(m) = 1, \Phi^-(m) = -1\}. \end{aligned}$$

Clearly, we have the relations
$M_\Phi^{(1,-1)} \subseteq M_\Phi^{(1,0)} \subseteq M_\Phi^{(0,0)}$, $M_\Phi^{(1,-1)} \subseteq M_\Phi^{(0,-1)} \subseteq M_\Phi^{(0,0)}$, $M_\Phi^{(1,0)} \cap M_\Phi^{(0,-1)} = M_\Phi^{(1,-1)}$.

Definition 3.22 A normal bipolar fuzzy BCK-submodule $\Phi = (M; \Phi^+, \Phi^-)$ is said to be completely normal if there exists $m \in M$ such that $\Phi(m) = (0, 0)$.

Example 3.23 Let N be a proper submodule of M. Then $\Phi_N = (M; \Phi_N^+, \Phi_N^-)$

is completely normal bipolar fuzzy BCK-submodule.

Denote by $\mathcal{C}(M)$ the set of all completely normal bipolar fuzzy BCK-submodules of M. Note that $\mathcal{C}(M) \subseteq \mathcal{N}(M)$ and the restriction of the partial ordering " $\subseteq$ " of $\mathcal{N}(M)$ gives a partial ordering of $\mathcal{C}(M)$.

Theorem 3.24 A non-constant maximal element of $(\mathcal{N}(M), \subseteq)$ is also a maximal element of $(\mathcal{C}(M), \subseteq)$.

Proof First, we show that if $\Phi = (M; \Phi^+, \Phi^-)$ is a non-constant maximal element of $(\mathcal{N}(M), \subseteq)$,then $\Phi \in \mathcal{C}(M)$. Suppose that there is no $m \in M$ with $\Phi(m) = (0, 0)$ i.e. $M_\Phi^{(0,0)} - \left(M_\Phi^{(1,0)} \cup M_\Phi^{(0,-1)}\right) = \varnothing$. Since Φ is non-constant normal, then Φ assumes the value $(1, 0)$ or (and) $(0, -1)$ at some points in M and so we have the following cases:

i. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} = \varnothing$.
ii. $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} = \varnothing$.
iii. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$.

For case i. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1, -\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}. \end{cases}$$

For case ii. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2}, -1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}. \end{cases}$$

For case iii. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1, -\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2}, -1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}. \end{cases}$$

Noting that $M_\Phi^{(1,-1)}$, $M_\Phi^{(1,0)}$, and $M_\Phi^{(0,-1)}$ are submodules of M, it is not difficult to show that $\Psi = (M; \Psi^+, \Psi^-) \in \mathcal{BFS}(M)$ in each case. obviously, Ψ is non-constant

normal and $\Phi \subset \Psi$. But this contradicts the fact that Φ is non-constant maximal in $(\mathcal{N}(M), \subseteq)$. Thus Φ should has the value $(0,0)$ at some points $m \in M$ and so $\Phi \in \mathcal{C}(M)$. Now let $\Phi' \in \mathcal{C}(M)$ such that $\Phi \subseteq \Phi'$. It follows that $\Phi \subseteq \Phi'$ in $\mathcal{N}(M)$. Since Φ is maximal in $(\mathcal{N}(M), \subseteq)$ and since Φ' is non-constant, then $\Phi = \Phi'$. Therefore Φ is maximal in $(\mathcal{C}(M), \subseteq)$.

Definition 3.25 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ is said to be maximal if it satisfies:
(i) Φ is non-constant
(ii) $\bar{\Phi}$ is maximal element in $(\mathcal{N}(M), \subseteq)$.

Theorem 3.26 A maximal bipolar fuzzy BCK-submodule is completely normal and equivalent to its normalization.

Proof If $\Phi = (M; \Phi^+, \Phi^-)$ is a maximal bipolar fuzzy BCK-submodule of M, then $\bar{\Phi}$ is non-constant maximal in $(N(M), \subseteq)$ and so it is maximal in $(\mathcal{C}(M), \subseteq)$. So for some $m \in M$,

$$\begin{aligned} 0 &= \bar{\Phi}^+(m) = \Phi^+(m) + 1 - \Phi^+(0), \\ 0 &= \bar{\Phi}^-(m) = \Phi^-(m) - 1 - \Phi^-(0). \end{aligned}$$

Which implies that

$$\begin{aligned} \Phi^+(m) &= \Phi^+(0) - 1 \leq 0, \\ \Phi^-(m) &= \Phi^-(0) + 1 \geq 0. \end{aligned}$$

Since $\Phi^+(m) \geq 0$ and $\Phi^-(m) \leq 0$, then $\Phi^+(0) = 1$ and $\Phi^-(0) = -1$. Therefore $\Phi = \bar{\Phi}$ and so Φ is completely normal.

Now we arrive at one of our main theorems

Theorem 3.27 A maximal bipolar fuzzy BCK-submodule takes exactly the values $(1,-1),\ (0,0)$.

Proof Assume that $\Phi = (M; \Phi^+, \Phi^-)$ is a maximal bipolar fuzzy BCK-submodule. Then Φ takes a value among $(0,0)$, $(1,0)$, $(0,-1)$ and $(1,-1)$ and it is completely normal. So $M_\Phi^{(0,0)} - \left(M_\Phi^{(1,0)} \cup M_\Phi^{(0,-1)}\right) \neq \varnothing$. The subsets $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}$ and $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}$ of M are empty. If not, then we have the following cases:
i. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing,\ M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} = \varnothing$
ii. $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing,\ M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} = \varnothing$
iii. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing,\ M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$
For case i. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1, -\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}, \\ (0,0) & \text{if } m \in M_\Phi^{(0,0)} - M_\Phi^{(1,0)}. \end{cases}$$

For case ii. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2}, -1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}, \\ (0,0) & \text{if } m \in M_\Phi^{(0,0)} - M_\Phi^{(0,-1)}. \end{cases} .$$

For case iii. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1,-\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2},-1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}, \\ (0,0) & \text{if } m \in M_\Phi^{(0,0)} - \left(M_\Phi^{(1,0)} \cup M_\Phi^{(0,-1)}\right). \end{cases}$$

In each case, Ψ is non-constant completely normal bipolar fuzzy BCK-submodules containing Φ. A contradiction. Hence $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} = \varnothing$ and $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} = \varnothing$, i.e. Φ does not assume the values $(1,0)$ and $(0,-1)$ at any point in M. Therefore Φ takes exactly the values $(0,0)$ and $(1,-1)$.

Clearly $M_\Phi^{(1,-1)} = M_\Phi$, for $\Phi \in \mathcal{N}(M)$. We are thus led to the following result.

Corollary 3.28 If $\Phi = (M; \Phi^+, \Phi^-)$ is a maximal bipolar fuzzy BCK-submodule, then $\Phi_{M_\Phi} = \Phi$.

Theorem 3.29 For a maximal bipolar fuzzy BCK-submodule $\Phi = (M; \Phi^+, \Phi^-)$, M_Φ is a maximal submodule of M.

Proof Let $\Phi = (M; \Phi^+, \Phi^-)$ be a maximal bipolar fuzzy BCK-submodule and suppose that N is a proper submodule of M such that $M_\Phi \subset N$. Consider the normal bipolar fuzzy BCK-submodule $\Psi_N = (M; \Psi_N^+, \Psi_N^-)$. If M_Φ is a proper submodule of N then Φ is proper subset of Ψ_N. A contradiction. Hence M_Φ is a maximal submodule of M.

4. Conclusion

The notion of bipolar-valued fuzzy set was introduced by K. M. Lee in 2000. Since then, bipolarity has been applied to various algebraic structures by many researchers. In this paper, we have applied the notion of bipolar-valued fuzzy sets to BCK-modules. We have characterized our new concept "bipolar fuzzy BCK-submodule " in several ways. Next, the homomorphic images and pre images of bipolar fuzzy BCK-submodules were discussed. The remainder of the paper was focused on the normal and completely normal bipolar fuzzy BCK-submodules which guided finally to the concept of maximality.

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[1]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia.
E-mail address: malghamdi0612@stu.kau.edu.sa

[2]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia.
E-mail address: nalshehrie@kau.edu.sa

[3]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia.
E-mail address: nmuthana@kau.edu.sa

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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Dipartimento di Matematica
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Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
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The University of Memphis
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Zeros of functions in weighted Dirichlet spaces and Carleson type measure

Ruishen Qian, Songxiao Li* and Yanhua Zhang

Abstract. In this paper, we study the zeros of functions in weighted Dirichlet space and a class of Carleson type measure.

MSC 2000: 30J99, 30H99

Keywords: Zero, weighted Dirichlet space, Carleson measure.

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, namely $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. Let H^∞ denote the space of bounded analytic function.

Throughout this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function. An $f \in H(\mathbb{D})$ is said to belong to the weighted Dirichlet space, denoted by D_K, if (see, e.g., [24])

$$\|f\|^2_{D_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K\left(1-|z|^2\right) dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue measure on $\mathbb{D}$. Clearly, D_K is a Hilbert space. When $K(t) = t^s$, $0 \le s < \infty$, the space D_K gives the usual Dirichlet type space D_s. In particular, if $s = 0$, this gives the classical Dirichlet space $\mathcal{D}$. We refer to [16, 19, 20] for the space D_s. The space D_K has been extensively studied. For example, under some conditions on K, Kerman and Sawyer [11] characterized Carleson measures and multipliers of D_K in terms of a maximal operator. Aleman [1] proved that each element of the space D_K can be written as a quotient of two bounded functions in the same space. See [2, 3, 8, 14, 18, 24] for more results on weighted Dirichlet spaces.

We say that $Z = \{z_n\} \subset \mathbb{D}$ is a zero set of an analytic function space X defined on $\mathbb{D}$ if there is a $f \in X$ that vanishes on Z and nowhere else. Describing the zero sets for an analytic function space is a difficult problem. See [4, 5, 6, 13, 17, 21] for more information about this topic.

Let μ denote a positive Borel measure on $\mathbb{D}$. For a subarc $I \subseteq \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \le |z| < 1 \text{ and } \frac{z}{|z|} \in I\}.$$

If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. Let $0 \leq s < \infty$. We say that μ is a (K, s)-Carleson measure on $\mathbb{D}$ if

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{K(|I|)|I|^s} < \infty.$$

Here and henceforth $\sup_{I\subseteq\partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. We say that μ is a vanishing (K, s)-Carleson measure, if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{K(|I|)|I|^s} = 0.$$

If $K(t)t^s = t$, then we get the classical Carleson measure and vanishing Carleson measure, respectively. Carleson measure was firstly introduced in [4] by Carleson and it has many applications, such as in the interpolating sequence, $\overline{\partial}$-equations, composition operators and integral operators. Hence Carleson measure is a very important tool for the function theory, harmonic analysis and operator theory. For more results on Carleson measure and its' generalization, we refer to [7, 10, 12, 22, 23, 25].

Recently, Pau and Peláez [13, Theorem 1] gave a nice characterization of zero sets of Dirichlet spaces D_s $(0 < s < 1)$. Motivated by [13, Theorem 7], in this paper we study the zero sets of the space D_K. Moreover, we will characterize (K, s)-Carleson measure and vanishing (K, s)-Carleson measure. In particular, we will characterize vanishing (K, s)-Carleson measure by functions in the space D_K.

Throughout this paper, we assume that $K(0) = 0$ such that

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{2}$$

where

$$\varphi_K(s) = \sup_{0\leq t\leq 1} K(st)/K(t), \quad 0 < s < \infty.$$

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2 Zero sets

In order to study the zero sets of the D_K space, we define the space S_K, which consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{S_K} = |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \frac{K\left(1-|z|^2\right)}{K\left(\frac{|1-\overline{a}z|^2}{1-|a|^2}\right)} dA(z) < \infty.$$

We say that a positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for D_K if there is a positive constant C such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\|f\|^2_{D_K}$$

for all $f \in D_K$.

We also need the following results.

Lemma 1. [11] *Let K be increasing, concave and $\lim_{x\to 0} x/K(x) = 0$. Then, μ is a Carleson measure for D_K if and only if there exists a constant $C > 0$ such that*

$$\int_I \sup_{\theta \in J \subset I} \frac{\mu(S(J))^2}{K(|J|)|J|} d\theta \leq C\mu(S(I)),$$

for all arcs $I \subset \partial\mathbb{D}$. Here the supremum is taken over all closed arcs $J \subset I$.

Remark 1. From [9, Lemma 2.3], we known that if K satisfy (1) and (2), there exists K_3, such that K_3 is increasing, concave, $\lim_{x\to 0} x/K_3(x) = 0$ and $K_3(t) \approx K(t)$, $0 < t < \infty$. As in [9], let $c \in (0, 1)$ be a small constant such that $\frac{K(t)}{t^c}$ and $\frac{t^{1-c}}{K(t)}$ are nondecreasing functions when $0 < t < 1$, moreover

$$\varphi_K(s) \lesssim s^{1-c}, \ \ s \geq 1 \tag{3}$$

and

$$\varphi_K(s) \lesssim s^c, \ \ s \leq 1. \tag{4}$$

Lemma 2. *Suppose that K satisfy (1) and (2). Then $f \in S_K$ if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}} \int_{S(I)} |f'(z)|^2 \frac{K(1-|z|^2)}{K(|I|)} dA(z) < \infty. \tag{5}$$

Proof. Assume that $f \in S_K$. For any $I \subseteq \partial\mathbb{D}$, let $b = (1-|I|)\eta \in \mathbb{D}$, where η is the center of I. Then

$$1 - |b| \approx |1 - \overline{b}z| \approx |I|, \ \ z \in S(I).$$

Thus,

$$K\left(\frac{|1-\overline{b}z|^2}{1-|b|^2}\right) \approx K(|I|), \ \ z \in S(I).$$

Therefore,

$$\begin{aligned}\int_{S(I)} |f'(z)|^2 \frac{K(1-|z|^2)}{K(|I|)} dA(z) &\lesssim \int_{\mathbb{D}} |f'(z)|^2 \frac{K\left(1-|z|^2\right)}{K\left(\frac{|1-\overline{b}z|^2}{1-|b|^2}\right)} dA(z) \\ &\lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \frac{K\left(1-|z|^2\right)}{K\left(\frac{|1-\overline{a}z|^2}{1-|a|^2}\right)} dA(z) < \infty,\end{aligned}$$

which implies the desired result.

Conversely, assume that (5) holds. Without loss of generality, we can assume $|a| > 1/2$. Let I be a subarc of $\partial\mathbb{D}$ such that $|I_n| = 2^n|I|$, $n = 0, 1, 2..., N-1$ and $I_N = \partial\mathbb{D}$. Then we have

$$\frac{|1-\overline{a}\eta|^2}{1-|a|^2} \approx |I|, \ \ \eta \in I,$$

and

$$\frac{|1-\overline{a}\eta|^2}{1-|a|^2} \approx 2^{2n}|I|, \ \ \eta \in I_{n+1}/I_n, \ \ n = 0, 1, 2..., N-1.$$

Since K satisfy (1) and (2), by Remark 1 we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^2 \frac{K\left(1-|z|^2\right)}{K\left(\frac{|1-\overline{a}z|^2}{1-|a|^2}\right)} dA(z) \\
\lesssim &\sum_{n=1}^{\infty} \frac{1}{K(2^{2n}|I|)} \int_{S(2^{n+1}I)\setminus S(2^nI)} |f'(z)|^2 K(1-|z|^2) dA(z) \\
&\qquad + \frac{1}{K(|I|)} \int_{S(2I)} |f'(z)|^2 K(1-|z|^2) dA(z) \\
\lesssim &\sum_{n=1}^{\infty} \frac{1}{K(2^{2n}|I|)} \int_{S(2^{n+1}I)} |f'(z)|^2 K(1-|z|^2) dA(z) + \frac{K(2|I|)}{K(|I|)} \\
\lesssim &\sum_{n=1}^{\infty} \frac{K(2^{n+1}|I|)}{K(2^{2n}|I|)} + \varphi_K(2) \\
\lesssim &\sum_{n=1}^{\infty} 2^{(1-n)c} + 2^{1-c} < \infty.
\end{aligned}$$

Here c is defined in Remark 1. Hence $f \in S_K$. □

From Lemma 2, we can easily obtain the following corollary.

Corollary 1. *Suppose that K satisfy the conditions (1) and (2). Then $f \in S_K$ if and only if $|f'(z)|^2 K(1-|z|^2)dA(z)$ is $(K, 0)$-Carleson measure.*

Let $M(D_K)$ denote the space of multipliers of D_K, that is,

$$M(D_K) = \{g \in H(\mathbb{D}) : gf \in D_K \text{ for all } f \in D_K\}.$$

Theorem 1. *Suppose that K satisfy the conditions (1) and (2). Then $M(D_K)$, $S_K \cap H^\infty$, S_K and D_K have the same zero sets.*

Proof. Since $M(D_K) \subseteq D_K$, we have any zero sets in $M(D_K)$ is zero sets in D_K. Next we prove that any zero set in D_K is zero set in $M(D_K)$.

Suppose that $\|f\|_{D_K} = 1$. Let $\{z_k\}$ be the zeros of f. Fix $z_0 \in \mathbb{D}$ such that $f(z_0) \neq 0$. Set $w_0 = \frac{f(z_0)}{\|R_{z_0}\|_{D_K}}$ and $w_j = 0$ $(j \geq 1)$, where

$$R_{z_j}(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{nK(\frac{1}{n})} \overline{z_j}^n z^n, \ z \in \mathbb{D}, \ j \geq 0.$$

From [3, Lemma 3.1], we know that $R_{z_j}(z)$ is the reproducing kernel of D_K space at z_j. Then, for each $n \geq 1$ and $a_0, a_1, ..., a_n \in \mathbb{C}$, we have

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=0}^{n} a_i\overline{a_j}(1-\overline{w_i}w_j)\langle R_{z_i}, R_{z_j}\rangle \\
&= \left\|\sum_{j=0}^{n} a_j R_{z_j}\right\|_{D_K}^2 - |a_0|^2|f(z_0)|^2 \\
&= \left\|\sum_{j=0}^{n} a_j R_{z_j}\right\|_{D_K}^2 - \left|\langle f, \sum_{j=0}^{n} a_j R_{z_j}\rangle\right|^2 \geq 0.
\end{aligned}$$

Combine with Lemma 3.2 of [3], we know that D_K has Pick property. Thus, there exists $F_n \in M(D_K)$ with $\|F_n\|_{M(D_K)} \leq 1$ such that

$$F_n(z_0) = \frac{f(z_0)}{\|R_{z_0}\|_{D_K}} \quad \text{and} \quad F_n(z_j) = 0 \ \ (j = 1, ..., n).$$

Then, for all n, we have $\|F_n\|_{H^\infty} \leq \|F_n\|_{M(D_K)} \leq 1$. So $\{F_n\}_{n\geq 1}$ is a normal family, the limit function F is also a multiplier with $\|F\|_{M(D_K)} \leq 1$,

$$F(z_0) = \frac{f(z_0)}{\|R_{z_0}\|_{D_K}} \neq 0 \quad \text{and} \quad F(z_j) = 0 \ \ (j = 1, ..., n).$$

By f-property of D_K space (see [15]), we have every function $f \in D_K$, there exist $F \in M(D_K)$ with the same zero set. That is, D_K and $M(D_K)$ have the same zero sets.

Note that $S_K \cap H^\infty \subseteq S_K \subseteq D_K$. We only need to prove that $M(D_K) \subseteq S_K \cap H^\infty$. Suppose that $f \in M(D_K)$. From [3, Theorem 4.6], we known that $|f'(z)|^2K(1-|z|^2)dA(z)$ is a Carleson measure for D_K. Let $|I| = |J|$ in Lemma 1, that is, if μ is a Carleson measure for D_K, we can deduce that $\mu(S(I)) \lesssim K_3(|I|) \approx K(|I|)$. Thus,

$$\int_{S(I)} |f'(z)|^2K(1-|z|^2)dA(z) \lesssim K(|I|).$$

Combine with Lemma 2, we deduce that $f \in S_K$. Notice that $M(D_K) \subseteq H^\infty$ (see [3, Theorem 4.6]). That is, $M(D_K) \subseteq S_K \cap H^\infty$. □

3 Carleson type measure

In this section, we give a characterization for (K, s)-Carleson measure and vanishing (K, s)-Carleson measure.

Theorem 2. *Suppose that K satisfy the conditions (1) and (2). Let μ be a positive Borel measure on $\mathbb{D}$, $0 \leq s < \infty$ such that $s + c > 1$. Then μ is a (K, s)-Carleson*

measure if and only if

$$\sup_{a\in\mathbb{D}}\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)<\infty. \tag{6}$$

Proof. Suppose that (6) holds. For any $I\subseteq\partial\mathbb{D}$, let $b=(1-|I|)\zeta\in\mathbb{D}$, where ζ is the center of I. Then, for any $z\in S(I)$, we have

$$1-|b|\approx|1-\overline{b}z|\approx|I| \quad\text{and}\quad K(1-|b|^2)\approx K(|I|).$$

Therefore,

$$\begin{aligned}\frac{\mu(S(I))}{K(|I|)|I|^s}\lesssim&\frac{1}{K(1-|b|^2)}\int_{S(I)}\frac{(1-|b|^2)^s}{|1-\overline{b}z|^{2s}}d\mu(z)\\ \leq&\frac{1}{K(1-|b|^2)}\int_{\mathbb{D}}\frac{(1-|b|^2)^s}{|1-\overline{b}z|^{2s}}d\mu(z)\\ \leq&\sup_{a\in\mathbb{D}}\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)<\infty,\end{aligned}$$

which implies that μ is a (K,s)-Carleson measure by the arbitrary of I.

Conversely, assume that μ is a (K,s)-Carleson measure. Without loss of generality, we assume $|a|>\frac{4}{5}$. Let I_n be the arc on $\partial\mathbb{D}$ such that $\frac{a}{|a|}$ is the center of I_n and $|I_n|=A^{(n-1)}(1-|a|)$, where $1<A<2^{\frac{1}{s}}$, $n=1,2,...,N$, where N is the smallest integer such that $A^{(N-1)}(1-|a|)\geq 1$. Since

$$\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\lesssim\frac{1}{A^{2(n-1)s}(1-|a|)^s}\lesssim\frac{1}{A^{2ns}(1-|a|)^s},\ \ z\in S(I_n)\setminus S(I_{n-1}),$$

and

$$\frac{|I_n|^s}{A^{ns}(1-|a|)^s}=A^{(n-1)s-ns}<1,\qquad\frac{K\left(A^{(n-1)}(1-|a|)\right)}{K(1-|a|^2)}\lesssim A^{(n-1)(1-c)},$$

we obtain

$$\begin{aligned}\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)\leq&\sum_{n=1}^{N}\frac{|I_n|^sK(|I_n|)}{A^{2ns}(1-|a|)^sK(1-|a|^2)}\\ \lesssim&\sum_{n=1}^{N}\frac{K\left(A^{(n-1)}(1-|a|)\right)}{A^{ns}K(1-|a|^2)}\\ \lesssim&\sum_{n=1}^{N}\frac{A^{(n-1)(1-c)}}{A^{ns}}<\sum_{n=1}^{\infty}\frac{A^{n(1-c)}}{A^{ns}}\\ <&\infty.\end{aligned}$$

Here we used the fact that $s+c>1$. □

Theorem 3. *Suppose that K satisfy the conditions (1) and (2). Let μ be a positive Borel measure on $\mathbb{D}$, $0 \leq s < \infty$ such that $s + c > 1$. Then μ is a vanishing (K, s)-Carleson measure if and only if*

$$\lim_{|a|\to 1} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} d\mu(z) = 0. \tag{7}$$

Proof. First we assume that μ is a vanishing (K, s)-Carleson measure. For any $\epsilon > 0$, there is a $\eta > 0$ such that for all arcs $I \subseteq \partial\mathbb{D}$ with $|I| \leq \eta$, such that

$$\frac{\mu(S(I))}{K(|I|)|I|^s} < \epsilon.$$

Assume $a = re^{i\theta}$ and $r > 1 - \eta$. Let $I_\eta \subseteq \partial\mathbb{D}$ such that $e^{i\theta}$ is the center of I_η and $|I_\eta| = \eta$. Then

$$\int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} d\mu(z) =: M_1 + M_2,$$

where

$$M_1 =: \int_{\mathbb{D}\setminus S(I_\eta)} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} d\mu(z)$$

and

$$M_2 =: \int_{S(I_\eta)} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} d\mu(z).$$

Suppose that $e^{i\theta}$ is also the center of $\{I_n\}$, $|I_n| = A^{n-1}(1-|a|)$, $\quad A > 1, \quad n = 1, 2, ..., N-1$ and N is the smallest integer such that $|I_N| > \eta$. Let $I_0 = \phi$. Note that

$$\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} \lesssim \frac{1}{A^{2ns}(1-|a|)^s}, \qquad z \in S(I_n) \setminus S(I_{n-1}).$$

We have

$$\begin{aligned} M_2 \leq & \sum_{n=1}^{N} \int_{S(I_n)\setminus S(I_{n-1})} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} d\mu(z) \\ \lesssim & \frac{1}{(1-|a|)^s} \sum_{n=1}^{N-1} \frac{\mu(S(I_n)\setminus S(I_{n-1}))}{A^{2ns}} + \frac{\mu(S(I_\eta)\setminus S(I_{N-1}))}{A^{2Ns}(1-|a|)^s} \\ \lesssim & \frac{1}{(1-|a|)^s} \sum_{n=1}^{N} \frac{\mu(S(I_n))}{A^{2ns}}. \end{aligned}$$

Note that $\frac{\mu(S(I))}{K(|I|)|I|^s} < \epsilon$ and $|I_n| = A^{n-1}(1-|a|)$, using the fact that $\varphi_K(t) \lesssim$

$t^{1-c},\ \ t \geq 1$, we deduce that

$$\begin{aligned} M_2 &\lesssim \frac{1}{(1-|a|)^s}\sum_{n=1}^{N}\frac{\epsilon K(|I_n|)|I_n|^s}{A^{2ns}} \\ &\lesssim \sum_{n=1}^{N}\frac{\epsilon K(A^{n-1}(1-|a|))}{K(1-|a|)A^{ns+s}}K(1-|a|) \\ &\lesssim \sum_{n=1}^{N}\varphi_K(A^{n-1})\frac{\epsilon K(1-|a|)}{A^{ns+s}} \\ &\lesssim \epsilon\sum_{n=1}^{N}A^{(n-1)(1-c)-ns-s}K(1-|a|) \lesssim \epsilon K(1-|a|). \end{aligned} \tag{8}$$

Now, we estimate M_1. Since $|1-\overline{a}z| \geq \eta,\ \ z \in \mathbb{D}\backslash S(I_\eta)$, and notice the fact that $\frac{t^{1-c}}{K(t)}$ is a nondecreasing function when $0 < t < 1$, we obtain

$$\begin{aligned} M_1 &\lesssim \frac{\mu(\mathbb{D})}{\eta^{2s}}\frac{(1-|a|)^s}{K(1-|a|)}K(1-|a|) \\ &\lesssim \frac{(1-|a|)^{1-c+s-1+c}}{K(1-|a|)}K(1-|a|) \lesssim (1-|a|)^{s+c-1}K(1-|a|). \end{aligned} \tag{9}$$

From (8) and (9) we see that (7) holds.

The proof for another side is similar to Theorem 2. Thus, we omit the details. □

Finally, we give another characterization of vanishing (K,s)-Carleson measure by using functions in D_K.

Theorem 4. *Suppose that K satisfy the conditions (1) and (2). Let μ be a positive Borel measure on $\mathbb{D}$, $0 \leq s < \infty$ such that $s+c > 1$. Let $\{g_n\}$ be a bounded sequence in D_K such that $g_n \to 0$ uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$. Then μ is a vanishing (K,s)-Carleson measure if and only if*

$$\lim_{n\to\infty}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g_n(a)|^2\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z) = 0. \tag{10}$$

Proof. First we assume that μ is a vanishing (K,s)-Carleson measure. Following the proof of Theorem 3, for any given $\epsilon > 0$, we may find $\kappa > 0$ such that

$$\sup_{a\in\mathbb{D}\backslash\overline{\mathbb{D}_\kappa}}\frac{1}{K(1-|a|)}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z) < \epsilon,$$

where $\mathbb{D}_\kappa = \{z \in \mathbb{D} : |z| < \kappa\}$. Since

$$|g(z)| \lesssim \frac{\|g\|_{D_K}}{\sqrt{K(1-|z|)}},\ \ g \in D_K,$$

we obtain

$$\begin{aligned} & \sup_{a\in\mathbb{D}\setminus\overline{\mathbb{D}_\kappa}}\int_{\mathbb{D}}|g_n(a)|^2\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z) \\ \lesssim & \sup_{a\in\mathbb{D}\setminus\overline{\mathbb{D}_\kappa}}\frac{1}{K(1-|a|)}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)<\epsilon. \end{aligned} \tag{11}$$

Also, since $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we see that for n sufficiently large,

$$\sup_{a\in\overline{\mathbb{D}_\kappa}}\int_{\mathbb{D}}|g_n(a)|^2\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)\leq\epsilon\sup_{a\in\overline{\mathbb{D}_\kappa}}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)\lesssim\epsilon. \tag{12}$$

From (11) and (12) we see that (10) holds.

Conversely, assume that (10) holds. For $a\in\mathbb{D}$, it is easy to check that

$$g_a(z)=\frac{1-|a|}{(1-\overline{a}z)\sqrt{K(1-|a|)}}\in D_K.$$

For any $I_n\subseteq\partial\mathbb{D}$ such that $|I_n|1$ as $n\to\infty$, let $a_n=(1-|I_n|)e^{i\theta_n}\in\mathbb{D}$, where $e^{i\theta_n}$ is the center of I_n. It is easy to check that $\{g_{a_n}\}$ is a bounded sequence in D_K and $g_{a_n}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. By (10), we have

$$\lim_{n\to\infty}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g_{a_n}(z)|^2\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}d\mu(z)=0.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{|I_n|^s}\int_{S(I_n)}|g_{a_n}(z)|^2d\mu(z)=0.$$

Notice the fact that $|g_{a_n}(z)|^2\gtrsim\frac{1}{K(|I_n|)}$, we get

$$\frac{\mu(S(I_n))}{K(|I_n|)|I_n|^s}\to 0,\ \ as\ \ n\to\infty,$$

as desired. □

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Ruishen Qian: School of Mathematics and Statistics, Lingnan normal University, 524048, Zhanjiang, Guangdong, P. R. China.
Email: qianruishen@sina.cn

Songxiao Li: Institute of System Engineering, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau.
Email: jyulsx@163.com

Yanhua Zhang: Department of Mathematics, Qufu Normal University, 273165, Qufu, ShanDong, P. R. China.
Email: qfuzhangyanhua@163.com

⋆ Corresponding author: Songxiao Li

Smarandache fuzzy BCI-algebras

Sun Shin Ahn[1] and Young Joo Seo[*,2]

[1]**Department of Mathematics Education, Dongguk University, Seoul 04620, Korea**
[2]**Research Institute for Natural Sciences, Department of Mathematics, Hanyang University, Seoul, 04763, Korea**

Abstract. The notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean(fresh) ideal of a Smarandache BCI-algebra are introduced. Examples are given, and several related properties are investigated.

1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A with a strong structure S which is embedded in A. In [4], R. Padilla showed that Smarandache semigroups are very important for the study of congruences. Y. B. Jun ([1,2]) introduced the notion of Smarandache BCI-algebras, Smarandache fresh and clean ideals of Smarandache BCI-algebras, and obtained many interesting results about them.

In this paper, we discuss a Smarandache fuzzy structure on BCI-algebras and introduce the notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache BCI-algebra are introduced, and we investigate their properties.

2. Preliminaries

An algebra $(X;*,0)$ of type (2,0) is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)(((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y\in X)((x*(x*(x*y))*y=0)$,
(III) $(\forall x\in X)((x*x=0)$,
(IV) $(\forall x,y\in X)(x*y=0$ and $y*x=0$ imply $x=y)$.

If a BCI-algebra X satisfies the following identity;

(V) $(\forall x\in X)(0*x=0)$,

then X is said to be a *BCK-algebra*. We can define a partial order " $\leq$ " on X by $x\leq y$ if and only if $x*y=0$.

Every BCI-algebra X has the following properties:

(a_1) $(\forall x\in X)(x*0=x)$,
(a_1) $(\forall x,y,z\in X)(x\leq y$ implies $x*z\leq y*z, z*y\leq z*x)$.

A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(i) $0\in I$,
(ii) $(\forall x\in X)(\forall y\in I)(x*y\in I$ implies $x\in I)$.

[0*] Correspondence: Tel: +82 10 9247 6575 (Y. J. Seo).

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Definition 2.1. ([1]) A *Smarandache BCI-algebra* is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,
(ii) Q is a BCK-algebra under the same operation of X.

By a *Smarandache positive implicative* (resp. *commutative* and *implicative*) *BCI-algebra*, we mean a BCI-algebra X which has a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,
(ii) Q is a positive implicative (resp. commutative and implicative) BCK-algebra under the same operation of X.

Let $(X; *, 0)$ be a Smarandache BCI-algebra and H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H; *, 0)$ is a Smarandache BCI-algebra.

A non-empty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in Q)(\forall y \in I)(x * y \in I \text{ implies } x \in I)$,

where Q is a BCK-algebra contained in X. If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 2.2. ([2]) A non-empty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, a *Q-Smarandache ideal*) of X if it satisfies:

(c_1) $0 \in I$,
(c_2) $(\forall x \in Q)(\forall y \in I)(x * y \in I \text{ implies } x \in I)$.

If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

Definition 2.3. ([2]) A non-empty subset I of X is called a *Smarandache fresh ideal* of X related to Q (or briefly, a *Q-Smarandache fresh ideal* of X) if it satisfies the conditions (c_1) and

(c_3) $(\forall x, y, z \in Q)(((x * y) * z) \in I \text{ and } y * z \in I \text{ imply } x * z \in I)$.

Theorem 2.4. ([2]) *Every Q-Smarandache fresh ideal which is contained in Q is a Q-Smarandache ideal.*

The converse of Theorem 2.4 need not be true in general.

Theorem 2.5. ([2]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache fresh ideal of X, then so is J.*

Definition 2.6. ([2]) A non-empty subset I of X is called a *Smarandache clean ideal* of X related to Q (or briefly, a *Q-Smarandache clean ideal* of X) if it satisfies the conditions (c_1) and

(c_4) $(\forall x, y \in Q)(z \in I)((x * (y * x)) * z \in I \text{ implies } x \in I)$.

Theorem 2.7. ([2]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache ideal.*

The converse of Theorem 2.7 need not be true in general.

Theorem 2.8. ([2]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache fresh ideal.*

Theorem 2.9. ([2]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache clean ideal of X, then so is J.*

A fuzzy set μ in X is called a *fuzzy subalgebra* of a BCI-algebra X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in X is called a *fuzzy ideal* of X if

(F_1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in X$.

Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ.

3. **Smarandache fuzzy ideals**

Definition 3.1. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy subalgebra* of X if it satisfies

(SF_1) $\mu(0) \geq \mu(x)$ for all $x \in P$,

(SF_2) $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$.

A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy ideal* of X if it satisfies (SF_1) and

(SF_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy subalgebra (ideal) is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a fuzzy subalgebra(ideal) of X.

Example 3.2. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,2,3\}, \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy subalgebra of X. It is verified that μ restricted to a subset $\{0,1,2,3\}$ which is a subalgebra of X is a fuzzy subalgebra of X, i.e., $\mu_{\{0,1,2,3\}} : \{0,1,2,3\} \to [0,1]$ is a fuzzy subalgebra of X. Thus $\mu : X \to [0,1]$ is a Smarandache fuzzy subalgebra of X. Note that $\mu : X \to [0,1]$ is not a fuzzy subalgebra of X, since $\mu(5*4) = \mu(0) = 0.5 \not\geq \min\{\mu(5), \mu(4)\} = 0.7$.

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Example 3.3. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	44	4	0	0	
5	5	4	4	5	1	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2\} \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy ideal of X. It is verified that μ restricted to a subset $\{0, 1, 2\}$ which is an ideal of X is a fuzzy ideal of X, i.e., $\mu_{\{0,1,2\}} : \{0, 1, 2\} \to [0, 1]$ is a fuzzy ideal of X. Thus $\mu : X \to [0, 1]$ is a Smarandache fuzzy ideal of X. Note that $\mu : X \to [0, 1]$ is not a fuzzy ideal of X, since $\mu(2) = 0.5 \not\geq \min\{\mu(2*4) = \mu(4), \mu(4)\} = \mu(4) = 0.7$.

Lemma 3.4. *Every Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X is order reversing.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x, y \in P$ with $x \leq y$, then $x * y = 0$. Hence we have $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. □

Theorem 3.5. *Any Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X must be a Smarandache fuzzy subalgebra of X.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|X| \geq 2$. Since $x * y \leq x$ for any $x, y \in P$, it follows from Lemma 3.4 that $\mu(x) \leq \mu(x * y)$, so by (SF_2) we obtain $\mu(x * y) \geq \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. This shows that μ is a Smarandache fuzzy subalgebra of X, proving the theorem. □

Proposition 3.6. *Let μ_P be a Smarandache fuzzy ideal of a Smarandache BCI-algebra X. If the inequality $x * y \leq z$ holds in P, then $\mu(x) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in P$.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x * y \leq z$ in P, then $(x * y) * z = 0$. Hence we have $\mu(x * y) \geq \min\{\mu((x * y) * z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)$. It follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}$. □

Theorem 3.7. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy ideal of X if and only if for all $x, y \in P$, the inequality $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

Proof. Suppose that μ_P is a Smarandache fuzzy subalgebra of X satisfying the condition $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$. Since $x * (x * y) \leq y$ for all $x, y \in P$, it follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Hence μ_P is a Smarandache fuzzy ideal of X. The converse follows from Proposition 3.6. □

Definition 3.8. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0, 1]$ is called a *Smarandache fuzzy clean ideal* of X if it satisfies (SF_1) and

(SF_3) $\mu(x) \geq \min\{\mu(x * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in P$,

where $P \subsetneq X$ and P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy clean ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy clean ideal of X.

Example 3.9. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	5
2	2	1	0	1	0	5
3	3	4	4	4	0	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.4 & \text{if } x \in \{0,1,2,3\} \\ 0.8 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy clean ideal of X, but μ is not a fuzzy clean ideal of X, since $\mu(3) = 0.4 \not\geq \min\{\mu((3*(0*3))*5), \mu(5)\} = \min\{\mu(5), \mu(5)\} = \mu(5) = 0.8$.

Theorem 3.10. *Let X be a Smarandache BCI-algebra. Any Smarandache fuzzy clean ideal μ_P of X must be a Smarandache fuzzy ideal of X.*

Proof. Let X be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Let $\mu_P : P \to [0,1]$ be a Smarndache fuzzy clean ideal of X. If we let $y := x$ in (SF_3), then $\mu(x) \geq \min\{\mu((x*(x*x))*z), \mu(z)\} = \min\{\mu((x*0)*z), \mu(z)\} = \min\{\mu(x*z), \mu(z)\}$, for all $x, y, z \in P$. This shows that μ satisfies (SF_2). Combining (SF_1), μ_P is a Smarandache fuzzy ideal of X, proving the theorem. □

Corollary 3.11. *Every Smarandache fuzzy clean ideal μ_P of a Smarndache BCI-algebra X must be a Smarandache fuzzy subalgebra of X.*

Proof. It follows from Theorem 3.5 and Theorem 3.10. □

The converse of Theorem 3.10 may not be true as shown in the following example.

Example 3.12. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	0	5
2	2	2	0	0	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

Let μ_P be a fuzzy set in $P = \{0,1,2,3,4\}$ defined by $\mu(0) = \mu(2) = 0.8$ and $\mu(1) = \mu(3) = \mu(4) = 0.3$. It is easy to check that μ_P is a fuzzy ideal of X. Hence $\mu : X \to [0,1]$ is a Smarandache fuzzy ideal of X. But it is not a Smarandache fuzzy clean ideal of X since $\mu(1) = 0.3 \not\geq \min\{\mu((1*(3*1))*2), \mu(2)\} = \min\{\mu(0), \mu(2)\} = 0.8$.

Theorem 3.13. *Let X be a Smarandache implicative BCI-algebra. Every Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy clean ideal of X.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Since X is a Smarandache implicative BCI-algebra, we have $x = x * (y * x)$ for all $x, y \in P$. Let μ_P be a Smarandache fuzzy ideal of X. It follows from (SF_2) that $\mu(x) \geq \min\{\mu(x * z), \mu(z)\} \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, for all $x, y, z \in P$. Hence μ_P is a Smarandache clean ideal of X. The proof is complete. □

In what follows, we give characterizations of fuzzy implicative ideals.

Theorem 3.14. *Let X be a Smarandache BCI-algebra. Suppose that μ_P is a Smarandache fuzzy ideal of X. Then the following equivalent:*

(i) *μ_P is Smarandache fuzzy clean,*
(ii) *$\mu(x) \geq \mu(x * (y * x))$ for all $x, y \in P$,*
(iii) *$\mu(x) = \mu(x * (y * x))$ for all $x, y \in P$.*

Proof. (i) ⇒ (ii): Let μ_P be a Smarandache fuzzy clean ideal of X. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * 0), \mu(0)\} = \min\{\mu(x * (y * x)), \mu(0)\} = \mu(x * (y * x)),\ \forall x, y \in P$. Hence the condition (ii) holds.
(ii) ⇒ (iii): Since X is a Smarnadache BCI-algebra, we have $x * (y * x) \leq x$ for all $x, y \in P$. It follows from Lemma 3.4 that $\mu(x) \leq \mu(x * (y * x))$. By (ii), $\mu(x) \geq \mu(x * (y * x))$. Thus the condition (iii) holds.
(iii) ⇒ (i): Suppose that the condition (iii) holds. Since μ_P is a Smarandache fuzzy ideal, by (SF_2), we have $\mu(x * (y * x)) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Combining (iii), we obtain $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Hence μ satisfies the condition (SF_3). Obviously, μ satisfies (SF_1). Therefore μ is a fuzzy clean ideal of X. Hence the condition (i) holds. The proof is complete. □

For any fuzzy sets μ and ν in X, we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for any $x \in X$.

Definition 3.15. Let X be a Smarandache BCI-algebra and let $\mu_P : P \to [0, 1]$ be a Smarandache fuzzy BCI-algebra of X. For $t \leq \mu(0)$, the set $\mu_t := \{x \in P | \mu(x) \geq t\}$ is called a *level subset* of μ_P.

Theorem 3.16. *A fuzzy set μ in P is a Smarandache fuzzy clean ideal of X if and only if, for all $t \in [0, 1]$, μ_t is either empty or a Smarandache clean ideal of X.*

Proof. Suppose that μ_P is a Smarandache fuzzy clean ideal of X and $\mu_t \neq \emptyset$ for any $t \in [0, 1]$. It is clear that $0 \in \mu_t$ since $\mu(0) \geq t$. Let $\mu((x * (y * x)) * z) \geq t$ and $\mu(z) \geq t$. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\} \geq t$, namely, $x \in \mu_t$. This shows that μ_t is a Smarandache clean ideal of X.

Conversely, assume that for each $t \in [0, 1]$, μ_t is either empty or a Smaranadche clean ideal of X. For any $x \in P$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a Smarandache clean ideal of X, therefore $0 \in \mu_t$ and hence $\mu(0) \geq \mu(x) = t$. Thus $\mu(0) \geq \mu(x)$ for all $x \in P$. Now we show that μ satisfies (SF_3). If not, then there exist $x', y', z' \in P$ such that $\mu(x') < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Taking $t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}\}$, we have $\mu(x') < t_0 < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Hence $x' \notin \mu_{t_0}$, $(x' * (y' * x')) * z \in \mu_{t_0}$, and $z' \in \mu_{t_0}$, i.e., μ_{t_0} is not a Smaraqndache clean of X, which is a contradiction. Therefore, μ_P is a Smarnadche fuzzy clean ideal, completing the proof. □

Theorem 3.17. ([2]) (Extension Property) *Let X be a Smarandache BCI-algebra. Let I and J be Q-Smarandache ideals of X and $I \subseteq J \subseteq Q$. If I is a Q-Smarandache clean ideal of X, then so is J.*

Next we give the extension theorem of Smarandache fuzzy clean ideals.

Theorem 3.18. *Let X be a Smarandache BCI-algebra. Let μ and ν be Smarandache fuzzy ideals of X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a Smarndache fuzzy clean ideal of X, then so is ν.*

Proof. It suffices to show that for any $t \in [0,1]$, ν_t is either empty or a Smarandache clean ideal of X. If the level subset ν_t is non-empty, then $\mu_t \neq \emptyset$ and $\mu_t \subseteq \nu_t$. In fact, if $x \in \mu_t$, then $t \leq \mu(x)$; hence $t \leq \nu(x)$, i.e, $x \in \nu_t$. So $\mu_t \subseteq \nu_t$. By the hypothesis, since μ is a Smarandache fuzzy clean ideal of X, μ_t is a Smarandache clean of X by Theorem 3.16. It follows from Theorem 3.17 that ν_t is a Smarandache clean ideal of X. Hence ν is a Smarandache fuzzy clean of X. The proof is complete. □

Definition 3.19. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy fresh ideal* of X if it satisfies (SF_1) and

(SF_4) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$ for all $x, y, z \in P$,

where P is a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. This Smarandache fuzzy ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy fresh ideal of X.

Example 3.20. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,3\}, \\ 0.9 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy fresh ideal of X. But it is not a fuzzy fresh ideal of X, since $\mu(2 * 4) = \mu(0) = 0.5 \not\geq \min\{\mu((2 * 5) * 4), \mu(5 * 4)\} = \mu(5) = 0.9$.

Theorem 3.21. *Any Smarandache fuzzy fresh ideal of a Smarandache BCI-algebra X must be a Smarandache fuzzy ideal of X.*

Proof. Taking $z := 0$ in (SF_4) and $x * 0 = x$, we have $\mu(x * 0) \geq \min\{\mu((x * y) * 0), \mu(y * 0)\}$. Hence $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Thus (SF_2) holds. □

The converse of Theorem 3.21 may not be true as show in the following example.

Example 3.22. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	1	5
2	2	1	0	1	2	5
3	3	1	1	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,4\}, \\ 0.4 & \text{otherwise} \end{cases}$$

Clearly $\mu(x)$ is a Samrandache fuzzy ideal of X. But $\mu(x)$ is not a Samrandache fuzzy fresh ideal of X, since $\mu(2*3) = \mu(1) = 0.4 \not\geq \min\{\mu((2*1)*3), \mu(1*3)\} = \min\{\mu(1*3), \mu(0)\} = \mu(0) = 0.5$.

Proposition 3.23. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh ideal of X if and only if it satisfies the condition $\mu(x*y) \geq \mu((x*y)*y)$ for all $x, y \in P$.*

Proof. Assume that μ_P is a Smarandache fuzzy fresh ideal of X. Putting $z := y$ in (SF_4), we have $\mu(x*y) \geq \min\{\mu((x*y)*y), \mu(y*y)\} = \min\{\mu((x*y)*y), \mu(0)\} = \mu((x*y)*y),\ \forall x, y \in P$.

Conversely, let μ_P be Smarandache fuzzy ideal of X such that $\mu(x*y) \geq \mu((x*y)*y)$. Since, for all $x, y, z \in P$, $((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$, we have $\mu((x*y)*z) \leq \mu(((x*z)*z)*(y*z))$. Hence $\mu(x*z) \geq \mu((x*z)*z) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y*z)\} \geq \min\{\mu((x*y)*z), \mu(y*z)\}$. This completes the proof. □

Since $(x*y)*y \leq x*y$, it follows from Lemma 3.4 that $\mu(x*y) \leq \mu((x*y)*y)$. Thus we have the following theorem.

Theorem 3.24. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh if and only if it satisfies the identity*

$$\mu(x*y) = \mu((x*y)*y),\ textfor\ all\ x, y \in X.$$

We give an equivalent condition for which a Smarandache fuzzy subalgebra of a Smarandache BCI-algebra to be a Smarandache fuzzy clean ideal of X.

Theorem 3.25. *A Smarandache fuzzy subalghebra μ_P of X is a Smarandache fuzzy clean ideal of X if and only if it satisfies*

$$(x*(y*x))*z \leq u \text{ implies } \mu(x) \geq \min\{\mu(z), \mu(u)\} \text{ for all} x, y, z, u \in P. \qquad (*)$$

Proof. Assume that μ_P is a Smarandache fuzzy clean ideal of X. Let $x, y, z, u \in P$ be such that $(x*(y*x))*z \leq u$. Since μ is a Smarandache fuzzy ideal of X, we have $\mu(x*(y*x)) \geq \min\{\mu(z), \mu(u)\}$ by Theorem 3.7. By Theorem 3.14-(iii), we obtain $\mu(x) \geq \min\{\mu(z), \mu(u)\}$.

Conversely, suppose that μ_P satisfies $(*)$. Obviously, μ_P satisfies (SF_1), since $(x*(y*x))*((x*(y*x))*z) \leq z$, by $(*)$, we obtain $\mu(x) \geq \min\{\mu((x*(y*x))*z), \mu(z)\}$, which shows that μ_P satisfies (SF_3). Hence μ_P is a Smarandache fuzzy clean ideal of X. The proof is complete. □

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On Fibonacci derivative equations

Hee Sik Kim[1], J. Neggers[2] and Keum Sook So[3,*]

[1]**Department of Mathematics, Hanyang University, Seoul, 133-791, Korea**
[2]**Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A**
[3,*]**Department of Mathematics, Hallym University, Chuncheon 200-702, Korea**

Abstract. In this paper, we introduce the notions of Fibonacci (co-)derivative of real-valued functions. We find general solutions of the equations $\triangle(f(x)) = g(x)$ and $(\triangle + I)(f(x)) = g(x)$.

1. Introduction

The theory of Fibonacci-numbers has been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. The most amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. Atanassov et al. [1] and Dunlap [2] provided general and fundamental surveys on the theory of Fibonacci numbers. Hyers-Ulam studied the stability of Fibonacci functional equations [5]. Han et al. [3] discussed Fibonacci sequences in both several groupoids and groups. The present authors [6] introduced the notion of generalized Fibonacci sequences over a groupoid, and investigated these in particular for the case of a groupoid containing idempotents and pre-idempotents.

Han et al. [4] studied Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1) + f(x)$, and they developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. The present authors [7] studied Fibonacci functions using the (ultimately) periodicity and also discussed the exponential Fibonacci functions. Especially, given a non-negative real-valued function, the present authors obtained several exponential Fibonacci functions.

In this paper, we introduce the notions of Fibonacci (co-)derivative of real-valued functions. We find general solutions of the equations $\triangle(f(x)) = g(x)$ and $(\triangle + I)(f(x)) = g(x)$.

2. Preliminaries

A function f defined on the real numbers is said to be a *Fibonacci function* ([4]) if it satisfies the formula

$$f(x+2) = f(x+1) + f(x)$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 2.1. ([4]) Let $f(x) := a^x$ be a Fibonacci function on $\mathbf{R}$ where $a > 0$. Then $a^x a^2 = f(x+2) = f(x+1) + f(x) = a^x(a+1)$. Since $a > 0$, we have $a^2 = a+1$ and $a = \frac{1+\sqrt{5}}{2}$. Hence $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, and the unique Fibonacci function of this type on $\mathbf{R}$.

If we let $u_0 = 0, u_1 = 1$, then we consider the *full Fibonacci sequence*: $\cdots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \cdots$, i.e., $u_{-n} = (-1)^n u_n$ for $n > 0$, and $u_n = F_n$, the nth Fibonacci number.

[0*] Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

Example 2.2. ([4]) Let $\{u_n\}_{n=-\infty}^{\infty}$ and $\{v_n\}_{n=-\infty}^{\infty}$ be full Fibonacci sequences. We define a function $f(x)$ by $f(x) := u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor} t$, where $t = x - \lfloor x \rfloor \in (0,1)$. Then $f(x+2) = u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor} t = u_{(\lfloor x \rfloor + 2)} + v_{(\lfloor x \rfloor + 2)} t = (u_{(\lfloor x \rfloor + 1)} + u_{\lfloor x \rfloor}) + (v_{(\lfloor x \rfloor + 1)} + v_{\lfloor x \rfloor})t = f(x+1) + f(x)$ for any $x \in \mathbf{R}$. This proves that f is a Fibonacci function.

Note that if a Fibonacci function is differentiable on $\mathbf{R}$, then its derivative is also a Fibonacci function.

Proposition 2.3. ([4]) *Let f be a Fibonacci function. If we define $g(x) := f(x+t)$ where $t \in \mathbf{R}$ for any $x \in \mathbf{R}$, then g is also a Fibonacci function.*

For example, since $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, $g(x) = (\frac{1+\sqrt{5}}{2})^{x+t} = (\frac{1+\sqrt{5}}{2})^t f(x)$ is also a Fibonacci function where $t \in \mathbf{R}$.

3. **Fibonacci derivatives**

Let $f : \mathbf{R} \to \mathbf{R}$ be a real-valued function. We shall consider the expression

$$(\triangle f)(x) := f(x+2) - f(x+1) - f(x)$$

to be the *Fibonacci derivative* of $f(x)$. For example, if $\Phi := \frac{1+\sqrt{5}}{2}$, then $f(x) = \Phi^x$ yields $(\triangle f)(x) = \Phi^{x+2} - \Phi^{x+1} - \Phi^x = \Phi^x(\Phi^2 - \Phi - 1) = 0$. If f is any Fibonacci function, then $(\triangle f)(x) = 0$ for all $x \in \mathbf{R}$ and conversely. Note that if $\triangle f = \triangle g$, then $f - g$ is a Fibonacci function.

Example 3.1. If $f(x) := ax + b$, then

$$\begin{aligned} \triangle(ax+b) &= [a(x+2)+b] - [a(x+1)+b] - [ax+b] \\ &= -ax + (a-b) \end{aligned}$$

and $\triangle(b) = -b,\ \triangle(x) = -x + 1$.

Simultaneously we shall also consider the *Fibonacci co-derivative* of f, denoted $(\triangle + I)(f)$, by the formula

$$(\triangle + I)(f)(x) = \triangle(f) + f(x) = f(x+2) - f(x+1)$$

Thus for example, if $f(x) = ax + b$, then $(\triangle + I)(ax+b) = [a(x+2)+b] - [a(x+1)+b] - [ax+b] = a$, which coincides with $\frac{d}{dx}(ax+b)$.

We pose a question: what is the "anti-derivative" of a function $f : \mathbf{R} \to \mathbf{R}$, i.e., given $f : \mathbf{R} \to \mathbf{R}$, find $g : \mathbf{R} \to \mathbf{R}$ such that $\triangle g = f$. For example, $\triangle(-x-1) = [-(x+2)-1] - [-(x+1)-1] - [-x-1] = x$. Hence the Fibonacci anti-derivative of x is $-x - 1 + \varphi$ where φ is a Fibonacci function.

Proposition 3.2. *Fibonacci functions are fixed points for Fibonacci co-derivative operator $\triangle + I$.*

Proof. Let $f(x)$ be a Fibonacci function. Then $(\triangle f)(x) = 0$ and hence $(\triangle + I)(f)(x) = (\triangle f)(x) + f(x) = f(x)$. □

Proposition 3.3. *If $(\triangle + I)(f)(x) = 0$, then $(\triangle^2 f)(x) = f(x)$.*

Proof. If $(\triangle + I)(f)(x) = 0$, then $(\triangle f)(x) = -f(x)$. It follows that $(\triangle^2 f)(x) = \triangle(-f(x)) = (-f)(x+2) - (-f)(x+1) - (-f)(x) = -(\triangle f)(x) = -(-f(x)) = f(x)$. □

Example 3.4. Suppose that $\triangle f = x^2$ and $f(x) = ax^2+bx+c$. Since $\triangle(x^2) = -x^2+2x+3$ and $\triangle(x) = -x+1$, we obtain

$$\begin{aligned} x^2 &= \triangle(ax^2+bx+c) \\ &= a\triangle(x^2)+b\triangle(x)+\triangle(c) \\ &= -ax^2+(2a-b)x+(3a+b-c) \end{aligned}$$

It follows that $a=-1, b=-2$ and $c=-5$, i.e., $\triangle(-x^2-2x-5) = x^2$. Thus, the general Fibonacci anti-derivative of x^2 is $-x^2-2x-5+\varphi$ where φ is a Fibonacci function.

Example 3.5. Suppose that $\triangle f = x^3$ and $f(x) = ax^3+bx^2+cx+d$. Since $\triangle(x^3) = -x^3+3x^2+9x+7$, we obtain $\triangle(-x^3-3x^2-15x-31) = x^3$ as in Example 3.4.

Theorem 3.6. *Let $\triangle f_n = x^n$ and let $f_0, f_1, \cdots, f_{n-1}$ be determined to yield particular solutions for $\triangle f_k = x^k$ $(k = 0, 1, \cdots, n-1)$. Then*

$$f_n = -x^n + \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1] f_k + \varphi$$

where φ is a Fibonacci function.

Proof. Let $\triangle f_n = x^n$ and let $f_n = -x^n + Q_n(x)$ where $Q_n(x)$ is a polynomial of x of degree $n-1$. Then

$$\begin{aligned} x^n &= \triangle(f_n) \\ &= \triangle(-x^n+Q_n(x)) \\ &= -\triangle(x^n)+\triangle(Q_n(x)) \\ &= -[(x+2)^n-(x+1)^n-x^n]+\triangle(Q_n(x)) \end{aligned}$$

It follows that $\triangle(Q_n(x)) = (x+2)^n-(x+1)^n = \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1]x^k$. Assume $f_0, f_1, \cdots, f_{n-1}$ are determined to have a particular solutions for $\triangle f_k = x^k$ $(k = 0, 1, \cdots, n-1)$. Then

$$\begin{aligned} \triangle(Q_n(x)) &= \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1] \triangle f_k \\ &= \triangle(\sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1] f_k) \end{aligned}$$

It follows that $Q_n(x) = \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1] f_k + \varphi$ for some Fibonacci function φ. Hence $f_n = -x^n + Q_n(x) = \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1] f_k + \varphi$. □

Example 3.7. In the above examples it was known that $f_0 = -1+\varphi, f_1 = -x-1+\varphi, f_2 = -x^2-2x-5+\varphi$ and $f_3 = -x^3-3x^2-15x-31+\varphi$ where φ is a Fibonacci function. We compute $Q_4(x)$ as follows:

$$\begin{aligned} Q_4(x) &= \sum_{k=0}^{3} \left(\begin{array}{c} 4 \\ k \end{array} \right) [2^{4-k}-1]f_k + \varphi \\ &= 15f_0 + 28f_1 + 18f_2 + 4f_3 + \varphi \\ &= 15(-1) + 28(-x-1) + 18(-x^2-2x-5) \\ &\quad +4(-x^3-3x^2-15x-31) \\ &= -4x^3 - 30x^2 - 124x - 257 + \varphi \end{aligned}$$

This shows that $f_4 = -x^4 + Q_4(x) + \varphi = -x^4 - 4x^3 - 30x^2 - 124x - 257 + \varphi$ where φ is a Fibonacci function.

Theorem 3.8. *Given a polynomial $g(x) := a_0 + a_1x + \cdots + a_nx^n$, we have a particular solution for $\triangle(f(x)) = g(x)$ as $f(x) = a_0f_0 + a_1f_1 + \cdots + a_nf_n$, where $\triangle f_k = x^k$ $(k = 0, 1, \cdots, n)$ and a general solution $f(x) + \varphi(x)$ where $\triangle(\varphi(x)) = 0$.*

Proof. It follows immediately from Theorem 3.6. □

4. **Fibonacci co-derivatives**

Let us consider the problem $(\triangle + I)^k(f(x)) = x^n$. We have $(\triangle + I)(1) = \triangle(1) + I(1) = 0, (\triangle + I)(x) = \triangle(x) + I(x)(-x+1) + x = 1$ and $(\triangle + I)(x^2) = \triangle(x^2) + I(x^2) = 2x + 3$. Using Theorem 3.6, we obtain the following proposition.

Proposition 4.1. *The Fibonacci co-derivative of x^n is*

$$(\triangle + I)(x^n) = \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1]x^k$$

Proof. Using Theorem 3.6, we obtain

$$\begin{aligned} (\triangle + I)(x^n) &= \triangle(x^n) + I(x^n) \\ &= (x+2)^n - (x+1)^n \\ &= \triangle(Q_n(x)) \\ &= \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [2^{n-k}-1]x^k, \end{aligned}$$

proving the proposition. □

Example 4.2. If we let $n := 4$ in Proposition 4.1, then

$$\begin{aligned}(\triangle + I)(x^4) &= \sum_{k=0}^{3} \left(\begin{array}{c} 4 \\ k \end{array} \right) [2^{4-k} - 1]x^k \\ &= \left(\begin{array}{c} 4 \\ 0 \end{array} \right) (2^4 - 1)x^0 + \left(\begin{array}{c} 4 \\ 1 \end{array} \right) (2^3 - 1)x^1 + \left(\begin{array}{c} 4 \\ 2 \end{array} \right) (2^2 - 1)x^2 \\ &\quad + \left(\begin{array}{c} 4 \\ 3 \end{array} \right) (2^3 - 1)x^3 \\ &= 4x^3 + 18x^2 + 28x + 15\end{aligned}$$

It follows that $\triangle(x^4) = -x^4 + 4x^3 + 18x^2 + 28x + 15$.

Consider now $(\triangle + I)^2(1) = (\triangle + I)[(\triangle + I)(1)] = (\triangle + I)(0) = 0$ and $(\triangle + I)^2(x) = (\triangle + I)[(\triangle + I)(x^2)] = (\triangle + I)(2x + 3) = 2$. Similarly we obtain $(\triangle + I)^2(x^3) = (\triangle + I)[3x^2 + 9x + 7] = 6x + 18$.

Using Proposition 4.1, we obtain the following formula.

Proposition 4.3. *For any natural number n, we have*

$$(\triangle + I)^2(x^n) = \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} k \\ j \end{array} \right) (2^{n-k} - 1)(2^{k-j} - 1)x^j$$

Proof. Using Proposition 4.1, we obtain the following.

$$\begin{aligned}(\triangle + I)^2(x^n) &= (\triangle + I)[\sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) (2^{n-k} - 1)x^k] \\ &= \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) (2^{n-k} - 1) \sum_{j=0}^{k-1} \left(\begin{array}{c} k \\ j \end{array} \right) (2^{k-j} - 1)x^j \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} k \\ j \end{array} \right) (2^{n-k} - 1)(2^{k-j} - 1)x^j\end{aligned}$$

□

Example 4.4. We compute $(\triangle + I)^2(x^4)$ as follows.

$$\begin{aligned}(\triangle + I)^2(x^4) &= (\triangle + I)[4x^3 + 18x^2 + 28x + 15] \\ &= 4(\triangle + I)(x^3) + 18(\triangle + I)(x^2) + 28(\triangle + I)(x) \\ &\quad +15(\triangle + I)(1) \\ &= 12x^2 + 72x + 110\end{aligned}$$

Upon checking Proposition 4.3 when $n = 4$, we find that

$$\begin{aligned}
(\triangle + I)^2(x^4) &= \sum_{k=0}^{3}\sum_{j=0}^{k-1} \begin{pmatrix} 4 \\ k \end{pmatrix} \begin{pmatrix} k \\ j \end{pmatrix} (2^{4-k} - 1)(2^{k-j} - 1)x^j \\
&= \sum_{j=0}^{0} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} (2^3 - 1)(2^{1-j} - 1)x^j \\
&+ \sum_{j=0}^{1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ j \end{pmatrix} (2^2 - 1)(2^{2-j} - 1)x^j \\
&+ \sum_{j=0}^{2} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ j \end{pmatrix} (2^1 - 1)(2^{3-j} - 1)x^j \\
&= 28x^0 + 54x^0 + 36x + 28x^0 + 36x + 12x^2 \\
&= 12x^2 + 72x + 110
\end{aligned}$$

Next, we want to obtain an exact analog of Theorem 3.6 for the Fibonacci co-derivative $\triangle + I$.

Example 4.5. Let $f_1(x) := ax^2 + bx + c$ be a polynomial satisfying $(\triangle + I)(f_1(x)) = x$. Then $x = (\triangle + I)(ax^2 + bx + c) = 2ax + 3a + b$. It follows that $2a = 1, 3a + b = 0$, i.e., $a = \frac{1}{2}, b = -\frac{3}{2}$ and c is arbitrary. Hence $(\triangle + I)(\frac{1}{2}x^2 - \frac{3}{2}x + c) = x$ where c is a constant. Similarly, we may find a polynomial $f_2(x)$ satisfying $(\triangle + I)(f_2(x)) = x^2$, i.e., $(\triangle + I)(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{13}{6}x + d) = x^2$ where d is a constant. In this fashion, we obtain a polynomial $f_n(x) = \frac{1}{n+1}x^{n+1} + q_{n+1}(x)$ which can be determined so that $(\triangle + I)(f_n(x)) = x^n$ where $q_{n+1}(x)$ is a polynomial of degree n.

Theorem 4.6. *Given a polynomial $g(x) := a_0 + a_1x + \cdots + a_nx^n$, a particular solution for $(\triangle + I)(f(x)) = g(x)$ as $f(x) = a_0f_0 + a_1f_1 + \cdots + a_nf_n$ is obtained, where $f_n(x) = \frac{1}{n+1}x^{n+1} + q_{n+1}(x)$ where $q_{n+1}(x)$ is a polynomial of degree n.*

Proof. The proof is similar to the proof of Theorem 3.8. □

5. **Solving the equation** $(\triangle + I)^n(f(x)) = q(x)$

Consider $(\triangle + I)(f(x)) = (\triangle + I)(g(x))$. It means that $(\triangle + I)(f(x) - g(x)) = 0$, i.e., $(f - g)(x + 2) - (f - g)(x + 1) = 0$ for all $x \in \mathbf{R}$. This shows that there exists a map $\psi : \mathbf{R} \to \mathbf{R}$ with $\psi(x+2) = \psi(x+1)$ for all $x \in \mathbf{R}$ such that $f = g + \psi$. If we let $\mathbf{B}_1 := \{\psi | (\triangle + I)(\psi(x)) = 0, \forall x \in \mathbf{R}\}$, then $\mathbf{B}_1$ consists of all functions $\varphi : \mathbf{R} \to \mathbf{R}$ such that φ is periodic of period 1. This means that

$$\varphi \in \mathbf{B}_1 \iff \varphi(x + 1) - \varphi(x) = 0, \forall x \in \mathbf{R}$$

Hence general solution of $(\triangle + I)^n(f(x)) = q(x)$ is $\{p(x) + \psi(x) \mid \triangle(p(x)) = q(x), \psi(x) \in \mathbf{B}_1\} = \{p(x) + \psi(x) \mid \triangle(p(x)) = q(x), \psi(x + 1) = \psi(x), \forall x \in \mathbf{R}\}$. Consider $(\triangle + I)^2(f(x)) = q(x)$. Let $p(x)$ be a polynomial in $\mathbf{R}[x]$ such that $(\triangle + I)^2(p(x)) = q(x)$. Then $(\triangle + I)^2(p(x)) = (\triangle + I)^2(f(x))$. It follows that $(\triangle + I)^2(f(x) - p(x)) = 0$, i.e.,

there exists a polynomial $\psi(x) \in \mathbf{R}[x]$ such that $f(x) - p(x) = \psi(x)$ and $(\triangle + I)^2(\psi(x)) = 0$. This means that $(\triangle + I)[(\triangle + I)(\psi(x))] = 0$, i.e., $(\triangle + I)(\psi(x)) \in \mathbf{B}_1$. If we let $\mathbf{B}_2 := \{\varphi(x) \in \mathbf{R}[x] \,|\, (\triangle + I)(\varphi(x)) \in \mathbf{B}_1\}$, then

$$\begin{aligned}
\varphi(x) \in \mathbf{B}_2 &\iff (\triangle + I)(\varphi(x)) \in \mathbf{B}_1 \\
&\iff \exists h(x) \in \mathbf{B}_1 \text{ such that } \varphi(x+2) - \varphi(x+1) = h(x) \\
&\iff h(x+1) - h(x) = 0, \varphi(x+2) - \varphi(x+1) = h(x) \\
&\iff \varphi(x+2) - 2\varphi(x+1) + \varphi(x) = 0
\end{aligned}$$

Hence the set of all general solutions of $(\triangle + I)^2(f(x)) = q(x)$ is $\{p(x) + \psi(x) \,|\, (\triangle + I)^2(p(x)) = q(x), \psi(x) \in \mathbf{B}_2\} = \{p(x) + \psi(x) \,|\, (\triangle + I)^2(p(x)) = q(x), \psi(x+2) - 2\psi(x+1) + \psi(x) = 0, \forall x \in \mathbf{R}\}$. Similarly, if we let $\mathbf{B}_3 := \{\varphi(x) \,|\, (\triangle + I)(\varphi(x)) \in \mathbf{B}_2\}$, then $\mathbf{B}_3 = \{\varphi \in \mathbf{R}[x] \,|\, \varphi(x+3) - 3\varphi(x+2) + 3\varphi(x+1) - \varphi(x) = 0\}$. We generalize this fact as follows:

Lemma 5.1. *If we let* $\mathbf{B}_n := \{\varphi(x) \,|\, (\triangle + I)(\varphi(x)) \in \mathbf{B}_{n-1}\}$, *then* $\mathbf{B}_n = \{\varphi \in \mathbf{R}[x] \,|\, \sum_{r=0}^{n} \binom{n}{r} \varphi(x+n-r) = 0, \forall x \in \mathbf{R}\}$.

Theorem 5.2. *Given a polynomial* $p(x) \in \mathbf{R}[x]$, *there exists a polynomial* $q_n(x) \in \mathbf{R}[x]$ *such that* $(\triangle + I)^n(q_n(x)) = p(x)$, *and its general solution* $f(x)$ *is of the form* $q_n(x) + \varphi(x)$ *where* $\varphi(x) \in \mathbf{B}_n$.

Proof. It follows from Theorem 4.6 and Lemma 5.1. □

6. **Concluding remark**

Given Theorem 3.6 and the fact the $\varphi(x) \equiv 0$ is a Fibonacci function, a particular solution to the Fibonacci derivative equation $\triangle f_n = x^n$, is given iteratively by the formula:

$$f_n = -x^n + \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] f_k$$

where if we set $f_0 = 1$, we obtain a sequence of polynomials of degree n for f_n, $n = 0, 1, 2, \cdots$. From the structure of the formula we may surmise the existence of many combinatorial properties of the sequence. Also upon rewriting:

$$f_n = \sum_{l=0}^{n} A_{ln} x^l, \;\; A_{nn} = -1,$$

the coefficients A_{ln}, thought of as analogs of binomial numbers, should illustrate a great number of combinatorial relations among themselves as well as with other families, including the binomial numbers (coefficients). Since $(\triangle + I)(x^n) = \sum_{k=0}^{n-1} \binom{n}{k} [2^{n-k} - 1] x^k$ exhibits a "similar" form, we expect there to be confirmation of the claim made above in a multitude of ways, above and beyond what has already been illustrated.

7. **Future works**

Given what has been done in this paper, it is clear that very much remains to be done. Thus, a much more detailed study of functions of the $F(x)$ type, as described above, remains to be done. Furthermore, as pointed out in the concluding remarks, there is much to be done still in completing the combinatorial grammar which is associated with the solution of a particular kind to the equation $\triangle^m f_{mn} = x^n$, of which some cases have been looked at above, but for which very significant gaps still remain to be explored. Also, as usual in this type of research, the law of natural growth of problems prevails, i.e., as one problem is successfully resolved, novel gaps noted present themselves for consideration and no finality is in sight (nor expected) for the area of study touched upon in this case as well, to the benefit of those engaged in furthering knowledge of this (as well as any other) subject.

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A NOTE ON SYMMETRIC IDENTITIES FOR TWISTED DAEHEE POLYNOMIALS

JONGKYUM KWON[1] AND JIN-WOO PARK[2,*]

Abstract. In this paper, we consider the twisted Daehee numbers and polynomials. We investigate some new and explicit symmetric identities for the twisted Daehee polynomials arising from p-adic invariant integral on $\mathbb{Z}_p$.

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the *p-adic invariant integral on $\mathbb{Z}_p$* is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_0(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)d\mu_0\left(x+p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x).\end{aligned} \tag{1.1}$$

Thus, by (1.1), we get

$$\int_{\mathbb{Z}_p} f_1(x)du_0(x) - \int_{\mathbb{Z}_p} f(x)du_0(x) = f'(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$ (see [1, 4, 9]).

From (1.2), we can derive

$$\int_{\mathbb{Z}_p} f_n(x)du_0(x) - \int_{\mathbb{Z}_p} f(x)du_0(x) = \sum_{l=0}^{n-1} f'(l),\ (n\in\mathbb{N}), \tag{1.3}$$

where $f_n(x) = f(x+n)$ (see [1, 5, 6]).

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\ \text{(see [1, 2, 4])}. \tag{1.4}$$

When $x=0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

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* corresponding author.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \left\{\omega | \omega^{p^n} = 1\right\}$ is the cyclic group of order p^n.

It is well known that for $\xi \in T_p$, the twisted Bernoulli polynomials are defined as

$$\left(\frac{t}{\xi e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x)\frac{t^n}{n!}, \text{ (see [1, 2])}. \tag{1.5}$$

When $x = 0$, $B_{n,\xi} = B_{n,\xi}(0)$ are called the twisted Bernoulli numbers.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, the Daehee polynomials are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{ (see [1, 2, 4 - 14, 16,17])}. \tag{1.6}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

From (1.4) and (1.6), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} &= \sum_{m=0}^{\infty} D_n(x)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} D_m(x)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{1.7}$$

where $S_2(n,m)$ is the Stirling number of the second kind which is given by the generating function to be

$$\frac{1}{m!}(e^t-1)^m = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}, \quad \text{(see [3, 15])}.$$

By (1.7), we get

$$B_n(x) = \sum_{m=0}^{n} D_m(x)S_2(n,m), (n \geq 0). \tag{1.8}$$

From (1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} &= \sum_{m=0}^{\infty} B_m(x)\frac{1}{m!}(\log(1+t))^m \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m(x)S_1(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{1.9}$$

where $S_1(n,m)$ is the Stirling number of the first kind which is defined by falling factorials as follows:

$$(x)_0 = 1, \ (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \in \mathbb{N}),$$

Thus, by (1.9), we get

$$D_n(x) = \sum_{m=0}^{n} B_m(x)S_1(n,m), \ (n \geq 0). \tag{1.10}$$

From (1.2), we derive Witt's formula for Daehee polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{x+y}du_0(y) &= \frac{\log(1+t)}{t}(1+t)^x \\ &= \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}.\end{aligned} \tag{1.11}$$

Thus, by (1.11), we get

$$\int_{\mathbb{Z}_p}(x+y)_n du_0(y) = D_n(x), (n \geq 0), \quad \text{(see [8])}. \tag{1.12}$$

Now, we consider the twisted Daehee polynomials defined by the generating function to be

$$\frac{\log(1+\xi t)}{\xi t}(1+\xi t)^x = \sum_{n=0}^{\infty} D_{n,\xi}(x)\frac{t^n}{n!}, \text{ (see [5, 13, 14])}. \tag{1.13}$$

When $x = 0$, $D_{n,\xi} = D_{n,\xi}(0)$ are called the twisted Daehee numbers.

In [5], authers define twisted λ-Daehee polynomials which are given by the p-adic invariant integral on $\mathbb{Z}_p$ to be

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\xi t)^{\lambda(x+y)}d\mu_0(y) &= \frac{\lambda\log(1+\xi t)}{(1+\xi t)^{\lambda}-1}(1+\xi t)^{\lambda x} \\ &= \sum_{n=0}^{\infty} D_{n,\lambda,\xi}(x)\frac{t^n}{n!}.\end{aligned} \tag{1.14}$$

In the special case, $\lambda = 1, \xi = 1$, we note that $D_{n,1,1}(x) = D_n(x)$. When $x = 0$, then $D_{n,\lambda,\xi} = D_{n,\lambda,\xi}(0)$ are called *twisted* λ *- Daehee numbers*.

Recently, several authors have researched twisted Daehee polynomials in the several areas (see [5, 13, 14]). In this paper, we investigate some explicit and new symmetric identities for the twisted Daehee polynomials which are derived from the p-adic invariant integral on $\mathbb{Z}_p$.

2. Symmetric identities for the twisted Daehee polynomials

Let $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. Now, we consider the following p-adic integral on $\mathbb{Z}_p$. From (1.3), we easily get

$$\begin{aligned}
&\frac{1}{\log(1+\xi t)}\left(\int_{\mathbb{Z}_p}(1+\xi t)^{n+x}du_0(x)-\int_{\mathbb{Z}_p}(1+\xi t)^{x}du_0(x)\right)\\
=&\frac{1}{\log(1+\xi t)}\sum_{l=0}^{n-1}(1+\xi t)^{l}\log(1+\xi t)\\
=&\sum_{l=0}^{n-1}(1+\xi t)^{l}\\
=&\sum_{i=0}^{n-1}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-1}i^{m}S_1(n,m)\right)\frac{\xi^n t^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_1(n,m)\sum_{i=0}^{n-1}i^{m}\right)\frac{\xi^n t^n}{n!}.
\end{aligned} \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned}
&\frac{1}{\log(1+\xi t)}\left(\int_{\mathbb{Z}_p}(1+\xi t)^{n+x}du_0(x)-\int_{\mathbb{Z}_p}(1+\xi t)^{x}du_0(x)\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_1(n,m)S_m(n-1)\right)\frac{\xi^n t^n}{n!},
\end{aligned} \tag{2.2}$$

where for given positive integer k, $S_k(n)=0^k+1^k+2^k+\cdots+n^k$.

From (1.2) and (1.3), we have

$$\begin{aligned}
&\frac{1}{\log(1+\xi t)}\left(\int_{\mathbb{Z}_p}(1+\xi t)^{n+x}du_0(x)-\int_{\mathbb{Z}_p}(1+\xi t)^{x}du_0(x)\right)\\
=&\frac{n\int_{\mathbb{Z}_p}(1+\xi t)^{x}du_0(x)}{\int_{\mathbb{Z}_p}(1+\xi t)^{n+x}du_0(x)}\\
=&\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}S_1(k,m)S_k(n-1)\right)\frac{\xi^n t^k}{k!}.
\end{aligned} \tag{2.3}$$

We recall that Cauchy numbers are defined by the generating function to be

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}. \tag{2.4}$$

By (2.3) and (2.4), we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\left(D_{m,\xi}(k)-D_{m,\xi}\right)C_{k-m}\binom{k}{m}\right)\frac{t^k}{k!}\\
=&\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k-1}S_1(k-1,m)S_{k-1}(n-1)\right)\frac{t^k}{k!}.
\end{aligned} \tag{2.5}$$

From (2.5), we have

$$\begin{aligned}&\frac{1}{k}\left(\sum_{m=0}^{k}(D_{m,\xi}(k)-D_{m,\xi})C_{k-m}\binom{k}{m}\right)\\&=\sum_{m=0}^{k-1}S_1(k-1,m)S_{k-1}(n-1),\ (k\in\mathbb{N},n\in\mathbb{N}).\end{aligned}\tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. *For $k,n\in\mathbb{N}$, we have*

$$\frac{1}{k}\left(\sum_{m=0}^{k}(D_{m,\xi}(k)-D_{m,\xi})C_{k-m}\binom{k}{m}\right)=\sum_{m=0}^{k-1}S_1(k-1,m)S_{k-1}(n-1).\tag{2.7}$$

Now, we consider symmetric identities for the twisted Daehee polynomials. Let $w_1,w_2\in\mathbb{N}$. Then, we easily see that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x_1+w_2x_2}du_0(x_1)du_0(x_2)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1w_2x}du_0(x)}&=\frac{\frac{w_1\log(1+\xi t)}{((1+\xi t)^{w_1}-1)}\frac{w_2\log(1+\xi t)}{((1+\xi t)^{w_2}-1)}}{\frac{w_1w_2\log(1+\xi t)}{((1+\xi t)^{w_1w_2}-1)}}\\&=\frac{\log(1+\xi t)((1+\xi t)^{w_1w_2}-1)}{((1+\xi t)^{w_1}-1)((1+\xi t)^{w_2}-1}.\end{aligned}\tag{2.8}$$

We consider the following double p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}I=&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x_1+w_2x_2+w_1w_2x}d\mu_0(x_1)d\mu_0(x_2)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1w_2x}d\mu_0(x)}\\=&\frac{\log(1+\xi t)(1+\xi t)^{w_1w_2x}((1+\xi t)^{w_1w_2}-1)}{((1+\xi t)^{w_1}-1)((1+\xi t)^{w_2}-1)}.\end{aligned}\tag{2.9}$$

From (1.2) and (1.3), we have

$$\begin{aligned}\frac{w_1\int_{\mathbb{Z}_p}(1+\xi t)^x du_0(x)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x}du_0(x)}&=\sum_{k=0}^{w_1-1}(1+\xi t)^k\\&=\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}S_1(l,m)S_m(w_1-1)\right)\frac{\xi^l t^l}{l!}\end{aligned}\tag{2.10}$$

From (2.10), we get

$$\begin{aligned}I=&\left(\frac{1}{w_1}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1(x_1+w_2x)}d\mu_0(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}(1+\xi t)^{w_2x_2}d\mu_0(x_2)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1w_2x}d\mu_0(x)}\right)\\=&\frac{1}{w_1}\left(\sum_{i=0}^{\infty}D_{i,w_1,\xi}(w_2x)\frac{t^i}{i!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}w_2^mS_1(k,m)S_m(w_1-1)\right)\frac{\xi^kt^k}{k!}\right)\\=&\frac{1}{w_1}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}D_{i,w_1,\xi}(w_2x)\sum_{m=0}^{n-i}w_2^mS_1(n-i,m)S_m(w_1-1)\right)\frac{\xi^{n-i}t^n}{n!}.\end{aligned}\tag{2.11}$$

On the other hand, by (2.10), we get

$$\begin{aligned} I =& \left(\frac{1}{w_2}\int_{\mathbb{Z}_p}(1+\xi t)^{w_2(x_2+w_1x)}d\mu_0(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x_1}d\mu_0(x_1)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1w_2x}d\mu_0(x)}\right)\\ =&\frac{1}{w_2}\left(\sum_{i=0}^{\infty}D_{i,w_2,\xi}(w_1x)\frac{t^i}{i!}\right)\left(\sum_{k=0}^{\infty}(\sum_{m=0}^{k}w_1^mS_1(k,m)S_m(w_2-1))\frac{\xi^kt^k}{k!}\right)\\ =&\frac{1}{w_2}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}D_{i,w_2,\xi}(w_1x)\sum_{m=0}^{n-i}w_1^mS_1(n-i,m)S_m(w_2-1)\right)\frac{\xi^{n-i}t^n}{n!}. \end{aligned} \tag{2.12}$$

Therefore, by (2.9), (2.11) and (2.12), we obtain the following theorem.

Theorem 2.2. *For $w_1, w_2 \in \mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$, we have*

$$\begin{aligned} &\frac{1}{w_1}\sum_{i=0}^{n}\binom{n}{i}D_{i,w_1,\xi}(w_2x)\ \xi^{n-i}\sum_{m=0}^{n-i}w_2^mS_1(n-i,m)S_m(w_1-1)\\ =&\frac{1}{w_2}\sum_{i=0}^{n}\binom{n}{i}D_{i,w_2,\xi}(w_1x)\ \xi^{n-i}\sum_{m=0}^{n-i}w_1^mS_1(n-i,m)S_m(w_2-1). \end{aligned}$$

Remark. By replacing t by $\frac{1}{\xi}\left(e^t-1\right)$ in (1.14), we get

$$\begin{aligned} \sum_{n=0}^{\infty}B_n(x)\lambda^n\frac{t^n}{n!} =& \sum_{n=0}^{\infty}D_{n,\lambda,\xi}(x)\frac{1}{n!}\left(\frac{1}{\xi}\left(e^t-1\right)\right)^n\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\xi^{-n}S_2(n,m)D_{m,\lambda,\xi}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

Thus, by (2.13), we have

$$\lambda^nB_n(x) = \sum_{m=0}^{n}\xi^{-n}S_2(n,m)D_{m,\lambda,\xi}(x),\ (n\geq 0). \tag{2.14}$$

By replacing t by $\log\left(t+\frac{1}{\xi}\right)$ in (1.5), we have

$$\begin{aligned} \sum_{n=0}^{\infty}D_{n,\xi}(x)\frac{t^n}{n!} =& \sum_{n=0}^{\infty}B_n(x)\frac{1}{n!}\left(\log\left(t+\frac{1}{\xi}\right)\right)^n\\ =&\sum_{n=0}^{\infty}B_n(x)\sum_{l=n}^{\infty}S_1(l,n)\frac{(\xi t)^n}{n!}\\ =&\sum_{n=0}^{\infty}\left(\xi^n\sum_{m=0}^{n}B_m(x)S_1(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.15}$$

Thus, by (2.15), we get

$$\xi^{-n}D_{n,\xi}(x) = \sum_{m=0}^{n}B_m(x)S_1(n,m),\ (n\geq 0). \tag{2.16}$$

From (2.9), we note that

$$\begin{aligned}
I &= \left(\frac{(1+\xi t)^{w_1w_2x}}{w_1}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x_1}d\mu_0(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}(1+\xi t)^{w_2x_2}d\mu_0(x_2)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1w_2x}d\mu_0(x)}\right)\\
&= \left(\frac{(1+\xi t)^{w_1w_2x}}{w_1}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x}d\mu_0(x_1)\right)\left(\sum_{l=0}^{w_1-1}(1+\xi t)^{w_2l}\right)\\
&= \left(\frac{1}{w_1}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x}d\mu_0(x_1)\right)\left(\sum_{l=0}^{w_1-1}(1+\xi t)^{w_1(w_2x+\frac{w_2}{w_1}l}\right)\\
&= \frac{1}{w_1}\sum_{l=0}^{w_1-1}\int_{\mathbb{Z}_p}(1+\xi t)^{w_1(x_1+w_2x+\frac{w_2}{w_1}l)}d\mu_0(x_1)\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{w_1}\sum_{l=0}^{w_1-1}D_{n,w_1,\xi}(w_2x+\frac{w_2}{w_1}l)\right)\frac{\xi^nt^n}{n!}.
\end{aligned} \tag{2.17}$$

On the other hand, we obtain the following equation by the symmetric property of p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}
I &= \left(\frac{(1+\xi t)^{w_1w_2x}}{w_2}\int_{\mathbb{Z}_p}(1+\xi t)^{w_2x_2}d\mu_0(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}(1+\xi t)^{w_1x_1}d\mu_0(x_1)}{\int_{\mathbb{Z}_p}(1+\xi t)^{w_1w_2x}d\mu_0(x)}\right)\\
&= \left(\frac{1}{w_2}\int_{\mathbb{Z}_p}(1+\xi t)^{w_2x_2}d\mu_0(x_2)\right)\left(\sum_{l=0}^{w_2-1}(1+\xi t)^{w_1l}\right)(1+\xi t)^{w_1w_2x}\\
&= \left(\frac{1}{w_2}\int_{\mathbb{Z}_p}(1+\xi t)^{w_2x_2}d\mu_0(x_2)\right)\left(\sum_{l=0}^{w_2-1}(1+\xi t)^{w_2(w_1x+\frac{w_1}{w_2}l)}\right)\\
&= \frac{1}{w_2}\sum_{l=0}^{w_2-1}\int_{\mathbb{Z}_p}(1+\xi t)^{w_2(x_2+w_1x+\frac{w_1}{w_2}l)}d\mu_0(x_2)\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{w_2}\sum_{l=0}^{w_2-1}D_{n,w_2,\xi}(w_1x+\frac{w_1}{w_2}l)\right)\frac{\xi^nt^n}{n!}.
\end{aligned} \tag{2.18}$$

Therefore, by comparing the coefficients on the both sides of (2.17) and (2.18), we obtain the following theorem.

Theorem 2.3. *For $w_1, w_2 \in \mathbb{N}$ and $n \geq 0$, we have*

$$\frac{1}{w_1}\sum_{l=0}^{w_1-1}D_{n,w_1,\xi}\left(w_2x+\frac{w_2}{w_1}l\right) = \frac{1}{w_2}\sum_{l=0}^{w_2-1}D_{n,w_2,\xi}\left(w_1x+\frac{w_1}{w_2}l\right).$$

Corollary 2.4. *For $w_1, w_2 \in \mathbb{N}$ and $n \geq 0$, we have*

$$\begin{aligned}
&\sum_{l=0}^{w_1-1}\sum_{m=0}^{n}w_1^{m-1}B_m\left(w_2x+\frac{w_2}{w_1}l\right)S_1(n,m)\\
&= \sum_{l=0}^{w_2-1}\sum_{m=0}^{n}w_2^{m-1}B_m\left(w_1x+\frac{w_1}{w_2}l\right)S_1(n,m).
\end{aligned}$$

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[1] Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea

E-mail address: mathkjk26@gnu.ac.kr

[2,*] Department of Mathematics Education, Daegu University, Gyeongsan-si, Kyungsangbukdo, 38453, Republic of Korea

E-mail address: a0417001@knu.ac.kr

Dynamics and Behavior of $x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}}$

M. M. El-Dessoky[1,2] and Aatef Hobiny[1,3]
[1]King Abdulaziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Mansoura University, Faculty of Science, Department of Mathematics,
Mansoura 35516, Egypt.
[3]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
E-mail: dessokym@mans.edu.eg; ahobany@kau.edu.sa

ABSTRACT

The main objective of this paper is to study the local and the global stability of the solutions, the periodic character and the boundedness of the difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}},$$

where the parameters $\beta,\ \alpha,\ a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are positive real numbers. Some numerical examples will be given to illustrate our results.

Keywords: Difference equations, Stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations or discrete dynamical systems are diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system $x_{n+1} = f(x_n)$ determines a difference equation and vice versa. Recently, there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in many applied sciences. The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, physiology, ecology, engineering, physics, economics, genetics, probability theory, psychology and resource management. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. Systems of rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such systems [1-33].

Many research have been done to study the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For example, Agarwal et al. [2] looked at the global stability, periodicity character and found the solution form of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ c,\ d$ and the initial conditions $x_{-r},\ x_{-r+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, while $x_i \neq b/c$ for $i = -r,\ -r+1,\ ...,\ 0$ where $r = \max\{l,\ k,\ s\}$.

Hamza and Morsy in [3] investigated the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\alpha,\ k \in (0, \infty)$ and the initial values x_{-1} and x_0 are arbitrary positive real numbers.

Elsayed et al. [4] studied the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{b+cx_{n-1}}{d+ex_{n-1}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

Zayed et al. [5] studied the behavior of the following rational recursive sequence

$$x_{n+1} = \frac{\left(A + \sum_{i=0}^{k} \alpha_i\ x_{n-i}\right)}{\sum_{i=0}^{k} \beta_i\ x_{n-i}}, \qquad n = 0,\ 1,\ 2,\ ...,$$

where the coefficients $A,\ \alpha_i,\ \beta_i$ and the initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, while k is a positive integer number.

Also, in [6] Zayed et al. obtained the global behavior of the difference equation

$$x_{n+1} = \frac{\sum_{i=0}^{k} \alpha_i\ x_{n-i}}{B + \sum_{i=0}^{k} \beta_i\ x_{n-i}}, \qquad n = 0,\ 1,\ 2,\ ...,$$

where the coefficients $B,\ \alpha_i,\ \beta_i$ and the initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_{-1},\ x_0$ are arbitrary positive real numbers, while k is a positive integer number.

In [7] El-Moneam investigated the periodicity, the boundedness and the global stability of the positive solutions of nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}, \qquad n = 0,\ 1,\ 2,\ ...,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and s are positive integers and the initial conditions $x_{-s},\ ...,\ x_{-l}, ...,\ x_{-k}, ...,\ x_{-1},\ x_0$ are arbitrary positive real numbers such that $k < l < s$.

Yalçınkaya [8] investigated the global behaviour of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}, \qquad n = 0,\ 1,\ ...,$$

where the parametere $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

Elabbasy et al. [9] studied the dynamics, the global stability, periodicity character and the solution of special case of the recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \qquad n = 0,\ 1,\ ...,$$

where the initial conditions $x_{-1},\ x_0$ are arbitrary real numbers and $a,\ b,\ c,\ d$ are positive constants.

El-Owaidy et al. [10] investigated local stability, oscillation and boundedness character of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \qquad n = 0,\ 1,\ ...,$$

under specified conditions.

Elsayed [11] studied some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{cx_n - dx_{n-1}}, \qquad n = 0,\ 1,\ ...,$$

where the initial conditions $x_{-1},\ x_0$ are arbitrary real numbers and $a,\ b,\ c,\ d$ are positive constants with $cx_0 - dx_{-1} \neq 0$.

Elsayed and El-Dessoky [12] investigated the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $a,\ b,\ c,\ d$ and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, ..., x_{-1},\ x_0$ are positive real numbers where $t = max\{s,\ l,\ k\}$.

This paper aims to study the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}}, \quad n = 0, 1, ..., \tag{1}$$

where the parameters $a,\ b,\ c,\ d,\ \alpha$ and β are positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are positive real numbers.

2. SOME BASIC PROPERTIES AND DEFINITIONS

In this section, we state some basic definitions and theorems that we need in this paper.

Let I be some interval of real numbers and let

$$F : I^3 \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-2},\ x_{-1},\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n,\ x_{n-1},\ ,\ x_{n-2}), \tag{2}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

Definition 1. (Equilibrium Point)

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = F(\overline{x},\ \overline{x},\ ,\ \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2.1. *(Stability)*
(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-2},\ x_{-1},\ x_0 \in I$ with

$$|x_{-2} - \overline{x}| + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-2},\ x_{-1},\ x_0 \in I$ with

$$|x_{-2} - \overline{x}| + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ *of Eq.(2) is global attractor if for all* $x_{-2},\ x_{-1},\ x_0 \in I$, *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ *of Eq.(2) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq.(2).*

(v) The equilibrium point $\overline{x}$ *of Eq.(2) is unstable if* $\overline{x}$ *is not locally stable.*

DEFINITION 2.2. *(Boundedness)*
A sequence $\{x_n\}_{n=-2}^{\infty}$ *is said to be bounded and persists if there exist posiyive constants* m *and* M *such that*

$$m \le x_n \le M \quad \text{for all } n \ge -2.$$

DEFINITION 2.3. *(Periodicity)*
A sequence $\{x_n\}_{n=-2}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \ge -1$. *A sequence* $\{x_n\}_{n=-2}^{\infty}$ *is said to be periodic with prime period* p *if* p *is the smallest positive integer having this property.*

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \frac{\partial F(\overline{x},\ \overline{x},\ ,\ \overline{x})}{\partial x_n} y_n + \frac{\partial F(\overline{x},\ \overline{x},\ ,\ \overline{x})}{\partial x_{n-1}} y_{n-1} + \frac{\partial F(\overline{x},\ \overline{x},\ ,\ \overline{x})}{\partial x_{n-2}} y_{n-2}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^2 + p_1\lambda + p_2 = 0, \tag{4}$$

where

$$p_0 = \frac{\partial F(\overline{x},\ \overline{x},\ ,\ \overline{x})}{\partial x_n},\ p_1 = \frac{\partial F(\overline{x},\ \overline{x},\ ,\ \overline{x})}{\partial x_{n-1}} \text{ and } p_2 = \frac{\partial F(\overline{x},\ \overline{x},\ ,\ \overline{x})}{\partial x_{n-2}}.$$

Theorem A [18]: Assume that $p_i \in R,\ i = 1,\ 2,\ 3$. Then

$$|p_1| + |p_2| + |p_3| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+3} + p_1 x_{n+2} + p_2 x_{n+1} + p_3 x_n = 0.$$

Theorem B [19]: Let $g : [a,b]^3 \to [a,b]$, be a continuous function, where 3 is a positive integer, and $[a,b]$ is an interval of real numbers and consider the difference equation

$$x_{n+1} = g(x_n,\ x_{n-1},\ x_{n-2}). \tag{5}$$

Suppose that g satisfies the following conditions:

(i) For every integer i with $1 \le i \le 3$, the function $g(z_1,\ z_2,\ z_3)$ is weakly monotonic in z_i, for fixed $z_1,\ z_2,\ z_3$.

(ii) If m, M is a solution of the system

$$m = g(m_1,\ m_2,\ m_3) \text{ and } M = g(M_1,\ M_2,\ M_3),$$

then $m = M$, where for each $i = 1,\ 2,\ 3$, we set

$$m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}.$$

Then, there exists exactly one equilibrium point $\overline{x}$ of the difference equation (5), and every solution of (5) converges to $\overline{x}$.

3. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In his section, we study the local stability character of the equilibrium point of Eq.(1).

Eq.(1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + b\overline{x} + \frac{\alpha + c\overline{x}}{\beta + d\overline{x}},$$

or

$$d(1-a-b)\overline{x}^2 + (\beta - \beta a - \beta b - c)\overline{x} - \alpha = 0.$$

Then if $a+b<1$, the only positive equilibrium point of Eq.(1) is given by

$$\overline{x} = \frac{(\beta a + \beta b + c - \beta) + \sqrt{(\beta a + \beta b + c - \beta)^2 + 4\alpha d(1-a-b)}}{2d(1-a-b)}.$$

THEOREM 3.1. *The equilibrium $\overline{x}$ of Eq. (1) is locally asymptotically stable if and only if*

$$(\beta + d\overline{x})^2 > \frac{|c\beta - d\alpha|}{(1-b-a)}. \tag{6}$$

Proof: Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a continuous function defined by

$$f(u,v,w) = au + bv + \frac{\alpha + cw}{\beta + dw}. \tag{7}$$

Therefore,

$$\frac{\partial f(u,v,w)}{\partial u} = a,\ \frac{\partial f(u,v,w)}{\partial v} = b,\ \frac{\partial f(u,v,w)}{\partial w} = \frac{(c\beta - d\alpha)}{(\beta + dw)^2}.$$

So, we can write

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial u} = a = p_1,\ \frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial v} = b = p_2,\ \frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial w} = \frac{(c\beta - d\alpha)}{(\beta + d\overline{x})^2} = p_3.$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} - p_1 y_{n-1} - p_2 y_n - p_3 y_{n-2} = 0, \tag{8}$$

It follows by Theorem A that, Eq.(1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|a| + \left|\frac{(c\beta - d\alpha)}{(\beta + d\overline{x})^2}\right| + |b| < 1,$$

and so

$$\begin{aligned} \left|\frac{(c\beta - d\alpha)}{(\beta + d\overline{x})^2}\right| &< 1 - b - a, \\ |c\beta - d\alpha| &< (\beta + d\overline{x})^2(1-b-a), \end{aligned}$$

or

$$\frac{|c\beta - d\alpha|}{(1-b-a)} < (\beta + d\overline{x})^2.$$

The proof is complete.

Example 1. The solution of the difference equation (1) is global stability if $a = 0.55,\ b = 0.3,\ c = 0.8,\ d = 0.5,\ \alpha = 7$ and $\beta = 2$ and the initial conditions $x_{-2} = 4,\ x_{-1} = 9$ and $x_0 = 0.3$ (See Fig. 1).

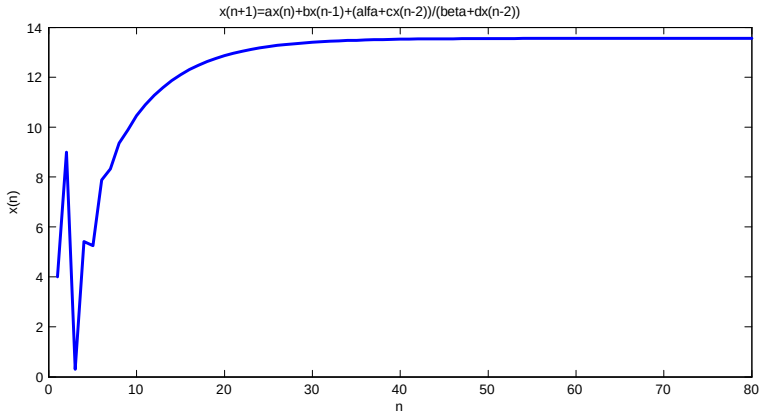


Figure 1. Plot the behavior of the solution of equation (1).

4. EXISTENCE OF BOUNDED AND UNBOUNDED SOLUTIONS OF EQ.(1)

Here we look at the boundedness nature of solutions of Eq.(1).

THEOREM 4.1. *Every solution of Eq.(1) is bounded if* $a + b < 1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}} = ax_n + bx_{n-1} + \frac{\alpha}{\beta + dx_{n-2}} + \frac{cx_{n-2}}{\beta + dx_{n-2}}.$$

Then

$$x_{n+1} < ax_n + bx_{n-1} + \frac{\alpha}{\beta} + \frac{cx_{n-2}}{dx_{n-2}} = ax_n + bx_{n-1} + \frac{\alpha}{\beta} + \frac{c}{d} \quad \text{for all} \quad n \geq 0.$$

By using a comparison, the right hand side can be written as follows

$$y_{n+1} = ay_n + by_{n-1} + \frac{\alpha}{\beta} + \frac{c}{d},$$

and this equation is locally asymptotically stable if $a + b < 1$, and converges to the equilibrium point $\overline{y} = \dfrac{\alpha d + c\beta}{\beta d(1 - a - b)}$.

Therefore

$$\limsup_{n\to\infty} x_n \leq \frac{\alpha d + c\beta}{\beta d(1 - a - b)}.$$

Hence, the solution is bounded.

THEOREM 4.2. *Every solution of Eq.(1) is unbounded if* $a > 1$ *or* $b > 1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then from Eq.(1) we see that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}} > ax_n \quad \text{for all} \quad n \geq 0.$$

The right hand side can be written as follows

$$y_{n+1} = ay_n \quad \Rightarrow \quad y_n = a^n y_0,$$

and this equation is unbounded because $a > 1$, and $\lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above.

Similarly from Equation (1) we see that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha + cx_{n-2}}{\beta + dx_{n-2}} > bx_{n-1} \quad \text{for all} \quad n \geq 0.$$

We see that the right hand side can be written as follows

$$y_{n+1} = by_{n-1} \quad \Rightarrow \quad y_{2n-1} = b^n y_{-1} \text{ and } y_{2n} = b^n y_0,$$

and this equation is unbounded because $b > 1$, and $\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} y_{2n} = \infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above.

Example 2. Figure (2) shows that behavior of the solution of the difference equation (1) is boundedness if we take $a = 0.3$, $b = 0.1$, $c = 0.8$, $d = 0.5$, $\alpha = 7$ and $\beta = 2$ and the initial conditions $x_{-2} = 4$, $x_{-1} = 9$ and $x_0 = 0.3$.

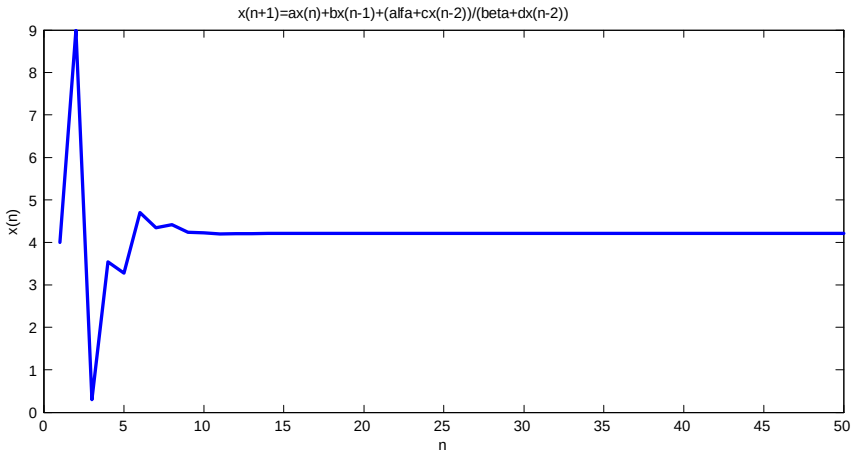


Figure 2. Show the boundedness of the solution of equation (1).

Example 3. Figure (3) shows the behavior of the solution of the difference equation (1) is undounded when we put $a = 1.5$, $b = 0.8$, $c = 2$, $d = 3$, $\alpha = 6$ and $\beta = 5$ and the initial conditions $x_{-2} = 0.4$, $x_{-1} = 0.9$ and $x_0 = 0.3$.

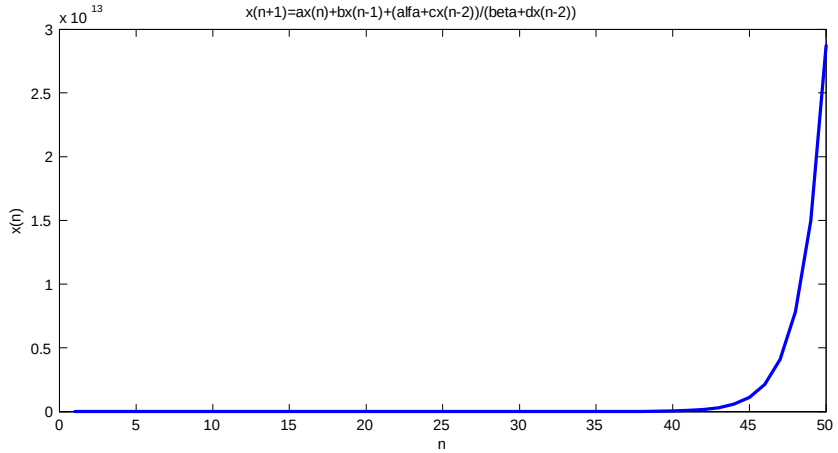


Figure 3. Show the unboundedness of the solution of equation (1).

5. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section, the global asymptotic stability of Eq.(1) is studied.

THEOREM 5.1. *The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if $a+b<1$.*

Proof: Suppose that ζ and η are real numbers and assume that $g:[\zeta,\eta]^3\longrightarrow[\zeta,\eta]$ is a function defined by

$$g(u,v,w)=au+bv+\frac{\alpha+cw}{\beta+dw}.$$

Then

$$\frac{\partial g(u,v,w)}{\partial u}=a,\ \frac{\partial g(u,v,w)}{\partial v}=b,\ \frac{\partial g(u,v,w)}{\partial w}=\frac{(c\beta-d\alpha)}{(\beta+dw)^2}.$$

Now, two cases must be considered :

Case (1): Let $c\beta-d\alpha<0$, then we can easily see that the function $g(u,v,w)$ increasing in $u,\ v$ and decreasing in v.

Let (m,M) be a solution of the system $M=g(M,M,m)$ and $m=g(m,m,M)$. Then from Eq.(1), we see that

$$M=aM+bM+\frac{\alpha+cm}{\beta+dm},\quad m=am+bm+\frac{\alpha+cM}{\beta+dM},$$

or

$$M(1-a-b)=\frac{\alpha+cm}{\beta+dm},\quad m(1-a-b)=\frac{\alpha+cM}{\beta+dM},$$

then

$$\beta(1-a-b)M+d(1-a-b)Mm=\alpha+cm,\ \beta(1-a-b)m+d(1-a-b)mM=\alpha+cM.$$

Subtracting we obtain

$$(M-m)\{\beta(1-a-b)+c\}=0,$$

under the condition $a+b<1$, we see that

$$M=m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1). This completes the proof of the theorem.

Case (2): Assume that $c\beta-d\alpha>0$ is true, then we can easily see that the function $g(u,v,w)$ increasing in $u,\ v,w$ and decreasing in v.

Let (m,M) be a solution of the system $M=g(M,M,M)$ and $m=g(m,m,m)$. Then from Eq.(1), we see that

$$M=aM+bM+\frac{\alpha+cM}{\beta+dM},\quad m=am+bM+\frac{\alpha+cm}{\beta+dm},$$

or

$$M(1-a-b)=\frac{\alpha+cM}{\beta+dM},\quad m(1-a-b)=\frac{\alpha+cm}{\beta+dm},$$

then

$$\begin{aligned}\beta(1-a-b)M+d(1-a-b)M^2&=\alpha+cM,\\ \beta(1-a-b)m+d(1-a-b)m^2&=\alpha+cm.\end{aligned}$$

Subtracting we obtain

$$\begin{aligned}\beta(1-a-b)(M-m)+d(1-a-b)(M^2-m^2)&=c(M-m)\\ (M-m)\{d(1-a-b)(M+m)+\beta(1-a-b)-c\}&=0,\end{aligned}$$

under the condition $a+b<1$ and $\beta(1-a-b)>c$ we see that

$$M=m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1).

6. EXISTENCE OF PERIODIC SOLUTIONS

In this section we investigate the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

THEOREM 6.1. *Eq.(1) has positive prime period two solutions if and only if*

$$(i)\ \left[\beta(b-1)-(\beta a+c)\right]^2(b-a-1)+4a\{\beta(1-b)[\beta(b-1)-(\beta a+c)]+\alpha ad\}>0.$$

Proof: Firstly, suppose that there exists a prime period two solution

$$...,p,q,p,q,...,$$

of Eq.(1). We will show that Condition (i) holds.

From Eq.(1), we get

$$p=aq+bp+\frac{\alpha+cq}{\beta+dq},$$

and

$$q=ap+bq+\frac{\alpha+cp}{\beta+dp}.$$

Therefore,

$$\beta p+dpq=a\beta q+adq^2+\beta bp+bdpq+\alpha+cq, \tag{9}$$

and

$$\beta q+dpq=a\beta p+adp^2+\beta bq+bdpq+\alpha+cp. \tag{10}$$

Subtracting (10) from (9) gives

$$\beta(p-q)+ad(p^2-q^2)=\beta b(p-q)-\beta a(p-q)-c(p-q).$$

Since $p\neq q$, it follows that

$$p+q=\frac{\beta(b-a-1)-c}{ad}. \tag{11}$$

Again, adding (9) and (10) yields

$$\begin{aligned}\beta(p+q)+2dpq &= (a\beta+\beta b)(p+q)+ad(p^2+q^2)+2bdpq+2\alpha+c(p+q),\\ ad(p^2+q^2) &= \beta(p+q)+2dpq-(a\beta+\beta b)(p+q)-2bdpq-2\alpha-c(p+q)\end{aligned}$$

$$ad(p^2+q^2)=(\beta-a\beta-\beta b-c)(p+q)+2dpq-2bdpq-2\alpha. \tag{12}$$

By using (11), (12) and the relation

$$p^2+q^2=(p+q)^2-2pq \quad \text{for all } p,q\in R,$$

we obtain

$$\begin{aligned}ad((p+q)^2-2pq) &= (\beta-a\beta-\beta b-c)(p+q)+2dpq-2bdpq-2\alpha\\ \tfrac{(\beta b-\beta a-\beta-c)^2}{ad}-2adpq &= \tfrac{(\beta-a\beta-\beta b-c)(\beta b-\beta a-\beta-c)}{ad}+2dpq-2bdpq-2\alpha\end{aligned}$$

Then,

$$pq=-\frac{\beta(b-1)[\beta(b-1)-(\beta a+c)]+\alpha ad}{ad^2(b-a-1)}. \tag{13}$$

Now it is obvious from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$t^2 - \frac{\beta(b-a-1)-c}{ad}t - \frac{\beta(b-1)[\beta(b-1)-(\beta a+c)]+\alpha ad}{ad^2(b-a-1)} = 0,$$
$$ad\,t^2 - (\beta b-\beta a-\beta-c)t - \frac{\beta(b-1)[\beta(b-1)-(\beta a+c)]+\alpha ad}{d(b-a-1)} = 0, \tag{14}$$

and so

$$[\beta(b-1)-(\beta a+c)]^2 + \frac{4a\{\beta(b-1)[\beta(b-1)-(\beta a+c)]+\alpha ad\}}{(b-a-1)} > 0,$$

or

$$[\beta(b-1)-(\beta a+c)]^2(b-a-1) + 4a\{\beta(b-1)[\beta(b-1)-(\beta a+c)]+\alpha ad\} > 0.$$

for $a+1<b$ then the inequalities (i) holds.

Conversely, suppose that inequality (i) is true. We will prove that Eq.(1) has a prime period two solution.

Suppose that

$$p = \frac{A+\zeta}{2ad},$$

and

$$q = \frac{A-\zeta}{2ad},$$

where $\zeta = \sqrt{A^2 + \frac{4a[\beta A(b-1)+\alpha ad]}{(b-a-1)}}$ and $A = \beta(b-1)-(\beta a+c)$.

We see from the inequality (i) that

$$A^2(b-a-1) + 4a\left[\beta A(b-1)+\alpha ad\right] > 0,$$

which equivalents to

$$A^2 + \frac{4a\left[\beta A(b-1)+\alpha ad\right]}{(b-a-1)} > 0.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-2} = p,\ x_{-1} = q \text{ and } x_0 = p.$$

We would like to show that

$$x_1 = x_{-2} = q \quad \text{and} \quad x_2 = x_{-1} = p.$$

It follows from Eq.(1) that

$$x_1 = ap + bq + \tfrac{\alpha+cp}{\beta+dp} = a\left(\frac{A+\zeta}{2ad}\right) + b\left(\frac{A-\zeta}{2ad}\right) + \frac{\alpha + c\left(\frac{A+\zeta}{2ad}\right)}{\beta + d\left(\frac{A+\zeta}{2ad}\right)}.$$

Dividing the denominator and numerator by $2ad$ we get

$$x_1 = a\left(\frac{A+\zeta}{2ad}\right) + b\left(\frac{A-\zeta}{2ad}\right) + \frac{2ad\alpha + c\,(A+\zeta)}{2ad\beta + d\,(A+\zeta)}.$$

Multiplying the denominator and numerator of the right side by $2ad\beta + d\,(A-\zeta)$ and by computation we obtain

$$x_1 = q.$$

Similarly as before, it is easy to show that

$$x_2 = p.$$

Then by induction we get

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -2.$$

Thus Eq.(1) has the prime period two solution

$$...,p,q,p,q,...,$$

where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

Example 4. Figure (4) shows the period two solution of equation (1) when $a = 0.2,\ b = 5,\ c = 0.4,\ d = 5,\ \alpha = 0.7$ and $\beta = 0.2$ and the initial conditions $x_{-2} = p,\ x_{-1} = q$ and $x_0 = p$ since p and q as in the previous theorem.

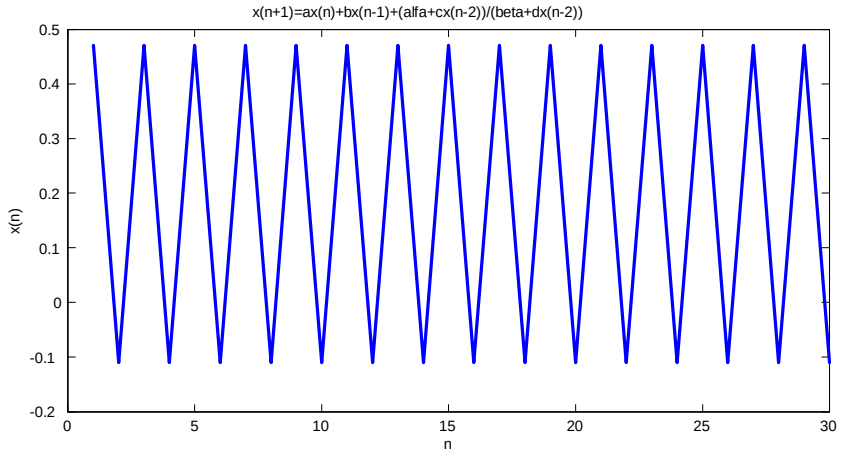


Figure 4. Plot the periodicity of the solution of equation (1).

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Algebraic and Order Properties of Tracy-Singh Products for Operator Matrices

Arnon Ploymukda[1], Pattrawut Chansangiam[1]*, Wicharn Lewkeeratiyutkul[2]
[1]Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.
[2]Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330, Thailand.

Abstract

We generalize the tensor product for operators to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces. This kind of operator product is compatible with algebraic operations and order relations for operators. It follows that this product preserves many structure properties of operators.

Keywords: tensor product, Tracy-Singh product, operator matrix, Moore-Penrose inverse
Mathematics Subject Classifications 2010: 15A69, 47A05, 47A80.

1 Introduction

In scientific computing, we consider a matrix to be a two-dimensional array for stacking data. A processing of such data can be performed using matrix products. One of extremely useful matrix products is the Kronecker product. For any complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the Kronecker product of A and B is given by the block matrix

$$A \hat{\otimes} B \;=\; [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

Equivalently, $A \hat{\otimes} B$ is the unique complex matrix of order $mp \times nq$ satisfying

$$(A \hat{\otimes} B)(x \hat{\otimes} y) \;=\; Ax \hat{\otimes} By \tag{1}$$

for all $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. This matrix product has wide applications in mathematics, computer science, statistics, physics, system theory, signal processing, and related fields. See [2, 5, 6, 12] for more information.

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

Kronecker product was generalized to the Tracy-Singh product of partitioned matrices by Tracy and Singh [10]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} of order $m_i \times n_j$ as the (i,j)th submatrix where $\sum_i m_i = m$ and $\sum_j n_j = n$. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} of order $p_k \times q_l$ as the (k,l)th submatrix where $\sum_k p_k = p$ and $\sum_l q_l = q$. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \big[\big[A_{ij} \hat{\otimes} B_{kl}\big]_{kl}\big]_{ij} \in M_{mp,nq}(\mathbb{C}),$$

where each block $A_{ij} \hat{\otimes} B_{kl}$ is of order $m_i p_k \times n_j q_l$. This kind of matrix product has several attractive properties in algebraic, order, and analytic points of views; see, *e.g.*, [3, 8, 9, 10]. The Tracy-Singh product can be applied widely in statistics, econometrics and related fields; see, *e.g.*, [9, 10].

As a natural generalization of a complex matrix, we consider a bounded linear operator between complex Hilbert spaces. The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Using the universal mapping property in the monoidal category of Hilbert spaces, the tensor product of $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into $\mathcal{H}' \otimes \mathcal{K}'$ such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$(A \otimes B)(x \otimes y) \;=\; Ax \otimes By. \tag{2}$$

A fundamental property of tensor product is the mixed product property:

$$(A \otimes B)(C \otimes D) \;=\; AC \otimes BD. \tag{3}$$

The theory of tensor product of operators has been continuously developed in the literature; see, *e.g.*, [4, 11].

From the previous discussion, it is natural to extend the notion of tensor product for operators to the "Tracy-Singh product"of operators. We shall propose a natural definition of such operator product. It turns out that this product is compatible with algebraic operations and order relations for operators. One of the most attractive properties, the mixed product property, also holds for Tracy-Singh products. It follows that this product preserves attractive properties of operators, such as being invertible, Hermitian, unitary, positive, and normal. Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators.

This paper is organized as follows. In section 2, we introduce the Tracy-Singh product for operator matrices and deduce its algebraic properties. In section 3, we show that the Tracy-Singh product is compatible with various kinds of operator inverses. We investigate the relationship between Tracy-Singh products and operator orderings in Section 4.

2 Tracy-Singh products and algebraic operations for operators

In this section, we introduce the Tracy-Singh product of operators on a Hilbert space. Then we will show that this product is compatible with addition, scalar multiplication, adjoint operation, usual multiplication, power, and direct sum of operator inverses.

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X},\mathcal{Y})$ the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X},\mathcal{X})$ to $\mathbb{B}(\mathcal{X})$.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. Such decompositions are fixed throughout the paper. For each $j = 1, \dots, n$, let E_j be the canonical embedding from $\mathcal{H}_j$ into $\mathcal{H}$, defined by

$$x_j \mapsto (0, \dots, 0, x_j, 0, \dots, 0).$$

Similarly, let F_l be the canonical embedding from $\mathcal{K}_l$ into $\mathcal{K}$ for each $l = 1, \dots, q$. For each $i = 1, \dots, m$ and $k = 1, \dots, p$, let $P'_i : \mathcal{H}' \to \mathcal{H}'_i$ and $Q'_k : \mathcal{K}' \to \mathcal{K}'_k$ be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and } B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} = P'_i A E_j$ and $B_{kl} = Q'_k B F_l$ for each i, j, k, l.

Definition 1. *Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ be operator matrices defined as above. We define the Tracy-Singh product of A and B to be the operator matrix*

$$A \boxtimes B = \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij} \tag{4}$$

which is a bounded linear operator from $\bigoplus_{j=1}^{n}\bigoplus_{l=1}^{q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i=1}^{m}\bigoplus_{k=1}^{p} \mathcal{H}'_i \otimes \mathcal{K}'_k$.

Note that if both A and B are 1×1 block operator matrices i.e. $m = n = p = q = 1$, then their Tracy-Singh product $A \boxtimes B$ is just the tensor product $A \otimes B$.

Next, we shall show that the Tracy-Singh product of two linear maps induced by two matrices is just the linear map induced by the Tracy-Singh product of these matrices. Recall that for each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the induced maps

$$L_A : \mathbb{C}^n \to \mathbb{C}^m,\ x \mapsto Ax \quad \text{and} \quad L_B : \mathbb{C}^q \to \mathbb{C}^p,\ y \mapsto By$$

are bounded linear operators. Using the universal mapping property, we identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq} \cong M_{n,q}(\mathbb{C})$ together with the canonical bilinear map $(x, y) \mapsto x \hat{\otimes} y$ for each $(x, y) \in \mathbb{C}^n \times \mathbb{C}^q$. It is similar for $\mathbb{C}^m \otimes \mathbb{C}^p$.

Lemma 2. *For each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, we have*

$$L_A \otimes L_B \;=\; L_{A \,\hat{\otimes}\, B}. \tag{5}$$

Proof. For any $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^q$, we obtain from the mixed product property of the Kronecker product (1) that

$$\begin{aligned}(L_A \otimes L_B)(x \otimes y) \;&=\; L_A(x) \otimes L_B(y) \;=\; L_A(x) \,\hat{\otimes}\, L_B(y)\\ &=\; Ax \,\hat{\otimes}\, By \;=\; (A \,\hat{\otimes}\, B)(x \,\hat{\otimes}\, y)\\ &=\; (A \,\hat{\otimes}\, B)(x \otimes y) \;=\; L_{A \,\hat{\otimes}\, B}(x \otimes y).\end{aligned}$$

Thus, by the uniqueness of tensor product, $L_A \otimes L_B = L_{A \,\hat{\otimes}\, B}$. □

Proposition 3. *For any complex matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ partitioned in block-matrix forms, we have*

$$L_A \boxtimes L_B \;=\; L_{A \,\hat{\boxtimes}\, B}. \tag{6}$$

Proof. Recall that the (i, j)th block of the matrix representation of L_A is the matrix A_{ij}. It follows from Lemma 2 that

$$L_A \boxtimes L_B \;=\; \left[\left[L_{A_{ij}} \otimes L_{B_{kl}}\right]_{kl}\right]_{ij} \;=\; \left[\left[L_{A_{ij}} \,\hat{\otimes}\, B_{kl}\right]_{kl}\right]_{ij} \;=\; L_{A \,\hat{\boxtimes}\, B}.$$

□

The next proposition shows that the Tracy-Singh product is compatible with the addition, the scalar multiplication and the adjoint operation of operators.

Proposition 4. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices, and let $\alpha \in \mathbb{C}$. Then*

$$(\alpha A) \boxtimes B \;=\; \alpha(A \boxtimes B) \;=\; A \boxtimes (\alpha B), \tag{7}$$

$$(A \boxtimes B)^* \;=\; A^* \boxtimes B^*, \tag{8}$$

$$A \boxtimes (B + C) \;=\; A \boxtimes B + A \boxtimes C, \tag{9}$$

$$(B + C) \boxtimes A \;=\; B \boxtimes A + C \boxtimes A. \tag{10}$$

Proof. Since each (i, j)th block of αA is given by $(\alpha A)_{ij} = \alpha A_{ij}$, we get

$$(\alpha A) \boxtimes B \;=\; \left[[(\alpha A_{ij}) \otimes B_{kl}]_{kl}\right]_{ij} \;=\; \left[[\alpha(A_{ij} \otimes B_{kl})]_{kl}\right]_{ij} \;=\; \alpha(A \boxtimes B).$$

Similarly, $A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$. Since $A^* = [A_{ji}^*]_{ij}$ and $B^* = [B_{lk}^*]_{kl}$ for all i, j, k, l, we obtain

$$(A \boxtimes B)^* \;=\; \left[[A_{ji} \otimes B_{kl}]_{kl}^*\right]_{ij} \;=\; \left[\left[A_{ji}^* \otimes B_{lk}^*\right]_{kl}\right]_{ij} \;=\; A^* \boxtimes B^*.$$

The proofs of (9) and (10) are done by using the fact that $(B+C)_{kl} = B_{kl} + C_{kl}$ for all k, l together with the left/right distributivity of the tensor product over the addition. □

Properties (7), (9) and (10) say that the map $(A, B) \mapsto A \boxtimes B$ is bilinear.

Proposition 5. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and let $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices. Then*

$$A \boxtimes B \;=\; [A_{ij} \boxtimes B]_{ij} \;=\; \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

That is, the (i, j)th block of $A \boxtimes B$ is just $A_{ij} \boxtimes B$, regardless of how to partition B.

Proof. It follows directly from the definition of the Tracy-Singh product. □

Remark 6. *It is not true in general that the (k, l)th block of $A \boxtimes B$ is $A \boxtimes B_{kl}$.*

When $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, the direct sum of $A_1 \in \mathbb{B}(\mathcal{H}_1, \mathcal{K}_1)$ and $A_2 \in \mathbb{B}(\mathcal{H}_2, \mathcal{K}_2)$ is defined to be the operator

$$A_1 \oplus A_2 \;=\; \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \;\in\; \mathbb{B}(\mathcal{H}, \mathcal{K}).$$

The next result gives a relation between the direct sum and the Tracy-Singh product.

Proposition 7. *The Tracy-Singh product is right distributive over the direct sum of operators. That is, for any operator matrices A, B and C, we have*

$$(A \oplus B) \boxtimes C \;=\; (A \boxtimes C) \oplus (B \boxtimes C). \tag{11}$$

Proof. It follows from Proposition 5 that

$$\begin{aligned} (A \oplus B) \boxtimes C \;&=\; \begin{bmatrix} A \boxtimes C & 0 \boxtimes C \\ 0 \boxtimes C & B \boxtimes C \end{bmatrix} \;=\; \begin{bmatrix} A \boxtimes C & 0 \\ 0 & B \boxtimes C \end{bmatrix} \\ &=\; (A \boxtimes C) \oplus (B \boxtimes C). \end{aligned}$$ □

It is not true in general that the Tracy-Singh product is left distributive over the direct sum of operators.

The next theorem shows that the Tracy-Singh product is compatible with the ordinary product of operators. This fundamental property, called the *mixed product property*, will be used many times in later discussions.

Theorem 8. *Let $\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{K}, \mathcal{K}'$ and $\mathcal{K}''$ be complex Hilbert spaces. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}', \mathcal{H}'')$, $C = [C_{ij}]_{i,j=1}^{n,r} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}', \mathcal{K}'')$ and $D = [D_{kl}]_{k,l=1}^{q,s} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ be operator matrices partitioned so that they are compatible with the decompositions of the corresponding Hilbert spaces. Then*

$$(A \boxtimes B)(C \boxtimes D) \;=\; AC \boxtimes BD. \tag{12}$$

Proof. Using block multiplication of operators and the mixed product property of the tensor product (3), we have

$$\begin{aligned}(A \boxtimes B)(C \boxtimes D) &= \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij} \left[[C_{ij} \otimes D_{kl}]_{kl}\right]_{ij} \\ &= \left[\left[\sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}(A_{i\alpha} \otimes B_{k\beta})(C_{\alpha j} \otimes D_{\beta l})\right]_{kl}\right]_{ij} \\ &= \left[\left[\sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}(A_{i\alpha}C_{\alpha j} \otimes B_{k\beta}D_{\beta l})\right]_{kl}\right]_{ij} \\ &= \left[\sum_{\alpha=1}^{n} A_{i\alpha}C_{\alpha j}\right]_{ij} \boxtimes \left[\sum_{\beta=1}^{q} B_{k\beta}D_{\beta l}\right]_{kl} \\ &= AC \boxtimes BD. \end{aligned}$$

□

Corollary 9. *For any operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$(A \boxtimes B)^r = A^r \boxtimes B^r \tag{13}$$

for any $r \in \mathbb{N}$.

In the rest of section, we investigate structure properties of operators under taking Tracy-Singh products. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *involutary* if $T^2 = I$, *idempotent* if $T^2 = T$, an *isometry* if $T^*T = I$, a *partial isometry* if the restriction of T to a closed subspace is an isometry, or equivalently, $TT^*T = T$.

Corollary 10. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: Hermitian, unitary, isometry, co-isometry, partial isometry, idempotent, involutary, projection.*

Proof. Applying Theorem 8 and Proposition 4, we get the results. □

If A and B are skew-Hermitian operators, then $A \boxtimes B$ is Hermitian. Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be *nilpotent* if there is a positive integer k such that $T^k = 0$. The smallest such integer k is called the degree of nilpotency of T. If $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ are nilpotent operators with degrees of nilpotency r and s, respectively, then $A \boxtimes B$ is also nilpotent with degree of nilpotency not exceed $\min\{r, s\}$.

3 Tracy-Singh products and operator inverses

Next, we discuss the invertibility of the Tracy-Singh product of operators. Recall that an operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is said to be *regular* if there is an operator $A^- \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ such that $AA^-A = A$. The operator A^- is called an *inner inverse* of A. An operator $X \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ is said to be an *outer inverse* of A if $XAX = X$.

Proposition 11. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$.*

(i) If A and B are left invertible with left inverses $\hat{A}$ and $\hat{B}$ respectively, then $A \boxtimes B$ is left invertible and $\hat{A} \boxtimes \hat{B}$ is its left inverse.

(ii) If A and B are right invertible with right inverses $\hat{A}$ and $\hat{B}$ respectively, then $A \boxtimes B$ is right invertible and $\hat{A} \boxtimes \hat{B}$ is its right inverse.

(iii) If A and B are regular with inner inverses A^- and B^- respectively, then $A \boxtimes B$ is regular with $A^- \boxtimes B^-$ as its inner inverse.

(iv) If A and B have A^- and B^- as their outer inverses respectively, then $A \boxtimes B$ has $A^- \boxtimes B^-$ as its outer inverse.

Proof. It follows from Theorem 8 and the facts that $I_{\mathcal{X}} \boxtimes I_{\mathcal{Y}} = I_{\mathcal{X} \otimes \mathcal{Y}}$ for any Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. □

As a consequence of (i) and (ii) in Proposition 11, we obtain the following result.

Corollary 12. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are invertible, then $A \boxtimes B$ is invertible and*

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}. \tag{14}$$

Next, we consider a kind of operator inverse, called Moore-Penrose inverse. Recall that a *Moore-Penrose inverse* of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ satisfying the following Penrose conditions ([7])

(i) $A^\dagger$ is an inner inverse of A ;

(ii) $A^\dagger$ is an outer inverse of A ;

(iii) $AA^\dagger$ is Hermitian ;

(iv) $A^\dagger A$ is Hermitian.

It is well known that the following statements are equivalent for $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ (see *e.g.* [1]):

(i) a Moore-Penrose inverse of A exists ;

(ii) a Moore-Penrose inverse of A is unique ;

(iii) the range of A is closed.

Theorem 13. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If A and B have closed ranges, then*

1. *the range of $A \boxtimes B$ is closed ;*
2. *$(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger$.*

Proof. Since the ranges of A and B are closed, the Moore-Penrose inverses $A^\dagger$ and $B^\dagger$ exist and are unique. Making use of Theorem 8 and Proposition 4, we can verify that $A^\dagger \boxtimes B^\dagger$ satisfies the following Penrose equations:

(i) $(A \boxtimes B)(A^\dagger \boxtimes B^\dagger)(A \boxtimes B) = A \boxtimes B$

(ii) $(A^\dagger \boxtimes B^\dagger)(A \boxtimes B)(A^\dagger \boxtimes B^\dagger) = A^\dagger \boxtimes B^\dagger$

(iii) $\big((A \boxtimes B)(A^\dagger \boxtimes B^\dagger)\big)^* = (A \boxtimes B)(A^\dagger \boxtimes B^\dagger)$

(iv) $\big((A^\dagger \boxtimes B^\dagger)(A \boxtimes B)\big)^* = (A^\dagger \boxtimes B^\dagger)(A \boxtimes B)$.

Hence, a Moore-Penrose inverse of $A \boxtimes B$ exists and it is uniquely determined by $A^\dagger \boxtimes B^\dagger$. It follows that $A \boxtimes B$ has a closed range. □

The results in this section indicate that the Tracy-Singh product is compatible with various kinds of operator inverses.

4 Tracy-Singh products and operator orderings

Now, we focus on order properties of Tracy-Singh products related to algebraic properties.

Theorem 14. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.*

(i) If $A, B \geqslant 0$, then $A \boxtimes B \geqslant 0$.

(ii) If $A, B > 0$, then $A \boxtimes B > 0$.

Proof. Assume $A, B \geqslant 0$. Using Theorem 8 and property (8), we obtain

$$\begin{aligned} A \boxtimes B &= A^{\frac{1}{2}} A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}} B^{\frac{1}{2}} = \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \\ &= \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)^*\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \geqslant 0. \end{aligned}$$

Consider the case $A, B > 0$. We have immediately by (i) that $A \boxtimes B \geqslant 0$. By Corollary 12, $A \boxtimes B$ is invertible. This implies that $A \boxtimes B > 0$. □

The next result provides the monotonicity of Tracy-Singh product.

Corollary 15. *Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$.*

(i) If $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$, then $A_1 \boxtimes B_1 \geqslant A_2 \boxtimes B_2$.

(ii) If $A_1 > A_2 > 0$ and $B_1 > B_2 > 0$, then $A_1 \boxtimes B_1 > A_2 \boxtimes B_2$.

Proof. Suppose that $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$. Applying Proposition 4 and Theorem 14 yields

$$\begin{aligned} A_1 \boxtimes B_1 - A_2 \boxtimes B_2 &= A_1 \boxtimes B_1 - A_2 \boxtimes B_1 + A_2 \boxtimes B_1 - A_2 \boxtimes B_2 \\ &= (A_1 - A_2) \boxtimes B_1 + A_2 \boxtimes (B_1 - B_2) \\ &\geqslant 0. \end{aligned}$$

The proof of (ii) is similar to that of (i). □

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Analytic Properties of Tracy-Singh Products for Operator Matrices

Arnon Ploymukda[1], Pattrawut Chansangiam[1]*, Wicharn Lewkeeratiyutkul[2]
[1]Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.
[2]Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330, Thailand.

Abstract

We show that the Tracy-Singh product of Hilbert space operators is continuous with respect to the operator-norm topology. The Tracy-Singh product of two nonzero operators is compact if and only if both factors are compact. We provide upper and lower bounds for certain Schatten p-norms of the Tracy-Singh product of operators. It turns out that this product is continuous with respect to the topologies on norm ideals of compact operators, trace class operators, and Hilbert-Schmidt class operators. Thus the Tracy-Singh product preserves such classes of operators.

Keywords: tensor product, Tracy-Singh product, operator matrix, compact operator, Schatten p-class operator
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1 Introduction

In matrix theory, one of useful matrix products is the Kronecker product. Recall that the Kronecker product of two complex matrices $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is given by the block matrix

$$A \hat{\otimes} B \;=\; [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

This matrix product was generalized to the Tracy-Singh product by Tracy and Singh [3]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} as the (i,j)th submatrix. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} as the (k,l)th submatrix. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \big[\big[A_{ij} \hat{\otimes} B_{kl}\big]_{kl}\big]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

This kind of matrix product has several attractive properties and can be applied widely in statistics, econometrics and related fields; see *e.g.*, [3, 5, 7, 8, 9].

The tensor product of Hilbert space operators is a natural extension of the Kronecker product to infinite-dimensional setting. Theory of Hilbert tensor product has been continuously investigated in the literature; see, *e.g.*, [2, 4, 10]. It is well known that the tensor product is continuous with respect to the operator-norm topology. Moreover, on the norm ideals of compact operators generated by Schatten p-norm for $p = 1, 2, \infty$, the tensor product are also continuous. Recently, the tensor product for operators was generalized to the Tracy-Singh product for operator matrices acting on the direct sum of Hilbert spaces in [6]. This kind of operator product satisfies certain pleasing algebraic and order properties.

In this paper, we discuss continuity, convergence, and compactness of the Tracy-Singh product for operators in the operator-norm topology. Then we obtain relations between Tracy-Singh product and certain analytic functions. We also investigate the Tracy-Singh product on norm ideals of compact operators generated by certain Schatten p-norms. In fact, this product is continuous with respect to the Schatten p-norm for $p = 1, 2, \infty$. Estimations by such norms for Tracy-Singh products are provided. It follows that trace class operators and Hilbert-Schmidt class operators are preserved under this product.

This paper is organized as follows. In section 2, we give preliminaries on Tracy-Singh products for operators on a Hilbert space. In section 3, we establish analytic properties of the Tracy-Singh product in the operator-norm topology. We investigate the Tracy-Singh product on the norm ideals of compact operators generated by certain Schatten p-norms in Section 4.

2 Preliminaries on Tracy-Singh products for operator matrices

Throughout, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces . When X and Y are Hilbert spaces, denote by $\mathbb{B}(X, Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathbb{B}(X, X)$ to $\mathbb{B}(X)$.

In order to define the Tracy-Singh product, we have to fix the decompositions of Hilbert spaces, namely,

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k$$

where each $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. For each $j = 1, \ldots, n$ and $l = 1, \ldots, q$, let $E_j : \mathcal{H}_j \to \mathcal{H}$ and $F_l : \mathcal{K}_l \to \mathcal{K}$ be the canonical embeddings. For each $i = 1, \ldots, m$ and $k = 1, \ldots, p$, let P'_i and Q'_k be the orthogonal projections. Thus, each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and B in $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} = P_i' A E_j : \mathcal{H}_j \to \mathcal{H}_i'$ and $B_{kl} = Q_k' B F_l : \mathcal{K}_l \to \mathcal{K}_k'$ for each i, j, k, l. We define the *Tracy-Singh product* of A and B to be a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}_i' \otimes \mathcal{K}_k'$ represented in the block-matrix form as follows:

$$A \boxtimes B \;=\; \big[\big[A_{ij} \otimes B_{kl}\big]_{kl}\big]_{ij}.$$

When $m = n = p = q = 1$, the Tracy-Singh product $A \boxtimes B$ becomes the tensor product $A \otimes B$.

Lemma 1 ([6]). *Fundamental properties of the Tracy-Singh product for operators are listed below (provided that each term is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *Compatibility with adjoints: $(A \boxtimes B)^* \;=\; A^* \boxtimes B^*$.*
3. *Mixed-product property: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*
4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*
5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} \;=\; A^{-1} \boxtimes B^{-1}$.*
6. *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*
7. *Strictly positivity: if $A > 0$ and $B > 0$, then $A \boxtimes B > 0$.*
8. *If A and B are partial isometries, then so is $A \boxtimes B$. Recall that an operator T is a partial isometry if and only if the restriction of T to a closed subspace is an isometry.*

3 Analytic properties of the Tracy-Singh product

In this section, we establish some analytic properties of the Tracy-Singh product involving operator norms. These properties involve continuity, convergence, norm estimates, and certain analytic functions. We denote the operator norm by $\|\cdot\|_\infty$.

In order to discuss the continuity of the Tracy-Singh product, recall the following bounds for the operator norm of operator matrices.

Lemma 2 ([1]). *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then*

$$n^{-2} \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2 \;\leqslant\; \|A\|_\infty^2 \;\leqslant\; \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2. \tag{1}$$

Lemma 3. *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be an operator matrix and let $(A_r)_{r=1}^{\infty}$ be a sequence in $\mathbb{B}(\mathcal{H})$ where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ if and only if $A_{ij}^{(r)} \to A_{ij}$ for all $i, j = 1, \ldots, n$.*

Proof. It is a direct consequence of Lemma 2. □

The next theorem explains that the Tracy-Singh product is (jointly) continuous with respect to the topology induced by the operator norm.

Theorem 4. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}] \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $(A_r)_{r=1}^{\infty}$ and $(B_r)_{r=1}^{\infty}$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxtimes B_r \to A \boxtimes B$.*

Proof. Suppose that $A_r \to A$ and $B_r \to B$. By Lemma 3, we have $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$ for each i, j, k, l. Since the tensor product is continuous, we have

$$A_{ij}^{(r)} \otimes B_{kl}^{(r)} \;\to\; A_{ij} \otimes B_{kl}$$

for each i, j, k, l. It follows that $A_r \boxtimes B_r \to A \boxtimes B$ by Lemma 3. □

The next theorem provides upper/lower bounds for the operator norm of the Tracy-Singh product.

Theorem 5. *For any operator matrices $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [A_{kl}]_{k,l=1}^{q,q} \in \mathbb{B}(\mathcal{K})$, we have*

$$\frac{1}{nq}\|A\|_\infty\|B\|_\infty \;\leqslant\; \|A \boxtimes B\|_\infty \;\leqslant\; nq\|A\|_\infty\|B\|_\infty. \tag{2}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned}\|A \boxtimes B\|_\infty^2 \;&\leqslant\; \sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_\infty^2 \;=\; \sum_{k,l}\sum_{i,j}\|A_{ij}\|_\infty^2\|B_{kl}\|_\infty^2 \\ &=\; \Big(\sum_{i,j}\|A_{ij}\|_\infty^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_\infty^2\Big) \;\leqslant\; (nq)^2\|A\|_\infty^2\|B\|_\infty^2.\end{aligned}$$

We also have

$$\begin{aligned}\|A \boxtimes B\|_\infty^2 \;&\geqslant\; (nq)^{-2}\sum_{k,l}\sum_{i,j}\|A_{ij} \otimes B_{kl}\|_\infty^2 \;=\; (nq)^{-2}\sum_{k,l}\sum_{i,j}\|A_{ij}\|_\infty^2\|B_{kl}\|_\infty^2 \\ &=\; (nq)^{-2}\Big(\sum_{i,j}\|A_{ij}\|_\infty^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_\infty^2\Big) \;\geqslant\; (nq)^{-2}\|A\|_\infty^2\|B\|_\infty^2.\end{aligned}$$

Hence, we obtain the bound (2). □

Theorem 6. *Let $A \in \mathbb{B}(\mathcal{H})$.*

(i) If f is an analytic function on a region containing the spectra of A and $I \boxtimes A$, then

$$f(I \boxtimes A) \;=\; I \boxtimes f(A). \tag{3}$$

(ii) If f is an analytic function on a region containing the spectra of A and $A \boxtimes I$, then

$$f(A \boxtimes I) \;=\; f(A) \boxtimes I. \tag{4}$$

Proof. (i) Since f is analytic on spectra of A and $I \boxtimes A$, we have the Taylor series expansion

$$f(z) = \sum_{r=0}^{\infty} \alpha_r z^r.$$

It follows that

$$f(A) = \sum_{r=0}^{\infty} \alpha_r A^r \quad \text{and} \quad f(I \boxtimes A) = \sum_{r=0}^{\infty} \alpha_r (I \boxtimes A)^r.$$

Making use of the bilinearity of Tracy-Singh product and Theorem 4 yields

$$\begin{aligned} f(I \boxtimes A) &= \sum_{r=0}^{\infty} \alpha_r \, (I \boxtimes A^r) = \sum_{r=0}^{\infty} (I \boxtimes \alpha_r A^r) \\ &= I \boxtimes \sum_{r=0}^{\infty} \alpha_r A^r = I \boxtimes f(A). \end{aligned}$$

Similarly, we obtain the assertion (ii). □

Theorem 7. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. For any $\alpha > 0$, we have*

$$(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha. \tag{5}$$

Proof. First, note that $A \boxtimes B$ is positive by property (6) of Lemma 1. It follows from the property (4) in Lemma 1 that for any $r, s \in \mathbb{N}$,

$$\left(A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}\right)^s = A^r \boxtimes B^r = (A \boxtimes B)^r,$$

and thus $(A \boxtimes B)^{\frac{r}{s}} = A^{\frac{r}{s}} \boxtimes B^{\frac{r}{s}}$. Now, for $\alpha > 0$, there is a sequence (q_n) of positive rational numbers such that $q_n \to \alpha$. It follows from the previous claim and the continuity of Tracy-Singh product (Theorem 4) that

$$\begin{aligned} (A \boxtimes B)^\alpha &= \lim_{n\to\infty} (A \boxtimes B)^{q_n} = \lim_{n\to\infty} A^{q_n} \boxtimes B^{q_n} \\ &= \lim_{n\to\infty} A^{q_n} \boxtimes \lim_{n\to\infty} B^{q_n} = A^\alpha \boxtimes B^\alpha. \end{aligned}$$ □

Corollary 8. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be strictly positive operators. For any real number α, we have*

$$(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha. \tag{6}$$

Proof. Note that $A \boxtimes B$ is strictly positive by property (7) of Lemma 1. For $\alpha < 0$, it follows from Theorem 7 and the property (5) in Lemma 1 that

$$\begin{aligned} (A \boxtimes B)^\alpha &= [(A \boxtimes B)^{-1}]^{-\alpha} = (A^{-1} \boxtimes B^{-1})^{-\alpha} \\ &= (A^{-1})^{-\alpha} \boxtimes (B^{-1})^{-\alpha} = A^\alpha \boxtimes B^\alpha. \end{aligned}$$ □

Corollary 9. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. Then*

$$|A \boxtimes B| \;=\; |A| \boxtimes |B|. \tag{7}$$

Proof. Applying Lemma 1 and property (5), we get

$$\begin{aligned} |A \boxtimes B| \;&=\; [(A \boxtimes B)^*(A \boxtimes B)]^{\frac{1}{2}} \;=\; [(A^* \boxtimes B^*)(A \boxtimes B)]^{\frac{1}{2}} \\ &=\; (A^*A \boxtimes B^*B)^{\frac{1}{2}} \;=\; (A^*A)^{\frac{1}{2}} \boxtimes (B^*B)^{\frac{1}{2}} \;=\; |A| \boxtimes |B|. \end{aligned}$$

□

Recall the polar decomposition theorem: for any $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$, there exists a partial isometry U such that $A = U|A|$. The next result is a polar decomposition for the Tracy-Singh product of operators.

Corollary 10. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If $A = U|A|$ and $B = V|B|$ are polar decompositions of A and B, respectively, then a polar decomposition of $A \boxtimes B$ is given by*

$$A \boxtimes B \;=\; (U \boxtimes V)|A \boxtimes B|. \tag{8}$$

Proof. Let U and V be partial isometries such that $A = U|A|$ and $B = V|B|$. It follows from Lemma 1(3) and Corollary 9 that

$$A \boxtimes B \;=\; U|A| \boxtimes V|B| \;=\; (U \boxtimes V)(|A| \boxtimes |B|) \;=\; (U \boxtimes V)|A \boxtimes B|.$$

Note that $U \boxtimes V$ is also a partial isometry, according to property (8) in Lemma 1. Hence, the decomposition (8) is a polar one. □

4 Tracy-Singh products on norm ideals of compact operators

In this section, we investigate the Tracy-Singh product on norm ideals of $\mathbb{B}(\mathcal{H})$. Recall that any proper ideal of $\mathbb{B}(\mathcal{H})$ is contained in the ideal $\mathcal{S}_\infty$ of compact operators. For any compact operator $A \in \mathbb{B}(\mathcal{H})$, let $(s_i(A))_{i=1}^{\infty}$ be the sequence of decreasingly-ordered singular values of A (i.e. eigenvalues of $|A|$). For each $1 \leqslant p < \infty$, the *Schatten p-norm* of A is defined by

$$\|A\|_p \;=\; \left(\sum_{i=1}^{\infty} s_i^p(A)\right)^{1/p}.$$

If $\|A\|_p$ is finite, we say that A is a *Schatten p-class operator*. The Schatten ∞-norm is just the operator norm. For each $1 \leqslant p \leqslant \infty$, let $\mathcal{S}_p$ be the Schatten p-class operators . In particular, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the trace class and the Hilbert-Schmidt class, respectively. Each Schatten p-norm induces a norm ideal of $\mathbb{B}(\mathcal{H})$ and this ideal is closed under the topology generated by this norm.

Lemma 11. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ be an operator matrix. Then A is compact if and only if A_{ij} is compact for all i, j.*

Proof. If A is compact, then $A_{ij} = P_i' A E_j$ is also compact for each i, j due to the fact that $\mathcal{S}_\infty$ is an ideal of $\mathbb{B}(\mathcal{H})$. Conversely, suppose that A_{ij} is compact for all i, j. Recall that a bounded linear operator is compact if and only if it maps a bounded sequence into a sequence having a convergent subsequence. Let $(x_r)_{r=1}^\infty$ be a bounded sequence in $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$. Write $x_r = [x_r^{(1)}\ x_r^{(2)}\ \dots\ x_r^{(n)}]^T \in \bigoplus_{i=1}^n \mathcal{H}_i$ for each $r \in \mathbb{N}$. Consider

$$Ax_r = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_r^{(1)} \\ \vdots \\ x_r^{(n)} \end{bmatrix} = \begin{bmatrix} A_{11}x_r^{(1)} + \cdots + A_{1n}x_r^{(n)} \\ \vdots \\ A_{n1}x_r^{(1)} + \cdots + A_{nn}x_r^{(n)} \end{bmatrix}.$$

For each $l = 1, 2, \dots, n$, since $(x_r^{(l)})_{r=1}^\infty$ is bounded, the sequence $(A_{ij}x_r^{(l)})_{r=1}^\infty$ has a convergent subsequence, namely, $(A_{ij}x_{r_k}^{(l)})_{k=1}^\infty$. Hence,

$$\begin{bmatrix} A_{11}x_{r_k}^{(1)} + \cdots + A_{1n}x_{r_k}^{(n)} \\ \vdots \\ A_{n1}x_{r_k}^{(1)} + \cdots + A_{nn}x_{r_k}^{(n)} \end{bmatrix}$$

is a desired convergent subsequence of $(Ax_r)_{r=1}^\infty$. □

Lemma 12 ([1])**.** *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the Schatten p-class.*

(i) For $1 \leqslant p \leqslant 2$, we have

$$\sum_{i,j=1}^n \|A_{ij}\|_p^2 \leqslant \|A\|_p^2 \leqslant n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{9}$$

(ii) For $2 \leqslant p < \infty$, we have

$$n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2 \leqslant \|A\|_p^2 \leqslant \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{10}$$

Lemma 13. *Let $1 \leqslant p < \infty$. An operator matrix $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ is a Schatten p-class operator if and only if A_{ij} is a Schatten p-class operator for all i, j.*

Proof. This is a direct consequence of the norm estimations in Lemma 12. □

Lemma 14. *Let $1 \leqslant p \leqslant \infty$. Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in the class $\mathcal{S}_p$ and let $(A_r)_{r=1}^\infty$ be a sequence in $\mathcal{S}_p$ where $A_r = \left[A_{ij}^{(r)}\right]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ in $\mathcal{S}_p$ if and only if $A_{ij}^{(r)} \to A_{ij}$ in $\mathcal{S}_p$ for all $i, j = 1, \dots, n$.*

Proof. Lemma 13 assures that A_{ij} and $A_{ij}^{(r)}$ belong to $\mathcal{S}_p$ for any $i,j=1,\ldots,n$ and $r\in\mathbb{N}$. Consider the case $1\leqslant p\leqslant 2$. Suppose that $A_r\to A$ in $\mathcal{S}_p$. For any fixed $i,j\in\{1,...,n\}$, we have from the estimation (9) that

$$\|A_{ij}^{(r)}-A_{ij}\|_p^2 \;\leqslant\; \sum_{i,j=1}^{n}\|A_{ij}^{(r)}-A_{ij}\|_p^2 \;\leqslant\; \|A_r-A\|_p^2.$$

Hence, $A_{ij}^{(r)}\to A_{ij}$ in $\mathcal{S}_p$. Conversely, suppose $A_{ij}^{(r)}\to A_{ij}$ in $\mathcal{S}_p$ for each i,j. Lemma 12 implies that

$$\|A_r-A\|_p^2 \;\leqslant\; n^{4/p-2}\sum_{i,j=1}^{n}\|A_{ij}^{(r)}-A_{ij}\|_p^2.$$

Hence, $A_r\to A$ in $\mathcal{S}_p$. The case $2<p<\infty$ and the case $p=\infty$ are done by using the norm estimations (10) and (1), respectively. □

Next, we discuss compactness of Tracy-Singh product of operators.

Lemma 15 ([10])**.** *Let $A\in\mathbb{B}(\mathcal{H})$ and $B\in\mathbb{B}(\mathcal{K})$ be nonzero operators. Then $A\otimes B$ is compact if and only if both A and B are compact.*

Theorem 16. *Let $A\in\mathbb{B}(\mathcal{H})$ and $B\in\mathbb{B}(\mathcal{K})$ be nonzero operator matrices. Then $A\boxtimes B$ is compact if and only if both A and B are compact.*

Proof. Write $A=[A_{ij}]$ and $B=[B_{kl}]$. For sufficiency, suppose that A and B are compact. By Lemma 11, we deduce that A_{ij} and B_{kl} are compact for all i,j,k,l. It follows from Lemma 15 that $A_{ij}\otimes B_{kl}$ is compact for all i,j,k,l. Lemma 11 ensures the compactness of $A\boxtimes B$. For necessity part, reverse the previous procedure. □

The following theorem supplies bounds for Schatten 1-norm of the Tracy-Singh product of operators.

Theorem 17. *For any nonzero compact operator $A=[A_{ij}]_{i,j=1}^{n,n}\in\mathbb{B}(\mathcal{H})$ and $B=[A_{kl}]_{k,l=1}^{q,q}\in\mathbb{B}(\mathcal{K})$, we have*

$$\frac{1}{nq}\|A\|_1\|B\|_1 \;\leqslant\; \|A\boxtimes B\|_1 \;\leqslant\; nq\|A\|_1\|B\|_1. \tag{11}$$

Hence, $A\boxtimes B$ is trace-class if and only if both A and B are trace-class.

Proof. Suppose that both A and B are nonzero and compact. Then the operator $A\boxtimes B$ is compact by Theorem 16. It follows from the norm bound (9) that

$$\begin{aligned}\|A\boxtimes B\|_1^2 \;&\leqslant\; (nq)^2\sum_{k,l}\sum_{i,j}\|A_{ij}\otimes B_{kl}\|_1^2 \;=\; (nq)^2\sum_{k,l}\sum_{i,j}\|A_{ij}\|_1^2\|B_{kl}\|_1^2\\ &=\; (nq)^2\Big(\sum_{i,j}\|A_{ij}\|_1^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_1^2\Big) \;\leqslant\; (nq)^2\|A\|_1^2\|B\|_1^2.\end{aligned}$$

We also have

$$\begin{aligned}\|A \boxtimes B\|_1^2 &\geqslant \sum_{k,l}\sum_{i,j}\|A_{ij}\otimes B_{kl}\|_1^2 = \sum_{k,l}\sum_{i,j}\|A_{ij}\|_1^2\|B_{kl}\|_1^2 \\ &= \Big(\sum_{i,j}\|A_{ij}\|_1^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_1^2\Big) \geqslant (nq)^{-2}\|A\|_1^2\|B\|_1^2.\end{aligned}$$

Hence, we obtain the bound (11). □

Theorem 18. *For any nonzero compact operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$\|A \boxtimes B\|_2 = \|A\|_2 \|B\|_2. \tag{12}$$

Hence, $A \boxtimes B$ is a Hilbert-Schmidt operator if and only if both A and B are Hilbert-Schmidt operators.

Proof. Since both A and B are nonzero and compact, the operator $A \boxtimes B$ is compact by Theorem 16. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. Then by Lemma 12(ii), we have

$$\begin{aligned}\|A \boxtimes B\|_2^2 &= \sum_{k,l}\sum_{i,j}\|A_{ij}\otimes B_{kl}\|_2^2 = \sum_{k,l}\sum_{i,j}\|A_{ij}\|_2^2\,\|B_{kl}\|_2^2 \\ &= \Big(\sum_{i,j}\|A_{ij}\|_2^2\Big)\Big(\sum_{k,l}\|B_{kl}\|_2^2\Big) = \|A\|_2^2\,\|B\|_2^2.\end{aligned}$$

Hence, we get the multiplicative property (12). □

The final result asserts that the Tracy-Singh product is continuous with respect to the topology induced by the Schatten p-norm for each $p \in \{1, 2, \infty\}$.

Theorem 19. *Let $p \in \{1, 2, \infty\}$. If a sequence $(A_r)_{r=1}^{\infty}$ converges to A and a sequence $(B_r)_{r=1}^{\infty}$ converges to B in the norm ideal $\mathcal{S}_p$, then $A_r \boxtimes B_r$ converges to $A \boxtimes B$ in $\mathcal{S}_p$.*

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. In the viewpoint of Lemma 14, it suffices to show that $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$ in $\mathcal{S}_p$ for all i, j, k, l. Since $A_r \to A$ and $B_r \to B$ in $\mathcal{S}_p$, we have by Lemma 14 that $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$ for all i, j, k, l. It follows that

$$\begin{aligned}\|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij} \otimes B_{kl}\|_p &= \|A_{ij}^{(r)} \otimes B_{kl}^{(r)} - A_{ij}^{(r)} \otimes B_{kl} + A_{ij}^{(r)} \otimes B_{kl} - A_{ij} \otimes B_{kl}\|_p \\ &\leqslant \|A_{ij}^{(r)} \otimes (B_{kl}^{(r)} - B_{kl})\|_p + \|(A_{ij}^{(r)} - A_{ij}) \otimes B_{kl}\|_p \\ &= \|A_{ij}^{(r)}\|_p \,\|B_{kl}^{(r)} - B_{kl}\|_p + \|A_{ij}^{(r)} - A_{ij}\|_p \,\|B_{kl}\|_p \\ &\to \|A_{ij}\|_p \cdot 0 + 0 \cdot \|B_{kl}\|_p = 0.\end{aligned}$$

Hence, $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$ in $\mathcal{S}_p$ for all i, j, k, l. □

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ON THE RADIAL DISTRIBUTION OF JULIA SET OF SOLUTIONS OF $f'' + Af' + Bf = 0$

JIANREN LONG

Abstract. The paper is devoted to study the dynamical properties of solutions of $f'' + A(z)f' + B(z)f = 0$, where $A(z)$ is nontrivial solution of $w'' + P(z)w = 0$, $P(z)$ is a polynomial, $B(z)$ is a transcendental entire function of lower order less than $\frac{1}{2}$. The lower bound of the size of the radial distribution of Julia sets of infinite order solutions of the equation are obtained. Another proof of the result in [9] is discussed in which the modified Phragmén-Lindelöf principle is needed.

1. Introduction and main results

For a function meromorphic f in the complex plane $\mathbb{C}$, the order of growth, lower order of growth and the convergence exponent of zero-sequence of f are given respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$$

and

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{f})}{\log r}.$$

In what follows, we assume that the reader is familiar with standard notation and basic results in Nevanlinna theory of meromorphic functions, such as $T(r,f)$, $m(r,f)$ and $N(r,f)$, see [10, 12, 23] for more details.

We define the nth iterate of meromorphic function f as follows:

$$f^0(z) = z, f^1(z) = f(z), \ldots, f^n(z) = f(f^{n-1}(z)), \quad n \in \mathbb{N}, \quad n \geq 2.$$

The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of $\mathbb{C}$ where the iterates $\{f^n(z)\}_{n=1}^{\infty}$ of f form a normal family, and its complement $J(f) = \mathbb{C}\backslash F(f)$ is called the Julia set of f. It is well known that $F(f)$ is open and completely invariant under f, and $J(f)$ is closed and

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non-empty, see [4]. We also need the following notations and definitions. For $\alpha < \beta$ and $r \in (0, \infty)$, set

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\},$$
$$S(\alpha, \beta, r) = \{z : |z| < r, \alpha < \arg z < \beta\},$$
$$S(r, \alpha, \beta) = \{z : |z| > r, \alpha < \arg z < \beta\}.$$

Let $\overline{F}$ denotes the closure of $F \subset \mathbb{C}$. Given $\theta \in [0, 2\pi)$, if $S(\theta-\varepsilon, \theta+\varepsilon) \cap J(f)$ is unbounded for any small $\varepsilon > 0$, then the ray $\arg z = \theta$ from the origin is called the radial distribution of $J(f)$. Define

$$\Delta(f) = \{\theta \in [0, 2\pi) : \arg z = \theta \text{ is the radial distribution of } J(f)\}.$$

Obviously, $\Delta(f)$ is closed and so measurable. Let $\mathrm{m}(\Delta(f))$ denotes the linear measure of $\Delta(f)$. What can we say according to $\mathrm{m}(\Delta(f))$ of any meromorphic function f in $\mathbb{C}$? It is interesting topic, many results have been obtained by several authors. Baker [1] considered the radial distribution of the Julia set and constructed an entire function with infinite lower order whose Julia set lies in a horizontal trip. Qiao [16] proved that if f is a transcendental entire function of finite lower order, then

$$\mathrm{m}(\Delta(f)) \begin{cases} = 2\pi, & \mu(f) < 1/2, \\ \geq \pi/\mu(f), & \mu(f) \geq 1/2. \end{cases}$$

Later, some observations on radial distribution of the Julia sets of transcendental meromorphic functions with finite lower order were made; see, for example, [25] and [17]. It seems that there are few work done on the case of meromorphic functions of infinite order. Recently, Huang-Wang [8, 9] studied the radial distribution of the Julia sets of entire functions of infinite lower order by using the tool of differential equations, i.e., for any nontrivial solutions f of (1.2) below, a lower bound of $\mathrm{m}(\Delta(f))$ is obtained. Zhang-Wang-Yang [24] also studied the radial distribution of the Julia sets of entire soutions f of (1.2), a lower bound of $\mathrm{m}(\Delta(f))$ is obtained when the coefficient $A(z)$ and $B(z)$ satisfy different conditions with [8, 9]. Our idea of this paper comes from [24], a new lower bound of $\mathrm{m}(\Delta(f))$ is found when $A(z)$ and $B(z)$ satisfy new conditions which are different with the conditions of [24].

Our starting point is a result which is related to the growth of solutions of (1.2).

Theorem 1.1 ([14]). *Let $A(z)$ be a nontrivial solution of the equation*

$$w'' + P(z)w = 0, \tag{1.1}$$

where $P(z) = a_n z^n + \cdots + a_0, a_n \neq 0$, *and let* $\lambda(A) < \rho(A)$. *Let* $B(z)$ *be a transcendental entire function with* $\mu(B) < \frac{1}{2}$. *Then every nontrivial solution of the equation*

$$f'' + A(z)f' + B(z)f = 0 \tag{1.2}$$

is of infinite order.

The following result shows that $\mathrm{m}(\Delta(f))$ has a lower bound when $A(z)$ and $B(z)$ satisfy the conditions of Theorem 1.1.

Theorem 1.2. *Let* $A(z)$ *and* $B(z)$ *be given as in Theorem 1.1. Then every nontrivial solution* f *of* (1.2) *satisfies* $\mathrm{m}(\Delta(f)) \geq \frac{2\pi}{n+2}$.

To state the following results, the definition of accumulation lines of zero-sequence is needed, which can be found in [13, 18, 21, 22].

Definition 1.3. Let f be a meromorphic function in $\mathbb{C}$, and let $\arg z = \theta \in [0, 2\pi)$ be a ray from the origin. We denote, for each $\varepsilon > 0$, the convergence exponent of zero-sequence of f in the region $S(\theta - \varepsilon, \theta + \varepsilon, r)$ by $\lambda_{\theta,\varepsilon}(f)$ and by $\lambda_\theta(f) = \lim\limits_{\varepsilon \to 0^+} \lambda_{\theta,\varepsilon}(f)$. That is,

$$\lambda_\theta(f) = \lim_{\varepsilon \to 0^+} \limsup_{r \to \infty} \frac{\log^+ n_{\theta-\varepsilon,\theta+\varepsilon}(r, 0, f)}{\log r},$$

where $n_{\theta-\varepsilon,\theta+\varepsilon}(r, 0, f)$ is the number of zeros of f, counting multiplicity in $S(\theta - \varepsilon, \theta + \varepsilon, r)$.

The ray $\arg z = \theta$ is called an accumulation line of the zero-sequence of f if $\lambda_\theta(f) = \rho(f)$. By Lemma 2.1 below, we know that the number of accumulation lines of zero-sequence of nontrivial solutions of (1.1) less than or equal to $n + 2$ and the set of the accumulation lines of zero-sequence of nontrivial solutions of (1.1) is the subset of $\{\arg z = \theta_j, 0 \leq j \leq n + 1\}$, where $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$. Let w be a nontrivial solution of (1.1), where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree $n \geq 1$, let $p(w)$ denotes the number of the rays $\arg z = \theta_j$, $j = 0, 1, \ldots, n + 1$, which are not accumulation lines of zero-sequence of w.

Remark 1.4. It follows from Lemma 2.1 that $p(w)$ must be an even number for every nontrivial solution w of (1.1).

Theorem 1.5. *Let* $A(z)$ *be a nontrivial solution of* (1.1), *and the number of accumulation lines of zero-sequence of* $A(z)$ *strictly less than* $n+2$. *Let* $B(z)$ *be a transcendental entire function with* $\mu(B) < \frac{1}{2}$. *Then every nontrivial solution* f *of* (1.2) *satisfies* $\rho(f) = \infty$ *and* $\mathrm{m}(\Delta(f)) \geq \frac{2\pi}{n+2}$.

Furthermore, we study the radial distribution of Julia set of the derivatives of nontrivial solutions of (1.2).

Theorem 1.6. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.1. Then every nontrivial solution f of (1.2) satisfies $\mathrm{m}\left(\Delta(f)\cap\Delta(f^{(k)})\right)\geq\frac{2\pi}{n+2}$, where $k\geq 1$ is an integer.*

By using similar reasoning in proving Theorems 1.5 and 1.6, we have the following result.

Theorem 1.7. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.5. Then every nontrivial solution f of (1.2) satisfies $\mathrm{m}\left(\Delta(f)\cap\Delta(f^{(k)})\right)\geq\frac{2\pi}{n+2}$, where $k\geq 1$ is an integer.*

Applying Theorems 1.6 and 1.7, we immediately obtain the following corollaries.

Corollary 1.8. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.6. Then $\mathrm{m}\left(\Delta(f^{(k)})\right)\geq\frac{2\pi}{n+2}$ for every nontrivial solution f of (1.2), where $k\geq 1$ is an integer.*

Corollary 1.9. *Let $A(z)$ and $B(z)$ be given as in Theorem 1.7. Then $\mathrm{m}\left(\Delta(f^{(k)})\right)\geq\frac{2\pi}{n+2}$ for every nontrivial solution f of (1.2), where $k\geq 1$ is an integer.*

Obviously, we can obtain Theorems 1.2 and 1.5 from Theorems 1.6 and 1.7, however, we need the results of Theorems 1.2 and 1.5 in proving Theorems 1.6 and 1.7. So we will give the proofs of Theorems 1.2 and 1.5 in Sections 3 and 4, respectively.

2. Auxiliary results

In this section, we will give some auxiliary results for proving our theorems. To this end, we introduce following notations. Let f be an entire function of order $\rho(f)\in(0,\infty)$. For simplicity, set $\rho(f)=\rho$ and $S=S(\alpha,\beta)$. If for any $\theta\in(\alpha,\beta)$,

$$\lim_{r\to\infty}\frac{\log\log|f(re^{i\theta})|}{\log r}=\rho,$$

then we say that f blows up exponentially in $\overline{S}$. If for any $\theta\in(\alpha,\beta)$,

$$\lim_{r\to\infty}\frac{\log\log|f(re^{i\theta})|^{-1}}{\log r}=\rho,$$

then we say that f decays to zero exponentially in $\overline{S}$.

The following lemma, originally due to Hille [11, Chapter 7.4], which also be found in [6, 19], plays an important role in proving our results.

Lemma 2.1. *Let w be a nontrivial solution of (1.1), where $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \ldots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then w has the following properties.*

(i) *In each sector S_j, w either blows up or decays to zero exponentially.*

(ii) *If, for some j, w decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for w to blow up in many adjacent sectors.*

(iii) *If w decays to zero in S_j, then w has at most finitely many zeros in any closed subsector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.*

(iv) *If w blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, w has infinitely many zeros in each sector $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, and furthermore, as $r \to \infty$,*

$$n\left(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, w\right) = (1 + o(1))\frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, w)$ is the number of zeros of w, counting multiplicity in $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$.

Before stating the next lemma, for $E \subset [0, \infty)$, we define the Lebesgue linear measure of E by $\mathrm{m}(E) = \int_E dt$, and the logarithmic measure of $F \subset [1, \infty)$ is $\mathrm{m_l}(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic density of $F \subset [1, \infty)$ are given by

$$\overline{\log \mathrm{dens}}(F) = \limsup_{r \to \infty} \frac{\mathrm{m_l}(F \cap [1, r])}{\log r}$$

and

$$\underline{\log \mathrm{dens}}(F) = \liminf_{r \to \infty} \frac{\mathrm{m_l}(F \cap [1, r])}{\log r},$$

respectively.

The following result is due to Barry [3].

Lemma 2.2. *Let f be an entire function with $0 \leq \mu(f) < 1$, and denote $m(r) = \inf_{|z|=r} \log|f(z)|$ and $M(r) = \sup_{|z|=r} \log|f(z)|$. Then, for every $\alpha \in (\mu(f), 1)$,*

$$\overline{\log \mathrm{dens}}\left(\{r \in [1, \infty) : m(r) > M(r)\cos \pi\alpha\}\right) \geq 1 - \frac{\mu(f)}{\alpha}.$$

We say that an open set is hyperbolic if it has at least three boundary points in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let W be a hyperbolic open set in $\mathbb{C}$. For an $a \in \overline{\mathbb{C}} \backslash W$, define

$$C_W(a) = \inf\{\lambda_W(z)|z - a| : z \in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. We know that if every component of W is simply connected, then $C_W(a) \geq \frac{1}{2}$. The following result was proved in [25, Lemma 2.2].

Lemma 2.3. *Let f be an analytic in $S(r_0, \theta_1, \theta_2)$, let U be a hyperbolic domain and $f : S(r_0, \theta_1, \theta_2) \to U$. If there exists a point $a \in \partial U \backslash \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $l > 0$ such that, for sufficiently small $\varepsilon > 0$, one has*

$$|f(z)| = O(|z|^l), \quad z \in S(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon), \quad |z| \to \infty.$$

The following lemma is related to the Nevanlinna theory in an angular domain. To the end, we recall some notations and properties of Nevanlinna theory in an angular domain, see [5] for more details. Let f be meromorphic in $\overline{S}(\alpha, \beta)$, where $0 < \alpha < \beta \leq 2\pi$. Then we have

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r (\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t},$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(te^{i\varphi})| \sin \omega(\varphi - \alpha) d\varphi,$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{1<|b_n|<r} (\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}}) \sin \omega(\theta_n - \alpha),$$

where $\omega = \frac{\pi}{\beta - \alpha}$ and $b_n = |b_n| e^{i\theta_n}$ are poles of f in $\overline{S}(\alpha, \beta)$ appearing according to their multiplicities. The sectorial Nevanlinna characteristic is defined by

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f).$$

We denote the order of f in an angular domain $S(\alpha, \beta)$ by

$$\sigma_{\alpha,\beta}(f) = \limsup_{r \to \infty} \frac{\log^+ S_{\alpha,\beta}(r, f)}{\log r}.$$

The definition of an $R-$set is needed, which can be found in [12]. Set $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$, then $\bigcup_{n=1}^{\infty} B(z_n, r_n)$ is called an $R-$set. Clearly, the $\{|z| : z \in \bigcup_{n=1}^{\infty} B(z_n, r_n)\}$ is of finite linear measure.

The next lemma shows an estimation for the logarithmic derivative of analytic functions in an angular domain. It is a combination of results in [15, 21] and [8, Lemma 7], which can be found in [9, Lemma 2.2].

Lemma 2.4. *Let $z = re^{i\psi}$, $r_0 + 1 < r$, and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that $n \geq 2$ is an integer and that f is analytic in $S(r_0, \alpha, \beta)$*

with $\sigma_{\alpha,\beta}(f) < \infty$. *Choose* $\alpha < \alpha_1 < \beta_1 < \beta$. *Then, for every* $\varepsilon_j \in (0, \frac{\beta_j - \alpha_j}{2})$, $j = 1, 2, \ldots, n-1$, *outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \ \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \ldots, n-1,$$

there exist $K > 0$ *and* $M > 0$ *only depending on* $f, \varepsilon_1, \ldots, \varepsilon_{n-1}$ *and* $S(\alpha_{n-1}, \beta_{n-1})$, *and not depending on* z, *such that*

$$\left| \frac{f'(z)}{f(z)} \right| \leq K r^M \left(\sin k(\psi - \alpha) \right)^{-2}$$

and

$$\left| \frac{f^{(n)}(z)}{f(z)} \right| \leq K r^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j) \right)^{-2}$$

for all $z \in S(\alpha_{n-1}, \beta_{n-1})$ *outside an* $R-$*set* H, *where* $k = \frac{\pi}{\beta - \alpha}$ *and* $k_{\varepsilon_j} = \frac{\pi}{\beta_j - \alpha_j}$, $j = 1, 2, \ldots, n-1$.

3. Proof of Theorem 1.2

Set $d = \frac{2\pi}{n+2}$. Suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.2) with $\mathrm{m}(\Delta(f)) < d$. We aim for a contradiction. Set $\eta = d - \mathrm{m}(\Delta(f))$. Since $\Delta(f)$ is closed, then $S = [0, 2\pi) \backslash \Delta(f)$ consists of at most countable many open intervals. Therefore, we choose finite many open intervals $I_i = (\alpha_i, \beta_i)$, $i = 1, 2, \ldots, m$, which satisfy $[\alpha_i, \beta_i] \subset S$ and $\mathrm{m}(S \backslash \cup_{i=1}^m I_i) < \frac{\eta}{4}$. For the sector domain $S(\alpha_i, \beta_i)$, and for sufficiently large r_i, we have

$$(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset, \quad S(r_i, \alpha_i, \beta_i) \cap J(f) = \emptyset.$$

This shows that, for each $i = 1, 2, \cdots, m$, there exist the corresponding r_i and unbounded Fatou component U_i of $F(f)$, such that $S(r_i, \alpha_i, \beta_i) \subset U_i$ (see [2]). In boundary of U_i, we take an unbounded and connected section $\gamma_i \subset \partial U_i$, then the mapping $f : S(r_i, \alpha_i, \beta_i) \to \mathbb{C} \backslash \gamma_i$ is analytic. According to the choose of γ_i, we know that $\mathbb{C} \backslash \gamma_i$ is simply connected, thus for any $a \in \gamma_i \backslash \{\infty\}$, $C_{\mathbb{C} \backslash \gamma_i}(a) \geq \frac{1}{2}$. In every $S(r_i, \alpha_i, \beta_i)$, applying Lemma 2.3 to f, there exists a positive constant l_1 such that

$$|f(z)| = O(|z|^{l_1}), \quad z \in \cup_{i=1}^m S(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon), \quad |z| \to \infty,$$

where $0 < \varepsilon < \min\{\frac{\eta}{16m}, \frac{\beta_i - \alpha_i}{8}\}$, $i = 1, 2, \ldots, m$. Hence we immediately get

$$S_{\alpha_i + \varepsilon, \beta_i - \varepsilon}(r, f) = O(1), \quad i = 1, 2, \ldots, m,$$

and then $\sigma_{\alpha_i+\varepsilon,\beta_i-\varepsilon}(f)$ is finite. Applying Lemma 2.4, there exist two constants $M>0$ and $K>0$ such that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq Kr^M, \quad k=1,2, \tag{3.1}$$

for all $z \in \cup_{i=1}^m S(r_i, \alpha_i+2\varepsilon, \beta_i-2\varepsilon)$, outside a $R-$set H.

Set $\theta_j = \frac{2j\pi-\arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j=0,1,2,\ldots,n+1$, if $j=n+1$, set $\theta_{j+1}=\theta_0+2\pi$. Since $\lambda(A)<\rho(A)$, by Lemma 2.1, there exists at least one sector of the $n+2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n+1$. This implies that for any $\theta \in (\theta_{j_0}+\varepsilon, \theta_{j_0+1}-\varepsilon)$,

$$\lim_{r\to\infty} \frac{\log\log \frac{1}{|A(re^{i\theta})|}}{\log r} = \frac{n+2}{2}. \tag{3.2}$$

Set $S'_{j_0} = \{\theta \in [0,2\pi) : re^{i\theta} \in S_{j_0}(\varepsilon)\}$, where $S_{j_0}(\varepsilon) = \{z : \theta_{j_0}+\varepsilon < \arg z < \theta_{j_0+1}-\varepsilon\}$, then $\mathrm{m}(S'_{j_0}) = \theta_{j_0+1}-\varepsilon-(\theta_{j_0}+\varepsilon) \geq d-\frac{\eta}{4}$. Thus

$$\mathrm{m}(S'_{j_0}\cap S) = \mathrm{m}\left(S'_{j_0}\backslash(S'_{j_0}\cap\Delta(f))\right) \geq \mathrm{m}(S'_{j_0}) - \mathrm{m}(\Delta(f))) > \frac{3\eta}{4} > 0.$$

So,

$$\begin{aligned}\mathrm{m}\left(S'_{j_0}\cap(\cup_{i=1}^m I_i)\right) &= \mathrm{m}(S'_{j_0}\cap S) - \mathrm{m}(S'_{j_0}\cap(S\backslash\cup_{i=1}^m I_i)) \\ &> \frac{3\eta}{4} - \mathrm{m}(S\backslash\cup_{i=1}^m I_i) \\ &> \frac{\eta}{2} > 0.\end{aligned}$$

Thus, there exists an open interval $I_{i_0} = (\alpha,\beta) \subset \bigcup_{i=1}^m I_i \subset S$, such that

$$\mathrm{m}\left(S'_{j_0}\cap I_{i_0}\right) > \frac{\eta}{2m} > 0. \tag{3.3}$$

By equation (1.2), we get

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.4}$$

We divide into two cases to $B(z)$ for finishing the proof.

Case 1. $0<\mu(B)<\frac{1}{2}$. By Lemma 2.2, there exists a set $E_1^* \subset [1,\infty)$ with $\overline{\log\mathrm{dens}}(E_1^*) \geq 1-\frac{\mu(B)}{\alpha_0}$, where $\alpha_0 = \frac{\mu(B)+\frac{1}{2}}{2}$, $E_1^* = \{r\in[1,\infty) : m(r) > M(r)\cos\pi\alpha_0\}$, $m(r) = \inf\limits_{|z|=r}\log|B(z)|$, $M(r) = \sup\limits_{|z|=r}\log|B(z)|$. Hence there exists a constant $R_0>1$, such that

$$|B(z)| > \exp(r^{\mu(B)-\varepsilon}) \tag{3.5}$$

for all $|z| = r \in E_1 = E_1^* \backslash [0, R_0)$. Then there exists a sequence of points $\{r_s e^{i\theta}\}$ outside H, $\theta \in I_{i_0}$, $r_s \in E_1$ satisfying $r_s \to \infty$ as $s \to \infty$, such that (3.1), (3.5) hold for $z = r_s e^{i\theta}$, and

$$\lim_{s\to\infty} \frac{\log\log \frac{1}{|A(r_s e^{i\theta})|}}{\log r_s} = \frac{n+2}{2}. \tag{3.6}$$

It follows from (3.1), (3.4), (3.5) and (3.6) that

$$\exp(r_s^{\mu(B)-\varepsilon}) < K r_s^M (1 + o(1))$$

for sufficiently large s. Obviously, this is a contradiction.

Case 2. $\mu(B) = 0$. By Lemma 2.2, there exists a set $E_2 \subset [1, \infty)$ with $\overline{\log \mathrm{dens}}(E_2) = 1$, such that for all z satisfying $|z| = r \in E_2$, we have

$$\log |B(z)| > \frac{\sqrt{2}}{2} \log M(r, B), \tag{3.7}$$

where $M(r, B) = \max\limits_{|z|=r} |B(z)|$. It follows from (3.3) and (3.7), there exists a sequence of points $\{r_s e^{i\theta}\}$ outside H, $\theta \in I_{i_0}$, $r_s \in E_2$ satisfying $r_s \to \infty$ as $s \to \infty$, such that (3.1), (3.6) and (3.7) hold for $z = r_s e^{i\theta}$. We deduce from (3.1), (3.4), (3.6) and (3.7) that

$$M(r_s, B)^{\frac{\sqrt{2}}{2}} \leq K r_s^M (1 + o(1)) \tag{3.8}$$

for sufficiently large s. However $B(z)$ is a transcendental entire function, we have

$$\liminf_{r\to\infty} \frac{\log M(r, B)}{\log r} = \infty. \tag{3.9}$$

We get a contradiction from (3.8) and (3.9). This completes the proof.

4. Proof of Theorem 1.5

We begin by recalling a lemma on logarithmic derivative due to Gundersen plays an important role in proving Theorem 1.5, it can be found in [7].

Lemma 4.1. *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be a given real constant, and let k and j be integers such that $k > j \geq 0$. Then there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi)\backslash E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

Firstly, we prove that every nontrivial solution of (1.2) is of infinite order. To the end, suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.2) with $\rho(f) < \infty$. We aim for a contradiction. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 0, 1, 2, \ldots, n+1$, if $j = n+1$, set $\theta_{j+1} = \theta_0 + 2\pi$. Since $p(A) \geq 2$, by Lemma 2.1, there exists at least one sector of the $n+2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n+1$. That is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, we have (3.2) holds.

In the following, we divide into two cases to $B(z)$.

Case 1. $0 < \mu(B) < \frac{1}{2}$. By using the similar reasoning as in the case 1 of proof of Theorem 1.2, we have (3.5) holds for all $|z| = r \in E_1$, where $\overline{\log \mathrm{dens}}(E_1) \geq 1 - \frac{\mu(B)}{\alpha_0}$, $\alpha_0 = \frac{\mu(B)+\frac{1}{2}}{2}$.

Applying Lemma 4.1, there exists a set $E_2 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi)\backslash E_2$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$,

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{2\rho(f)}, \quad k = 1, 2. \tag{4.1}$$

Thus, there exists a sequence of points $z_s = r_s e^{i\theta}$ with $r_s \to \infty$ as $s \to \infty$, $r_s \in E_1$ and $\theta \in (\theta_{j_0}, \theta_{j_0+1})\backslash E_2$, such that (3.2), (3.5) and (4.1) hold. It follows from (1.2), (3.2), (3.5) and (4.1) that

$$\begin{aligned} \exp(r_s^{\mu(B)-\varepsilon}) &\leq |B(r_s e^{i\theta})| \\ &\leq \left|\frac{f''(r_s e^{i\theta})}{f(r_s e^{i\theta})}\right| + |A(r_s e^{i\theta})| \left|\frac{f'(r_s e^{i\theta})}{f(r_s e^{i\theta})}\right| \\ &\leq r_s^{2\rho(f)}(1 + o(1)) \end{aligned} \tag{4.2}$$

for sufficiently large s. Obviously, this is a contradiction for arbitrary small ε. Hence we have $\rho(f) = \infty$ for every nontrivial solutions f of (1.2).

Case 2. $\mu(B) = 0$. By Lemma 2.2, there exists a set $E_3 \subset [0, \infty)$ with $\overline{\log \mathrm{dens}}(E_3) = 1$, such that for all z satisfying $|z| = r \in E_3$, we have (3.7) holds. Thus, there exists a sequence of points $z_s = r_s e^{i\theta}$ with $r_s \to \infty$ as $s \to \infty$, $r_s \in E_3$ and $\theta \in (\theta_{j_0}, \theta_{j_0+1})\backslash E_2$, such that (3.2), (3.7) and (4.1) hold. Therefore, we deduce from (1.2), (3.2), (3.7) and (4.1) that

$$M(r_s, B)^{\frac{\sqrt{2}}{2}} \leq r_s^{2\rho(f)}(1 + o(1)) \tag{4.3}$$

for sufficiently large s. But $B(z)$ is a transcendental entire function, we have (3.9) holds. We obtain a contradiction from (3.9) and (4.3). So, $\rho(f) = \infty$ for every nontrivial solutions f of (1.2).

Secondly, we can prove $\mathrm{m}(\Delta(f)) \geq \frac{2\pi}{n+2}$ by using the similar reasoning in proving Theorem 1.2, we omit the details. This completes the proof.

5. Proof of Theorem 1.6

Set $d = \frac{2\pi}{n+2}$. Suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.2) with $\mathrm{m}(\Delta(f) \cap \Delta(f^{(k)})) < d$. We aim for a contradiction. Set $\eta = d - \mathrm{m}(\Delta(f) \cap \Delta(f^{(k)}))$. By using the idea as in proving Theorem 1.2, in order to finish the proof, we need find an open interval $I = (\alpha, \beta) \subset \Delta(f^{(k)})^c$, $0 < \beta - \alpha < d$, such that

$$\mathrm{m}\left(\Delta(f) \cap S'_{j_0} \cap I\right) > 0, \tag{5.1}$$

where $\Delta(f^{(k)})^c = [0, 2\pi)\backslash\Delta(f^{(k)})$, S'_{j_0} is defined as in the proof of Theorem 1.2. First, we claim that

$$\mathrm{m}\left(S'_{j_0}\backslash\Delta(f)\right) = 0. \tag{5.2}$$

If it is not true, then there exist $\phi_0 \in \Delta(f)^c$ and $\zeta > 0$ satisfying

$$\mathrm{m}\left((\phi_0 - \zeta, \phi_0 + \zeta) \cap (S'_{j_0}\backslash\Delta(f))\right) > 0. \tag{5.3}$$

Since $\arg z = \phi_0$ is not the radial distribution of $J(f)$, there exists a constant $r_0 > 0$ such that

$$S(r_0, \phi_0 - \zeta, \phi_0 + \zeta) \cap J(f) = \emptyset.$$

It follows that there exists an unbounded component U of Fatou set $F(f)$, such that $S(r_0, \phi_0 - \zeta, \phi_0 + \zeta) \subset U$. In boundary of U, we take an unbounded and connected set $\gamma \subset \partial U$, then the mapping $f : S(r_0, \phi_0 - \zeta, \phi_0 + \zeta) \to \mathbb{C}\backslash\gamma$ is analytic. Since $\mathbb{C}\backslash\gamma$ is simply connected, then, for any $a \in \gamma\backslash\{\infty\}$, we get $C_{\mathbb{C}\backslash\gamma}(a) \geq \frac{1}{2}$. For any $0 < \xi < \frac{\zeta}{4}$, applying Lemma 2.3 to f in $S(r_0, \phi_0 - \zeta, \phi_0 + \zeta)$, we get

$$|f(z)| = O(|z|^{l_1}), \quad z \in S(r_0, \phi_0 - \zeta + \xi, \phi_0 + \zeta - \xi), \quad |z| \to \infty, \tag{5.4}$$

where l_1 is a positive constant. Hence we get

$$S_{\phi_0 - \zeta + \xi, \phi_0 + \zeta - \xi}(r, f) = O(1),$$

and then $\sigma_{\phi_0 - \zeta + \xi, \phi_0 + \zeta - \xi}(f)$ is finite. Applying Lemma 2.4, there exist two constants $M > 0$ and $K > 0$ such that (3.1) holds for all $z \in S(r_0, \phi_0 - \zeta + 2\xi, \phi_0 + \zeta - 2\xi)$, outside a $R-$set H. Since ξ is arbitrary small, from (5.3), we have

$$\mathrm{m}\left((\phi_0 - \zeta + 2\xi, \phi_0 + \zeta - 2\xi) \cap S'_{j_0}\right) > 0.$$

By the similar reasoning as in the cases 1 and 2 of the proof of Theorem 1.2, then there exists a sequence of points $\{r_s e^{i\phi}\}$, where $\phi \in (\phi_0 - \zeta + 2\xi, \phi_0 + \zeta - 2\xi)$ and $r_s \to \infty$ as $s \to \infty$, such that (3.5), (3.6) and (3.7) hold for $z = r_s e^{i\phi}$. Combining (1.2), (3.1), (3.5), (3.6) and (3.7), we get a contradiction. Therefore, (5.2) is valid.

We know that $\mathrm{m}(\Delta(f)) \geq d$ from Theorem 1.2. It follows from the definition of S'_{j_0} and Lemma 2.1 that $\mathrm{m}(S'_{j_0}) \geq d - 2\varepsilon$ for any small $\varepsilon > 0$. From this and (5.2), we have

$$\mathrm{m}\left(\Delta(f) \cap S'_{j_0}\right) \geq d - \frac{\eta}{4}. \tag{5.5}$$

Since $\Delta(f^{(k)})$ is closed, then $\Delta(f^{(k)})^c$ consists of at most countable many open intervals. We can choose finite many open intervals I_i such that

$$I_i \subset \Delta(f^{(k)})^c,\ \mathrm{m}\left(\Delta(f^{(k)})^c \backslash \bigcup_{i=1}^{m} I_i\right) < \frac{\eta}{4}, \quad i = 1, 2, \ldots, m.$$

Since

$$\begin{aligned} &\mathrm{m}(\Delta(f) \cap S'_{j_0} \cap (\bigcup_{i=1}^{m} I_i)) + \mathrm{m}(\Delta(f) \cap S'_{j_0} \cap \Delta(f^{(k)})) \\ &= \mathrm{m}\left(\Delta(f) \cap S'_{j_0} \cap (\Delta(f^{(k)}) \cup (\bigcup_{i=1}^{m} I_i))\right) \geq d - \frac{\eta}{2}, \end{aligned}$$

then

$$\begin{aligned} &\mathrm{m}\left(\Delta(f) \cap S'_{j_0} \cap (\bigcup_{i=1}^{m} I_i)\right) \\ &\geq d - \frac{\eta}{2} - \mathrm{m}(\Delta(f) \cap S'_{j_0} \cap \Delta(f^{(k)})) \\ &\geq d - \frac{\eta}{2} - \mathrm{m}(\Delta(f) \cap \Delta(f^{(k)})) = \frac{\eta}{2} > 0. \end{aligned}$$

Thus, there exists an open interval $I_{j_0} = (\alpha, \beta) \subset \bigcup_{i=1}^{m} I_i \subset \Delta(f^{(k)})^c$, such that

$$\mathrm{m}\left(\Delta(f) \cap S'_{j_0} \cap I_{j_0}\right) \geq \frac{\eta}{2m} > 0.$$

Thus (5.1) is valid. From (5.1), we know that there are $\widetilde{\phi_0}$ and $\widetilde{\zeta} > 0$, such that $(\widetilde{\phi_0} - \widetilde{\zeta}, \widetilde{\phi_0} + \widetilde{\zeta}) \subset I_{j_0}$, and $\mathrm{m}(\Delta(f) \cap S'_{j_0} \cap (\widetilde{\phi_0} - \widetilde{\zeta}, \widetilde{\phi_0} + \widetilde{\zeta})) > 0$. Then there exists $\widetilde{r_0} > 0$, such that $S(\widetilde{r_0}, \widetilde{\phi_0} - \widetilde{\zeta}, \widetilde{\phi_0} + \widetilde{\zeta}) \cap J(f^{(k)}) = \emptyset$. By using similar reasoning as in proving (5.4), for any $0 < \widetilde{\xi} < \frac{\widetilde{\zeta}}{6}$, we have

$$|f^{(k)}(z)| = O(|z|^{l_2}), \quad z \in S(\widetilde{r_0}, \widetilde{\phi_0} - \widetilde{\zeta} + \widetilde{\xi}, \widetilde{\phi_0} + \widetilde{\zeta} - \widetilde{\xi}), \quad |z| \to \infty, \tag{5.6}$$

where l_2 is a positive constant.

Fix $r_* e^{i\phi_*}$, where $r_* > \widetilde{r_0}$ and $\phi_* \in (\widetilde{\phi_0} - \widetilde{\zeta} + \widetilde{\xi}, \widetilde{\phi_0} + \widetilde{\zeta} - \widetilde{\xi})$, and for any $z = re^{i\phi} \in S(\widetilde{r_0}, \widetilde{\phi_0} - \widetilde{\zeta} + \widetilde{\xi}, \widetilde{\phi_0} + \widetilde{\zeta} - \widetilde{\xi})$. Take a simple Jordan arc γ_z in $S(\widetilde{r_0}, \widetilde{\phi_0} - \widetilde{\zeta} + \widetilde{\xi}, \widetilde{\phi_0} + \widetilde{\zeta} - \widetilde{\xi})$ which connects $r_* e^{i\phi_*}$ to $r_* e^{i\phi}$ along $|z| = r_*$ and connects $r_* e^{i\phi}$ to $re^{i\phi}$ along $\arg z = \phi$. It follows from (5.6) and Cauchy

integral formula that

$$|f^{(k-1)}(z)| \leq \int_{\gamma_z} |f^{(k)}(z)||dz| + c_k \leq O(|z|^{l_2+1}), \quad |z| \to \infty.$$

Similarly,

$$|f^{(k-2)}(z)| \leq \int_{\gamma_z} |f^{(k-1)}(z)||dz| + c_{k-1} \leq O(|z|^{l_2+2}), \quad |z| \to \infty.$$

By induction, we have

$$|f(z)| \leq \int_{\gamma_z} |f'(z)||dz| + c_1 \leq O(|z|^{l_2+k}), \quad |z| \to \infty,$$

where c_i, $i = 1, 2, \ldots, k$, are positive constants. Therefore,

$$S_{\widetilde{\phi_0}-\widetilde{\zeta}+\widetilde{\xi},\widetilde{\phi_0}+\widetilde{\zeta}-\widetilde{\xi}}(r, f) = O(1),$$

and then $\sigma_{\widetilde{\phi_0}-\widetilde{\zeta}+\widetilde{\xi},\widetilde{\phi_0}+\widetilde{\zeta}-\widetilde{\xi}}(f) < \infty$. Applying Lemma 2.3, we know that (3.1) holds for all $z \in S(\widetilde{r_0}, \widetilde{\phi_0}-\widetilde{\zeta}+2\widetilde{\xi}, \widetilde{\phi_0}+\widetilde{\zeta}-2\widetilde{\xi})$, outside a $R-$set H. By applying similar reasoning as in the cases 1 and 2 of the proof of Theorem 1.2, we can get a contradiction. Therefore, we have

$$\mathrm{m}\left(\Delta(f) \cap \Delta(f^{(k)})\right) \geq d.$$

This completes the proof.

6. Annex remarks

In [9], Huang and Wang proved the following result by using the spread relation and Pólya peaks of meromorphic functions.

Theorem 6.1. *Suppose that $A_i(z)$, $i = 0, 1, \ldots, k-1$, are entire functions satisfying $\rho(A_j) < \mu(A_0) < \infty$, $j = 1, 2, \ldots, k-1$. Then every nontrivial solution f of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0 \tag{6.1}$$

satisfies $\mathrm{m}(\Delta(f)) \geq \min\{2\pi, \frac{\pi}{\mu(A_0)}\}$.

In this section, we will give a simple proof of Theorem 6.1 than original way which is given in [9], which rely heavily on the following modified Phragmén-Lindelöf principle.

Lemma 6.2 ([20])**.** *Let f be an entire function of lower order $\mu(f) \in [\frac{1}{2}, \infty)$. Then there exists a sector domain $S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with $\beta - \alpha \geq \frac{\pi}{\mu(f)}$, where $0 \leq \alpha < \beta \leq 2\pi$, such that*

$$\limsup_{r\to\infty} \frac{\log\log|f(re^{i\theta})|}{\log r} \geq \mu(f)$$

for all the ray $\arg z = \theta \in (\alpha, \beta)$.

Another proof of Theorem 6.1. We divide into two cases to finish our proof.

Case 1. $\mu(A_0) \geq \frac{1}{2}$. We need prove that $\mathrm{m}(\Delta(f)) \geq \frac{\pi}{\mu(A_0)}$ for every nontrivial solution f of (6.1). To the end, suppose on the contrary to the assertion that there exists a nontrivial solution f of (6.1) with $\mathrm{m}(\Delta(f)) < \frac{\pi}{\mu(A_0)}$. Set $S = (0, 2\pi)\backslash\Delta(f)$. By Lemma 6.2 to $A_0(z)$, there exists a sector domain $S(\alpha, \beta)$ with $\beta - \alpha \geq \frac{\pi}{\mu(A_0)}$, such that

$$\limsup_{r\to\infty} \frac{\log\log|A_0(re^{i\theta})|}{\log r} \geq \mu(A_0) \tag{6.2}$$

for any $\theta \in (\alpha, \beta)$.

Since $\mathrm{m}(\Delta(f)) < \frac{\pi}{\mu(A_0)}$ and $\beta - \alpha \geq \frac{\pi}{\mu(A_0)}$, then there exists a sector domain $S(\alpha', \beta')$, such that $\alpha < \alpha' < \beta' < \beta$ and $(\alpha', \beta') \subset S$. Then for any $\alpha' < \theta < \beta'$, we have (6.2) holds. For the sector domain $S(\alpha', \beta')$, it is easy to see that

$$(\alpha', \beta') \cap \Delta(f) = \emptyset, \quad S(r, \alpha', \beta') \cap J(f) = \emptyset$$

for sufficiently large r. This implies that there exists $r_0 > 0$ and unbounded Fatou component U of $F(f)$ such that $S(r_0, \alpha', \beta') \subset U$. We take a unbounded and connected section γ of ∂U, then the mapping $f : S(r_0, \alpha', \beta') \to \mathbb{C}\backslash\gamma$ is analytic. Since we have chosen γ such that $\mathbb{C}\backslash\gamma$ is simply connected, for any $a \in \gamma\backslash\{\infty\}$, we have $C_{\mathbb{C}\backslash\gamma}(a) \geq \frac{1}{2}$. Applying Lemma 2.3 to f, there exists a positive constant l such that

$$|f(z)| = O(|z|^l), \quad z \in S(r_0, \alpha' + \varepsilon, \beta' - \varepsilon), \quad |z| \to \infty, \tag{6.3}$$

where $0 < \varepsilon < \frac{\beta'-\alpha'}{8}$. Thus we immediately obtain $S_{\alpha'+\varepsilon,\beta'-\varepsilon}(r, f) = O(1)$, and then $\sigma_{\alpha'+\varepsilon,\beta'-\varepsilon}(f) < \infty$. Applying Lemma 2.4, there exist two constants $M > 0$ and $K > 0$, such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Kr^M, \quad j = 1, 2, \ldots, k, \tag{6.4}$$

for all $z \in S(\alpha' + 2\varepsilon, \beta' - 2\varepsilon)$, outside a $R-$set H.

Let $\max\limits_{1\leq j\leq k-1}\{\rho(A_j)\} = \eta$, and $\delta \in (\eta + \varepsilon, \mu(A_0) - 2\varepsilon)$ be a constant. Since $\rho(A_j) < \mu(A_0)$, $j = 1, 2, \ldots, k-1$, then there exists a constant $r_1 > r_0$, such that for any $|z| = r > r_1$, we have

$$|A_j(z)| \leq \exp(r^\delta), \quad j = 1, 2, \ldots, k-1. \tag{6.5}$$

Thus there exists a sequence of points $z_n = r_n e^{i\theta}$ outside H, $r_n \to \infty$ as $n \to \infty$, $\alpha' < \theta < \beta'$, such that (6.2), (6.4) and (6.5) hold. It follows from

(6.1), (6.2), (6.4) and (6.5) that

$$\begin{aligned}\exp(r_n^{\mu(A_0)-\varepsilon}) &\leq |A_0(r_ne^{i\theta})| \\ &\leq \left|\frac{f^{(k)}(r_ne^{i\theta})}{f(r_ne^{i\theta})}\right| + \cdots + |A_1(r_ne^{i\theta})|\left|\frac{f'(r_ne^{i\theta})}{f(r_ne^{i\theta})}\right| \\ &\leq Kr_n^M(k-1)\exp(r_n^\delta). \qquad (6.6)\end{aligned}$$

Obviously, this is a contradiction for sufficiently large n.

Case 2. $\mu(A_0) < \frac{1}{2}$. We need prove that $\mathrm{m}(\Delta(f)) = 2\pi$ for every nontrivial solution f of (6.1). Suppose on the contrary to the assertion that there exists a nontrivial solution f of (6.1) with $\mathrm{m}(\Delta(f)) < 2\pi$. Then, by similar reasoning as in case 1 above, there exist $(\alpha, \beta) \subset [0, 2\pi)\backslash\Delta(f)$ and constant $r_0 > 1$ such that $S(r_0, \alpha, \beta) \subset F(f)$, and (6.3) holds for $z \in S(r, \alpha+\varepsilon, \beta-\varepsilon)$. Hence we have $S_{\alpha+\varepsilon,\beta-\varepsilon}(r, f) = O(1)$, and then $\sigma_{\alpha+\varepsilon,\beta-\varepsilon}(f) < \infty$. Applying Lemma 2.4, we see that (6.4) holds for all $z \in S(r, \alpha + 2\varepsilon, \beta - 2\varepsilon)$ outside a $R-$set H.

Since $\mu(A_0) < \frac{1}{2}$, applying Lemma 2.2 to $A_0(z)$, there exists a set $E_1^* \subset [1, \infty)$ with $\overline{\log \mathrm{dens}}(E_1^*) \geq 1-\frac{\mu(A_0)}{\alpha_0}$, where $\alpha_0 = \frac{\mu(A_0)+\frac{1}{2}}{2}$, $E_1^* = \{r \in [1, \infty) : m(r) > M(r)\cos\pi\alpha_0\}$, $m(r) = \inf\limits_{|z|=r} \log|A_0(z)|$, and $M(r) = \sup\limits_{|z|=r} \log|A_0(z)|$. Thus, there exists a constant $R_0 > 1$ such that

$$|A_0(z)| > \exp(r^{\mu(A_0)-\varepsilon}) \qquad (6.7)$$

for all $|z| = r \in E_1 = E_1^*\backslash[0, R_0]$. Thus, there exists a sequence of points $z_n = r_ne^{i\theta}$ outside H, $r_n \to \infty$ as $n \to \infty$, $\alpha < \theta < \beta$, such that (6.4), (6.5) and (6.7) hold. It follows from (6.1), (6.4), (6.5) and (6.7) that

$$\exp(r_n^{\mu(A_0)-\varepsilon}) \leq Kr_n^M(k-1)\exp(r_n^\delta).$$

Obviously, this is a contradiction for sufficiently large n. This completes the proof.

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Jianren Long
School of Mathematicial Sciences, Guizhou Normal University, Guiyang, 550001, P.R. China.
E-mail address: longjianren2004@163.com, jrlong@gznu.edu.cn

Common Fixed Point Results for the Family of Multivalued Mappings Satisfying Contractions on a Sequence in Hausdorff Fuzzy Metric Space

Abdullah Shoaib[1], Akbar Azam[2], and Aqeel Shahzad[3]

Abstract: The aim of this paper is to establish common fixed point results on a sequence contained in a closed ball for family of multivalued mapping in complete fuzzy metric space. Simple and different technique has been used. Example has been constructed to demonstrate the novelty of our results. Our results unify, extend and generalize several results in the existing literature.

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1 Introduction and Preliminaries

The notion of fuzzy sets was first introduced by Zadeh [5]. Kramosil et al. [10] introduced the concept of fuzzy metric space and obtained many fixed point results. Later on many authors [7, 8, 9, 11] used this concept and prove many fixed point results using the different contractive conditions. Lopez et al. [11] discuss the method for constructing a Hausdorff fuzzy metric on nonempty compact subsets of a given fuzzy metric space.
Sometimes, it happens that the fixed point of a mapping exists, but the contraction does not hold. Recently, Shoaib et al. [1, 2, 3, 4, 6, 13] obtained the necessary and sufficient conditions for the existence of a fixed point of such self mapping. In this paper, we prove the existence of a common fixed point of a family of such multivalued mappings which are contractive on a sequence contained in a closed ball instead of the whole space, by using the concept of Hausdorff fuzzy metric space. We also present an example to support our results.
Definition 1.1 [7] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it is satisfies the following conditions:
i) $*$ is associative and commutative;
ii) $*$ is continuous;
iii) $a * 1 = a$ for all $a \in [0,1]$;
iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.
Definition 1.2 [10] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and F is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

F1) $F(x, y, 0) = 0$;
F2) $F(x, y, t) = 1$ if and only if $x = y$;
F3) $F(x, y, t) = F(y, x, t)$;
F4) $F(x, z, t + s) \geq F(x, y, t) * F(y, z, s)$;
F5) $F(x, y, .) : (0, \infty) \to [0, 1]$ is left-continuous.
Example 1.3 [7] Let (X, d) be a metric space. Define $a * b = ab$ and

$$F(x, y, t) = \frac{kt^n}{kt^n + md(x, y)},$$

for all $x, y \in X$ and $k, m, n \in \mathbb{R}^+$. Then $(X, F, *)$ is a fuzzy metric space.
Definition 1.4 [9] Let $(X, F, *)$ be a fuzzy metric space. Then, we have
i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ denoted $x_n \to x$, if $\lim_{n\to\infty} F(x_n, x, t) = 1$ for each $t > 0$.
ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if $\lim_{n\to\infty} F(x_n, x_{n+p}, t) = 1$ for each $t > 0$, $p > 0$.
iii) A fuzzy metric space $(X, F, *)$ in which every Cauchy sequence is convergent is called a complete fuzzy metric space.
Definition 1.5 [11] Let $(X, F, *)$ be a fuzzy metric space. Define a function H_{FM} on $\hat{C}_0(X) \times \hat{C}_0(X) \times (0, \infty)$ by

$$H_{FM}(A, B, t) = \min\left\{\inf_{a\in A} F(a, B, t), \inf_{b\in B} F(A, b, t)\right\},$$

for all $A, B \in \hat{C}_0(X)$ and $t > 0$, where $\hat{C}_0(X)$ is the collection of all nonempty compact subsets of X.
Definition 1.6 [7] Let $(X, F, *)$ be a fuzzy metric space. Then,

$$B_F(x, r, t) = \{y \in X : F(x, y, t) > 1 - r\}$$

and

$$\overline{B_F(x, r, t)} = \{y \in X : F(x, y, t) \geq 1 - r\}$$

are called open and closed balls respectively, with centre $x \in X$ and radius r for $0 < r < 1$, $t > 0$.
Lemma 1.7 [11] Let $(X, F, *)$ be a complete fuzzy metric space. Then, for each $a \in X$, $B \in \hat{C}_0(X)$ and for all $t > 0$ there is $b_o \in B$ such that

$$F(a, b_o, t) = F(a, B, t).$$

Lemma 1.8 [12] Let $(X, F, *)$ be a complete fuzzy metric space. $(\hat{C}_0(X), H_{FM}, *)$ is a hausdorff fuzzy metric space on $\hat{C}_0(X)$. Then, for all $A, B \in \hat{C}_0(X)$, for each $a \in A$ and for all $t > 0$ there exists $b_a \in B$, satisfies $F(a, B, t) = F(a, b_a, t)$, then

$$H_{FM}(A, B, t) \leq F(a, b_a, t).$$

2 Main Results

Let $(X, F, *)$ be a fuzzy metric space, $x_0 \in X$ and let $\{S_\beta : \beta \in \Omega\}$ be a family of multivalued mappings from X to $\hat{C}_0(X)$. Then, there exists $x_1 \in S_a x_0$ for some $a \in \Omega$, such that $F(x_0, S_a x_0, t) = F(x_0, x_1, t)$, for all $t > 0$. Let $x_2 \in S_b x_1$ be such that $F(x_1, S_b x_1, t) = F(x_1, x_2, t)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in S_\gamma x_n$, $F(x_n, S_\gamma x_n, t) = F(x_n, x_{n+1}, t)$, for all $t > 0$. We denote this iterative sequence $\{XS_\beta(x_n) : \beta \in \Omega\}$ and say that $\{XS_\beta(x_n) : \beta \in \Omega\}$ is a sequence in X generated by x_0.

Theorem 2.1 Let $(X, F, *)$ be a complete fuzzy metric space, where $*$ be a continuous t-norm, defined as $a*a \geq a$ or $a*b = \min\{a, b\}$. Let $(\hat{C}_0(X), H_{FM}, *)$ be a Hausdorff fuzzy metric space on $\hat{C}_0(X)$, $\{S_\beta : \beta \in \Omega\}$ be a family of multivalued mappings from X to $\hat{C}_0(X)$ and $\{XS_\beta(x_n) : \beta \in \Omega\}$ be a sequence in X generated by x_0. Assume that, for some $0 < \alpha_{i,j} \leq k < 1$, for all $t > 0$, $x_0 \in X$, for all $x, y \in \overline{B_F(x_0, r, t)} \cap \{XS_\beta(x_n) : \beta \in \Omega\}$, with $x \neq y$ and for all $i, j \in \Omega$ with $i \neq j$, we have

$$H_{FM}(S_i x, S_j y, \alpha_{i,j} t) \geq F(x, y, t) \tag{2.1}$$

and, for some $t > 0$

$$F(x_0, x_1, (1-k)t)) \geq 1 - r. \tag{2.2}$$

Then, $\{XS_\beta(x_n) : \beta \in \Omega\}$ is a sequence in $\overline{B_F(x_0, r, t)}$ and $\{XS_\beta(x_n) : \beta \in \Omega\} \to z \in \overline{B_F(x_0, r, t)}$. Also, if (2.1) holds for z, then there exists a common fixed point for the family of multivalued mappings $\{S_\beta : \beta \in \Omega\}$ in $\overline{B_F(x_0, r, t)}$.

Proof: Let $\{XS_\beta(x_n) : \beta \in \Omega\}$ be a sequence in X generated by x_0. If $x_0 = x_1$, then x_0 is a common fixed point of S_a for all $a \in \Omega$. Let $x_0 \neq x_1$ and by Lemma 1.8, we have

$$F(x_1, x_2, t) \geq H_{FM}(S_a x_0, S_b x_1, t).$$

By induction, we have by Lemma 1.8, we have

$$F(x_n, x_{n+1}, t) \geq H_{FM}(S_i x_{n-1}, S_\gamma x_n, t). \tag{2.3}$$

First, we will show that $x_n \in \overline{B_F(x_0, r, t)}$. By (2.2), we get

$$\begin{aligned} F(x_0, x_1, t) &= F(x_0, S_a x_0, t) > F(x_0, x_1, (1-k)t) \geq 1 - r \\ F(x_0, x_1, t) &> 1 - r. \end{aligned}$$

This shows that $x_1 \in \overline{B_F(x_0, r, t)}$. Let $x_2, \cdots, x_j \in \overline{B_F(x_0, r, t)}$. Now, we have

$$\begin{aligned}
F(x_j, x_{j+1}, t) &\geq H_{FM}(S_\delta x_{j-1}, S_\eta x_j, t) \geq F(x_{j-1}, x_j, \frac{t}{\alpha_{\delta,\eta}}) \\
&\geq H_{FM}(S_\rho x_{j-2}, S_\delta x_{j-1}, \frac{t}{\alpha_{\delta,\eta}}) \quad \text{(by Lemma 1.8)} \\
&\geq F(x_{j-2}, x_{j-1}, \frac{t}{\alpha_{\rho,m}, \alpha_{\delta,\eta}}) \\
&\geq F(x_{j-2}, x_{j-1}, \frac{t}{k^2}) \geq \geq F(x_0, x_1, \frac{t}{k^j}) \\
F(x_j, x_{j+1}, t) &\geq F(x_0, x_1, \frac{t}{k^j}) \qquad (2.4)
\end{aligned}$$

Now,

$$\begin{aligned}
F(x_0, x_{j+1}, t) &\geq F(x_0, x_{j+1}, (1-k^{j+1})t) \\
&\geq F(x_0, x_1, (1-k)t) * F(x_1, x_2, (1-k)kt) * \\
&\quad * F(x_j, x_{j+1}, (1-k)k^j t) \\
&\geq F(x_0, x_1, (1-k)t) * F(x_0, x_1, (1-k)t) * \\
&\quad * F(x_0, x_1, (1-k)t) \qquad \text{(by (2.4))} \\
&\geq 1-r * 1-r * .. * 1-r = 1-r \\
F(x_0, x_{j+1}, t) &\geq 1-r.
\end{aligned}$$

This implies that $x_{j+1} \in \overline{B_F(x_0, r, t)}$. Now, inequality (2.4) can be written as

$$F(x_n, x_{n+1}, t) \geq F(x_0, x_1, \frac{t}{k^n}). \qquad (2.5)$$

Let $n, m \in N$ with $m > n$. Assume that $m = n + p$, we have

$$\begin{aligned}
F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1-k)t) * F(x_{n+1}, x_{n+p}, kt) \\
&\geq F(x_n, x_{n+1}, (1-k)t) * H_{FM}(S_j x_n, S_k x_{n+p-1}, kt) \\
&\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+p-1}, \frac{kt}{\alpha_{j,k}}) \\
&\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+p-1}, t) \\
&\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) \\
&\quad * F(x_{n+1}, x_{n+p-1}, (1-k)t) \\
&\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) \\
&\quad * H_{FM}(S_j x_n, S_l x_{n+p-2}, kt) \\
&\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) \\
&\quad * F(x_n, x_{n+p-2}, \frac{kt}{\alpha_{j,l}}) \\
F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) \\
&\quad * F(x_n, x_{n+p-2}, t).
\end{aligned}$$

Using the above, we have

$$\begin{aligned} F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) * ... * \\ &\quad F(x_n, x_{n+1}, t) \\ F(x_n, x_{n+p}, t) &\geq F(x_n, x_{n+1}, (1-k)t) * F(x_n, x_{n+1}, (1-k)t) * ... * \\ &\quad F(x_n, x_{n+1}, (1-k)t) \\ &\geq F(x_0, x_1, \frac{(1-k)t}{k^n}) * F(x_0, x_1, \frac{(1-k)t}{k^n}) * ... * \\ &\quad F(x_0, x_1, \frac{(1-k)t}{k^n}) \qquad \text{(by (2.5))} \\ F(x_n, x_{n+p}, t) &\geq F(x_0, x_1, \frac{(1-k)t}{k^n}). \end{aligned}$$

As, we have

$$\lim_{t\to\infty} F(x, y, t) = 1 \text{ for all } x, y \in X.$$

In particular

$$F(x_0, x_1, \frac{(1-k)t}{k^n}) = 1 \text{ as } n \to \infty.$$

By using above, we get

$$F(x_n, x_m, t) = 1 \text{ as } n \to \infty.$$

Hence, $\{XS_\beta(x_n)\}$ is a Cauchy sequence in $\overline{B_F(x_0, r, t)}$. As every closed ball in a complete fuzzy metric space is complete. So, $\overline{B_F(x_0, r, t)}$ is complete. Then, there exists $z \in \overline{B_F(x_0, r, t)}$, such that $x_n \to z$ as $n \to \infty$. Now, for some $q \in \Omega$, we have

$$F(z, S_q z, t) \geq F(z, x_n, (1-k)t) * F(x_n, S_q z, kt).$$

By Lemma 1.8, we have

$$\begin{aligned} F(z, S_q z, t) &\geq F(z, x_n, (1-k)t) * H_{FM}(S_r x_{n-1}, S_q z, kt) \\ &\geq F(z, x_n, (1-k)t) * F(x_{n-1}, z, \frac{kt}{\alpha_{r,q}}) \\ &\geq F(z, x_n, (1-k)t) * F(x_{n-1}, z, t). \end{aligned}$$

Letting $n \to \infty$, we have

$$F(z, S_q z, t) \geq 1 * 1 = 1.$$

This implies that $z \in S_q z$. Hence, $z \in \bigcap_{q\in\Omega} S_q z$. This completes the proof.

Let $(X, F, *)$ be a fuzzy metric space, $x_0 \in X$ and let S be a multivalued mapping from X to $\hat{C}_0(X)$. Then, there exists $x_1 \in Sx_0$, such that $F(x_0, Sx_0, t) = F(x_0, x_1, t)$, for all $t > 0$. Let $x_2 \in Sx_1$ be such that $F(x_1, Sx_1, t) = F(x_1, x_2, t)$.

Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Sx_n$, $F(x_n, Sx_n, t) = F(x_n, x_{n+1}, t)$, for all $t > 0$. We denote this iterative sequence $\{XS(x_n)\}$ and say that $\{XS(x_n)\}$ is a sequence in X generated by x_0.

Corollary 2.2 Let $(X, F, *)$ be a complete fuzzy metric space, where $*$ be a continuous t-norm, defined as $a*a \geq a$ or $a*a = \min\{a, b\}$. Let $(\hat{C}_0(X), H_{FM}, *)$ is Hausdorff fuzzy metric space on $\hat{C}_0(X)$, $x_0 \in X$, $S : X \to \hat{C}_0(X)$ be a multivalued mapping and $\{XS(x_n)\}$ be a sequence in X generated by x_o. Assume that for some $k \in (0, 1)$ $t > 0$, and $x_o \in X$, we have

$$H_{FM}(Sx, Sy, kt) \geq F(x, y, t) \text{ for all } x, y \in \overline{B_F(x_0, r, t)} \cap \{XS(x_n)\} \tag{2.6}$$

and

$$F(x_0, Sx_0, (1-k)t)) \geq 1 - r$$

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_F(x_0, r, t)}$ and $\{XS(x_n)\} \to z \in \overline{B_F(x_0, r, t)}$. Also, if (2.6) holds for z, then there exists a fixed point for S in $\overline{B_F(x_0, r, t)}$.

Proof: By using the similar steps as we have used in Theorem 2.1, it can be proved easily.

Corollary 2.3 Let $(X, F, *)$ be a complete fuzzy metric space, where $*$ be a continuous t-norm, defined as $a * a \geq a$ or $a * a = \min\{a, b\}$. Let $x_0 \in X$ and $S : X \to X$ be a self mapping. Assume that for some $k \in (0, 1)$, $t > 0$ and $x_o \in X$, we have

$$F(Sx, Sy, kt) \geq F(x, y, t) \text{ for all } x, y \in \overline{B_F(x_0, r, t)}$$

and

$$F(x_0, Sx_0, (1-k)t)) \geq 1 - r.$$

Then S has a fixed point in $\overline{B_F(x_0, r, t)}$.

Example 2.4 Let $X = [0, 5]$ and $d : X \times X \to \mathbb{R}$ be a complete metric space defined by,

$$d(x, y) = |x - y| \text{ for all } x, y \in X$$

Denote $a*b = ab$ or $a*b = min\{a, b\}$ for all $a, b \in [0, 1]$ and $F(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then, we can find that $(X, F, *)$ is a complete fuzzy metric space. Consider the multivalued mappings $S_\beta : X \to \hat{C}_0(X)$ where $\beta = a, 1, 2, 3, \cdots$. defined as,

$$S_n x = \begin{cases} [\frac{x}{3n}, \frac{x}{2n}] & \text{if } x \in [0, \frac{7}{2}] \\ [2nx, 3nx] & \text{if } x \in (\frac{7}{2}, 5] \end{cases}, \text{ where } n = 1, 2, \cdots,$$

and

$$S_a x = \begin{cases} [\frac{x}{3}, \frac{5x}{12}] & \text{if } x \in [0, \frac{7}{2}] \\ [2x, 3x] & \text{if } x \in (\frac{7}{2}, 5] \end{cases}.$$

Considering, $x_0 = \frac{1}{2}$ and $r = \frac{3}{4}$. then, $\overline{B_F(x_0, r, t)} = [0, \frac{7}{2}]$. Now,

$$\begin{aligned} F(x_0, S_a x_0, t) &= F(\frac{1}{2}, S_a \frac{1}{2}, t) = F(\frac{1}{2}, \frac{5}{24}, t) \\ F(x_1, S_1 x_1, t) &= F(\frac{5}{24}, S_1 \frac{5}{24}, t) = F(\frac{5}{24}, \frac{5}{48}, t) \end{aligned}$$

So, we obtain a sequence $\{XS_\beta(x_n)\} = \{\frac{1}{2}, \frac{5}{24}, \frac{5}{48}, \frac{5}{192}, ...\}$ in X generated by x_0. Now, for $x = 4$, $y = 5$, $k = \alpha_{1,a} = \frac{5}{6}$ and $t = 1$, we have

$$\begin{aligned} H_{FM}(S_1 4, S_a 5, \frac{5}{6}) &= \min\left\{\inf_{a \in S_1 4} F(a, S_a 5, \frac{5}{6}), \inf_{b \in S_a 5} F(S_1 4, b, \frac{5}{6})\right\} = 0.22 \\ F(4, 5, 1) &= \frac{1}{1 + |4 - 5|} = \frac{1}{2} = 0.5 \end{aligned}$$

So, we have

$$H_{FM}(S_1 4, S_a 5, \frac{5}{6}) \not\geq F(4, 5, 1)$$

So, the contractive condition does not hold on X. Now, for all $x, y \in \overline{B_F(x_0, r, t)} \cap \{XS_\beta(x_n)\}$, we have

$$\begin{aligned} H_{FM}(S_n x, S_a y, kt) &= \min\left\{\inf_{a \in S_n x} F(a, S_a y, kt), \inf_{b \in S_a y} F(S_n x, b, kt)\right\} \\ &= \min\left\{\inf_{a \in S_n x} F(a, [\frac{y}{3}, \frac{5y}{12}], \frac{5}{6}t), \inf_{b \in S_a y} F([\frac{x}{3n}, \frac{x}{2n}], b, \frac{5}{6}t)\right\} \\ &= \min\left\{F(\frac{x}{2n}, \frac{5y}{12}, \frac{5}{6}t), F(\frac{x}{3n}, \frac{y}{3}, \frac{5}{6}t)\right\} \\ &= \min\left\{\frac{(5/6)t}{(5/6)t + |x/3 - y/3|}, \frac{(5/6)t}{(5/6)t + |y/3 - x/3|}\right\} \\ H_{FM}(Sx, Sy, kt) &= \frac{(5/6)t}{(5/6)t + |x/3 - y/3|} \geq \frac{t}{t + |x - y|} = F(x, y, t) \end{aligned}$$

So, the contractive condition holds on $\overline{B_F(x_0, r, t)} \cap \{XS_\beta(x_n)\}$. Also, for $t = 1$, we have

$$\begin{aligned} F(x_0, x_1, (1 - k)t)) &= F(\frac{1}{2}, \frac{5}{24}, \frac{1}{6}) \\ &= \frac{4}{11} > \frac{1}{4} = 1 - r \end{aligned}$$

Hence, all the conditions of above theorem are satisfied. Now, we have $\{XS_\beta(x_n)\}$ is a sequence in $\overline{B_F(x_0, r, t)}$, and $\{XS_\beta(x_n)\} \to 0 \in \overline{B_F(x_0, r, t)}$. Moreover, $\{S_\beta : \beta = a, 1, 2 \cdots\}$ has a common fixed point 0.

Competing interests

The authors declare that they have no competing interests.

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[1,3]Department of Mathematics and Statistics, Riphah International University, Islamabad - 44000, Pakistan. [1]E-mail address: abdullahshoaib15@yahoo.com. [3]E-mail address: aqeel4all84@gmail.com.

[2]Department of Mathematics, COMSATS Institute of Information Technology, Chack Shahzad, Islamabad - 44000, Pakistan. E-mail address:akbarazam@yahoo.com.

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

CHOONKIL PARK[1], JUNG RYE LEE[2], AND DONG YUN SHIN[3*]

Abstract. Let

$$\begin{aligned} M_1f(x,y): &= \frac{3}{4}f(x+y)-\frac{1}{4}f(-x-y)\\ &\quad +\frac{1}{4}f(x-y)+\frac{1}{4}f(y-x)-f(x)-f(y),\\ M_2f(x,y): &= 2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y). \end{aligned}$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1f(x,y)\| \le \|\rho M_2f(x,y)\|, \tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho|<1$, and

$$\|M_2f(x,y)\| \le \|\rho M_1f(x,y)\|, \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\le |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([8]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

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*Corresponding author: Dong Yun Shin (email: dyshin@uos.ac.kr).

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;
(iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [18] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right) = f(x)+f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 10, 11, 13, 14, 15, 16, 17, 20, 21]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Additive-quadratic ρ-functional inequality (0.1) in non-Archimedean normed spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive-quadratic ρ-functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 2.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \leq \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1)*, then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|\|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$\|f(0)\| \leq \|2\rho f(0)\| = |2| \cdot |\rho| \cdot \|f(0)\|.$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\left\|\frac{1}{2} f(2x) - 2f(x)\right\| \leq 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}&\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\\&\quad\le\left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\|\\&\quad=|\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping such that*

$$\|M_1 f(x,y)\| \quad\le\quad \|\rho M_2 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \le 2\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So $\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.7)\\&\quad\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\quad= \max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\quad\le \max\left\{\frac{|2|^l}{|2|^{rl+r}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)+r}}\right\} 2\theta\|x\|^r = \frac{2\theta}{|2|^{(r-1)l+r}}\|x\|^r\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}\|M_1A(x,y)\| &= \lim_{n\to\infty}|2|^n\left\|M_1f\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty}|2|^n|\rho|\left\|M_2f\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty}\frac{|2|^n\theta}{|2|^{nr}}(\|x\|^r+\|y\|^r) \\ &= |\rho|\,\|M_2A(x,y)\| = \|\rho M_2A(x,y)\|\end{aligned}$$

for all $x, y \in X$. So

$$\|M_1A(x,y)\| \le \|\rho M_2A(x,y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive .

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|2^qA\left(\frac{x}{2^q}\right)-2^qT\left(\frac{x}{2^q}\right)\right\| \\ &\le \max\left\{\left\|2^qA\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|,\left\|2^qT\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|\right\} \le \frac{2\theta}{|2|^{(r-1)q+r}}\|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.5). □

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping satisfying* (2.4)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{2\theta}{|2|}\|x\|^r \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{2}{|2|}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}&\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^m}f\left(2^mx\right)\right\| \\ &\quad\le \max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|\right\} \\ &\quad= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^mx\right)\right\|\right\} \\ &\quad\le \max\left\{\frac{|2|^{lr}}{|2|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}2\theta\|x\|^r = \frac{2\theta}{|2|^{(1-r)l+1}}\|x\|^r\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 2.4. *Let $r < 2$ and θ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying* (2.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|}{|2|^r} 2\theta \|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\left\| \frac{1}{2} f(2x) - 2f(x) \right\| \leq 2\theta \|x\|^r \tag{2.10}$$

for all $x \in X$. So $\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \leq \frac{|2|}{|2|^r} 2\theta \|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} &\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \\ &\quad \leq \max\left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \cdots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &\quad = \max\left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \cdots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &\quad \leq \max\left\{ \frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \frac{|2|}{|2|^r} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l}} \frac{|2|}{|2|^r} \|x\|^r \end{aligned} \tag{2.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.11) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.9).

It follows from (2.4) that

$$\begin{aligned} \|M_1 Q(x,y)\| &= \lim_{n\to\infty} |4|^n \left\| M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} |4|^n |\rho| \left\| M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| + \lim_{n\to\infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) \\ &= |\rho| \, \|M_2 Q(x,y)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|M_1 Q(x,y)\| \leq \|\rho M_2 Q(x,y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.9). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ &\leq \max\left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \frac{|2|}{|2|^{(r-2)q+r}} 2\theta \|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.9). □

Theorem 2.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (2.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|}\|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.10) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{2\theta}{|2|}\|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.4. □

3. Additive-quadratic ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the additive-quadratic ρ-functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and* (3.1)*, then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\ &\leq |\rho|\|f(x+y) - f(x) - f(y)\|\end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}&\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\\&\quad=\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\|\\&\quad\le|\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \theta\|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta\|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned}&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.7)\\&\quad\le\max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\quad=\max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\quad\le\max\left\{\frac{|2|^l}{|2|^{rl}},\cdots,\frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}}\|x\|^r\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.4). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{|2|^r\theta}{|2|}\|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{|2|^r\theta}{|2|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \qquad (3.9)\\
&\leq \max\left\{\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|\right\}\\
&= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\|\right\}\\
&\leq \max\left\{\frac{|2|^{rl}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 3.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \theta\|x\|^r \tag{3.10}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.11}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.12)\\
&\quad \le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad = \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.12) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{|2|^r\theta}{|4|}\|x\|^r \qquad (3.13)$$

for all $x \in X$.

Proof. It follows from (3.11) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \qquad (3.14)\\
&\quad \le \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\
&\quad = \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|2|^{rl}}{|4|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(2-r)l+2}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.13).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.4. □

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[1]Research Institute for Natural Sciences,
Hanyang University, Seoul 04763,
Republic of Korea
E-mail address: baak@hanyang.ac.kr

[2]Department of Mathematics,
Daejin University, Kyunggi 11159,
Republic of Korea
E-mail address: jrlee@hdaejin.ac.kr

[3]Department of Mathematics,
University of Seoul, Seoul 02504,
Republic of Korea
E-mail address: dyshin@uos.ac.kr

Differential equations associated with the generalized Euler polynomials of the second kind

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper, we study linear differential equations arising from the generating functions of the generalized Euler polynomials of the second kind. We give explicit identities for the second kind Euler polynomials.

Key words : linear differential equations, the second kind Euler numbers, generalized Euler polynomials of the second kind.

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1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and the second kind Euler numbers(see [1, 2, 3, 4, 6, 7]). The generalized Euler polynomials $\mathcal{E}_n(x)(n \geq 0)$ of the second kind, were introduced by Ryoo(see [5, 6]). The generalized Euler polynomials $\mathcal{E}_n(x)$ of the second kind are defined by the generating function:

$$F = F(t,x) = \left(\frac{2e^t}{e^{2t}+1}\right)^x = \sum_{n=0}^{\infty} \mathcal{E}_n(x)\frac{t^n}{n!}. \tag{1.1}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [8])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{1.2}$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.4}$$

If x is a variable, we use the following notation:

$$< x >_k = x(x+1)\cdots(x+k-1), \quad \binom{x}{k} = \frac{(x)_k}{k!}, \quad (1+t)^x = \sum_{k=0}^{\infty}\binom{x}{k}t^k. \tag{1.5}$$

Nonlinear differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [3, 4]). In this paper, we study linear differential equations arising from the generating functions of the generalized Euler polynomials of the second kind. We give explicit identities for the generalized Euler polynomials of the second kind.

2. Differential equations associated with the generalized Euler polynomials of the second kind

In this section, we study linear differential equations arising from the generating functions of the generalized Euler polynomials $\mathcal{E}_n(x)$ of the second kind.

Let

$$F = F(t,x) = \left(\frac{2e^t}{e^{2t}+1}\right)^x. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x) = \frac{d}{dt}\left(\frac{2e^t}{e^{2t}+1}\right)^x = x\left(\frac{2e^t}{e^{2t}+1}\right)^{x-1}\left(\frac{2e^t}{e^{2t}+1} - \left(\frac{2e^t}{e^{2t}+1}\right)^2 e^t\right) \\ &= xF(t,x) - xF(t,x+1)e^t, \\ F^{(2)} &= \frac{d}{dt}F^{(1)} = xF(t,x)^{(1)} - xF^{(1)}(t,x+1)e^t - xF(t,x+1)e^t \\ &= x\left(xF(t,x) - xF(t,x+1)e^t\right) \\ &\quad - x\left((x+1)F(t,x+1) - (x+1)F(t,x+2)e^t\right)e^t - xF(t,x+1)e^t, \\ &= x^2F(t,x) - (2x^2+2x)F(t,x+1)e^t + x(x+1)F(t,x+2)e^{2t}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(3)} = \frac{d}{dt}F^{(2)} &= x^2F^{(1)}(t,x) - (2x^2+2x)F^{(1)}(t,x+1)e^t - (2x^2+2x)(t,x+1)e^t \\ &\quad + x(x+1)F^{(1)}(t,x+2)e^{2t} + 2x(x+1)F(t,x+2)e^{2t} \\ &= x^3F(t,x) - (3x^3+6x^2+4x)F(t,x+1)e^t \\ &\quad - (3x^3+9x^2+6x)F(t,x+2)e^{2t} - (x^3+3x^2+2x)F(t,x+3)e^{3t}. \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \sum_{i=0}^{N} a_i(N,x)F(t,x+i)e^{it}, \quad (N=0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we have

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= \sum_{i=0}^{N} a_i(N,x)ie^{it}F(t,x+i) + \sum_{i=0}^{N} a_i(N,x)e^{it}F^{(1)}(t,x+i) \\ &= \sum_{i=0}^{N} a_i(N,x)(x+2i)e^{it}F(t,x+i) - \sum_{i=1}^{N+1} a_{i-1}(N,x)(x+i-1)e^{it}F(t,x+i). \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x)e^{it}F(t,x+i) \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned} &\sum_{i=0}^{N} a_i(N,x)(x+2i)e^{it}F(t,x+i) - \sum_{i=1}^{N+1} a_{i-1}(N,x)(x+i-1)e^{it}F(t,x+i) \\ &= \sum_{i=0}^{N+1} a_i(N+1,x)e^{it}F(t,x+i). \end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$a_0(N+1,x) = xa_0(N,x), \quad a_{N+1}(N+1,x) = -(x+N)a_N(N,x), \tag{2.8}$$

and

$$a_i(N+1,x) = (x+2i)a_i(N,x) - (x+i-1)a_{i-1}(N,x), (1 \le i \le N). \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F^{(0)} = F^{(0)}(t,x) = a_0(0,x) = F(t,x). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,x) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} xF(t,x) - xF(t,x+1)e^t &= \sum_{i=0}^{1} a_i(1,x)e^{it}F(t,x+i) \\ &= a_0(1,x)F(t,x) + a_1(1,x)F(t,x+1)e^t. \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,x) = x, \quad a_1(1,x) = -x. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x) = xa_0(N,x) = \cdots = x^{N+1}a_0(0,x) = x^{N+1},$$

and

$$\begin{aligned} a_{N+1}(N+1,x) &= -(x+N)a_N(N,x) = \cdots \\ &= (-1)^{N+1}(x+N)(x+N-1)\cdots(x+1)x. \end{aligned} \tag{2.14}$$

For $i = 1, 2, 3$ in (2.9), we get

$$a_1(N+1,x) = -x\sum_{k=0}^{N}(x+2\cdot 1)^k a_0(N-k,x),$$

$$a_2(N+1,x) = -(x+1)\sum_{k=0}^{N-1}(x+2\cdot 2)^k a_1(N-k,x),$$

and

$$a_3(N+1,x) = -(x+2)\sum_{k=0}^{N-2}(x+2\cdot 3)^k a_2(N-k,x).$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x) = -(x+i-1)\sum_{k=0}^{N+1-i}(x+2\cdot i)^k a_{i-1}(N-k,x). \tag{2.15}$$

Here, we note that the matrix $a_i(j,x)_{0\le i,j\le N}$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^N \\ 0 & (-1)^{<x>_1} & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & (-1)^{<x>_2} & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & (-1)^{<x>_3} & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{<x>_N} \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,x)$. By (2.14) and (2.15), we get

$$\begin{aligned} a_1(N+1,x) &= -x\sum_{k_1=0}^{N}(x+2)^{k_1}a_0(N-k_1,x) \\ &= -x\sum_{k_1=0}^{N-1}(x+2)^{k_1}x^{N-k_1} \end{aligned}$$

$$\begin{aligned} a_2(N+1,x) &= -(x+1)\sum_{k_2=0}^{N-1}(x+2\cdot 2)^{k_2}a_1(N-k_2,x) \\ &= (-1)^2x(x+1)\sum_{k_2=0}^{N-1}(x+2\cdot 2)^{k_2}(x+2)^{k_1}x^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} &a_3(N+1) \\ &= -(x+2)\sum_{k_3=0}^{N-2}(x+4)^{k_3}a_2(N-k_3,x) \\ &= (-1)^3x(x+1)(x+2) \\ &\quad\times\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}(x+2\cdot 3)^{k_3}(x+2\cdot 2)^{k_2}(x+2)^{k_1}x^{N-k_3-k_2-k_1-2}. \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} &a_i(N+1) \\ &= (-1)^i < x >_i \sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1}\prod_{l=1}^{i}(x+2l)^{k_l}x^{N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1}. \end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N=0,1,2,\ldots$, the linear functional equations

$$F^{(N)} = \left(\sum_{i=0}^{N}a_i(N,x)\left(\frac{2e^t}{e^{2t}+1}\right)^i e^{it}\right)F$$

have a solution

$$F = F(t,x) = \left(\frac{2e^t}{e^{2t}+1}\right)^x,$$

where

$$\begin{aligned} &a_0(N,x) = x^N, \\ &a_N(N,x) = (-1)^N < x >_N, \\ &a_i(N) = (-1)^i < x >_i \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}\prod_{l=1}^{i}(x+2l)^{k_l}x^{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}, \\ &(1\le i\le N-1). \end{aligned}$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \sum_{k=0}^{\infty}\mathcal{E}_{k+N}(x)\frac{t^k}{k!}. \tag{2.17}$$

From Theorem 1 and (2.17), we can derive the following equation:

$$\begin{aligned}\sum_{k=0}^{\infty}\mathcal{E}_{k+N}(x)\frac{t^k}{k!} &= F^{(N)} \\ &= \sum_{i=0}^{N}a_i(N,x)e^{it}\left(\frac{2e^t}{e^{2t}+1}\right)^{x+i} \\ &= \sum_{i=0}^{N}a_i(N,x)\left(\sum_{k=0}^{\infty}i^k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}\mathcal{E}_k(x+i)\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{l=0}^{k}\binom{k}{l}a_i(N,x)i^{k-l}\mathcal{E}_l(x+i)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \ldots$, and $N = 0, 1, 2, \ldots$, we have

$$\mathcal{E}_{k+N}(x) = \sum_{i=0}^{N}\sum_{l=0}^{k}\binom{k}{l}a_i(N,x)i^{k-l}\mathcal{E}_l(x+i), \tag{1.19}$$

$a_0(N,x) = x^N$,

$a_N(N,x) = (-1)^N < x >_N$,

$a_i(N) = (-1)^i < x >_i \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}\prod_{l=1}^{i}(x+2l)^{k_l}x^{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}$,

$(1 \le i \le N-1)$.

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \ldots$, we have

$$\mathcal{E}_N(x) = \sum_{i=0}^{N}a_i(N,x).$$

The first few of them are

$$\begin{aligned}&\mathcal{E}_0(x) = 1, \\ &\mathcal{E}_1(x) = 0, \quad \mathcal{E}_2^{(x)} = -x, \\ &\mathcal{E}_3(x) = 0, \quad \mathcal{E}_4(x) = 2x + 3x^2, \\ &\mathcal{E}_5(x) = 0, \quad \mathcal{E}_6(x) = -16x - 30x^2 - 15x^3, \\ &\mathcal{E}_7(x) = 0, \quad \mathcal{E}_8(x) = 272x + 588x^2 + 420x^3 + 105x^4, \\ &\mathcal{E}_9(x) = 0, \quad \mathcal{E}_{10}(x) = -7936x - 18960x^2 - 16380x^3 - 6300x^4 - 945x^5.\end{aligned}$$

For $N = 0, 1, 2, \ldots$, the linear functional equations

$$F^{(N)} = \left(\sum_{i=0}^{N}a_i(N,x)\left(\frac{2e^t}{e^{2t}+1}\right)^i e^{it}\right)F$$

have a solution

$$F = F(t,x) = \left(\frac{2e^t}{e^{2t}+1}\right)^x.$$

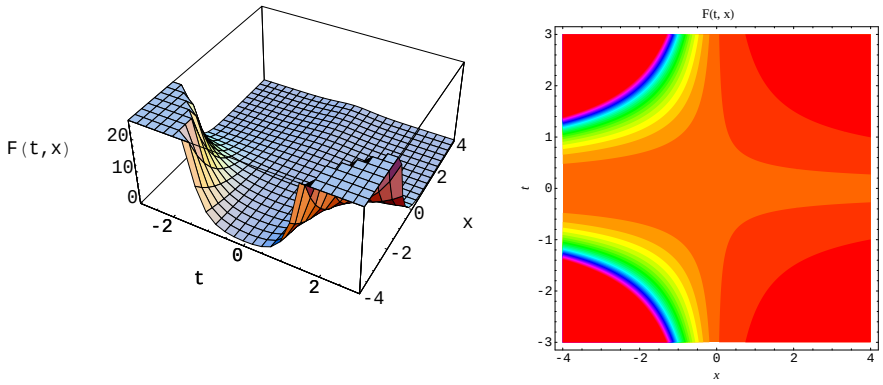


Figure 1: The surface for the solution $F(t,x)$

Here is a plot of the surface for this solution. In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

The author has no doubt that investigations along this line will lead to a new approach employing numerical method in the research field of the generalized Euler polynomials of the second kind to appear in mathematics and physics(see [5, 6, 7]).

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Chlodowsky Variant of Bernstein-Schurer Operators Based on (p,q)-Integers

Eser Gemikonakli · **Tuba Vedi-Dilek**

Abstract In this paper, we introduce the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers. By obtaining first few moments of these operators, we prove well-known Korovkin-type approximation theorems in different function spaces. We also compute the error of approximation by using modulus of continuity and Lipschitz-type functionals. Moreover, we study the generalization of the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers and investigate their approximations. Finally, numerical results are presented in detail.

Keywords (p,q)-integers, q-Bernstein operators, q-Bernstein-Schurer operators.

1 Introduction

The classical Bernstein-Chlodowsky operators were defined by Chlodowsky [4] as

$$C_n(f;x)=\sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r},$$

where the function f is defined on $[0,\infty)$ and (b_n) is a positive increasing sequence with $b_n\to\infty$ and $\dfrac{b_n}{n}\to 0$ as $n\to\infty$.

In 2008, Karsh and Gupta [9] defined q-analogue of Chlodowsky operators as follows:

$$C_n(f;q;x)=\sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}b_n\right)\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right),\qquad 0\le x\le b_n$$

where (b_n) has the same properties of Bernstein-Chlodowsky operators.

Eser Gemikonakli
Institute of Applied Sciences, University of Kyrenia, Girne, Mersin 10, Turkey
E-mail: eser.gemikonakli@kyrenia.edu.tr

Tuba Vedi-Dilek
Institute of Applied Sciences, University of Kyrenia, Girne, Mersin 10, Turkey
E-mail: tuba.vedi@kyrenia.edu.tr

In 1987, Lupaş [11] defined the q-based Bernstein operators and obtained the Korovkin-type approximation theorem. Over the past several years, there has been a considerable amount of research on the q-based operators (see [2], [3], [7], [12], [13],[16],[17], [20], [21], [22], [24]). To date, the focus of published work has been largely on (p,q) based operators.

In 2015, (p,q)-analogue of Bernstein operators were introduced by Mursaleen et al [14] as

$$B_{n,p,q}(f;x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^{k}\prod_{s=0}^{n-k-1}(p^{s}-q^{s}x)\,f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right),\quad x\in[0,1]. \tag{1.1}$$

For $0<q\leq p<1$, the (p,q)-numbers are given as [6]

$$[k]_{p,q}=\frac{p^{k}-q^{k}}{p-q}.$$

For each $k\in\mathbb{N}_0$ the (p,q)-factorial is represented by

$$[k]_{p,q}!=\begin{cases}[k][k-1]...[1], & k=1,2,3,...,\\ 1 \quad , & k=0\end{cases}$$

and (p,q)-binomial coefficients are defined as

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}$$

where $n\geq k\geq 0$. Note that, as it is introduced in [18], the operators are reduced to the q-Bernstein operators for $p=1$ in Eq.(1.1).

Recently, the (p,q)-analogue of Berntein-Schurer operators have been introduced by Sidharth and Agrawal [1] as

$$\bar{B}_{n,s}(f;p,q;x)=\sum_{k=0}^{n+s}p^{\frac{k(k-1)}{2}-\frac{(n+s)(n+s-1)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}x^{k}(1-x)_{p,q}^{n+s-k}f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\right)$$

where $f\in C[0,1+s],s\in\mathbb{N}_0$ and $n\in\mathbb{N}$.

Corollary 1 *If we use the properties of (p,q) integers, we have*

$$[n]_{p,q}=p^{n-1}[n]_{q/p}$$

and

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=p^{k(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_{q/p}.$$

This paper is structured in the following way;
The next section introduces the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers and investigate the moments of the operator. Section 3 discusses several Korovkin-type theorems in different function spaces. In section 4, we obtain the order of convergence of the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers by means of Lipschitz class functions and the first modulus of continuity. Section 5 provides the generalization of the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers and investigate their approximations. Finally, in section 6, numerical results to illustrate the contribution of the Chlodowsky Variant of Bernstein-Schurer Operators based on (p; q)-integers are presented.

2 Construction of the Operators

We construct the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers as

$$\bar{C}_{n,s}(f;p,q;x) \tag{2.1}$$
$$:=\sum_{k=0}^{n+s} f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\right)p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right),$$

where $n,s\in\mathbb{N}$, $0\le x\le b_n$ and $0<q<p\le 1$. Note that, in case p=1 in Eq.(2.1), Chlodowsky variant of Bernstein-Schurer reduces to the Chlodowsky variant of q-Bernstein-Schurer operators.
First of all, we obtained the following lemma and used it throughout the paper.

Lemma 1 *Let $\bar{C}_{n,s}(f;p,q;x)$ be given in Eq.(2.1). Then we get,*

(i) $\bar{C}_{n,s}(1;p,q;x)=1$,

(ii) $\bar{C}_{n,s}(t;p,q;x)=\dfrac{[n+s]_{p,q}}{p^s[n]_{p,q}}x$,

(iii) $\bar{C}_{n,s}\left(t^2;p,q;x\right)=\dfrac{p^{1-2s}q[n+s-1]_{p,q}[n+s]_{p,q}}{[n]_{p,q}^2}x^2+\dfrac{p^{n-s-1}[n+s]_{p,q}}{[n]_{p,q}^2}b_nx$,

$(iv)\bar{C}_{n,s}\left((t-x);p,q;x\right)=\left(\dfrac{[n+s]_{p,q}}{p^s[n]_{p,q}}-1\right)x$,

(v) $\bar{C}_{n,s}\left((t-x)^2;p,q;x\right)=\left(\dfrac{p^{1-2s}q[n+s-1]_{p,q}[n+s]_{p,q}}{[n]_{p,q}^2}-2\dfrac{[n+s]_{p,q}}{[n]_{p,q}}+1\right)x^2+\dfrac{p^{n-s-1}[n+s]_{p,q}}{[n]_{p,q}^2}b_nx.$

Proof Applying the Corollary 1, we have

$$\begin{aligned}\bar{C}_{n,s}(1;p,q;x)&=\sum_{k=0}^{n+s}p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k(1-x)_{p,q}^{n+s-k}\\&=\sum_{k=0}^{n+s}p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}p^{k(n+s-k)}\begin{bmatrix} n+s \\ k \end{bmatrix}_{q/p}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q/p}^{n+s-k}\\&=1\end{aligned}$$

when $0<\dfrac{q}{p}\le 1$.

Using the linearity of the operators and Corollary 1 with some basic calculations, we can obtain the assertions (ii), (iii). Then, from (i) and (ii), we have

$$\begin{aligned}\bar{C}_{n,s}\left((t-x);p,q;x\right)&=\bar{C}_{n,s}(t;p,q;x)-x\bar{C}_{n,s}(t;p,q;x)\\&=\left(\frac{[n+s]_{p,q}}{p^s[n]_{p,q}}-1\right)x.\end{aligned}$$

It is known that

$$\bar{C}_{n,s}\left((t-x)^2;p,q;x\right)=\bar{C}_{n,s}(x^2;p,q;x)-2x\bar{C}_{n,s}(x;p,q;x)+x^2\bar{C}_{n,s}(1;p,q;x).$$

Hence, the result proposed is acquired.

Lemma 2 *For the second central moment we obtain the following estimate*

$$\sup_{0\leq x\leq b_n}\bar{C}_{n,s}\left((t-x)^2;p,q;x\right)$$
$$\leq\left(\frac{p^{1-2s}q\left[n+s-1\right]_{p,q}\left[n+s\right]_{p,q}+p^{n-s-1}\left[n+s\right]_{p,q}}{\left[n\right]_{p,q}^2}-2\frac{\left[n+s\right]_{p,q}}{\left[n\right]_{p,q}}+1\right)b_n^2.$$

3 Korovkin Type Approximation Theorem

In this section, we give Korovkin-type approximation theorem for the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers. Let us denote C_ρ as the space of all continuous functions f, which satisfies the following condition.

$$\left|f\left(x\right)\right|\leq M_f\rho\left(x\right),\qquad -\infty<x<\infty.$$

Therefore, C_ρ is a linear normed space with the norm

$$\left\|f\right\|_\rho=\sup_{-\infty<x<\infty}\frac{\left|f\left(x\right)\right|}{\rho\left(x\right)}.$$

Theorem 1 *(See [10]) There exists a sequence of positive linear operators Q_n, acting from C_ρ to C_ρ, satisfying the conditions.*

$$\lim_{n\to\infty}\left\|Q_n\left(1;x\right)-1\right\|_\rho=0 \tag{3.1}$$
$$\lim_{n\to\infty}\left\|Q_n\left(\phi;x\right)-\phi\right\|_\rho=0 \tag{3.2}$$
$$\lim_{n\to\infty}\left\|Q_n\left(\phi^2;x\right)-\phi^2\right\|_\rho=0 \tag{3.3}$$

where $\phi\left(x\right)$ is continuous and increasing function on $(-\infty,\infty)$ such that $\lim_{x\to\pm\infty}\phi\left(x\right)=\pm\infty$ and $\rho\left(x\right)=1+\phi^2$ and there exists a function $f^*\in C_\rho$ for which $\overline{\lim_{n\to\infty}}\left\|Q_nf^*-f^*\right\|_\rho>0$.

The following theorem has been given in [10] and can be used in the investigation of approximation properties of $\bar{C}_{n,s}\left(f;q;x\right)$ in weighted spaces.

Theorem 2 *(See [10]) The conditions Eqs.(3.1),(3.2) and (3.3) imply $\lim_{n\to\infty}\left\|Q_nf-f\right\|_\rho=0$ for any function f belonging to the subset C_ρ^0 of C_ρ for which*

$$\lim_{\left|x\right|\to\infty}\frac{f\left(x\right)}{\rho\left(x\right)}$$

exists finitely.

Particularly, choosing $\rho\left(x\right)=1+x^2$ and performing Theorem 2 to the operators

$$Q_n\left(f;p,q;x\right)=\begin{cases}\bar{C}_{n,s}\left(f;p,q;x\right) & \text{if } 0\leq x\leq b_n\\ f\left(x\right) & \text{if } x\notin\left[0,b_n\right]\end{cases},$$

we can state the following theorem.

Theorem 3 *For all $f \in C^0_{1+x^2}$ we have*

$$\lim_{n\to\infty} \sup_{0\le x\le b_n} \frac{|Q_n(f;q_n;x)-f(x)|}{1+x^2} = 0$$

provided that $q := (q_n)$, $p := (p_n)$ with $0 < q_n < p_n \le 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$ as $n \to \infty$.

Proof In view of Theorem 2, by using Lemma 1 (i), (ii) and (iii) we get the following inequalities, respectively,

$$\sup_{0\le x\le b_n} \frac{|Q_n(t;p_n,q_n;x)-x|}{1+x^2} \le \sup_{0\le x\le b_n} \frac{\left(\frac{[n+s]_{p,q}}{p^s[n]_{p,q}}-1\right)x}{(1+x^2)}$$
$$\le \left|\frac{[n+s]_{p,q}}{p^s[n]_{p,q}}-1\right| \to 0$$

and

$$\sup_{0\le x\le b_n} \frac{\left|Q_n\left(t^2;p_n,q_n;x\right)-x^2\right|}{1+x^2}$$
$$\le \sup_{0\le x\le b_n} \frac{\left|\left(\frac{[n+s]_{p,q}}{[n]_{p,q}}-1\right)^2 x^2 + \frac{p^{n-s-1}[n+s]_{p,q}}{[n]^2_{p,q}} b_n x\right|}{(1+x^2)}$$
$$\le \left|\left(\frac{[n+s]_{p,q}}{[n]_{p,q}}-1\right)^2 + \frac{p^{n-s-1}[n+s]_{p,q}}{[n]^2_{p,q}} b_n\right| \to 0$$

is satisfied since $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} p_n = 1$ and $\frac{b_n}{[n]} \to 0$ as $n \to \infty$.

Lemma 3 *Let B be a positive real number independent of n and f be a continuous function which vanishes on $[B,\infty]$. Assume that $q := (q_n)$, $p := (p_n)$ with $0 < q_n < p_n \le 1$, $\lim_{n\to\infty} p_n^n = N < \infty$, $\lim_{n\to\infty} q_n^n = K < \infty$ and $\lim_{n\to\infty} \frac{b_n^2}{[n]} = 0$. Then we have*

$$\lim_{n\to\infty} \sup_{0\le x\le b_n} \left|\bar{C}_{n,s}(f;p_n,q_n;x) - f(x)\right| = 0.$$

Proof From the hypothesis on f, one can write $|f(x)| \le M$ $(M > 0)$. For arbitrary small $\varepsilon > 0$, we have

$$\left|f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\right) - f(x)\right| < \varepsilon + \frac{2M}{\delta^2}\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n - x\right)^2,$$

where $x \in [0,b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n. With the help of the following equality

$$\sum_{k=0}^{n+s}\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n - x\right)^2 p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} \left(\frac{x}{b_n}\right)^k \prod_{j=0}^{n+s-k-1}\left(p^j - q^j\frac{x}{b_n}\right)$$
$$= \bar{C}_{n,s}(f;p,q;x),$$

we get by Theorem 3 and Lemma 1 that

$$\sup_{0\le x\le b_n}\left|\bar{C}_{n,s}(f;p,q;x)-f(x)\right|$$
$$\le \varepsilon+\frac{2M}{\delta^2}\left[\left(\frac{p^{1-2s}q\,[n+s-1]_{p,q}\,[n+s]_{p,q}}{[n]_{p,q}^2}-2\frac{[n+s]_{p,q}}{[n]_{p,q}}+1\right)b_n^2+\frac{p^{n-s-1}\,[n+s]_{p,q}}{[n]_{p,q}^2}b_n^2\right].$$

Since $\dfrac{b_n^2}{[n]}\to 0$ as $n\to\infty$, we have the desired result.

4 Order of Convergence

In this section, we compute the rate of convergence of the operators in terms of the elements of Lipschitz classes and the modulus of continuity of the function. Now, we give the rate of convergence of the operators $\bar{C}_{n,s}$ in terms of the Lipschitz class $Lip_M(\gamma)$, for $0<\gamma\le 1$. Let $C_B[0,\infty)$ denotes the space of bounded continuous functions on $[0,\infty)$. A function $f\in C_B[0,\infty)$ belongs to $Lip_M(\gamma)$ if

$$|f(t)-f(x)|\le M\,|t-x|^{\gamma}\qquad (t,x\in[0,\infty))$$

is satisfied.

Theorem 4 *Let $f\in Lip_M(\gamma)$*

$$|\bar{C}_{n,s}(f;p,q;x)-f(x)|\le M\,(\lambda_n(x))^{\gamma/2}$$

where $\lambda_{n,q}(x)=\left(\dfrac{p^{1-2s}q\,[n+s-1]_{p,q}\,[n+s]_{p,q}}{[n]_{p,q}^2}-2\dfrac{[n+s]_{p,q}}{[n]_{p,q}}+1\right)x^2+\dfrac{p^{n-s-1}\,[n+s]_{p,q}}{[n]_{p,q}^2}b_n x.$

Proof Considering the monotonicity and the linearity of the operators, and taking into account that $f\in Lip_M(\gamma)$

$$|\bar{C}_{n,s}(f;p,q;x)-f(x)|$$
$$=\Big|\sum_{k=0}^{n+s}\Big(f\Big(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\Big)-f(x)\Big)p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)\Big|$$

$$\le\sum_{k=0}^{n+s}\left|f\Big(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\Big)-f(x)\right|p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)$$
$$\le M\sum_{k=0}^{n+s}\left|p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n-x\right|^{\gamma}p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right).$$

Using Hölder's inequality with $p=\dfrac{2}{\gamma}$ and $q=\dfrac{2}{2-\gamma}$, we get by the statement (Lemma 2)

$$|\bar{C}_{n,s}(f;p,q;x)-f(x)|$$
$$\le M\sum_{k=0}^{n+s}\left\{\left[\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n-x\right)^2p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)\right]^{\frac{\gamma}{2}}\right.$$
$$\left.\times\left[p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)\right]^{\frac{2-\gamma}{2}}\right\}$$

$$\leq M\left[\{\sum_{k=0}^{n+s}[(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n-x)^2 p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)]\}^{\frac{\gamma}{2}}\right.$$
$$\left.\times\{\sum_{k=0}^{n+s}[p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)]\}^{\frac{2-\gamma}{\gamma}}\right]$$

$$=M[\bar{C}_{n,s}\left((t-x)^2;p,q;x\right)]^{\frac{\gamma}{2}}$$
$$\leq M(\lambda_{n,q}(x))^{\frac{\gamma}{2}}.$$

Now we give the rate of convergence of the operators by means of the modulus of continuity which is denoted by $\omega(f;\delta)$. Let $f\in C_B[0,\infty)$ and $x\geq 0$. Then the definition of the modulus of continuity of f is given by

$$\omega(f;\delta)=\max_{\substack{|t-x|\leq\delta\\ t,x\in[0,\infty)}}|f(t)-f(x)|. \tag{4.1}$$

It is following that for any $\delta>0$ the following inequality

$$|f(x)-f(y)|\leq\omega(f;\delta)\left(\frac{|x-y|}{\delta}+1\right) \tag{4.2}$$

is satisfied ([5]).

Theorem 5 *If $f\in C_B[0,\infty)$, we have*

$$\left|\bar{C}_{n,s}(f;p,q;x)-f(x)\right|\leq 2\omega\left(f;\sqrt{\lambda_{n,p,q}(x)}\right)$$

where $\omega(f;\cdot)$ is modulus of continuity of f which is defined in Eq.(4.1) and $\lambda_{n,q}(x)$ be the same as in Theorem 4.

Proof By triangular inequality, we get

$$\left|\bar{C}_{n,s}(f;p,q;x)-f(x)\right|$$
$$=\left|\sum_{k=0}^{n+s}f(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n)p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)-f(x)\right|$$
$$\leq\sum_{k=0}^{n+s}\left|(f(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n)-f(x)\right|p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right).$$

Now, using Eq.(4.2) and Hölder inequality, we can write

$$\left|\bar{C}_{n,s}(f;p,q;x)-f(x)\right|$$
$$\leq\sum_{k=0}^{n+s}\left(\frac{\left|p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n-x\right|}{\lambda}+1\right)\omega(f;\lambda)p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)$$
$$=\omega(f;\lambda)\sum_{k=0}^{n+s}p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)$$
$$+\frac{\omega(f;\lambda)}{\lambda}\sum_{k=0}^{n+s}\left|p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n-x\right|p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix}n+s\\k\end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\prod_{j=0}^{n+s-k-1}\left(p^j-q^j\frac{x}{b_n}\right)$$

$$= \omega(f;\lambda)$$
$$+ \frac{\omega(f;\lambda)}{\lambda}\left\{\sum_{k=0}^{n+s}\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n - x\right)^2 p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k \prod_{j=0}^{n+s-k-1}\left(p^j - q^j\frac{x}{b_n}\right)\right\}^{1/2}$$
$$= \omega(f;\lambda) + \frac{\omega(f;\lambda)}{\lambda}\left\{\bar{C}_{n,s}\left((t-x)^2\right);p,q;x\right\}^{1/2}.$$

Now, choosing $\lambda_{n,q}(x)$ same as in Theorem 4, then we have

$$\left|\bar{C}_{n,s}(f;p,q;x) - f(x)\right| \leq 2\omega\left(f;\sqrt{\lambda_{n,p,q}(x)}\right).$$

5 Generalization of the Chlodowsky Variant of q-Bernstein-Schurer-Stancu Operators

In this section, we introduce generalization of Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers and this provides us to obtain approximate continuous functions on more general weighted spaces. For $x \geq 0$, consider any continuous function $\omega(x) \geq 1$ and define

$$G_f(t) = f(t)\frac{1+t^2}{\omega(t)}.$$

Let us consider the generalization of the $\bar{C}_{n,s}(f;p,q;x)$ as follows

$$L_n(f;p,q;x)$$
$$= \frac{\omega(x)}{1+x^2}\sum_{k=0}^{n+s} G_f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\right)p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k \prod_{j=0}^{n+s-k-1}\left(p^j - q^j\frac{x}{b_n}\right),$$

where $0 \leq x \leq b_n$ and (b_n) has the same properties of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators.

Theorem 6 *For the continuous functions satisfying*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)} = K_f < \infty,$$

we have

$$\lim_{n\to\infty}\sup_{0\leq x\leq b_n}\frac{|L_n(f;p,q;x) - f(x)|}{\omega(x)} = 0.$$

Proof Clearly,

$$L_n(f;p,q;x) - f(x)$$
$$= \frac{\omega(x)}{1+x^2}\left(\sum_{k=0}^{n+p} G_f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\right)p^{\frac{k(k-1)}{2}-\frac{(n+s-1)(n+s)}{2}}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k\right.$$
$$\left.\times \prod_{j=0}^{n+s-k-1}\left(p^j - q^j\frac{x}{b_n}\right) - G_f(x)\right)$$

thus

$$\sup_{0\leq x\leq b_n}\frac{|L_n(f;p,q;x)-f(x)|}{\omega(x)}=\sup_{0\leq x\leq b_n}\frac{\left|\bar{C}_{n,s}\left(G_f;p,q;x\right)-G_f(x)\right|}{1+x^2}.$$

By using $|f(x)|\leq M_f\omega(x)$ and continuity of the function f, we get that $\left|G_f(x)\right|\leq M_f\left(1+x^2\right)$ for $x\geq 0$ and $G_f(x)$ is continuous function on $[0,\infty)$. Thus, from the Theorem 1 we get the result.

Finally note that, the operators $L_n(f;p,q;x)$ reduces to $\bar{C}_{n,s}\left(G_f;p,q;x\right)$ by taking $\omega(x)=1+x^2$.

6 Numerical Results and Discussions

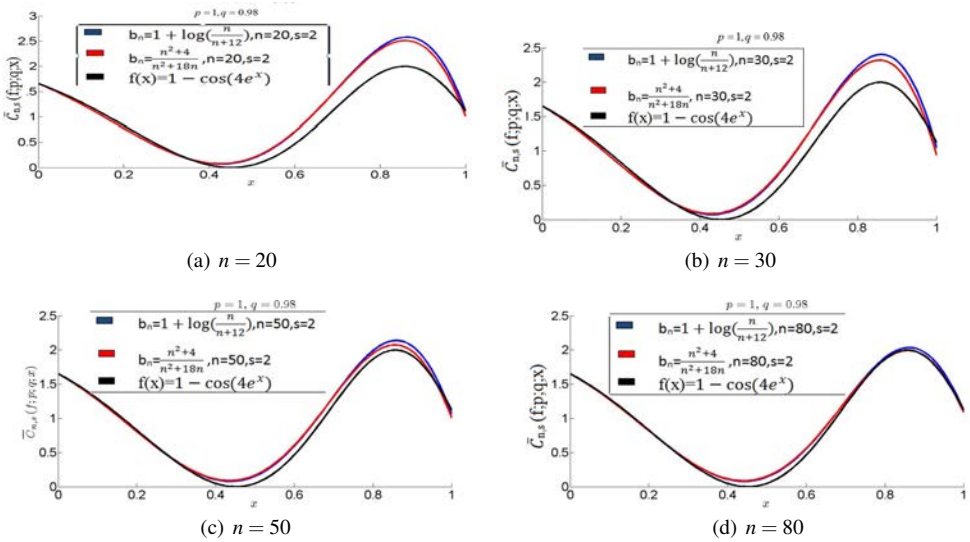
In order to show the effectiveness and accuracy of $\bar{C}_{n,s}(f;p,q;x)$ to f(x) with different values of parameters, numerical results are presented in this section. Sensitivity analysis is carried out to minimise the error of approximation of $\bar{C}_{n,s}(f;p,q;x)$ to the function $f(x)=1-\cos(4e^x)$ for minimum n and s values by taken into account different p and q values.

In Figure 1, $\bar{C}_{n,s}(f;p,q;x)$ results are given as a function of x for different n values. To minimise the error of the approximation of $\bar{C}_{n,s}(f;p,q;x)$ to $f(x)$, two different sequences $b_n=1+\log(\frac{n}{n+12})$ and $b_n=\frac{n^2+4}{n^2+18n}$ are considered respectively. It is evident that the $\bar{C}_{n,s}(f;p;q;x)$ converges to $f(x)$ for both sequences as the value of q and p approaches towards 1, while $s=2$. However, using $b_n=1+\log(\frac{n}{n+12})$ for $\bar{C}_{n,s}(f;p,q;x)$ rather than $b_n=\frac{n^2+4}{n^2+18n}$ results better approximation results. Therefore, the effect of increasing n further than 20 is less evident for $x<0.5$ for the convergence of $\bar{C}_{n,s}(f;p,q;x)$ to the function $f(x)$. On the other hand, it is required to increase the value of n further than 50 for $x>0.5$ in order to have more accurate results for each sequences. Comparative results are given in *Table* 1 and *Table* 2, for the errors of the approximation of $\bar{C}_{n,s}(f;p,q;x)$ to $f(x)$, considering each sequences for different n values.

x	$f(x)$	$\left\|f(x)-\bar{C}_{20,2}(f;p,q;x)\right\|$	$\left\|f(x)-\bar{C}_{30,2}(f;p,q;x)\right\|$	$\left\|f(x)-\bar{C}_{50,2}(f;p,q;x)\right\|$	$\left\|f(x)-\bar{C}_{80,2}(f;p,q;x)\right\|$
0.1	1.2876	0.0460	0.0310	0.0190	0.0125
0.2	0.8276	0.0720	0.0432	0.0207	0.0090
0.3	0.3657	0.0543	0.0200	0.0049	0.0172
0.4	0.0494	0.0180	0.0411	0.0544	0.0595
0.5	0.0482	0.1369	0.1224	0.1050	0.0933
0.6	0.4642	0.2913	0.1998	0.1280	0.0898
0.7	1.1997	0.4942	0.2723	0.1210	0.0489
0.8	1.8665	0.7628	0.3627	0.1242	0.0224
0.9	1.9157	0.9275	0.3925	0.1406	0.0521

Table 1 Errors of approximation $\bar{C}_{n,s}(f;p,q;x)$ to $f(x)$ $\left(s=2,b_n=\frac{n^2+4}{n^2+18n},p=1,q=0.98\right)$

Figure 2 demonstrates the convergence of $\bar{C}_{n,s}(f;p,q;x)$ to $f(x)$ but this time considering different p and q values, when $n=50$ for $bn=1+\log(\frac{n}{n+12})$. In Figures 2(a), and 2(b), as q values are increased, the errors of the approximation of $\bar{C}_{n,s}(f;p,q;x)$ to $f(x)$ is



(a) $n = 20$ (b) $n = 30$

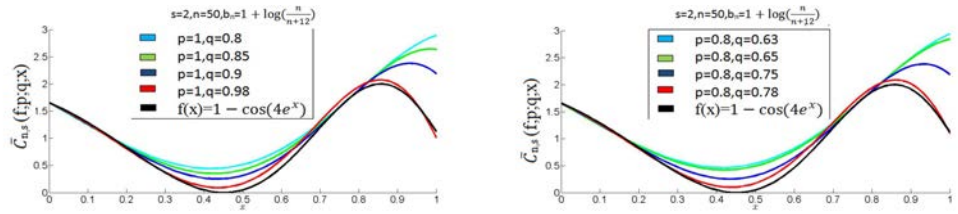
(c) $n = 50$ (d) $n = 80$

Fig. 1 Convergence of $\bar{C}_{n,s}(f;p,q;x)$ for different n values

x	$f(x)$	$\lvert f(x)-\bar{C}_{20,2}(f;p,q;x)\rvert$	$\lvert f(x)-\bar{C}_{30,2}(f;p,q;x)\rvert$	$\lvert f(x)-\bar{C}_{50,2}(f;p,q;x)\rvert$	$\lvert f(x)-\bar{C}_{80,2}(f;p,q;x)\rvert$
0.1	1.2876	0.0459	0.0317	0.0196	0.0130
0.2	0.8276	0.0721	0.0427	0.0204	0.0089
0.3	0.3657	0.0546	0.0156	0.0088	0.0198
0.4	0.0494	0.0173	0.0520	0.0644	0.0663
0.5	0.0482	0.1358	0.1390	0.1203	0.1040
0.6	0.4642	0.2903	0.2187	0.1412	0.0993
0.7	1.1997	0.4947	0.2650	0.1157	0.0461
0.8	1.8665	0.7665	0.3110	0.0825	0.0045
0.9	1.9157	0.9361	0.2897	0.0640	0.0038

Table 2 Errors of approximation $\bar{C}_{n,s}(f;p,q;x)$ to $f(x)$ $\left(s=2, b_n=1+\log(\frac{n}{n+12}), p=1, q=0.98\right)$

minimised for $x < 0.5$ and $x > 0.8$ for any given p values. On the other hand, the results show that decreasing q values has an effect on the convergence of $\bar{C}_{n,s}(f;p,q;x)$ which provide better approximate results for $x > 0.6$ and $x < 0.8$ (See *Table* 2).



(a) $p = 1$ (b) $p = 0.8$

Fig. 2 Convergence of $\bar{C}_{n,s}(f;p,q;x)$ for different p and q values

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Dynamical behavior of a general HIV-1 infection model with HAART and cellular reservoirs

Ahmed M. Elaiw, Abdullah M. Althiabi, Mohammed A. Alghamdi and Nicola Bellomo
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email: a_m_elaiw@yahoo.com (A. Elaiw)

Abstract

This paper study the dynamical behavior of a general HIV-1 infection model under the effect of Highly active antiretroviral therapies (HAART). The model includes three types of infected cells (i) long-lived productively infected cells which live for long time and creat small amount of HIV-1 particles, (ii) latently infected cells which do not creat HIV-1 until they have been activated (iii) short-lived productively infected cells which live for long time and creat large amount of HIV-1. The model incorporates humoral immune response and general nonlinear forms for the incidence rate of infection, the generation and removal rates of all compartments. The nonnegativity and boundedness of the solutions of the model as well as global stability of the steady states are studied. The global stability are established using Lyapunov method. Using MATLAB we conduct some numerical simulations to confirm our results.

Keywords: HIV-1 infection; HAART; global stability; humoral immune response; latency; viral reservoirs

1 Introduction

Human immunodeficiency virus type 1 (HIV-1) infects the $CD4^+$ T cells which play the central role in the immune system of the human body. HIV-1 causes gradual depletion in the concentration of the uninfected $CD4^+$ T cells which decreases the efficiency of the immune system against other infections. During the last decades, substantial efforts have been paid to propose treatment strategies for HIV-1 [1], [2]. Highly active antiretroviral therapies (HAART) which combines two classes of antiviral drugs, reverse transcriptase inhibitor (RTI) and protease inhibitor (PI), can rapidly decrease the concentration of the HIV-1 and increase the concentration of the healthy $CD4^+$ T cells in the plasma. However, HAART can not eradicate the HIV-1 completely due to the presence of viral reservoirs such as latently infected cells. Mathematical modeling and analysis of the dynamics of HIV-1 are helpful in understanding the virus dynamics and improving diagnosis and treatment strategies [3]-[23]. Modeling the HIV-1 dynamics with latent infection has been studied by several researchers [24]-[29]. The HIV-1 dynamics model with latently infected cells consists of four compartments: uninfected $CD4^+$ T cells, latently infected cells, actively infected cells and free HIV-1 particles [24].

$$\dot{x} = \rho - dx - (1-\varepsilon_r)\beta x v, \tag{1}$$

$$\dot{w} = h(1-\varepsilon_r)\beta x v - (a_1+\delta_1)w, \tag{2}$$

$$\dot{y} = (1-h)(1-\varepsilon_r)\beta x v + a_1 w - \delta_2 y, \tag{3}$$

$$\dot{v} = \overline{N}\delta_2 y - \delta_4 v, \tag{4}$$

where x, w, y, v represent the concentrations of the uninfected $CD4^+$ T cells, latently infected cells, actively infected cells and free HIV-1 particles, respectively. $\rho > 0$ is the replenished rate of uninfected $CD4^+$ T cells from

body's sources such as bone marrow and thymus. The parameters d, δ_1, δ_2 and δ_4 are the death rate constants of the uninfected CD4$^+$ T cells, latently infected cells, actively infected cells and HIV-1 particles, respectively. The uninfected CD4$^+$ T cells become infected by viruses with infectivity β. The efficacy of the RTI drugs is given by ε_r, where $\varepsilon_r \in [0,1]$. The latently infected cells are activated at rate $a_1 w$. The parameter $\overline{N}$ is the average number of HIV-1 particles generated in the lifetime of the actively infected. A fraction $h \in (0,1)$ of infection events result in latent infection. The global stability analysis of model (1)-(4) has been studied by Wang et al. in [26].

As reported in [24], [2] and [30] three are two types of productively infected cells, the first is short-lived productively infected cells which live for short time and produce hight numbers of HIV-1 particles, and the second is the long-lived productively infected cells which live for long time and produce small numbers of HIV-1 particles. Long-lived productively infected cells can be seen as another reservoirs which a major obstacle to eliminate the HIV-1 completely by HAART. Model (1)-(4) has been modified by including: (i) mitotic proliferation of the uninfected CD4$^+$ T cells, (ii) three types of infected cells, latently infected cells (w), short-lived productively infected cells (y), and long-lived productively infected cells (u) [30].

$$\dot{x} = \rho - dx + px\left(1 - \frac{x}{x_{\max}}\right) - (1-\varepsilon_r)(\overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3)xv, \tag{5}$$

$$\dot{w} = (1-\varepsilon_r)\overline{\beta}_1 xv - (a_1 + \delta_1)w, \tag{6}$$

$$\dot{y} = (1-\varepsilon_r)\overline{\beta}_2 xv + a_1 w - \delta_2 y, \tag{7}$$

$$\dot{u} = (1-\varepsilon_r)\overline{\beta}_3 xv - \delta_3 u, \tag{8}$$

$$\dot{v} = (1-\varepsilon_p)\overline{N}\delta_2 y + (1-\varepsilon_p)\overline{M}\delta_3 u - \delta_4 v. \tag{9}$$

Uninfected CD4$^+$ T cells can be produced by proliferation of existing healthy cells in the body. The parameter $p > 0$ is the maximum proliferation rate of uninfected cells. The parameter $x_{\max} > 0$ is the maximum level of uninfected cell concentration in the body. If the concentration arrives at $x_{\max}$, it should decreases. The parameters δ_2 and δ_3 are the death rate constants of the short-lived productively infected cells and long-lived productively infected cells, respectively. The uninfected CD4$^+$ T cells become infected by viruses with infectivity $\overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3$. The efficacy of the PI drugs is given by ε_p, where $\varepsilon_p \in [0,1]$. The parameter $\overline{M}$ is the average number of HIV-1 particles generated in the lifetime of the long-lived productively infected cells.

In model (5)-(9) we note the following (i) the infection rate is given by bilinear incidence which may not describe the virus dynamics accurately, (ii) the death rate of all compartments, the production rate of viruses and the latent-to-active transmission rate are given by linear functions, however, these rates are generally not known, (iii) the effect of immune response has been neglecting. The aim of this paper is to propose an HIV-1 infection model which improves model (5)-(9) by taking into account the humoral immune response and by assuming that the intrinsic growth rate of uninfected CD4$^+$ T cells as well as the death rate of HIV-1 and infected cells are given by general nonlinear functions. We study the qualitative behavior of the proposed model. The existence and global stability of all the steady states of the model is established. Lyapunov functionals and LaSalle's invariance principle are used to prove the global stability of the model.

2 Mathematical HIV-1 dynamics model

Based on the above discussion we formulate a general nonlinear HIV dynamics model with humoral immunity. The model can be considered as a generalization of several existing HIV-1 models.

$$\dot{x} = \pi(x) - (1-\varepsilon_r)(\overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3)\xi(x,v), \tag{10}$$

$$\dot{w} = (1-\varepsilon_r)\overline{\beta}_1\xi(x,v) - (a_1+\delta_1)g_1(w), \tag{11}$$

$$\dot{y} = (1-\varepsilon_r)\overline{\beta}_2\xi(x,v) + a_1 g_1(w) - \delta_2 g_2(y), \tag{12}$$

$$\dot{u} = (1-\varepsilon_r)\overline{\beta}_3\xi(x,v) - \delta_3 g_3(u), \tag{13}$$

$$\dot{v} = (1-\varepsilon_p)\overline{N}\delta_2 g_2(y) + (1-\varepsilon_p)\overline{M}\delta_3 g_3(u) - \delta_4 g_4(v) - q g_4(v) g_5(z), \tag{14}$$

$$\dot{z} = r g_4(v) g_5(z) - \delta_5 g_5(z), \tag{15}$$

where z represents the concentration of the B cells. Function $\pi(x)$ represents the intrinsic growth rate of uninfected CD4$^+$ T cells accounting for both production and natular mortality. The viruses are neutralized at rate $qg_5(z)g_4(v)$ and die at rate $\delta_4 g_4(v)$, where q and δ_4 are positive constants. The B cells are activated at rate $rg_5(z)g_4(v)$ and die at rate $\delta_5 g_5(z)$. All the parameters are positive. Let us define $\beta_i = (1-\varepsilon_r)\overline{\beta}_i$, $i = 1,2,3$, $N = (1-\varepsilon_p)\overline{N}$ and $M = (1-\varepsilon_p)\overline{M}$. Functions π, ξ, g_i, $i = 1,...,5$ are continuously differentiable, moreover, they satisfy some hypotheses:

(H1). (i) there exists x_0 such that $\pi(x_0) = 0$, $\pi(x) > 0$ for $x \in [0, x_0)$,
(ii) $\pi'(x) < 0$ for $x \in (0,\infty)$,
(iii) there are $b > 0$ and $\overline{b} > 0$ such that $\pi(x) \le b - \overline{b}x$ for $x \in [0,\infty)$.

(H2). (i) $\xi(x,v) > 0$ and $\xi(0,v) = \xi(x,0) = 0$ for $x, v \in (0,\infty)$,
(ii) $\frac{\partial\xi(x,v)}{\partial x} > 0$, $\frac{\partial\xi(x,v)}{\partial v} > 0$ and $\frac{\partial\xi(x,0)}{\partial v} > 0$ for all $x, v \in (0,\infty)$,
(iii) $\left(\dfrac{\partial\xi(x,0)}{\partial v}\right)' > 0$ for $x \in (0,\infty)$.

(H3). (i) $g_j(\eta) > 0$ for $\eta \in (0,\infty)$, $g_j(0) = 0$, $j = 1,...,5$
(ii) $g_j'(\eta) > 0$ for $\eta \in (0,\infty)$, $j = 1,2,3,5$, $g_4'(\eta) > 0$, for $\eta \in [0,\infty)$,
(iii) there are $\alpha_j > 0$, $j = 1,...,5$ such that $g_j(\eta) \ge \alpha_j\eta$ for $\eta \in [0,\infty)$.

(H4). $\frac{\partial}{\partial v}\left(\dfrac{\xi(x,v)}{g_4(v)}\right) \le 0$ for $v \in (0,\infty)$.

3 Basic properties

In this section we study the basic properties of model (10)-(15). The non-negativity and boundedness of the solutions of the model will be established in the next theorem:

Theorem 1. Let Hypotheses (H1)-(H3) be hold true, then there exist a set

$$\Delta = \left\{(x,w,y,u,v,z) \in \mathbb{R}^6_{\ge 0} : 0 \le x,w,y,u \le \kappa_1, 0 \le v \le \kappa_2, 0 \le z \le \kappa_3\right\}$$

which is positively invariant with respect to system (10)-(15), where κ_1, κ_2 and κ_3 are positive numbers.

Proof. First, we show that $\mathbb{R}^6_{\ge 0}$ is positively invariant for system (10)-(15) as:

$$\begin{aligned}
&\dot{x}\mid_{x=0} = \pi(0) > 0,\\
&\dot{w}\mid_{w=0} = \beta_1\xi(x,v) \ge 0 && \text{for} \quad x,v \in [0,\infty),\\
&\dot{y}\mid_{y=0} = \beta_2\xi(x,v) + a_1 g_1(w) \ge 0 && \text{for} \quad x,w,v \in [0,\infty),\\
&\dot{u}\mid_{u=0} = \beta_3\xi(x,v) \ge 0 && \text{for} \quad x,v \in [0,\infty),\\
&\dot{v}\mid_{v=0} = N\delta_2 g_2(y) + M\delta_3 g_3(u) \ge 0 && \text{for } y,u \in [0,\infty),\\
&\dot{z}\mid_{z=0} = 0.
\end{aligned}$$

Hence, all the state variables of system (10)-(15) are non-negative.

Let $T_1(t) = x(t) + w(t) + y(t) + u(t)$, then

$$\begin{aligned}\dot{T}_1 &= \pi(x) - \delta_1 g_1(w) - \delta_2 g_2(y) - \delta_3 g_3(u)\\ &\leq b - \bar{b}x - \delta_1\alpha_1 w - \delta_2\alpha_2 y - \delta_3\alpha_3 u.\\ &\leq b - \sigma_1\left(x + w + y + u\right) = b - \sigma_1 T_1,\end{aligned}$$

where $\sigma_1 = \min\{\bar{b}, \delta_1\alpha_1, \delta_2\alpha_2, \delta_3\alpha_3\}$. Hence, $T_1(t) \leq \kappa_1$, if $T_1(0) \leq \kappa_1$, where $\kappa_1 = \dfrac{b}{\sigma_1}$. The non-negativity of $x(t)$, $w(t)$, $y(t)$ and $u(t)$ implies $0 \leq x(t), w(t), y(t), u(t) \leq \kappa_1$, if $0 \leq x(0) + w(0) + y(0) + u(0) \leq \kappa_1$. Moreover, we let $T_2(t) = v(t) + \frac{q}{r}z(t)$. Then

$$\begin{aligned}\dot{T}_2 &= N\delta_2 g_2(y) + M\delta_3 g_3(u) - \delta_4 g_4(v) - \frac{q\delta_5}{r}g_5(z)\\ &\leq N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) - \delta_4\alpha_4 v - \frac{q\delta_5}{r}\alpha_5 z\\ &\leq N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) - \sigma_2 T_2,\end{aligned}$$

where $\sigma_2 = \min\{\delta_4\alpha_4, \delta_5\alpha_5\}$. Hence, $T_2(t) \leq \kappa_2$ if $T_2(0) \leq \kappa_2$, where $\kappa_2 = \dfrac{N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1)}{\sigma_2}$. The non-negativity of $v(t)$ and $z(t)$ implies $0 \leq v(t) \leq \kappa_2$ and $0 \leq z(t) \leq \kappa_3$ if $0 \leq v(0) + \frac{q}{r}z(0) \leq \kappa_2$, where $\kappa_3 = \dfrac{r\kappa_2}{q}$.

Theorem 2. Suppose that Hypotheses (H1)-(H4) are valid, then there exist two bifurcation parameters $\mathcal{R}_0$ and $\mathcal{R}_1$ with $\mathcal{R}_0 > \mathcal{R}_1 > 0$ such that

(i) if $\mathcal{R}_0 \leq 1$, then the system has only one positive steady state $S_0 \in \Delta$,

(ii) if $\mathcal{R}_1 \leq 1 < \mathcal{R}_0$, then the system has only two positive steady states $S_0 \in \Delta$ and $S_1 \in \Delta$,

(iii) if $\mathcal{R}_1 > 1$, then the system has three positive steady states $S_0 \in \Delta$, $S_1 \in \Delta$ and $S_2 \in \overset{\circ}{\Delta}$.

Proof. Let $S(x, w, y, u, v, z)$ be any steady state of (10)-(15) satisfying the following equations:

$$0 = \pi(x) - (\beta_1 + \beta_2 + \beta_3)\xi(x, v), \tag{16}$$

$$0 = \beta_1\xi(x, v) - (a_1 + \delta_1)g_1(w), \tag{17}$$

$$0 = \beta_2\xi(x, v) + a_1 g_1(w) - \delta_2 g_2(y), \tag{18}$$

$$0 = \beta_3\xi(x, v) - \delta_3 g_3(u), \tag{19}$$

$$0 = N\delta_2 g_2(y) + M\delta_3 g_3(u) - \delta_4 g_4(v) - q g_4(v) g_5(z), \tag{20}$$

$$0 = r g_4(v) g_5(z) - \delta_5 g_5(z). \tag{21}$$

From Eq. (21) we have two possible solutions, $g_5(z) = 0$ and $g_4(v) = \delta_5/r$. Let us consider the case $g_5(z) = 0$, then from Hypothese (H3) we get $z = 0$. Hypothese (H3) implies that g_i^{-1}, $i = 1, ..., 5$ exist, strictly increasing and $g_i^{-1}(0) = 0$. Let us define

$$\begin{aligned}\theta(x) &= g_1^{-1}\left(\frac{\beta_1}{\beta(a_1 + \delta_1)}\pi(x)\right), \qquad \psi(x) = g_2^{-1}\left(\frac{a_1\beta_1 + (a_1 + \delta_1)\beta_2}{\delta_2\beta(a_1 + \delta_1)}\pi(x)\right),\\ \mu(x) &= g_3^{-1}\left(\frac{\beta_3}{\delta_3\beta}\pi(x)\right), \qquad\qquad \ell(x) = g_4^{-1}\left(\frac{\gamma}{\beta}\pi(x)\right),\end{aligned} \tag{22}$$

where $\beta = \beta_1 + \beta_2 + \beta_3$ and $\gamma = \frac{N(a_1\beta_1 + (a_1 + \delta_1)\beta_2) + M\beta_3(a_1 + \delta_1)}{\delta_4(a_1 + \delta_1)}$. It follows from Eqs. (16)-(21) that:

$$w = \theta(x), \quad y = \psi(x), \quad u = \mu(x), \quad v = \ell(x). \tag{23}$$

Obviousely, $\theta(x), \psi(x), \mu(x), \ell(x) > 0$ for $x \in [0, x_0)$ and $\theta(x_0) = \psi(x_0) = \mu(x_0) = \ell(x_0) = 0$. From Eqs. (16), (22) and (23) we obtain

$$\gamma\xi(x, \ell(x)) - g_4(\ell(x)) = 0. \tag{24}$$

Eq. (24) admits a solution $x = x_0$ which gives the infection-free steady state $S_0(x_0, 0, 0, 0, 0, 0)$. Let

$$\Psi_1(x) = \gamma\xi(x, \ell(x)) - g_4(\ell(x)) = 0.$$

It is clear from Hypotheses (H1) and (H2) that, $\Psi_1(0) = -g_4(\ell(0)) < 0$ and $\Psi_1(x_0) = 0$. Moreover,

$$\Psi_1'(x_0) = \gamma\left[\frac{\partial\xi(x_0,0)}{\partial x} + \ell'(x_0)\frac{\partial\xi(x_0,0)}{\partial v}\right] - g_4'(0)\ell'(x_0).$$

We note from Hypothese (H2) that $\frac{\partial\xi(x_0,0)}{\partial x} = 0$. Then,

$$\Psi_1'(x_0) = \ell'(x_0)g_4'(0)\left(\frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_0,0)}{\partial v} - 1\right).$$

From Eq. (22), we get

$$\Psi_1'(x_0) = \frac{\gamma}{\beta}\pi'(x_0)\left(\frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_0,0)}{\partial v} - 1\right).$$

Therefore, from Hypothese (H1), we have $\pi'(x_0) < 0$. Therefore, if $\frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_0,0)}{\partial v} > 1$, then $\Psi_1'(x_0) < 0$ and there exists $x_1 \in (0, x_0)$ such that $\Psi_1(x_1) = 0$. Hypotheses (H1)-(H3) imply that

$$w_1 = \theta(x_1) > 0, \quad y_1 = \psi(x_1) > 0, \quad u_1 = \mu(x_1) > 0, \quad v_1 = \ell(x_1) > 0. \tag{25}$$

It means that, a humoral-inactivated infection steady state $S_1(x_1, w_1, y_1, u_1, v_1, 0)$ exists when $\frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_0,0)}{\partial v} > 1$. Let us define

$$\mathcal{R}_0 = \frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_0,0)}{\partial v},$$

The other solution of Eq. (21) is $g_4(v_2) = \dfrac{\delta_5}{r}$ which yields $v_2 = g_4^{-1}\left(\dfrac{\delta_5}{r}\right) > 0$. Substituting $v = v_2$ in Eq. (16) and letting $\Psi_2(x) = \pi(x) - \beta\xi(x, v_2) = 0$. According to Hypotheses (H1) and (H2), Ψ_2 is strictly decreasing, $\Psi_2(0) = \pi(0) > 0$ and $\Psi_2(x_0) = -\beta\xi(x_0, v_2) < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_2(x_2) = 0$. It follows from Eqs. (20) and (23) that,

$$w_2 = \theta(x_2) > 0, \quad y_2 = \psi(x_2) > 0, \quad u_2 = \mu(x_2) > 0, \quad v_2 = g_4^{-1}\left(\frac{\delta_5}{r}\right) > 0,$$
$$z_2 = g_5^{-1}\left(\frac{\delta_4}{q}\left(\gamma\frac{\xi(x_2, v_2)}{g_4(v_2)} - 1\right)\right).$$

Thus, $z_2 > 0$ when $\gamma\frac{\xi(x_2,v_2)}{g_4(v_2)} > 1$. Now we define the paramater $\mathcal{R}_1$ as:

$$\mathcal{R}_1 = \gamma\frac{\xi(x_2, v_2)}{g_4(v_2)}.$$

If $\mathcal{R}_1 > 1$, then $z_2 = g_5^{-1}\left(\frac{\delta_4}{q}(\mathcal{R}_1 - 1)\right) > 0$ and exists a humoral-activated infection steady state $S_2(x_2, w_2, y_2, u_2, v_2, z_2)$.

Now we show that $S_0 \in \Delta$, $S_1 \in \Delta$ and $S_2 \in \overset{\circ}{\Delta}$. Clearly, $S_0 \in \Delta$. Now we show that $S_1 \in \Delta$. We have $x_1 \in (0, x_0)$, then from Hypothese (H1) we obtain

$$0 = \pi(x_0) < \pi(x_1) \leq b - \bar{b}x_1.$$

It follows that

$$0 < x_1 < \frac{b}{\bar{b}} \leq \frac{b}{\sigma_1} = \kappa_1.$$

From Hypothese (H3) and Eqs. (22)-(23), we get the following:

$$\begin{aligned}
\alpha_1 w_1 \leq g_1(w_1) &= \frac{\beta_1}{\beta(a_1+\delta_1)}\pi(x_1) < \frac{\beta_1}{\beta(a_1+\delta_1)}\pi(0) < \frac{\beta_1 b}{\beta(a_1+\delta_1)} \\
&\Rightarrow 0 < w_1 < \frac{\beta_1 b}{\beta\alpha_1(a_1+\delta_1)} < \frac{\beta_1 b}{\beta\alpha_1\delta_1} < \frac{b}{\alpha_1\delta_1} \leq \kappa_1, \\
\alpha_2 y_1 \leq g_2(y_1) &= \frac{a_1\beta_1 + (a_1+\delta_1)\beta_2}{\delta_2\beta(a_1+\delta_1)}\pi(x_1) < \frac{a_1\beta_1 + (a_1+\delta_1)\beta_2}{\delta_2\beta(a_1+\delta_1)}\pi(0) \\
&\leq \frac{(a_1\beta_1 + (a_1+\delta_1)\beta_2)b}{\delta_2\beta(a_1+\delta_1)} < \frac{(a_1+\delta_1)(\beta_1+\beta_2)b}{\delta_2\beta(a_1+\delta_1)} < \frac{b}{\delta_2} \\
&\Rightarrow 0 < y_1 < \frac{b}{\alpha_2\delta_2} \leq \kappa_1, \\
\alpha_3 u_1 \leq g_3(u_1) &= \frac{\beta_3}{\delta_3\beta}\pi(x_1) < \frac{\beta_3}{\delta_3\beta}\pi(0) < \frac{\beta_3 b}{\delta_3\beta} < \frac{b}{\delta_3} \\
&\Rightarrow 0 < u_1 < \frac{b}{\alpha_3\delta_3} \leq \kappa_1,
\end{aligned}$$

Eq. (20) implies that

$$\begin{aligned}
\delta_4\alpha_4 v_1 \leq \delta_4 g_4(v_1) &= N\delta_2 g_2(y_1) + M\delta_3 g_3(u_1) < N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) \\
&\Rightarrow 0 < v_1 < \frac{N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1)}{\delta_4\alpha_4} \leq \kappa_2.
\end{aligned}$$

We have $z_1 = 0$ then, $S_1 \in \Delta$.

It is clear that $0 < x_2, w_2, y_2, u_2 < \kappa_1$. Next we show that $0 < v_2 < \kappa_2$ and $0 < z_2 < \kappa_3$ when $\mathcal{R}_1 > 1$. From the steady state conditions of S_2 we have,

$$\delta_4 g_4(v_2) + q g_4(v_2) g_5(z_2) = N\delta_2 g_2(y_2) + M\delta_3 g_3(u_2).$$

Then if $\mathcal{R}_1 > 1$ we get

$$\begin{aligned}
\delta_4 g_4(v_2) &< N\delta_2 g_2(y_2) + M\delta_3 g_3(u_2) \\
&\Rightarrow \delta_4\alpha_4 v_2 < N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) \\
&\Rightarrow 0 < v_2 < \frac{N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1)}{\delta_4\alpha_4} \leq \kappa_2,
\end{aligned}$$

and

$$\begin{aligned}
q g_4(v_2) g_5(z_2) &\leq N\delta_2 g_2(y_2) + M\delta_3 g_3(u_2) \\
&\Rightarrow \frac{q\delta_5}{r}\alpha_5 z_2 \leq N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1) \\
&\Rightarrow 0 < z_2 < \frac{r\left[N\delta_2 g_2(\kappa_1) + M\delta_3 g_3(\kappa_1)\right]}{q\delta_5\alpha_5} \leq \kappa_3.
\end{aligned}$$

Then, $S_2 \in \overset{\circ}{\Delta}$. Clearly from Hypotheses (H2) and (H4), we have

$$\mathcal{R}_1 = \gamma\frac{\xi(x_2, v_2)}{g_4(v_2)} \leq \gamma \lim_{v\to 0^+}\frac{\xi(x_2, v)}{g_4(v)} = \frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_2, 0)}{\partial v} < \frac{\gamma}{g_4'(0)}\frac{\partial\xi(x_0, 0)}{\partial v} = \mathcal{R}_0. \ \square$$

4 Global properties

In this section, we established the global stability of the three steady states by of system (10)-(15) by constructing suitable Lyapunov functions.

Theorem 3. Suppose that Hypotheses (H1)-(H4) are valid and $\mathcal{R}_0 \leq 1$, then S_0 is globally asymptotically stable in Δ.

Proof. Define

$$W_0 = x - x_0 - \int_{x_0}^{x} \lim_{v\to 0^+} \frac{\xi(x_0,v)}{\xi(\eta,v)} d\eta + k_1 w + k_2 y + k_3 u + k_4 v + k_5 z, \tag{26}$$

where

$$\begin{aligned} \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 = \beta, \qquad & (a_1+\delta_1)k_1 = a_1 k_2, \\ k_2 = N k_4, \qquad k_3 = M k_4, \qquad & q k_4 = r k_5. \end{aligned} \tag{27}$$

The solution of Eqs. (27) is given by

$$k_1 = \frac{a_1 N \beta}{\gamma \delta_4 (a_1+\delta_1)},\ k_2 = \frac{N\beta}{\gamma\delta_4},\ k_3 = \frac{M\beta}{\gamma\delta_4},\ k_4 = \frac{\beta}{\gamma\delta_4},\ k_5 = \frac{q\beta}{r\gamma\delta_4}. \tag{28}$$

We evaluate $\frac{dW_0}{dt}$ along the solutions of (10)-(15) as:

$$\begin{aligned} \frac{dW_0}{dt} = & \left(1 - \lim_{v\to 0^+} \frac{\xi(x_0,v)}{\xi(x,v)}\right)(\pi(x) - \beta\xi(x,v)) \\ & + k_1\left(\beta_1\xi(x,v) - (a_1+\delta_1)g_1(w)\right) + k_2\left(\beta_2\xi(x,v) + a_1 g_1(w) - \delta_2 g_2(y)\right) \\ & + k_3\left(\beta_3\xi(x,v) - \delta_3 g_3(u)\right) + k_4\left(N\delta_2 g_2(y) + M\delta_3 g_3(u) - \delta_4 g_4(v) - q g_4(v) g_5(z)\right) \\ & + k_5(r g_4(v) g_5(z) - \delta_5 g_5(z)). \end{aligned} \tag{29}$$

Collecting terms of Eq. (29) and using $\pi(x_0) = 0$, we obtian

$$\begin{aligned} \frac{dW_0}{dt} &= (\pi(x) - \pi(x_0))\left(1 - \lim_{v\to 0^+}\frac{\xi(x_0,v)}{\xi(x,v)}\right) + \left(\frac{\beta\xi(x,v)}{g_4(v)}\lim_{v\to 0^+}\frac{\xi(x_0,v)}{\xi(x,v)} - k_4\delta_4\right) g_4(v) - k_5\delta_5 g_5(z) \\ &\leq (\pi(x) - \pi(x_0))\left(1 - \lim_{v\to 0^+}\frac{\xi(x_0,v)}{\xi(x,v)}\right) + \left(\lim_{v\to 0^+}\frac{\beta\xi(x,v)}{g_4(v)}\lim_{v\to 0^+}\frac{\xi(x_0,v)}{\xi(x,v)} - k_4\delta_4\right) g_4(v) - k_5\delta_5 g_5(z) \\ &= (\pi(x) - \pi(x_0))\left(1 - \frac{\partial\xi(x_0,0)/\partial v}{\partial\xi(x,0)/\partial v}\right) + k_4\delta_4\left(\frac{\beta}{k_4\delta_4 g_4'(0)}\frac{\partial\xi(x_0,0)}{\partial v} - 1\right) g_4(v) - k_5\delta_5 g_5(z) \\ &= (\pi(x) - \pi(x_0))\left(1 - \frac{\partial\xi(x_0,0)/\partial v}{\partial\xi(x,0)/\partial v}\right) + k_4\delta_4(\mathcal{R}_0 - 1) g_4(v) - k_5\delta_5 g_5(z). \end{aligned} \tag{30}$$

From Hypotheses (H1) and (H2), we have

$$(\pi(x) - \pi(x_0))\left(1 - \frac{\partial\xi(x_0,0,0)/\partial v}{\partial\xi(x,0,0)/\partial v}\right) \leq 0.$$

Therefore, if $\mathcal{R}_0 \leq 1$, then $\frac{dW_0}{dt} \leq 0$ for $x, v, z \in (0,\infty)$. Moreover, $\frac{dW_0}{dt} = 0$ if and only if $x(t) = x_0$, $v(t) = 0$ and $z(t) = 0$ for all t. It easy to show that, the largest invariant set $\Gamma_0 \subseteq \Gamma = \left\{(x,w,y,u,v,z) : \frac{dW_0}{dt} = 0\right\}$ is the singlton $\{S_0\}$ [31]. LaSalle's invariance principle provide that S_0 is globally asymptotically stable. □

Lemma 1. Let $\mathcal{R}_0 > 1$ and Hypotheses (H1)-(H4) are satisfied, then

$$sgn(\mathcal{R}_1 - 1) = sgn(v_1 - v_2) = sgn(x_2 - x_1).$$

Proof. Using Hypotheses (H1) and (H2), that for $x_1, x_2, v_1, v_2 > 0$, we get

$$(x_1 - x_2)\left(\pi(x_2) - \pi(x_1)\right) > 0, \tag{31}$$

$$(x_2 - x_1)(\xi(x_2,v_2) - \xi(x_1,v_2)) > 0, \tag{32}$$

$$(v_2 - v_1)\left(\xi(x_1, v_2) - \xi(x_1, v_1)\right) > 0, \tag{33}$$

and from Hypothese (H4), we obtain

$$(v_1 - v_2)\left(\frac{\xi(x_1, v_2)}{g_2(v_2)} - \frac{\xi(x_1, v_1)}{g_2(v_1)}\right) > 0. \tag{34}$$

First, we show that $sgn(v_1 - v_2) = sgn(x_2 - x_1)$. Suppose that $sgn(v_2 - v_1) = sgn(x_2 - x_1)$. Using the steady states conditions of S_1 and S_2 we obtain

$$\pi(x_2) - \pi(x_1) = \beta\left[\xi(x_2, v_2) - \xi(x_1, v_1)\right] = \beta\left[\left(\xi(x_2, v_2) - \xi(x_1, v_2)\right) + \left(\xi(x_1, v_2) - \xi(x_1, v_1)\right)\right].$$

Therefore, from inequalities (31)-(33) we obtain $sgn\left(x_2 - x_1\right) = sgn\left(x_1 - x_2\right)$, which is a contradiction, hence, $sgn\left(v_1 - v_2\right) = sgn\left(x_2 - x_1\right)$. Using Eq. (25) and the definition of $\mathcal{R}_1$ we get

$$\mathcal{R}_1 - 1 = \gamma\left(\frac{\xi(x_2, v_2)}{g_4(v_2)} - \frac{\xi(x_1, v_1)}{g_4(v_1)}\right) = \gamma\left[\frac{1}{g_4(v_2)}\left(\xi(x_2, v_2) - \xi(x_1, v_2)\right) + \frac{\xi(x_1, v_2)}{g_4(v_2)} - \frac{\xi(x_1, v_1)}{g_4(v_1)}\right].$$

Thus, from Eqs. (32) and (34) we obtain $sgn(\mathcal{R}_1 - 1) = sgn(v_1 - v_2)$.

Theorem 4. Suppose that Hypotheses (H1)-(H4) are satisfied and $\mathcal{R}_1 \leq 1 < \mathcal{R}_0$, then S_1 is globally asymptotically stable in Δ.

Proof. We introduce Lyapunov function

$$\begin{aligned} W_1 = x - x_1 - \int_{x_1}^{x} \frac{\xi(x_1, v_1)}{\xi(\eta, v_1)} d\eta + k_1\left(w - w_1 - \int_{w_1}^{w} \frac{g_1(w_1)}{g_1(\eta)} d\eta\right) + k_2\left(y - y_1 - \int_{y_1}^{y} \frac{g_2(y_1)}{g_2(\eta)} d\eta\right) \\ + k_3\left(u - u_1 - \int_{u_1}^{u} \frac{g_3(u_1)}{g_3(\eta)} d\eta\right) + k_4\left(v - v_1 - \int_{v_1}^{v} \frac{g_4(v_1)}{g_4(\eta)} d\eta\right) + k_5 z, \end{aligned}$$

and evaluate $\frac{dW_1}{dt}$ along the trajectories of (10)-(15):

$$\begin{aligned} \frac{dW_1}{dt} = &\left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)}\right)\left(\pi(x) - \beta\xi(x, v)\right) + k_1\left(1 - \frac{g_1(w_1)}{g_1(w)}\right)\left(\beta_1\xi(x, v) - (a_1 + \delta_1)g_1(w)\right) \\ &+ k_2\left(1 - \frac{g_2(y_1)}{g_2(y)}\right)\left(\beta_2\xi(x, v) + a_1 g_1(w) - \delta_2 g_2(y)\right) + k_3\left(1 - \frac{g_3(u_1)}{g_3(u)}\right)\left(\beta_3\xi(x, v) - \delta_3 g_3(u)\right) \\ &+ k_4\left(1 - \frac{g_4(v_1)}{g_4(v)}\right)\left(N\delta_2 g_2(y) + M\delta_3 g_3(u) - \delta_4 g_4(v) - q g_4(v) g_5(z)\right) \\ &+ k_5\left(r g_4(v) g_5(z) - \delta_5 g_5(z)\right). \end{aligned} \tag{35}$$

Collecting terms of Eq. (35) and applying $\pi(x_1) = \beta\xi(x_1, v_1)$ we get

$$\begin{aligned} \frac{dW_1}{dt} = &\left(\pi(x) - \pi(x_1)\right)\left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)}\right) + \beta\xi(x_1, v_1)\left(1 - \frac{\xi(x_1, v_1)}{\xi(x, v_1)}\right) \\ &+ \beta\xi(x, v)\frac{\xi(x_1, v_1)}{\xi(x, v_1)} - k_1\beta_1\xi(x, v)\frac{g_1(w_1)}{g_1(w)} + k_1(a_1 + \delta_1)g_1(w_1) - k_2\beta_2\xi(x, v)\frac{g_2(y_1)}{g_2(y)} \\ &- k_2 a_1\frac{g_2(y_1)g_1(w)}{g_2(y)} + k_2\delta_2 g_2(y_1) - k_3\beta_3\xi(x, v)\frac{g_3(u_1)}{g_3(u)} + k_3\delta_3 g_3(u_1) - k_4 N\delta_2 g_2(y)\frac{g_4(v_1)}{g_4(v)} \\ &- k_4 M\delta_3 g_3(u)\frac{g_4(v_1)}{g_4(v)} - k_4\delta_4 g_4(v) + k_4\delta_4 g_4(v_1) + k_4 q g_4(v_1) g_5(z) - k_5\delta_5 g_5(z). \end{aligned} \tag{36}$$

Utilizing conditions of the steady state S_1, we obtain

$$\begin{aligned} (a_1 + \delta_1)g_1(w_1) = \beta_1\xi(x_1, v_1), \quad k_2\delta_2 g_2(y_1) = (k_1\beta_1 + k_2\beta_2)\xi(x_1, v_1), \\ \delta_3 g_3(u_1) = \beta_3\xi(x_1, v_1), \quad k_4\delta_4 g_4(v_1) = \beta\xi(x_1, v_1), \end{aligned}$$

then, we have

$$\begin{aligned}\frac{dW_1}{dt} &= (\pi(x)-\pi(x_1))\left(1-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}\right)+\beta\xi(x_1,v_1)\left(1-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}\right)\\ &+\beta\xi(x_1,v_1)\left(\frac{\xi(x,v)}{\xi(x,v_1)}-\frac{g_4(v)}{g_4(v_1)}\right)-k_1\beta_1\xi(x_1,v_1)\frac{\xi(x,v)g_1(w_1)}{\xi(x_1,v_1)g_1(w)}+k_1\beta_1\xi(x_1,v_1)\\ &-k_2\beta_2\xi(x_1,v_1)\frac{\xi(x,v)g_2(y_1)}{\xi(x_1,v_1)g_2(y)}-k_1\beta_1\xi(x_1,v_1)\frac{g_2(y_1)g_1(w)}{g_2(y)g_1(w_1)}+(k_1\beta_1+k_2\beta_2)\xi(x_1,v_1)\\ &-k_3\beta_3\xi(x_1,v_1)\frac{\xi(x,v)g_3(u_1)}{\xi(x_1,v_1)g_3(u)}+k_3\beta_3\xi(x_1,v_1)-(k_1\beta_1+k_2\beta_2)\xi(x_1,v_1)\frac{g_2(y)g_4(v_1)}{g_2(y_1)g_4(v)}\\ &-k_3\beta_3\xi(x_1,v_1)\frac{g_3(u)g_4(v_1)}{g_3(u_1)g_4(v)}+\beta\xi(x_1,v_1)+k_5r\left(g_4(v_1)-\frac{\delta_5}{r}\right)g_5(z). \end{aligned}\tag{37}$$

Equation (37) can be simplified as:

$$\begin{aligned}\frac{dW_1}{dt} &= (\pi(x)-\pi(x_1))\left(1-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}\right)+\frac{\beta\xi(x_1,v_1)g_4(v)}{\xi(x,v)\xi(x,v_1))}\left(\frac{\xi(x,v)}{g_4(v)}-\frac{\xi(x,v_1)}{g_4(v_1)}\right)(\xi(x,v)-\xi(x,v_1))\\ &+k_1\beta_1\xi(x_1,v_1)\left[5-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}-\frac{\xi(x,v)g_1(w_1)}{\xi(x_1,v_1)g_1(w)}-\frac{g_2(y_1)g_1(w)}{g_2(y)g_1(w_1)}-\frac{g_2(y)g_4(v_1)}{g_2(y_1)g_4(v)}-\frac{g_4(v)\xi(x,v_1)}{g_4(v_1)\xi(x,v)}\right]\\ &+k_2\beta_2\xi(x_1,v_1)\left[4-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}-\frac{\xi(x,v)g_2(y_1)}{\xi(x_1,v_1)g_2(y)}-\frac{g_2(y)g_4(v_1)}{g_2(y_1)g_4(v)}-\frac{g_4(v)\xi(x,v_1)}{g_4(v_1)\xi(x,v)}\right]\\ &+k_3\beta_3\xi(x_1,v_1)\left[4-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}-\frac{\xi(x,v)g_3(u_1)}{\xi(x_1,v_1)g_3(u)}-\frac{g_3(u)g_4(v_1)}{g_3(u_1)g_4(v)}-\frac{g_4(v)\xi(x,v_1)}{g_4(v_1)\xi(x,v)}\right]\\ &+k_5r\left(g_4(v_1)-g_4(v_2)\right)g_5(z). \end{aligned}\tag{38}$$

Hypotheses (H1), (H2), (H4), Lemma 1 and the condition $\mathcal{R}_1\le 1$ imply that

$$\begin{aligned}(\pi(x)-\pi(x_1))\left(1-\frac{\xi(x_1,v_1)}{\xi(x,v_1)}\right)&\le 0,\\ \left(\frac{\xi(x,v)}{g_4(v)}-\frac{\xi(x,v_1)}{g_4(v_1)}\right)(\xi(x,v)-\xi(x,v_1))&\le 0,\\ g_4(v_1)-g_4(v_2)&\le 0.\end{aligned}$$

It is known that the arithmetical mean is greater than or equal to the geometrical mean. It follows that for all $x,y,v,z>0$ we have $\frac{dW_1}{dt}\le 0$. Clearly, the largest invariant set $\Gamma_0\subseteq\Gamma=\left\{(x,w,y,u,v,z):\frac{dW_1}{dt}=0\right\}$ is the singlton $\{S_1\}$. By LaSalle's invariance principle, S_1 is globally asymptotically stable. □

Theorem 5. Let $\mathcal{R}_1>1$ and Hypotheses (H1)-(H4) are satisfied, then S_2 is globally asymptotically stable in $\overset{\circ}{\Delta}$.

Proof. Define a Lyapunov functional

$$\begin{aligned}W_2 &= x-x_2-\int_{x_2}^{x}\frac{\xi(x_2,v_2)}{\xi(\eta,v_2)}d\eta+k_1\left(w-w_2-\int_{w_2}^{w}\frac{g_1(w_2)}{g_1(\eta)}d\eta\right)+k_2\left(y-y_2-\int_{y_2}^{y}\frac{g_2(y_2)}{g_2(\eta)}d\eta\right)\\ &+k_3\left(u-u_2-\int_{u_2}^{u}\frac{g_3(u_2)}{g_3(\eta)}d\eta\right)+k_4\left(v-v_2-\int_{v_2}^{v}\frac{g_4(v_2)}{g_4(\eta)}d\eta\right)+k_5\left(z-z_2-\int_{z_2}^{z}\frac{g_5(z_2)}{g_5(\eta)}d\eta\right).\end{aligned}$$

Calculating $\frac{dW_2}{dt}$ along the solutions of model (10)-(15), we get

$$\begin{aligned}\frac{dW_2}{dt} &= \left(1-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}\right)(\pi(x)-\beta\xi(x,v)) + k_1\left(1-\frac{g_1(w_2)}{g_1(w)}\right)(\beta_1\xi(x,v)-(a_1+\delta_1)g_1(w))\\ &+ k_2\left(1-\frac{g_2(y_2)}{g_2(y)}\right)(\beta_2\xi(x,v)+a_1g_1(w)-\delta_2g_2(y)) + k_3\left(1-\frac{g_3(u_2)}{g_3(u)}\right)(\beta_3\xi(x,v)-\delta_3g_3(u))\\ &+ k_4\left(1-\frac{g_4(v_2)}{g_4(v)}\right)(N\delta_2g_2(y)+M\delta_3g_3(u)-\delta_4g_4(v)-qg_4(v)g_5(z))\\ &+ k_5\left(1-\frac{g_5(z_2)}{g_5(z)}\right)(rg_4(v)g_5(z)-\delta_5g_5(z)). \end{aligned} \tag{39}$$

Collecting terms of Eq. (39) and applying $\pi(x_2)=\beta\xi(x_2,v_2)$ we get

$$\begin{aligned}\frac{dW_1}{dt} &= (\pi(x)-\pi(x_2))\left(1-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}\right)+\beta\xi(x_2,v_2)\left(1-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}\right)\\ &+\beta\xi(x,v)\frac{\xi(x_2,v_2)}{\xi(x,v_2)}-k_1\beta_1\xi(x,v)\frac{g_1(w_2)}{g_1(w)}+k_1(a_1+\delta_1)g_1(w_2)\\ &-k_2\beta_2\xi(x,v)\frac{g_2(y_2)}{g_2(y)}-k_2a_1\frac{g_2(y_2)g_1(w)}{g_2(y)}+k_2\delta_2g_2(y_2)-k_3\beta_3\xi(x,v)\frac{g_3(u_2)}{g_3(u)}\\ &+k_3\delta_3g_3(u_2)-k_4N\delta_2g_2(y)\frac{g_4(v_2)}{g_4(v)}-k_4M\delta_3g_3(u)\frac{g_4(v_2)}{g_4(v)}-k_4\delta_4g_4(v)+k_4\delta_4g_4(v_2)\\ &+k_4qg_4(v_2)g_5(z)-k_5\delta_5g_5(z)-k_5rg_5(z_2)g_4(v)+k_5\delta_5g_5(z_2)\end{aligned}$$

Using the following steady state conditions for S_1:

$$\begin{aligned}(a_1+\delta_1)g_1(w_2)=\beta_1\xi(x_2,v_2), &\quad k_2\delta_2g_2(y_2)=(k_1\beta_1+k_2\beta_2)\xi(x_2,v_2),\\ \delta_3g_3(u_2)=\beta_3\xi(x_2,v_2), &\quad k_4\delta_4g_4(v_2)=\beta\xi(x_2,v_2)-k_4qg_5(z_2)g_4(v_2),\end{aligned}$$

we obtain

$$\begin{aligned}\frac{dW_2}{dt} &= (\pi(x)-\pi(x_2))\left(1-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}\right)+\beta\xi(x_2,v_2)\left(1-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}\right)\\ &+\beta\xi(x_2,v_2)\left(\frac{\xi(x,v)}{\xi(x,v_2)}-\frac{g_4(v)}{g_4(v_2)}\right)-k_1\beta_1\xi(x_2,v_2)\frac{\xi(x,v)g_1(w_2)}{\xi(x_2,v_2)g_1(w)}+k_1\beta_1\xi(x_2,v_2)\\ &-k_2\beta_2\xi(x_2,v_2)\frac{\xi(x,v)g_2(y_2)}{\xi(x_2,v_2)g_2(y)}-k_1\beta_1\xi(x_2,v_2)\frac{g_2(y_2)g_1(w)}{g_2(y)g_1(w_2)}+(k_1\beta_1+k_2\beta_2)\xi(x_2,v_2)\\ &-k_3\beta_3\xi(x_2,v_2)\frac{\xi(x,v)g_3(u_2)}{\xi(x_2,v_2)g_3(u)}+k_3\beta_3\xi(x_2,v_2)\\ &-(k_1\beta_1+k_2\beta_2)\xi(x_2,v_2)\frac{g_2(y)g_4(v_2)}{g_2(y_2)g_4(v)}-k_3\beta_3\xi(x_2,v_2)\frac{g_3(u)g_4(v_2)}{g_3(u_2)g_4(v)}+\beta\xi(x_2,v_2).\end{aligned} \tag{40}$$

Equation (40) can be simplified as:

$$\begin{aligned}\frac{dW_2}{dt} &= (\pi(x)-\pi(x_2))\left(1-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}\right)+\frac{\beta\xi(x_2,v_2)g_4(v)}{\xi(x,v)\xi(x,v_2))}\left(\frac{\xi(x,v)}{g_4(v)}-\frac{\xi(x,v_2)}{g_4(v_2)}\right)(\xi(x,v)-\xi(x,v_2))\\ &+k_1\beta_1\xi(x_2,v_2)\left[5-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}-\frac{\xi(x,v)g_1(w_2)}{\xi(x_2,v_2)g_1(w)}-\frac{g_2(y_2)g_1(w)}{g_2(y)g_1(w_2)}-\frac{g_2(y)g_4(v_2)}{g_2(y_2)g_4(v)}-\frac{g_4(v)\xi(x,v_2)}{g_4(v_2)\xi(x,v)}\right]\\ &+k_2\beta_2\xi(x_2,v_2)\left[4-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}-\frac{\xi(x,v)g_2(y_2)}{\xi(x_2,v_2)g_2(y)}-\frac{g_2(y)g_4(v_2)}{g_2(y_2)g_4(v)}-\frac{g_4(v)\xi(x,v_2)}{g_4(v_2)\xi(x,v)}\right]\\ &+k_3\beta_3\xi(x_2,v_2)\left[4-\frac{\xi(x_2,v_2)}{\xi(x,v_2)}-\frac{\xi(x,v)g_3(u_2)}{\xi(x_2,v_2)g_3(u)}-\frac{g_3(u)g_4(v_2)}{g_3(u_2)g_4(v)}-\frac{g_4(v)\xi(x,v_2)}{g_4(v_2)\xi(x,v)}\right].\end{aligned} \tag{41}$$

According to Hypotheses (H1), (H2) and (H4) and the relation between the geometrical and arithmetical means we get $\frac{dW_2}{dt}\le 0$. Clearly, the largest invariant set $\Gamma_0\subseteq\Gamma=\left\{(x,w,y,u,v,z):\frac{dW_2}{dt}=0\right\}$ is the singlton $\{S_2\}$. By LaSalle's invariance principle S_2 is globally asymptotically stable. □

5 Numerical simulations

We now perform some computer simulations on the following application:

$$\dot{x} = \rho - dx + px\left(1 - \frac{x}{x_{\max}}\right) - \frac{(1-\varepsilon_r)\overline{\beta}xv}{1+\eta_1 x+\eta_2 v}, \tag{42}$$

$$\dot{w} = \frac{(1-\varepsilon_r)\overline{\beta}_1 xv}{1+\eta_1 x+\eta_2 v} - (a_1+\delta_1)w, \tag{43}$$

$$\dot{y} = \frac{(1-\varepsilon_r)\overline{\beta}_2 xv}{1+\eta_1 x+\eta_2 v} + a_1 w - \delta_2 y, \tag{44}$$

$$\dot{u} = \frac{(1-\varepsilon_r)\overline{\beta}_3 xv}{1+\eta_1 x+\eta_2 v} - \delta_3 u, \tag{45}$$

$$\dot{v} = (1-\varepsilon_p)\overline{N}\delta_2 y + (1-\varepsilon_p)\overline{M}\delta_3 u - \delta_4 v - qvz, \tag{46}$$

$$\dot{z} = rvz - \delta_5 z. \tag{47}$$

We assume that $p < d$. In this application, we consider the following specific forms of the general functions:

$$\pi(x) = \rho - dx + px\left(1 - \frac{x}{x_{\max}}\right), \quad \xi(x,v) = \frac{xv}{1+\eta_1 x+\eta_2 v}, \quad g_i(\theta) = \theta, \quad i = 1,...,5.$$

First we verify Hypotheses (H1)-(H4) for the chosen forms, then we solve the system using MATLAB. Clearly, $\pi(0) = \rho > 0$ and $\pi(x_0) = 0$, where

$$x_0 = \frac{x_{\max}}{2p}\left(p - d + \sqrt{(p-d)^2 + \frac{4\rho p}{x_{\max}}}\right).$$

We have

$$\pi'(x) = -d + p - \frac{2px}{x_{\max}} < 0. \tag{48}$$

Clearly, $\pi(x) > 0$, for $x \in [0, x_0)$ and

$$\pi(x) = \rho - (d-p)x - p\frac{x^2}{x_{\max}} \leq \rho - (d-p)x$$

Then Hypothese (H1) is satisfied. We also have $\xi(x,v) > 0$, $\xi(0,v) = \xi(x,0) = 0$ for x,$v \in (0,\infty)$, and

$$\frac{\partial \xi(x,v)}{\partial x} = \frac{v\,(1+\delta v)}{(1+\eta_1 x+\eta_2 v)^2}, \qquad \frac{\partial \xi(x,v)}{\partial v} = \frac{x(1+\eta_1 x)}{(1+\eta_1 x+\eta_2 v)^2}, \qquad \frac{\partial \xi(x,0)}{\partial v} = \frac{x}{1+\eta_1 x}.$$

Then, $\frac{\partial \xi(x,v)}{\partial x} > 0$, $\frac{\partial \xi(x,v)}{\partial v} > 0$ and $\frac{\partial \xi(x,0)}{\partial v} > 0$ for $x, v \in (0,\infty)$. Therefore, Hypothese (H1) is satisfied. In addition

$$\xi(x,v) = \frac{xv}{1+\eta_1 x+\eta_2 v} \leq \frac{xv}{1+\eta_1 x} = v\frac{\partial \xi(x,0)}{\partial v},$$

$$\left(\frac{\partial \xi(x,0)}{\partial v}\right)' = \frac{1}{(1+\eta_1 x)^2} > 0 \text{ for all } x > 0.$$

It follows that, (H2) is satisfied. Clearly Hypothese (H3) holds true. Moreover,

$$\frac{\partial}{\partial v}\left(\frac{\xi(x,v)}{g_4(v)}\right) = \frac{-\eta_2 x}{(1+\eta_1 x+\eta_2 v)} < 0.$$

Therefore, Hypothese (H4) hold true and Theorems 3-5 are applicable. The parameters $\mathcal{R}_0$ and $\mathcal{R}_1$ for this application are given by:

$$\mathcal{R}_0 = \frac{(1-\varepsilon_r)(1-\varepsilon_p)\left\{\overline{N}(a_1\overline{\beta}_1 + (a_1+\delta_1)\overline{\beta}_2) + \overline{M}\overline{\beta}_3(a_1+\delta_1)\right\}}{\delta_4(a_1+\delta_1)}\frac{x_0}{1+\eta_1 x_0},$$

$$\mathcal{R}_1 = \frac{(1-\varepsilon_r)(1-\varepsilon_p)\left\{\overline{N}(a_1\overline{\beta}_1 + (a_1+\delta_1)\overline{\beta}_2) + \overline{M}\overline{\beta}_3(a_1+\delta_1)\right\}}{\delta_4(a_1+\delta_1)}\frac{x_2}{1+\eta_1 x_2+\eta_2 v_2}.$$

Now we are ready to perform some numerical simulations for system (42)-(47). The data of system (42)-(47) are provided in Table 1.

• **Effect of the drug efficacy on the stability of the steady states**

Now we verify our theoretical results given in Theorems 3-5 by numerical simulation. To discuss our global results we choose three different initial conditions:

IC1: $(x(0), w(0), y(0), u(0), v(0), z(0)) = (900, 10, 12, 60, 40, 1.6)$.

IC2: $(x(0), w(0), y(0), u(0), v(0), z(0)) = (700, 7, 8, 30, 25, 1.0)$.

IC3: $(x(0), w(0), y(0), u(0), v(0), z(0)) = (500, 4, 5, 10, 15, 0.6)$.

Let us address three scenarios for three different groups of the parameters ε_r, ε_p and r.

Scenario (I): In this case we choose $\varepsilon_r = 0.6$, $\varepsilon_p = 0.6$ and $r = 0.001$ which gives $\mathcal{R}_0 = 0.4941 < 1$ and $\mathcal{R}_1 = 0.4430 < 1$. Therefore, based on Theorems 2 and 3, the system has unique steady state, that is S_0 and it is globally asymptotically stable. As we can see from Figures 1-6 that the concentration of the uninfected $CD4^+$ T cells is increased and approached its normal value before infection that is $x_0 = 1083.9$, while concentrations of the other compartments converge to zero for all the three initial conditions. This case corresponds to the uninfected state where the HIV-1 is removed from the plasma.

Scenario (II): By taking $\varepsilon_r = 0.2$, $\varepsilon_p = 0.5$ and $r = 0.001$. With such choice we get, $\mathcal{R}_1 = 0.9351 < 1 < \mathcal{R}_0 = 1.2352$. Consequently, based on Theorems 2 and 4, the humoral-inactivated infection steady state S_1 is positive and is globally asymptotically stable. Figures 1-6 confirm that the numerical results support the theoretical results presented in Theorem 4. It can be observed that, the variables of the model eventually converge to $S_1 = (309.165, 13.2492, 15.4574, 94.0263, 72.0754, 0.0)$ for all the three initial conditions. This case corresponds to a chronic HIV-1 infection in the absense of immune response.

Scenario (III): $\varepsilon_r = 0.2$, $\varepsilon_p = 0.5$ and $r = 0.003$. Then, we calculate $\mathcal{R}_0 = 1.2352 > 1$ and $\mathcal{R}_1 = 1.19604 > 1$. According to Lemma 1 and Theorem 3, the humoral-activated infection steady state S_2 is positive and is globally asymptotically stable. We can see from Figures 1-6 that, there is a consistency between the numerical results and theoretical results of Theorem 5. The states of the system converge to $S_2 = \{820.603, 5.8629, 6.8401, 41.6079, 26.6667, 1.1762)$ for all the three initial conditions, in the same time frame. In this case the humoral immune response is activated and can control the disease.

• **Effect of the HAART on the basic reproduction number:**

Let us define the overall HAART effect as $\varepsilon_e = \varepsilon_r + \varepsilon_p - \varepsilon_r\varepsilon_p$ [9]. If $\varepsilon_e = 0$, then the HAART has no effect, if $\varepsilon_e = 1$, the HIV-1 growth is completely halted. Consequently, the parameter $\mathcal{R}_0$ is given by

$$\mathcal{R}_0(\varepsilon_e) = \frac{(1-\varepsilon_e)\left\{\overline{N}(a_1\overline{\beta}_1 + (a_1+\delta_1)\overline{\beta}_2) + \overline{M\beta}_3(a_1+\delta_1)\right\}}{\delta_4(a_1+\delta_1)} \frac{x_0}{1+\eta_1 x_0}.$$

We note that, the value of $\mathcal{R}_0(\varepsilon_e)$ does not depend on the values of the parameters q, r and δ_5. This means that, humoral immune response can play a significant role in reducing the infection progress but do not play a role in clearing the HIV-1 from the body. Since the goal is to clear the HIV-1 from the body, then we have to determine the drug efficacies that make $\mathcal{R}_0(\varepsilon_e) \leq 1$ for system (42)-(47). Now, we calculate the critical overall treatment effect ε_e^{crit} (i.e, the minimum overall treatment effect required to stabilize the system around the infection-free steady state). Let $\mathcal{R}_0(\varepsilon_e) \leq 1$, then

$$\varepsilon_e^{crit} \leq \varepsilon_e < 1, \quad \varepsilon_e^{crit} = \max\left\{0, \frac{\mathcal{R}_0(0)-1}{\mathcal{R}_0(0)}\right\},$$

Figure 15 shows the effect of the HAART on the basic reproduction number $\mathcal{R}_0(\varepsilon)$. We note that, if $\varepsilon_e^{crit} \leq \varepsilon_e < 1$, then $\mathcal{R}_0(\varepsilon_e) \leq 1$ and S_0 is globally asymptotically stable. Moreover, if $0 \leq \varepsilon_e < \varepsilon_e^{crit}$, then S_0 is unstable.

Table 1: The values of the parameters of example (42)-(47).

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
ρ	10	δ_1	0.02	$\overline{\beta}_2$	0.0625	μ	0.08
d	0.01	δ_2	0.36	$\overline{\beta}_3$	0.0625	$\overline{N}$	62
p	0.008	δ_3	0.031	a_1	0.2	$\overline{M}$	30
$x_{\max}$	1200	δ_4	3.0	η_1	1	ε_r, ε_p	Varied
q	0.5	$\overline{\beta}_1$	0.0625	η_2	1	r	Varied

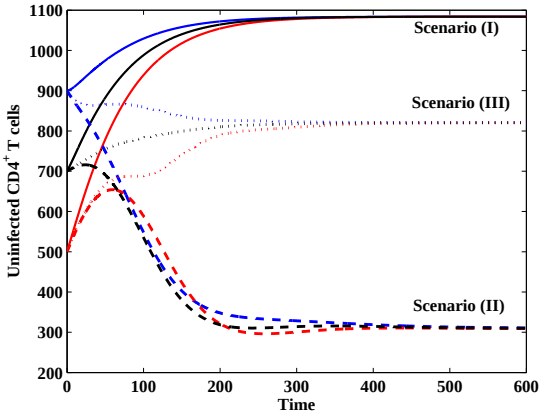


Figure 1: The concentration of uninfected CD4$^+$ T cells for system (42)-(47).

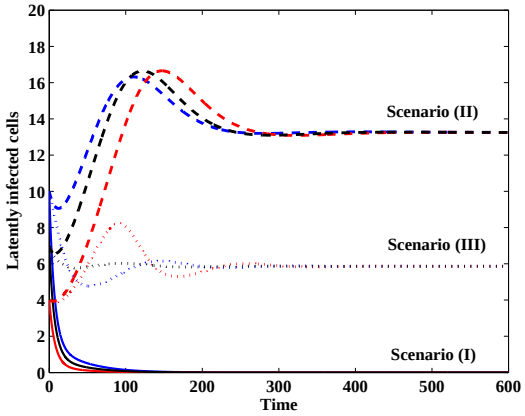


Figure 2: The concentration of latently infected cells for system (42)-(47).

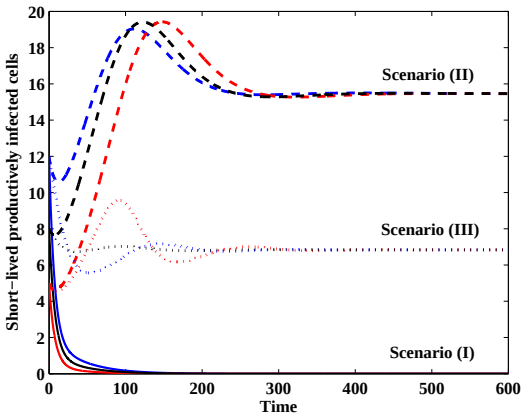


Figure 3: The concentration of short-lived productively infected cells for system (42)-(47).

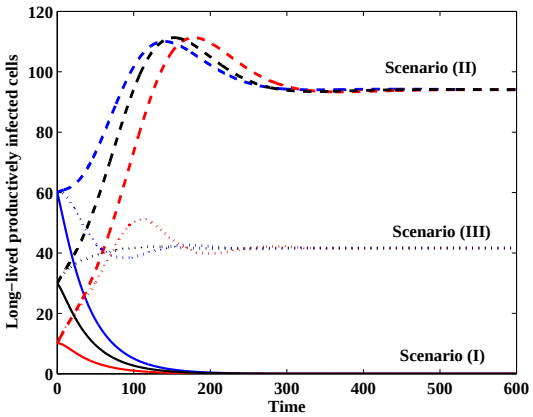


Figure 4: The concentration of long-lived productively infected cells for system (42)-(47).

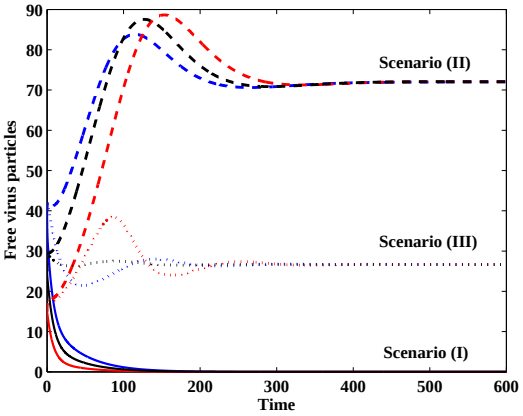


Figure 5: The concentration of free virus particles for system (42)-(47).

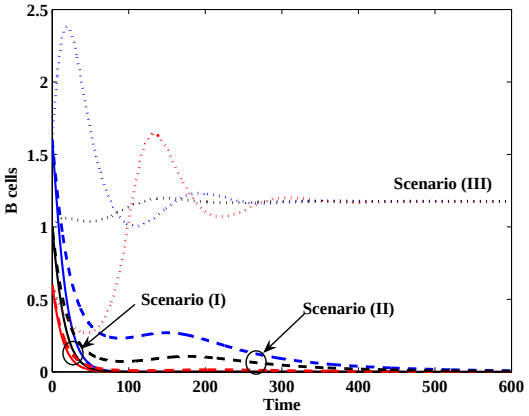


Figure 6: The concentration of B cells for system (42)-(47).

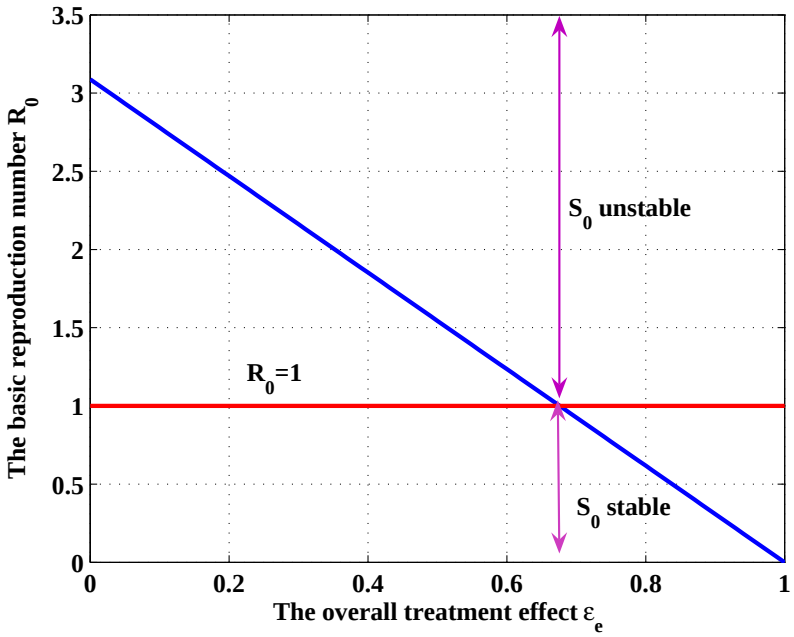


Figure 7: The basic reproduction number as a function of the overall treatment effect ε_e of system (42)-(47).

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Ideal theory of pre-logics based on the theory of falling shadows

Young Bae Jun[1] and Sun Shin Ahn[2,*]

[1] **Department of Mathematics Education, Gyeongsang National University, Jinju, 52828, Korea**

[2]**Department of Mathematics Education, Dongguk University, Seoul 04620, Korea**

Abstract. Based on the theory of a falling shadow which was first formulated by Wang [8], a theoretical approach of the ideal structure in pre-logics is established. The notions of a falling subalgebra, a falling and a positive implicative falling ideal of a pre-logic are introduced. Some fundamental properties are investigated. Relations among a falling subalgebra, a falling ideal and a positive implicative falling ideal are stated. Characterizations of falling deals and positive implicative falling ideals are discussed.

1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [3] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [7] introduced the theory of falling shadows which directly relates probability concepts to the membership function of fuzzy sets. Falling shadow representation theory shows us a method of selection relied on the joint degree distributions. It is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [8]. Y. B. Jun and C. H. Park [5] discussed the notion of a falling fuzzy subalgebra/ideal of a BCK/BCI-algebra. Y. B. Jun and M. S. Kang [4] established a theoretical approach for defining a fuzzy positive implicative ideal in a BCK-algebra based on the theory of falling shadows. I. Chajda and R. Halas [2] introduced the concept of a pre-logic which is an algebra weaker than a Hilbert algebra (an algebraic counterpart of intuitionistic logic) but strong enough to have deductive systems. Y. B. Jun and S. S. Ahn [1] defined the notion of pseudo-valuations (valuation) on pre-logics and induced a pseudo-metric by using a pseudo-valuation on pre-logics.

In this paper, we introduce the notions of a falling subalgebra, a falling ideal and a positive implicative falling ideal of a pre-logic. We investigate some fundamental properties. Also we give relations among a falling subalgebra, a falling ideal and a positive implicative falling ideal. We establish characterizations of falling ideals and positive implicative falling ideals.

2. Preliminaries

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[0]**Keywords**: falling shadow; falling subalgebra; falling ideal; positive implicative falling ideal.

* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409

[0]**E-mail**: skywine@gmail.com (Y. B. Jun); sunshine@dongguk.edu (S. S. Ahn)

We recall some definitions and results (see [1, 2, 6]).

Definition 2.1. ([2]) By a *pre-logic*, we mean a triple $(X; *, 1)$ where X is a non-empty set, $*$ is a binary operation on X and $1 \in X$ is a constant such that the following identities hold:

(P1) $(\forall x \in X)\ (x * x = 1)$,
(P2) $(\forall x \in X)\ (1 * x = x)$,
(P3) $(\forall x \in X)\ (x * (y * z) = (x * y) * (x * z))$,
(P4) $(\forall x, y, z \in X)\ (x * (y * z) = y * (x * z))$.

In what follows, let X denote a pre-logic unless otherwise specified.

Lemma 2.2. ([2]) *Let X be a pre-logic. Then the following hold:*

(a) $(\forall x \in X)\ (x * 1 = 1)$;
(b) $(\forall x, y \in X)\ (x * (y * x) = 1)$;
(c) *an order relation $\leq$ on X defined by*

$$(\forall x, y \in X)\,(x \leq y \text{ if and only if } x * y = 1)$$

is a quasiorder on X (i.e., a reflexive and transitive order relation on X);
(d) *$1 \leq x$ for all $x \in X$ implies $x = 1$.*

Remark 2.3. ([2]) The quasiorder $\leq$ of Lemma 2.2(c) is called the *induced quasiorder* of a pre-logic X.

Lemma 2.4. ([2]) *Let $\leq$ be the induced quasiorder of a pre-logic X and let $x, y, z \in X$. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.*

Definition 2.5. ([2]) Let $X = (X; *, 1)$ be a pre-logic. A non-empty subset D of X is called a *deductive system* of X if the following conditions hold:

(d1) $1 \in D$,
(d2) if $x \in D$ and $x * y \in D$, then $y \in D$.

Definition 2.6. ([2]) Let X be a pre-logic. A non-empty subset I of X is called an *ideal* of X if the following conditions are satisfied:

(I1) $x \in X$ and $y \in I$ imply $x * y \in I$;
(I2) $x \in X$ and $y_1, y_2 \in I$ imply $(y_2 * (y_1 * x)) * x \in I$.

Lemma 2.7. ([2]) *Let X be a pre-logic and $\leq$ its induced quasiorder. The the following hold:*

(a) $(\forall x, y \in X)\ (x * ((x * y) * y) = 1)$,
(b) $(\forall x, y, z \in X)\ ((y * z) * ((x * y) * (x * z)) = 1)$,
(c) *if D is a deductive system of X, $a \in D$, and $a \leq b$, then $b \in D$.*

Theorem 2.8. ([1]) *A non-empty subset I of a pre-logic X is an ideal of X if and only if it satisfies the following two conditions:*

(I1′) $(1 \in I)$;
(I2′) $(\forall x, z \in X)(\forall y \in I)\ (x * (y * z) \in I \Rightarrow x * z \in I)$.

Definition 2.9. ([6]) A non-empty subset I of a pre-logic X is a *positive implicative ideal* of X if it satisfies (I1′) and

(I3) $(\forall y, z \in X)(\forall x \in I)\ (x * ((y * z) * y) \in I \Rightarrow y \in I)$.

Theorem 2.10. ([6]) *Every positive implicative ideal of a pre-logic X is an ideal of X.*

We now display the basic theory on falling shadows. We refer the reader to the papers [3, 4, 5, 7, 8] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathscr{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.1}$$

and for each $E \in \mathscr{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \tag{2.2}$$

An ordered pair $(\mathscr{P}(U), \mathscr{B})$ is said to be a hyper-measurable structure on U if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$ and $\dot{U} \subseteq \mathscr{B}$. Given a probability space $(\Omega, \mathscr{A}, P)$ and a hyper-measurable structure $(\mathscr{P}(U), \mathscr{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathscr{P}(U)$ which is $\mathscr{A}$-$\mathscr{B}$ measurable, that is,

$$(\forall C \in \mathscr{B})\,(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathscr{A}). \tag{2.3}$$

Suppose that ξ is a random set on U. Let $\tilde{H}(u) := P(\omega \mid u \in \xi(\omega))$ for each $u \in U$. Then $\tilde{H}$ is a kind of fuzzy set in U. We call $\tilde{H}$ a *falling shadow* of the random set ξ, and ξ is called a *cloud* of $\tilde{H}$.

For example, $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t-cut of $\tilde{H}$. Then $\xi : [0,1] \to \mathscr{P}(U),\ t \mapsto \tilde{H}_t$ is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the *cut-cloud* of $\tilde{H}$ (see [3]).

3. **Falling subalgebras and falling ideals**

Definition 3.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let $\xi : \Omega \to \mathscr{P}(X)$ be a random set, where X is a pre-logic. If $\xi(\omega)$ is a subalgebra (resp. ideal) of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\tilde{H}$ of the random set ξ, i.e., $\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$ is called a *falling subalgebra* (resp. *falling ideal*) of X.

In what follows, let $\tilde{H}$ denote a falling shadow of the random set $\xi : \Omega \to \mathscr{P}(X)$.

Example 3.2. (1) Let $X := \{1, a, b, c, d\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; *, 1)$ is a pre-logic (see [6]). Let $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$ and define a random set $\xi : \Omega \to \mathscr{P}(X)$ as follows:

$$\xi(\omega) := \begin{cases} \{1, a, b\} & \text{if} \omega \in [0, 0.6) \\ \emptyset & \text{if } \omega \in [0.6, 0.7), \\ X & \text{if } \omega \in [0.7, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is both a falling subalgebra of X and a falling ideal of X.

Define a random set $\eta : \Omega \to \mathscr{P}(X)$ as follows:

$$\eta(\omega) := \begin{cases} \emptyset & \text{if } \omega \in [0, 0.3), \\ \{1, b, c\} & \text{if} \omega \in [0.3, 0.8), \\ X & \text{if } \omega \in [0.8, 1]. \end{cases}$$

Then $\eta(\omega)$ is a subalgebra of X for all $\omega \in \Omega$ with $\eta(\omega) \neq \emptyset$, but not an ideal of X, since $(b * (a * a)) * a = (b * 1) * a = 1 * a = a \notin \{1, b, c\}$. Hence the falling shadow $\tilde{H}$ of ξ is a falling subalgebra of X, but not a falling ideal of X.

For a probability space $(\Omega, \mathscr{A}, P)$ and any element x of a BCC-algebra X, let

$$\Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}. \tag{3.1}$$

Then $\Omega(x; \xi) \in \mathscr{A}$.

Lemma 3.3. *If $\tilde{H}$ is a falling subalgebra of a pre-logic X, then $(\forall x \in X)\,(\Omega(x; \xi) \subseteq \Omega(1; \xi))$.*

Proof. If $\Omega(x; \xi) = \emptyset$, then it is clear. Assume that $\Omega(x; \xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$, and so $1 = x * x \in \xi(\omega)$ since $\xi(\omega)$ is a subalgebra of X. Hence $\omega \in \Omega(1; \xi)$, and therefore $\Omega(x; \xi) \subseteq \Omega(1; \xi)$ for all $x \in X$. □

Proposition 3.4. *Every falling ideal of a pre-logic X is a falling subalgebra of X.*

Proof. Let $\tilde{H}$ be a falling ideal of X. Then $\xi(\omega)$ is an ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y \in \xi(\omega)$. Using (I1), we have $x * y \in \xi(\omega)$. Hence $\xi(\omega)$ is a subalgebra of X. Thus $\tilde{H}$ is a falling subalgebra of X. □

The converse of Proposition 3.4 is not true in general (see Example 3.2). We provide a characterization of a falling ideal.

Theorem 3.5. *Let X be a pre-logic. Then $\tilde{H}$ is a falling ideal of X if and only if the following conditions are valid:*

(i) $(\forall x, y \in X)\,(\Omega(x*(y*z);\xi) \cap \Omega(y;\xi) \subseteq \Omega(x*z;\xi))$,
(ii) $(\forall x \in X)\,(\Omega(x;\xi) \subseteq \Omega(1;\xi))$.

Proof. Assume that $\tilde{H}$ is a falling ideal of X. For any $x, y, z \in X$, if $\omega \in \Omega(x*(y*z);\xi) \cap \Omega(y;\xi)$, then $x*(y*z) \in \xi(\omega)$ and $y \in \xi(\omega)$. It follows from (I2′) that $x*z \in \xi(\omega)$ since $\xi(\omega)$ is an ideal of X. Hence $\omega \in \Omega(x*z;\xi)$. Therefore $\Omega(x*(y*z);\xi) \cap \Omega(y;\xi) \subseteq \Omega(x*z;\xi)$, for any $x, y, z \in X$. Thus (i) is valid. The second condition (ii) follows from Lemma 3.3 and Proposition 3.4.

Conversely, suppose that two conditions (i) and (ii) are valid. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x*(y*z) \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x*(y*z);\xi)$ and $\omega \in \Omega(y;\xi)$. If follows from (i) that $\omega \in \Omega(x*z;\xi)$. Hence $x*z \in \xi(\omega)$. Now, assume that $x \in \xi(\omega)$ for every $x \in X$ and for all $\omega \in \Omega$. Then $\omega \in \Omega(x;\xi) \subseteq \Omega(1;\xi)$ and so $1 \in \xi(\omega)$ for all $\omega \in \Omega$. Therefore $\xi(\omega)$ is an ideal of X for all $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence $\tilde{H}$ is a falling ideal of X. □

Theorem 3.6. *Let X be a pre-logic. Then $\tilde{H}$ is a falling ideal of X if and only if the following conditions are valid:*

(i) $(\forall x, y \in X)\,(\Omega(y;\xi) \subseteq \Omega(x*y;\xi))$,
(ii) $(\forall x, y, z \in X)\,(\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega((x*(y*z))*z;\xi))$.

Proof. Assume $\tilde{H}$ satisfies two conditions (i) and (ii). Let $x, y \in X$ and $\omega \in \Omega$ such that $y \in \xi(\omega)$. Then $\omega \in \Omega(y;\xi)$. Using (i), we have $\omega \in \Omega(x*y;\xi)$. Hence $x*y \in \xi(\omega)$. Now, let $x, y, z \in X$ and $\omega \in \Omega$ such that $x, y \in \xi(\omega)$. Then $\omega \in \Omega(x;\xi)$ and $\omega \in \Omega(y;\xi)$ and so $\omega \in \Omega(x;\xi) \cap \Omega(y;\xi)$. It follows from (ii) that $\omega \in \Omega((x*(y*z))*z;\xi)$. Hence $(x*(y*z))*z \in \xi(\omega)$ and so $\xi(\omega)$ is an ideal of X. Therefore $\tilde{H}$ is a falling ideal of X.

Conversely, suppose that $\tilde{H}$ is a falling ideal of X. Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(y;\xi)$. Then $y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X, we have $x*y \in \xi(\omega)$. Hence $\omega \in \Omega(x*y;\xi)$. Therefore (i) is valid. For any $x, y, z \in X$, if $\omega \in \Omega(x;\xi) \cap \Omega(y;\xi)$, then $x \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X, we get $(x*(y*z))*z \in \xi(\omega)$. Therefore $\omega \in \Omega((x*(y*z))*z;\xi)$. Thus (ii) is true. □

Proposition 3.7. *Every falling ideal of a pre-logic satisfies the following assertions:*

(i) $(\forall x \in X)\,(\Omega(x;\xi) \subseteq \Omega(1;\xi))$,
(ii) $(\forall x, y \in X)\,(\Omega(x;\xi) \subseteq \Omega((x*y)*y;\xi))$,
(iii) $(\forall x, y \in X)\,(x \leq y \Rightarrow \Omega(x;\xi) \subseteq \Omega(y;\xi))$.

Proof. (i) Using (P1) and Theorem 3.6(i), we have $\Omega(x;\xi) \subseteq \Omega(1;\xi)$.

(ii) Taking $x := x, y := 1$, and $z := y$ in Theorem 3.6(ii) and using (P2) and (i), we get $\Omega(x;\xi) = \Omega(x;\xi) \cap \Omega(1;\xi) \subseteq \Omega((x*(1*y))*y;\xi) = \Omega((x*y)*y;\xi)$.
(iii) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$. Using (P2), we have $\Omega(x;\xi) \subseteq \Omega((x*y)*y;\xi) = \Omega(1*y;\xi) = \Omega(y;\xi)$. □

Lemma 3.8. *Every falling ideal of $\tilde{H}$ of a pre-logic X satisfies the following property:*

$$(\forall x, y \in X)(\Omega(x*y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)). \tag{3.2}$$

Proof. Using (P1), (P2), and Theorem 3.6(ii), we have $\Omega(x*y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(((x*y)*(x*y))*y;\xi) = \Omega(1*y;\xi) = \Omega(y;\xi)$ for all $x, y \in X$. □

Corollary 3.9. *Let X be a pre-logic. Then $\tilde{H}$ is a falling ideal of X if and only if it satisfies the condition* (3.2) *and*

(i) $(\forall x \in X)\,(\Omega(x;\xi) \subseteq \Omega(1;\xi))$.

Proof. Assume that $\tilde{H}$ is a falling ideal of X. Using Proposition 3.7, (i) holds. By Lemma 3.8, the condition (3.2) holds.

Conversely, suppose that $\tilde{H}$ satisfies two conditions (3.2) and (i). Using (3.2), we have $\Omega(y*(x*z);\xi) \cap \Omega(y;\xi) \subseteq \Omega(x*z;\xi)$. Using (P4), we have $\Omega(x*(y*z);\xi) \cap \Omega(y;\xi) \subseteq \Omega(x*z;\xi)$. By Theorem 3.5, $\tilde{H}$ is a falling ideal of X. □

Lemma 3.10. *For any falling ideal $\tilde{H}$ of a pre-logic X, the following are equivalent:*

(i) $(\forall x, y \in X)\,(\Omega(x*y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi))$.
(ii) $(\forall x, y, z \in X)\,(\Omega(x*(y*z);\xi) \cap \Omega(x*y;\xi) \subseteq \Omega(x*z;\xi))$.

Proof. Assume that $\tilde{H}$ satisfies (i). For any $x, y, z \in X$, using (P3), we have $\Omega(x*(y*z);\xi) \cap \Omega(x*y;\xi) = \Omega((x*y)*(x*z);\xi) \cap \Omega(x*y;\xi)) \subseteq \Omega(x*z;\xi)$. Thus (ii) is valid.

Conversely, suppose that $\tilde{H}$ satisfies (ii). Putting $x := 1$ in (ii) and using (P2), we have $\Omega(y*z;\xi) \cap \Omega(y;\xi) = \Omega(1*(y*z));\xi) \cap \Omega(1*y;\xi)) \subseteq \Omega(1*z;\xi) = \Omega(z;\xi)$. Thus (i) is true. □

Proposition 3.11. *Let X be a pre-logic. Then $\tilde{H}$ is a falling ideal of X if and only if the following conditions are valid:*

(i) $(\forall x \in X)\,(\Omega(x;\xi) \subseteq \Omega(1;\xi))$.
(ii) $(\forall x, y, z \in X)\,(\Omega(x*(y*z);\xi) \cap \Omega(x*y;\xi) \subseteq \Omega(x*z;\xi))$.

Proof. It follows from Corollary 3.9 and Lemma 3.10. □

Corollary 3.12. *Every falling ideal $\tilde{H}$ of a pre-logic X satisfies the following property:*

$$(\forall x, y \in X)(\Omega(x*(x*y);\xi) \subseteq \Omega(x*y;\xi)).$$

Proof. Putting $x := x, z := y$ and $y := x$ in Proposition 3.11(ii), we have $\Omega(x*(x*y);\xi) = \Omega(x*(x*y);\xi) \cap \Omega(1;\xi) = \Omega(x*(x*y);\xi) \cap \Omega(x*x;\xi) \subseteq \Omega(x*y;\xi)$, for all $x, y \in X$. □

4. **Positive implicative falling ideals**

Definition 4.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let $\xi : \Omega \to \mathscr{P}(X)$ be a random set, where X is a pre-logic. If $\xi(\omega)$ is a positive implicative ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\tilde{H}$ of the random set ξ, i.e., $\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$ is called a *positive implicative falling ideal* of X.

Example 4.2. Let $X = \{1, a, b, c, d\}$ be a pre-logic as in Example 3.2.
(1) Consider a random set ξ as in Example 3.2. Then the falling shadow $\tilde{H}$ of ξ is a positive implicative falling ideal of X, since $\{1, a, b\}$ is a positive implicative ideal of X.
(2) Define a random set $\eta : \Omega \to \mathscr{P}(X)$ as follows:

$$\eta(\omega) := \begin{cases} \emptyset & \text{if } \omega \in [0, 0.3), \\ \{1, b\} & \text{if } \omega \in [0.3, 0.7), \\ X & \text{if } \omega \in [0.7, 1]. \end{cases}$$

Note that $J := \{1, b\}$ is an ideal of X but not a positive implicative ideal of X since $b*((a*d)*a) = b*(d*a) = b*1 = 1 \in J$ and $b \in J$ but $a \notin J$. Hence $\tilde{H}$ is a falling ideal of X, but not a positive implicative falling ideal of X.

Proposition 4.3. *Every positive implicative falling ideal of a pre-logic X is a falling ideal of X.*

Proof. Straightforward by Definition 4.1 and Theorem 2.10. □

The converse of Proposition 4.3 is not true in general (see Example 4.2(2)).

Theorem 4.4. *Let X be a pre-logic. Then $\tilde{H}$ is a positive implicative falling ideal of X if and only if $\tilde{H}$ satisfies the following two conditions:*

(i) $(\forall x \in X)(\Omega(x;\xi) \subseteq \Omega(1;\xi))$,
(ii) $(\forall x, y, z \in X)(\Omega(x*((y*z)*y);\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi))$.

Proof. Assume that $\tilde{H}$ satisfies two conditions (i) and (ii). Let $x \in \xi(\omega)$ for every $x \in X$ and for all $\omega \in \Omega$. Then $\omega \in \Omega(x;\xi) \subseteq \Omega(1;\xi)$ and so $1 \in \xi(\omega)$. Let $x, y, z \in X$ be such that $x \in \xi(\omega)$ and $x*((y*z)*y) \in \xi(\omega)$. Then $\omega \in \Omega(x;\xi)$ and $\omega \in \Omega(x*((y*z)*y);\xi)$. Using (ii), we have $\omega \in \Omega(y;\xi)$. Hence $y \in \xi(\omega)$ and so $\xi(\omega)$ is a positive implicative ideal of X. Therefore $\tilde{H}$ is a positive implicative falling ideal of X.

Conversely, suppose that $\tilde{H}$ is a positive implicative falling ideal of X. The first condition (i) follows from Proposition 3.7(i) and Proposition 4.3. For any $x, y, z \in X$, if $\omega \in \Omega(x*((y*z)*y);\xi) \cap \Omega(x;\xi)$, then $x*((y*z)*y) \in \xi(\omega)$ and $x \in \xi(\omega)$. Since $\xi(\omega)$ is a positive implicative ideal

of X, we have $\omega \in \Omega(y;\xi)$. Therefore $\Omega(x*((y*z)*y);\xi)\cap\Omega(x;\xi)\subseteq\Omega(y;\xi)$ for any $x,y,z\in X$. Thus (ii) holds. □

Theorem 4.5. *Let $\tilde{H}$ be a falling ideal of a pre-logic X. Then the following are equivalent:*

(i) *$\tilde{H}$ is a positive implicative falling ideal of X.*

(ii) $(\forall x,y\in X)(\Omega((x*y)*x;\xi)\subseteq\Omega(x;\xi))$.

Proof. Assume that $\tilde{H}$ is a positive implicative ideal of X. Putting $x:=1, y:=x$, and $z:=y$ in Theorem 4.4(ii), we have $\Omega(1*((x*y)*x);\xi)\cap\Omega(1;\xi)=\Omega((x*y)*x;\xi)\cap\Omega(1;\xi)=\Omega((x*y)*x;\xi)\subseteq\Omega(x;\xi)$. Hence (ii) holds.

Conversely, suppose that a falling ideal $\tilde{H}$ satisfies (ii). By Lemma 3.8, for any $x,y,z\in X$, we have $\Omega(x*((y*z)*y);\xi)\cap\Omega(x;\xi)\subseteq\Omega((y*z)*y;\xi)\subseteq\Omega(y;\xi)$. By Theorem 4.4, $\tilde{H}$ is a positive implicative falling ideal of X. Thus (i) is true. □

Corollary 4.6. *Any positive implicative falling ideal of a pre-logic X satisfies the following property:*

$$(\forall x,y\in X)(\Omega((x*y)*y;\xi)\subseteq\Omega((y*x)*x;\xi)).$$

Proof. Since $x\leq(y*x)*x$ for all $x,y\in X$, it follows from Lemma 2.4 that $((y*x)*x)*y\leq x*y$. Then $(x*y)*y\leq(y*x)*((x*y)*x)=(x*y)*((y*x)*x)\leq(((y*x)*x)*y)*((y*x)*x)$. By Proposition 3.7(iii) and Proposition 4.5, we have $\Omega((x*y)*y;\xi)\subseteq\Omega((((y*x)*x)*y)*((y*x)*x);\xi)\subseteq\Omega((y*x)*x;\xi)$, for any $x,y\in X$. This completes the proof. □

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QUADRATIC ρ-FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN BANACH SPACES

CHOONKIL PARK, GANG LU, YINHUA CUI, AND MING FANG*

ABSTRACT. In this paper, we solve the quadratic ρ-functional equations

$$f(x+y)+f(x-y)-2f(x)-2f(y)=\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right), \quad (0.1)$$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$, and

$$4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)=\rho(f(x+y)+f(x-y)-2f(x)-2f(y)), \quad (0.2)$$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\leq|r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\leq\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([8]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

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*Corresponding author.

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;
(iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [18] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [17] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $4f\left(\frac{x+y}{2}\right) + (x - y) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 9, 10, 12, 13, 14, 15, 16, 19, 20]).

In Section 2, we solve the quadratic ρ-functional equation (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic ρ-functional equation (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional equation (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Quadratic ρ-functional equation (0.1) in non-Archimedean normed spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the quadratic ρ-functional equation (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *If a mapping $f : G \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &\quad = \rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right) \end{aligned} \tag{2.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &\quad = \rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right) \\ &\quad = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $r < 2$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} &\left\|f(x+y) + f(x-y) - 2f(x) - 2f(y) - \rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &\qquad \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2}{|2|^r}\theta\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.3), we get $-2f(0) = \rho f(0)$. Since $|\rho| < |2|$, $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So $\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.6)\\
&\leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leq \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\frac{2}{|2|^r}\theta\|x\|^r = \frac{2\theta}{|2|^{(r-2)l}}\frac{1}{|2|^r}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned}
&\left\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) - \rho\left(4Q\left(\frac{x+y}{2}\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right)\right\|\\
&= \lim_{n\to\infty}|4|^n\left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right.\\
&\left.-\rho\left(4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right\| \leq \lim_{n\to\infty}\frac{|4|^n\theta}{|2|^{nr}}(\|x\|^r + \|y\|^r) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y) = \rho\left(4Q\left(\frac{x+y}{2}\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned}
&\|Q(x) - T(x)\| = \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\|\\
&\leq \max\left\{\left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|, \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|\right\} \leq \frac{2}{|2|^{(r-2)q+r}}\theta\|x\|^r,
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.4). □

Theorem 2.3. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.3). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|4|}\|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{2\theta}{|4|}\|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Quadratic ρ-functional equation (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the quadratic ρ-functional equation (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *If a mapping $f : G \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \\ = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{3.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &= 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) - \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\right\| \leq \theta(\|x\|^r + \|y\|^r) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \theta\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.3), we get $f(0) = 2\rho f(0)$. Since $|\rho| < |2|$, $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.6)\\
&\leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leq \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.3)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x)-\frac{1}{4^m}f(2^m x)\right\| \qquad (3.8)\\
&\le \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right)-\frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\
&= \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right)-\frac{1}{4}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{4}f\left(2^m x\right)\right\|\right\}\\
&\le \max\left\{\frac{|2|^{rl}}{|4|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r=\frac{|2|^r\theta}{|2|^{(2-r)l+2}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

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Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

Gang Lu
Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110870, P.R. China; Department of Mathematics, Zhejiang University, Hangzhou 310027, P.R. China
E-mail address: lvgang1234@hanmail.net

Yinhua Cui
Department of Mathematics, Yanbian University, Yanji 133001, P.R. China
E-mail address: cuiyh@ybu.edu.cn

Ming Fang
Department of Mathematics, Yanbian University, Yanji 133001, P.R. China
E-mail address: fangming@ybu.edu.cn

The Incomplete Global GMERR Algorithm to Solve $AX = B^*$

Yu-Hui Zheng[a]†, Jian-Lei Li[b], Dong-Xu Cheng[a], Ling-Ling Lv[b]

[a] *College of Science, Zhongyuan University of Technology, Zhengzhou, Henan, 450007, PR China.*

[b] *College of Mathematics and Information Science, North China University of, Water Resources and Electric Power, Zhengzhou, Henan, 450011, PR China.*

Abstract

To reduce the computational cost and storage requirement of global generalized minimal error(GLGMERR) method, in this paper, we propose a truncated version of GLGMERR method, which is termed as incomplete global generalized minimal error method. The proposed approach uses only a few rather than all of the prior computed matrices in recurrences to generate the next matrix. Moreover a quasi-minimum error solution is obtained as well. Finally, we present the numerical results by comparing with the traditional global GMERR method in CPU time and storage requirements to show the effectiveness and advantages of our method.

Key words:matrix equation; incomplete global generalized minimal error.

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1 Introduction

Consider the following problem:

$$Ax^{(i)} = b^{(i)}, i = 1, 2, \cdots s, \tag{1.1}$$

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†Corresponding author. E-mail:zhuoer2008@sohu.com.

where A is a $n \times n$ unsymmetric matrix, $x^{(i)},\ b^{(i)}$ are all $n \times 1$ real vectors, $s \leq n$. In our daily life, sometimes we have to solve this problem. Therefore, it is of importance that researchers are interested in the study of the numerical solutions, algorithms design and software development for solving problem (1.1).

Krylov subspace method, as one of the effective method for solving $Ax = b$, can be used to solve s linear systems one by one. However, when the order of A is large, it is not enough to use this method to solve this problem. Therefore, we have to find the other new method to solve it. It should be noticed that when all $b^{(i)}$ do impact on the whole system, the problem (1.1) can be rewritten as

$$AX = B, \tag{1.2}$$

where $X = [x^{(1)},\ x^{(2)},\ \cdots x^{(s)}]^T,\ B = [b^{(1)},\ b^{(2)},\ \cdots b^{(s)}]^T$.

In the past decades, some related works have been achieved to solve the problem (1.2). In 1999, Jbilou [1] et al proposed the global Arnoldi method and moreover, they proposed global FOM and GMRES methods based on global Arnoldi method, which extended the Krylov subspace method. Among all Krylov subspace methods, GMERR method is one of the most effective methods, because it can minimize the error norm of this method on Krylov subspace. The literature [2] presented the global GMRES method for solving unsymmetric linear systems, which maps the initial residual matrix to the Krylov subspace. In some sense, global GMERR and global GMERS [1] methods have similar structures.

The global generalized minimal error(GMERR) algorithm is an effective Krylov subspace method to solve the linear equations with multiple right-hand sides. As the global GMERR method and the GMERR method have the long recurrence, which result in the dramatic increase of the calculation and storage along with the increase of the step numbers. At present, there are many truncation strategies. For example, Young [3] presented the truncated forms of the orthogonal direction method and the orthogonal residual method. In [4, 5], a truncated forms of FOM method has been given. The truncated forms of IGMRES method or QGMRES method are presented in [6–10]. In this paper, we use the truncation strategy to improve the global GMERR algorithm, and propose a incomplete global GMERR algorithm, which use a few of the previously generated matrix to construct the new basis matrix, and we also give the quasi global minimum error solution on the Krylov subspace.

The remainder of this paper is organized as follows. In Section 2, we present the incomplete global GMERR algorithm. Section 3 and 4 give some numerical experiments to test the effectiveness of the incomplete global GMERR algorithm and conclusions, respectively

2 Incomplete global GMERR algorithm

The incomplete global GMERR algorithm, which is based on the incomplete orthogonality of the Krylov subspace matrix, is to seek the quasi global minimum error solution. The basis matrices $\{V_i\}(i = 1, 2, ..., m)$ of the Krylov subspace $K_m(A^T, R_0)$ can be obtained through the incomplete orthogonal process. $A^T V_i(i = 1, 2, ..., m)$ is carried out in the orthogonal process with the first $q(q < m)$ matrices $V_{i_0}, ...V_i(i_0 = max\{1, i-q+1\})$, The incomplete global GMERR algorithm is to seek the approximate solution $X_m = X_0 + Z_m, Z_m \in A^T K_m(A^T, R_0)$. Moreover $R_m = B - AX_m \perp K_m(A^T, R_0)$, i.e.,

$$R_0 - AZ_m \perp K_m(A^T, R_0). \tag{2.1}$$

Note that $U_m = [V_1, V_2, ..., V_m]$. let $Z_m = A^T U_m * y_m$, then we can have

$$X_m = X_0 + A^T U_m * y_m, \;\; R_m = R_0 - AA^T U_m * y_m.$$

Since $R_0 - AZ_m$ is orthogonal to $K_m(A^T, R_0)$ from equation (2.1). Therefore, for $i = 1, 2, ..., m$, we can obtain

$$< V_i, R_0 >=< V_i, AA^T U_m * y_m > . \tag{2.2}$$

Let $V_1 = R_0/ \parallel R_0 \parallel_F$, then for the formula (2.2), when $i = 1$, it means $tr(V_1^T R_0) = tr(V_1^T AA^T U_m * y_m)$, i.e.,

$$\parallel R_0 \parallel_F = (tr(V_1^T AA^T V_1), tr(V_1^T AA^T V_2), ...tr(V_1^T AA^T V_m))y_m;$$

when $i = q + 2, ..., m$, $(tr(V_i^T AA^T V_1), tr(V_i^T AA^T V_2), ...tr(V_i^T AA^T V_m))y_m = 0$;
when $i = 2, ..., q + 1$, $tr(V_i^T R_0) = (tr(V_i^T AA^T V_1), tr(V_i^T AA^T V_2), ...tr(V_i^T AA^T V_m))y_m$.

Therefore, through the above discussion, we can obtain y_m by solving the following linear system:

$$\begin{pmatrix} tr(V_1^T AA^T V_1) & tr(V_1^T AA^T V_2) & \cdots & tr(V_1^T AA^T V_2) \\ tr(V_2^T AA^T V_1) & tr(V_2^T AA^T V_2) & \cdots & tr(V_2^T AA^T V_2) \\ \vdots & \vdots & \ddots & \vdots \\ tr(V_m^T AA^T V_1) & tr(V_m^T AA^T V_2) & \cdots & tr(V_m^T AA^T V_2) \end{pmatrix} y_m = \begin{pmatrix} \parallel R_0 \parallel_F \\ 0 \\ \vdots \\ 0 \\ tr(V_{q+2}^T R_0) \\ \vdots \\ tr(V_m^T R_0) \end{pmatrix} \tag{2.3}$$

To sum up, we obtain the following restarting incomplete GMERR Algorithm.

Algorithm 1 (The restarting in complete GMERR Algorithm)

Step 1. Choose the restarting step number m, let $2 \le q \le m$, *set the precision tol and initial estimation* $n \times s$ *moment* X_0. *Then calculate* $R_0 = B - AX_0$, *Let* $V_1 = R_0/ \parallel R_0 \parallel_F$;

Step 2. For $i = 1, 2, ..., m$ *, do the following incomplete orthogonal process*

2.1 $W = A^T V_i$,

2.2 For $j = max(1, i - q + 1), ..., i$, *calculate*

$$h_{j,i} = tr(V_j^T A^T V_i),\ W = W - h_{j,i} V_j,$$

2.3 $h_{i+1,i} =\| W \|_F,\ V_{i+1} = W/h_{i+1,i}$;

Step 3. Solve the linear system (2.3) *to get* y_m;

Step 4. Calculate $X_m = X_0 + A^T U_m * y_m$;

Step 5. If $\| R_m \|_F =\| B - AX_0 \|_F \leq tol$, *stop; otherwise, let* $X_m = X_0$, *calculate* $R_0 = B - AX_0$, $V_1 = R_0 / \| R_0 \|_F$, *go to step 2.*

It is not difficult to find that the matrices and Hessenberg matrix H_m produced by the above incomplete orthogonal process satisfy the following theorem.

Theorem 1 *If the incomplete global GMERR algorithm doesnt interrupt before the mth step, i.e.,* $h_{i+1,i} \neq 0,\ (i = 1, 2, \cdots, m)$, *then* $\{V_i\}(i = 1, 2, \cdots, m)$, *which are produced by the incomplete orthogonal process, constitute a basis of the Krylov subspace. In addition, we have*

$$A^T U_m = U_m * H_m + S_{m+1},$$

$$tr(V_i^T V_j) = 0 (i \neq j, \mid i - j \mid \leq q), \quad tr(V_i^T V_i) = 1,\ (i, j = 1, 2, \cdots, m),$$

where $S_{m+1} = h_{m+1,m}[0_{n\times s}, 0_{n\times s} ... V_{k+1}]$.

By analyzing the above theorem, we can achieve H_m and $B_m = (tr(V_i^T V_j))_{m\times m}$, $(i, j = 1, 2, \cdots, m)$ in detail

$$H_m = \begin{pmatrix} tr(V_1^T A V_1) & \cdots & tr(V_1^T A V_q) & & & \\ tr(V_2^T A V_1) & \ddots & & 0 & & \\ & & & \ddots & & tr(V_{m-q+1}^T A V_m) \\ & 0 & & & \ddots & \vdots \\ & & & tr(V_m^T A V_{m-1}) & & tr(V_m^T A V_m) \end{pmatrix},$$

$$B_m = \begin{pmatrix} 1 & 0 & \cdots & 0 & & * \\ 0 & 1 & \ddots & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ 0 & & \ddots & \ddots & \ddots & \vdots \\ & \ddots & & \ddots & \ddots & 0 \\ * & & 0 & \cdots & 0 & 1 \end{pmatrix},$$

Now, let us state some results which are indispensable for our subsequent discussions.

Lemma 1 *For inner product* $< X, Y >$, *we have* $< X, Y > \leq \| X \|_F \| Y \|_F$, *where* $X,\ Y \in \mathbb{Z}_{n\times s}$, $\mathbb{Z}_{n\times s}$ *represents the* $n \times s$ *matrix space over* $\mathbb{R}$

Proof. Obviously, $< X, Y >= tr(X^T Y) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{s} x_{ij}y_{ij}$, By Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{s} x_{ij}y_{ij} &\leq \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{s} x_{ij}^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{s} y_{ij}^2\right)^{\frac{1}{2}}\right] \\ &\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{s} x_{ij}^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij}^2\right)^{\frac{1}{2}} \\ &= \| X \|_F \| Y \|_F .\end{aligned}$$

Hence, the proof of the theorem is completed. □

Lemma 2 *If* $j + 2 \leq i \leq m + 1$, *then* $tr(V_i^T A^T V_j) = \sum\limits_{k=j-q+1}^{min\{i-q-1,j+1\}} h_{k,j} tr(V_i^T V_k)$.

Proof. By analyzing the algorithm and components of H_m, we have $A^T V_j = \sum\limits_{k=1}^{m+1} h_{k,j} V_k$, where $j = 1, 2, \cdots, m$. Multiplying V_i^T left to the two sides of the above formula, we have $V_i^T A^T V_j = \sum\limits_{k=1}^{min\{i-q-1,j+1\}} h_{k,j} V_i^T V_k$.

Taking the trace, we can have

$$\begin{aligned}tr(V_i^T A^T V_j) &= \sum_{k=1}^{i-q-1} h_{k,j} tr(V_i^T V_k) + h_{i,j} + \sum_{k=i+q+1}^{m+1} h_{k,j} tr(V_i^T V_k) \\ &= \sum_{k=1}^{i-q-1} h_{k,j} tr(V_i^T V_k) + h_{i,j} + \sum_{k=i+q+1}^{j+1} h_{k,j} tr(V_i^T V_k)\ (k > 1, h_{k,j} = 0) \\ &= \sum_{k=1}^{min\{i-q-1,j+1\}} tr(h_{k,j} V_i^T V_k) \\ &= \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} h_{k,j} tr(V_i^T V_k)\end{aligned}$$

If $k \leq 0$, let $tr(V_i^T V_k)$, we have $h_{k,j} = 0$. □

Theorem 2 *Suppose* $q \geq 2, i \leq m + 1$ *and* $i - j \geq q + 1$, *if the incomplete global*

GMERR algorithm doesnt interrupt before the step, we have

$$| tr(V_j^T V_i) | \leq c_i \parallel A^T - A \parallel_F / h_{i,i-1}, \tag{2.4}$$

where $c_{i+1} = \max\limits_{1 \leq j \leq i-q} \gamma_{i+1,j}$,

$$\gamma_{i+1,j} = 1 + c_i \left(\sum_{k=j-q+1}^{min\{i-q-1,j+1\}} | h_{k,j} | \right) / h_{i,i-1} + \left(\sum_{k=j+q+1}^{i} c_k \mid h_{k,i} \mid \right) / h_{k,k-1}$$

and if $k \leq 0$, *then* $\gamma_{i+1,j} = 1$.

Proof. Let $U_{m+1} = [V_1, V_2, \ldots, V_{m+1}]$, $B_{m+1} = (tr(V_i^T V_j))_{(m+1)\times(m+1)}$. From Algorithm 1 and Lemma 2, when $i + 1 \leq m + 1$ and $i + 1 - j \geq q + 1$, we obtain that

$$h_{i+1,i} V_{i+1} = A^T V_i - \sum_{k=i_0}^{i} h_{k,i} V_k, \tag{2.5}$$

Left-multiplying V_j^T to both sides of equation (2.5) and taking trace, we can get

$$\begin{aligned} tr(V_j^T V_{i+1} h_{i+1,i}) &= tr\left(V_j^T \left(A^T V_i - \sum_{k=i_0}^{i} h_{k,i} V_k \right) \right) \\ &= tr\left(V_j^T \left(A^T - A \right) V_i \right) + \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} h_{k,j} tr(V_k^T V_i) - \sum_{k=i_0}^{i} h_{k,i} tr(V_j^T V_k), \end{aligned} \tag{2.6}$$

where $i_0 = \max\{1, i - q + 1\}$. In the following part, the inductive method is used to prove our theorem.

When $i + 1 = q + 2 \leq m + 1$, $j = 1$, from equation (2.2) and Lemma 2, we can obtain

$$h_{q+2,q+1} tr(V_1^T V_{q+2}) = tr\left(V_1^T \left(A^T - A \right) V_{q+1} \right).$$

Assume that the incomplete global GMERR algorithm doesnt interrupt, and then we can have

$$\begin{aligned} \left| tr(V_1^T V_{q+2}) \right| &= \left| tr\left(V_1^T \left(A^T - A \right) V_{q+1} \right) \right| / h_{q+2,q+1} \\ &= \left| < V_1^T, \left(A^T - A \right) V_{q+1} > \right| / h_{q+2,q+1} \\ &\leq \parallel V_1 \parallel_F \parallel A^T - A \parallel_F \parallel V_{q+1} \parallel_F / h_{q+2,q+1} \\ &= \parallel A^T - A \parallel_F / h_{q+2,q+1}, \end{aligned}$$

which shows that it satisfies formula (2.5), where $c_{q+2} = 1$.

Assume the first i columns in the upper right corner of matrix B_{m+1} satisfy equation (2.5). In the following we process the $(i+1)th$ column, where $1 \leq j \leq j-q,\ i.e.\ i-j \geq q$. For formula (2.6), we can separately get the following inequations

$$\left| \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} h_{k,j} tr(V_k^T V_i) \right| \leq \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} \left| h_{k,j} tr(V_k^T V_i) \right| \leq c_i \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} |h_{k,j}| \parallel A^T - A \parallel_F / h_{i,i-1},$$

and

$$\left| \sum_{k=i_0}^{i} h_{k,i} tr(V_j^T V_k) \right| \leq \sum_{k=i_0}^{i} \left| h_{k,i} tr(V_j^T V_k) \right| = \sum_{k=j+q+1}^{i} \left| h_{k,i} tr(V_j^T V_k) \right| \leq \sum_{k=j+q+1}^{i} c_k |h_{k,i}| \parallel A^T - A \parallel_F / h_{k,k-1}.$$

Then, we can conclude that

$$\begin{aligned} \left| tr(V_j^T V_{i+1}) \right| &\leq \frac{\left(\parallel V_j \parallel_F \parallel A^T - A \parallel_F \parallel V_i \parallel_F + \left| \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} h_{k,j} tr(V_k^T V_i) \right| + \left| \sum_{k=i_0}^{i} h_{k,i} tr(V_j^T V_k) \right| \right)}{h_{i+1,i}} \\ &\leq \frac{\left[1 + \left(c_i \sum_{k=j-q+1}^{min\{i-q-1,j+1\}} |h_{k,j}| \right) / h_{i,i-1} + \left(\sum_{k=j+q+1}^{i} c_k |h_{k,i}| / h_{k,k-1} \right) \right] \parallel A^T - A \parallel_F}{h_{i+1,i}} \\ &= \gamma_{i+1,j} \parallel A^T - A \parallel_F / h_{i+1,i}, \end{aligned}$$

where $c_{i+1} = \max\limits_{1 \leq j \leq i-q} \gamma_{i+1,j}$. The proof the theorem is completed. □

Theorem 3 *Any singular value $\sigma(B_m)$ of $B_m = \left(tr(V_I^T V_j) \right)_{m \times m}$ satisfies*

$$max\left\{0, 1 - (m-q-1)c \parallel A^T - A \parallel_F\right\} \leq \sigma(B_m) \leq 1 + (m-q-1)c \parallel A^T - A \parallel_F$$

where c is a function generated by H_m.

Proof. By Gerschgorin Circular disc Theorem and Theorem 1, for any singular value $\sigma(B_m)$ of B_m, there must exist i, satisfying

$$|\sigma(B_m) - 1| \leq \sum_{j \neq i} |tr(V_i^T V_j)| = \sum_{|i-j| \geq q+1} |tr(V_i^T V_j)|.$$

Therefore, we can get that

$$1 - \sum_{|i-j|\geq q+1} |tr(V_i^T V_j)| \leq \sigma(B_m) \leq 1 + \sum_{|i-j|\geq q+1} |tr(V_i^T V_j)|.$$

By analying the algorithm and the structure of B_m, we have

$$\sum_{|i-j|\geq q+1} |tr(V_i^T V_j)| = \sum_{j=1}^{i-q-1} |tr(V_i^T V_j)| + \sum_{j=i+q+1}^{m} |tr(V_i^T V_j)|$$

If $i \leq q+1$, $\sum\limits_{j=1}^{i-q-1} |tr(V_i^T V_j)| = 0$, then

$$\begin{aligned}\sum_{|i-j|\geq q+1} |tr(V_i^T V_j)| &\leq (m-i-q) \max_{i+q+1\leq j\leq m} |tr(V_i^T V_j)| \\ &\leq (m-q-1) \max_{q+2\leq j\leq m} |tr(V_i^T V_j)| \\ &\leq (m-q-1)c \parallel A^T - A \parallel_F,\end{aligned}$$

where $c = \max\limits_{q+2\leq j\leq m} c_j/h_{j,j-1}$.

If $i \geq m-q$, $\sum\limits_{j=i+q+1}^{m} |tr(V_i^T V_j)| = 0$, then

$$\begin{aligned}\sum_{|i-j|\geq q+1} |tr(V_i^T V_j)| &\leq (i-q-1) \max_{1\leq j\leq i-q-1} |tr(V_i^T V_j)| \\ &\leq (m-q-1) \max_{1\leq j\leq m-q-1} |tr(V_j^T V_i)| \\ &\leq (m-q-1)c \parallel A^T - A \parallel_F,\end{aligned}$$

where $c = \max\limits_{m-q\leq i\leq m} c_i/h_{i,i-1}$.

If $q+2 \leq i \leq m-q-1$, based on Lemma 2, we have

$$\begin{aligned}\sum_{|i-j|\geq q+1} |tr(V_i^T V_j)| &\leq (i-q-1) \max_{1\leq j\leq i-q-1} |tr(V_i^T V_j)| + (m-i-q) \max_{i+q+1\leq j\leq m} |tr(V_i^T V_j)| \\ &\leq (i-q-1) \parallel A^T - A \parallel_F \max_{q+2\leq i\leq m-q-1} c_i/h_{i,i-1} \\ &\quad + (m-i-q) \parallel A^T - A \parallel_F \max_{i+q+1\leq j\leq m} c_j/h_{j,j-1} \\ &\leq (m-2q-1)c \parallel A^T - A \parallel_F \leq (m-q-1)c \parallel A^T - A \parallel_F,\end{aligned}$$

where $c = \max\left\{ \max\limits_{q+2\leq i\leq m-q-1} c_i/h_{i,i-1}, \max\limits_{i+q+1\leq j\leq m} c_j/h_{j,j-1} \right\}$. Based the above disscussion, the proof the theorem is completed. □

Theorem 3 presents that the orthogonal degree of basis matrices is determined by the symmetry degree of the coefficient matrix. By the above theorems, we can obtain the conclusion on the algorithm convergence. If the algorithm is interrupted, which means $h_{1+i,i} = 0(1 \leq i \leq m)$ at ith step, the invariant subspace of A^T could be generated, moreover the approximate solution X_i generated by incomplete global GMERR algorithm is the exact solution of $AX = B$. Meanwhile, the error $R_i = 0$.

During the incomplete orthogonal process, the generated basis matrix may lose the orthogonality to some extent. From theorem 2 and theorem 3, we can find that the incomplete global GMERR algorithm can not control the orthogonal degree of the generated basis matrix when the coefficient matrix is far away from the symmetric property. And so, the algorithm may not converge.

3 Numerical experiments

In this section, we give numerical experiments to test the effectiveness of the incomplete global GMERR algorithm. Moreover we compare it with the global GMERR algorithm and find that our proposed method is more effective than the traditional method when they are set in the same accuracy.

Example Consider the two-dimension Convection-Diffusion Equation which is defined on the domain $\Omega = [0, 1] \times [0, 1]$

$$\begin{cases} -\Delta u(x, y) + \alpha \dfrac{\partial}{\partial x} u(x, y) = f(x, y), \\ u(x, y) = 0. \end{cases}$$

In this paper, we use the central difference method with grid length $h = 1/(l + 1)$ to discrete the above equation, and then we obtain the $n = l^2$ order non-symmetric matrix $A(\alpha)$

$$A(\alpha) = \begin{pmatrix} B(\alpha) & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & B(\alpha) \end{pmatrix},$$

where $B(\alpha) = \begin{pmatrix} 4 & a & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & a \\ & & b & 4 \end{pmatrix}$ is l order matrix, $a = -1 + \frac{\alpha}{2(l+1)}$, $b = -1 - \frac{\alpha}{2(l+1)}$, I is l order identity matrix, parameter α control the $A(\alpha)$ and deviation of symmetry.

Table 1: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 0.25$, $\| A(0.25)^T - A(0.25) \|_F = 0.3431$

m	q	CPU	$ratio$	IT	$\| R \|_F$
40	2	43.8590	0.7221	28	9.2951e-7
	5	45.7030	0.7725	28	9.2951e-7
	10	50.7810	0.8361	28	9.2951e-7
	20	55.6400	0.9161	28	9.2951e-7
	30	57.4530	0.9459	28	9.2951e-7
	40	60.7350	1	28	9.2951e-7
50	2	34.4850	0.6876	15	7.3342e-7
	5	37.0470	0.7386	15	7.3342e-7
	10	39.9530	0.7966	15	7.3342e-7
	20	43.9370	0.8760	15	7.3342e-7
	30	47.7030	0.9511	15	7.3342e-7
	40	49.4220	0.9854	15	7.3342e-7
	50	50.1560	1	15	7.3342e-7

Table 2: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 2.5$, $\| A(2.5)^T - A(2.5) \|_F = 3.4314$

m	q	CPU	$ratio$	IT	$\| R \|_F$
40	2	44.6410	0.7431	24	9.1938e-7
	5	47.2960	0.7872	24	9.1938e-7
	10	48.1100	0.8008	24	9.1938e-7
	20	58.7340	0.9778	24	9.1938e-7
	30	59.6250	0.9812	24	9.1938e-7
	40	60.0780	1	24	9.1938e-7
50	2	34.4690	0.7204	13	6.5037e-7
	5	33.4690	0.6995	13	6.5037e-7
	10	38.4060	0.8027	13	6.5037e-7
	20	39.2660	0.8207	13	6.5037e-7
	30	42.0780	0.8795	13	6.5037e-7
	40	44.0630	0.9209	13	6.5037e-7
	50	47.8440	1	13	6.5037e-7

Table 3: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 25$, $\| A(25)^T - A(25) \|_F = 34.3137$

m	q	CPU	$ratio$	IT	$\| R \|_F$
40	2	393.7810	0.7567	208	8.2432e-7
	5	381.4370	0.7330	208	9.6001e-7
	10	420.1400	0.8074	208	8.4919e-7
	20	461.5940	0.8871	208	9.3441e-7
	30	489.7580	0.9412	208	9.6380e-7
	40	520.3590	1	209	8.2174e-7
50	2	431.6090	0.8871	192	7.7878e-7
	5	324.9540	0.6679	137	8.8358e-7
	10	352.1880	0.7238	137	8.4919e-7
	20	392.1410	0.8059	137	9.1390e-7
	30	427.0160	0.8776	137	9.0901e-7
	40	442.0160	0.9084	136	9.7113e-7
	50	486.5620	1	136	9.1955e-7

Table 4: The incomplete global GMERR(m) algorithm with $n = 2500, \alpha = 2500$, $\| A(\alpha)^T - A(\alpha) \|_F = 3.4314e + 3$

m	q	CPU	$ratio$	IT	$\| R \|_F$
40	2	153.0620	0.7777	90	9.6087e-7
	5	145.2810	0.7382	92	9.8770e-7
	10	162.4370	0.8253	92	9.8773e-7
	20	180.1560	0.9154	92	9.8773e-7
	30	191.8900	0.9750	92	9.8773e-7
	40	196.8130	1	92	9.8773e-7
50	2	196.8280	0.7695	71	9.5397e-7
	5	183.8900	0.7189	71	9.3412e-7
	10	205.8750	0.8049	71	9.5319e-7
	20	228.0320	0.8915	71	9.5319e-7
	30	256.4370	1.0026	71	9.5319e-7
	40	244.1870	0.9547	71	9.5319e-7
	50	255.7810	1	71	9.5319e-7

The purpose of this section is to demonstrate that, with the same accuracy, the incomplete global GMERR algorithm is more effective than the global GMERR algorithm when we solve the large linear equations with multiple right-hand sides. Without loss of generality, we set $s = 2$, which means the two right-hand sides. Assuming $B = rand(n, s), X_0 = 0, l = 50, n = 2500, tol = 10^{-6}$, we test the incomplete global GMERR algorithm for numerical analysis for $m = 40,\ 50$, Moreover, we find that the incomplete global GMERR algorithm degenerate to global GMERR algorithm when $q = m$, Table 1-4 show us the numerical results, where CPU is denoted as the algorithm running time (in seconds), IT represents the iterate times, Ratio means the running time ratio of the incomplete global GMERR algorithm to the GMERR algorithm with the same accuracy requirements.

For $\alpha = 0.25$, $A(\alpha)$ is approximatly symmetric. Thus, the loss of the orthogonality of the basis matrices is not serious. The CPU time of incomplete global GMERR algorithm is shorter than the global GMERR algorithm, which shows effective of our proposed method.

For $\alpha = 2.5$, we can see the incomplete global GMERR algorithm is more effective than the global GMERR algorithm from table 2 and table 3. For $\alpha = 2500$, although $A(\alpha)$ is far away from the symmetric property and the loss of the orthogonality of the basis matrices is serious, we can find that the incomplete global GMERR algorithm is still effective than the global GMERR algorithm from table 4.

The experimental results show that, with the same accuracy, the incomplete global GMERR algorithm is more effective than the global GMERR algorithm. With the same computational cost, operation time and storage of our method is less than these of traditional method.

4 Conclusion

The incomplete global GMERR algorithm can overcome the long recurrence of the global GMERR algorithm by truncation strategy, which can save the computation and storage requirements effectively. In this paper, we present the incomplete global GMERR algorithm theoretically. Finally, the experimental results show effectiveness of the incomplete global GMERR algorithm by comparing with the traditional global GMERR algorithm.

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DOUBLE DIFFERENCE SPACES OF ALMOST NULL AND ALMOST CONVERGENT SEQUENCES FOR ORLICZ FUNCTION

KULDIP RAJ AND RENU ANAND

Abstract. The objective of this paper is to introduce and study some double difference spaces of almost null and almost convergent sequences defined by a Musielak-Orlicz function. We prove that these spaces are Banach, Barreled and Bornological spaces. An attempt is also made to prove that these spaces are BDK spaces and prove some interrelationship between these spaces.

1. Introduction and Preliminaries

The initial work on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [10], Móricz [15], Móricz and Rhoades [16], Tripathy ([27],[28]), Başarır and Sonalcan [2] and many others. Hardy [10] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [30] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [21] and Mursaleen and Edely [18] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{kl})$ into one whose core is a subset of the M-core of x. The set of all complex valued double sequences is a vector space with coordinatewise addition and scalar multiplication which is denoted by Ω.

By the convergence of a double sequence we mean the *convergence* in the Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > n_0$. We shall write more briefly as P-convergent. The space of all convergent double sequences in Pringsheim's sense is denoted by C_p.

A double sequence $x = (x_{kl})$ of complex numbers is said to be *bounded* if $||x||_\infty = \sup_{k,l \in \mathbb{N}} |x_{kl}| < \infty$, where $\mathbb{N} = \{0, 1, 2, ...\}$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$, which is a Banach space with the norm $||.||_\infty$.

It is well known that there are such sequences in the space C_p but not in the space $\mathcal{M}_u$. Indeed, if we define the sequence $x = (x_{kl})$ by

$$x_{kl} = \begin{cases} k, & k \in \mathbb{N} \\ l, & l \in \mathbb{N} \\ 0, & k, l \in \mathbb{N} \setminus \{0\}, \end{cases}$$

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for all $k, l \in \mathbb{N}$, then, it is trivial that $x \in Cp \backslash \mathcal{M}_u$, since $P-\lim_{k,l\to\infty} x_{kl} = 0$ but $||x||_\infty = \infty$. Therefore, we can consider the space C_{bp} of the double sequences that are both convergent in Pringsheim's sense and bounded which we write $C_{bp} = C_p \cap \mathcal{M}_u$.
A sequence in the space C_p is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the space of all such sequences by C_r. Also by C_{bp0} and C_{r0}, we denote the spaces of all double sequences converging to 0 contained in the sequence spaces C_{bp} and C_r, respectively. Moricz [15] proved that C_{bp}, C_{bp0}, C_r and C_{r0} are Banach spaces with the norm $||.||_\infty$.
The concept of almost convergence for single sequences was introduced by Lorentz [13] and for double sequences by Móricz and Rhoades [16]. A double sequence $x = (x_{kl})$ of complex numbers is said to be almost convergent to a generalized limit α if

$$P-\lim_{q,r\to\infty} \sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} (x_{kl} - \alpha) \Big| = 0. \qquad \text{(see [29])}$$

Here, α is called the f_2- limit of x. The space of all almost convergent double sequences is denoted by C_f. A $P-$ convergent double sequence need not to be almost convergent. However, every bounded convergent double sequence is almost convergent and every almost convergent double sequence is also bounded.

Definition 1.1. [6] A bounded double sequence $x = (x_{kl})$ of real numbers is said to $\sigma-$ convergent to a limit L if

$$P-\lim_{q,r} \tau_{qrst}(x) = L \text{ uniformly in } s, t \in \mathbb{N},$$

where

$$\tau_{qrst}(x) = \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} x_{\sigma^k(s),\sigma^l(t)}.$$

In this case, we write $\sigma_2 - \lim x = L$. The set of all bounded $\sigma-$ convergent double sequences is denoted by V_σ^2. Clearly, $C_{bp} \subset V_\sigma^2$.

Definition 1.2. [26] A topological vector space λ over $\mathbb{R}$ or $\mathbb{C}$ is called locally convex if it is a Hausdorff space such that every neighbourhood of any $x \in \lambda$ contains a convex neighbourhood of x.

Definition 1.3. [30] A locally convex double sequence space λ is called a $DK-$ space if all of the seminorms $r_{kl} : \lambda \to \mathbb{R}, x = (x_{kl}) \to |x_{kl}|$ for all $k, l \in \mathbb{N}$ are continuous. A $DK-$ space with a Frechet topology is called an $FDK-$ space. A normed $FDK-$ space is called a $BDK-$ space.

Definition 1.4. [26] Let λ be a vector space over the field $\mathbb{C}$ and let A, B be subsets of λ. Then A absorbs B if there exists $\alpha_0 \in \mathbb{C}$ such that $B \subset \alpha A$ whenever $|\alpha| > |\alpha_0|$. A subset C of λ is circled if $\alpha C \subset C$ whenever $|\alpha| \leq 1$.

Definition 1.5. [26] A locally convex space λ is bornological if every circled, convex subset $A \subset \lambda$ that absorbs every bounded set in λ is a neighbourhood of 0 in λ.

Definition 1.6. [5] Let λ be a locally convex space. Then a subset is called barrel if it is absolutely convex, absorbing and closed in λ. Moreover, λ is called a barreled space if each barrel is a neighbourhood of zero.

Lemma 1.7. [26] *Every Banach space and every Fréchet space is a barreled space.*

Lemma 1.8. [26] *Every Fréchet space and hence every Banach space is a bornological.*

Lemma 1.9. [5] *Let (X,p) be a seminormed space and q be a seminorm on X. Then the following are equivalent:*
(a) q is continuous.
(b) q is continuous at zero.
(c) There exists $M>0$ such that $q(x)\leq Mp(x)$ for all $x\in X$.

Altay and Başar [1] introduced the space BS of bounded series as follows:

$$BS=\Big\{x=(x_{kl})\in\Omega:||x||_{BS}=\sup_{m,n\in\mathbb{N}}\Big|\sum_{k,l=0}^{m,n}x_{kl}\Big|<\infty\Big\}.$$

The space is also a Banach space with the norm $\|.\|_{BS}$.
One can refer to Mursaleen and Mohiuddine [19] for relevant terminology and required details on the spaces of double sequences and related topics.

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later the concept have been studied by Bektaş et al. [3] and Et et al. [8]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [27] who studied the spaces $l_\infty(\Delta_v)$, $c(\Delta_v)$ and $c_0(\Delta_v)$ where m,v are non-negative integers. Now, for $Z=c,c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^m)=\{x=(x_k)\in\Omega:(\Delta^m x_k)\in Z\},$$

where $\Delta^m x=(\Delta^m x_k)=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})$ and $\Delta^0 x_k=x_k$ for all $k\in\mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m x_k=\sum_{v=0}^{m}(-1)^v\begin{pmatrix}m\\v\end{pmatrix}x_{k+v}.$$

Taking $m=1$, we get the spaces studied by Et and Çolak [7].

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\longrightarrow\infty$ as $x\longrightarrow\infty$.
Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x=(x_k)$, then

$$\ell_M=\Big\{x\in w:\sum_{k=1}^{\infty}M\Big(\frac{|x_k|}{\rho}\Big)<\infty,\ \text{ for some }\ \rho>0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x||=\inf\Big\{\rho>0:\sum_{k=1}^{\infty}M\Big(\frac{|x_k|}{\rho}\Big)\leq 1\Big\}.$$

It is shown in [12] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(p\geq 1)$. The Δ_2-condition is equivalent to $M(Lx)\leq kLM(x)$ for all values of $x\geq 0$ and for $L>1$. For more details about sequence spaces (see [9], [20], [23], [24], [25]) and references therein .
A sequence $\mathcal{M}=(M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [14],[22]). A sequence $\mathcal{N}=(N_k)$ is defined by

$$N_k(v)=\sup\{|v|u-(M_k):u\geq 0\},\ k=1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. In the present paper we define the following classes of sequences:

$C_f(\mathcal{M}, u, \Delta^m, p) =$

$$\Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl}\Big(\frac{u_{kl}\Delta^m x_{kl} - \alpha}{\varrho}\Big)\Big|^{p_{kl}} = 0,$$

for some $\varrho > 0\Big\}$ and

$C_{f_0}(\mathcal{M}, u, \Delta^m, p) =$

$$\Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}} =$$

$0,$ for some $\varrho > 0\Big\}$.

Remark 1.10. Let us consider a few special cases of the above sequence spaces:
(i) If we take $\mathcal{M}(x) = x$, then the above classes of sequences reduces to following sequence spaces:

$C_f(u, \Delta^m, p) =$

$$\Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} \Big(\frac{u_{kl}\Delta^m x_{kl} - \alpha}{\varrho}\Big)\Big|^{p_{kl}} = 0,$$

for some $\varrho > 0\Big\}$ and

$C_{f_0}(u, \Delta^m, p) =$

$$\Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} \Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}} = 0,$$

for some $\varrho > 0\Big\}$.
(ii) If we take $p = (p_{kl}) = 1$, then the above sequence space becomes

$C_f(\mathcal{M}, u, \Delta^m) =$

$$\Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} M_{kl}\Big(\frac{u_{kl}\Delta^m x_{kl} - \alpha}{\varrho}\Big)\Big| = 0,$$

for some $\varrho > 0\Big\}$ and

$C_{f_0}(\mathcal{M}, u, \Delta^m) =$

$$\Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big| = 0,$$

for some $\varrho > 0\Big\}$.

(iii) If we take $\mathcal{M}(x) = x$, $p = (p_{kl}) = 1$, $u = (u_{kl}) = 1$ $m = 0$ and $\varrho = 1$, then we have

$$C_f = \Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=s}^{s+q} \sum_{l=t}^{t+r} \Big(x_{kl} - \alpha\Big)\Big| = 0 \text{ and}$$

$$C_{f_0} = \Big\{x = (x_{kl}) \in \Omega : P - \lim_{q,r\to\infty} \sup_{s,t>0} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} \Big(x_{k+s,l+t}\Big)\Big| = 0\Big\}.$$

which were introduced and studied by Yeşilkayagiil and Başar [29].

The main purpose of the present paper is to show that the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are BDK-spaces, Barreled and Bornological. Furthermore, we also studied some inclusion relations between these spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are Banach spaces with the supremum norm.*

Proof. We are going to prove this for the space $C_f(\mathcal{M}, u, \Delta^m, p)$ and the other can be proved in the similar way. Define norm $||.||$ on $C_f(\mathcal{M}, u, \Delta^m, p)$ as:

$$||x||_{C_f}^{\mathcal{M},u,\Delta^m,p} = \sup_{q,r,s,t\in\mathbb{N}} \Big|\frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}.$$

Clearly, $C_f(\mathcal{M}, u, \Delta^m, p)$ is a normed linear space by the above defined norm. Now, we have to prove that $C_f(\mathcal{M}, u, \Delta^m, p)$ is complete. For this, let $(\Delta^m x^{(b)})$ be a Cauchy sequence in $C_f(\mathcal{M}, u, \Delta^m, p)$. Then $(\Delta^m x_{kl}^{(b)})$ is a Cauchy sequence in $\mathbb{C}$, for each k, l. Therefore, $\Delta^m x_{kl}^{(b)} \to \Delta^m x_{kl}$ (say). Put $\Delta^m x = (\Delta^m x_{kl})$. For given ϵ, there exists an integer $N(\epsilon) = N$ (say) such that for each $b, n > N$

$$||\Delta^m x^{(b)} - \Delta^m x^{(n)}|| < \frac{\epsilon}{2}.$$

Hence,

$$\sup_{q,r,s,t} |\tau_{qrst}(\Delta^m x^{(b)} - \Delta^m x^{(n)})| < \frac{\epsilon}{2}.$$

Then, for each q, r, s, t and $b, n > N$, we have

$$|\tau_{qrst}(\Delta^m x^{(b)} - \Delta^m x^{(n)})| < \frac{\epsilon}{2}. \tag{2.1}$$

Now, for fixed b, the above inequality holds. Since for fixed $b, \Delta^m x^{(b)} \in C_f(\mathcal{M}, u, \Delta^m, p)$, we get

$$\lim_{q,r\to\infty} \tau_{qrst}(\Delta^m x^{(b)}) = L,$$

uniformly in s,t.
For given $\epsilon > 0$, there exists positive integers q_0, r_0 such that

$$|\tau_{qrst}(\Delta^m x^{(b)}) - L| < \frac{\epsilon}{2}, \tag{2.2}$$

for $q \geq q_0, r \geq r_0$ and for all s,t. Here q_0, r_0 are independent of s,t but depend upon ϵ.
Now, by using (2.1) and (2.2), we obtain

$$\begin{aligned} |\tau_{qrst}(\Delta^m x) - L| &= |\tau_{qrst}(\Delta^m x) - \tau_{qrst}(\Delta^m x^{(b)}) + \tau_{qrst}(\Delta^m x^{(b)}) - L| \\ &\leq |\tau_{qrst}(\Delta^m x) - \tau_{qrst}(\Delta^m x^{(b)})| + |\tau_{qrst}(\Delta^m x^{(b)}) - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for $q \geq q_0, r \geq r_0$ and for all s,t.
Hence, $\Delta^m x = (\Delta^m x_{kl}) \in C_f(\mathcal{M}, u, \Delta^m, p)$ and so $C_f(\mathcal{M}, u, \Delta^m, p)$ is complete. This completes the proof. □

Corollary 2.2. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are barreled spaces.*

Proof. Since $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are Banach spaces with the supremum norm and every Banach space is a barreled space, it follows that $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are also barreled spaces. □

Corollary 2.3. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are bornological spaces.*

Proof. By the same arguement given in the above corollary, the above spaces are bornological spaces. □

Theorem 2.4. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the sequence spaces $C_f(\mathcal{M}, u, \Delta^m, p)$ and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ are $BDK-$spaces with the norm defined as:*

$$||x||_{C_f}^{\mathcal{M},u,\Delta^m,p} = \sup_{q,r,s,t\in\mathbb{N}} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}.$$

Proof. We will prove it for the space $C_f(\mathcal{M}, u, \Delta^m, p)$ and rest can be proved in a similar way. Since every normed space is a seminormed space, it follows that $C_f(\mathcal{M}, u, \Delta^m, p)$ is also a seminormed space with respect to the given norm. Now for $x = (x_{kl})$ in $C_f(\mathcal{M}, u, \Delta^m, p)$, we define some new seminorm $r_{kl} : C_f(\mathcal{M}, u, \Delta^m, p) \to \mathbb{R}$ as $r_{kl}(x_{kl}) = |x_{kl}|$ for all $k,l \in \mathbb{N}$. We have to show that the each one is continuous. By Lemma (1.9), there exists $T > 0$ for all $x \in C_f(\mathcal{M}, u, \Delta^m, p)$ such that

$$r_{kl}(x) = |x_{kl}| \leq T||x||_{C_f}^{\mathcal{M},u,\Delta^m,p} \ \ \forall \ \ k,l \in \mathbb{N}.$$

So, the seminorm is continuous. Therefore, $C_f(\mathcal{M}, u, \Delta^m, p)$ is a $DK-$ space and so is Banach space, it follows that $C_f(\mathcal{M}, u, \Delta^m, p)$ has Frechet topology. Thus, it is $BDK-$space with the above given norm. Hence, the proof is complete. □

Theorem 2.5. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Suppose that $\beta = \lim\limits_{w\to\infty} \frac{M_{kl(w)}}{w} < \infty$. Then, $C_{f_0}(u, \Delta^m, p) = C_{f_0}(\mathcal{M}, u, \Delta^m, p)$.*

Proof. In order to prove that $C_{f_0}(u, \Delta^m, p) = C_{f_0}(\mathcal{M}, u, \Delta^m, p)$, it is sufficient to show that $C_{f_0}(\mathcal{M}, u, \Delta^m, p) \subset C_{f_0}(u, \Delta^m, p)$. Now, let $\beta > 0$. By definition of β, we have $w \leq \frac{1}{\beta} M_{kl}(w), \ \forall \ w \geq 0$. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Thus, we have

$$\lim_{q,r\to\infty} \sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{p_{kl}}$$

$$\leq \frac{1}{\beta} \lim_{q,r\to\infty} \sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{p_{kl}}$$

which implies that $x = (x_{kl}) \in C_{f_0}(u, \Delta^m, p)$. □

Theorem 2.6. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. If $p = (p_{kl})$ and $v = (v_{kl})$ are bounded sequences of positive real numbers with $0 \leq p_{kl} \leq v_{kl} < \infty \ \forall \ k, l$; then*

$$C_{f_0}(\mathcal{M}, u, \Delta^m, p) \subset C_{f_0}(\mathcal{M}, u, \Delta^m, v).$$

Proof. Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Then

$$\sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{p_{kl}} \to 0 \ \text{ as } \ q, r \to \infty.$$

This implies that

$$\Big| M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{p_{kl}} \leq 1,$$ for sufficiently large values of k and l. Since (M_{kl}) is increasing and $p_{kl} \leq v_{kl}$, we have

$$\sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{v_{kl}}$$

$$\leq \sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{p_{kl}}.$$

Thus, $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, v)$. This completes the proof. □

Theorem 2.7. *Let $\mathcal{M} = (M_{kl})$ be Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of positive real numbers and $u = (u_{kl})$ be a double sequence of strictly positive real numbers. Then the following inclusions hold:*
(i) If $0 < \inf p_{kl} < p_{kl} \leq 1$ then $C_{f_0}(\mathcal{M}, u, \Delta^m, p) \subset C_{f_0}(\mathcal{M}, u, \Delta^m)$;
(ii) If $1 \leq p_{kl} \leq \sup p_{kl} < \infty$ then $C_{f_0}(\mathcal{M}, u, \Delta^m) \subset C_{f_0}(\mathcal{M}, u, \Delta^m, p)$.

Proof. (i) Let $x = (x_{kl}) \in C_{f_0}(\mathcal{M}, u, \Delta^m, p)$. Then, since $0 < \inf p_{kl} < p_{kl} \leq 1$, we obtain the following:

$$\lim_{q,r\to\infty} \sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|$$

$$\leq \lim_{q,r\to\infty} \sup_{s,t>0} \Big| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q} \sum_{l=0}^{r} M_{kl} \Big(\frac{u_{kl} \Delta^m x_{k+s,l+t}}{\varrho} \Big) \Big|^{p_{kl}}.$$

Thus, $x=(x_{kl})\in C_{f_0}(\mathcal{M},u,\Delta^m)$.
(ii) Let $p=(p_{kl})\geq 1$ for each k and l and $\sup p_{kl}<\infty$. Let $x=(x_{kl})\in C_{f_0}(\mathcal{M},u,\Delta^m)$. Then for each $0<\epsilon<1$, there exists a positive integer N such that

$$\lim_{q,r\to\infty}\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|\leq\epsilon<1\ \ \forall\ \ q,r\geq N.$$

This implies that

$$\lim_{q,r\to\infty}\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}$$

$$\leq\ \lim_{q,r\to\infty}\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|.$$

Therefore, $x=(x_{kl})\in C_{f_0}(\mathcal{M},u,\Delta^m,p)$. This concludes the proof. □

Theorem 2.8. *Let $\mathcal{M}=(M_{kl})$ and $\mathcal{M}'=(M'_{kl})$ be two Musielak-Orlicz functions, $p=(p_{kl})$ be a bounded sequence of positive real numbers and $u=(u_{kl})$ be a double sequence of strictly positive real numbers. Then,*

$$C_{f_0}(\mathcal{M},u,\Delta^m,p)\cap C_{f_0}(\mathcal{M}',u,\Delta^m,p)\subset C_{f_0}(\mathcal{M}+\mathcal{M}',u,\Delta^m,p).$$

Proof. Let $x=(x_{kl})\in C_{f_0}(\mathcal{M},u,\Delta^m,p)\cap C_{f_0}(\mathcal{M}',u,\Delta^m,p)$. Then

$$\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}\to 0\ \text{ as }\ q,r\to\infty$$

and

$$\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M'_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}\to 0\ \text{ as }\ q,r\to\infty.$$

Then, we have

$$\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}(M_{kl}+M'_{kl})\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}$$

$$\leq\ K\Big[\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}\Big]$$

$$+\ K\Big[\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M'_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}\Big]\to 0\ \text{ as }\ q,r\to\infty.$$

Thus, $\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}(M_{kl}+M'_{kl})\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}\to 0\ \text{ as }\ q,r\to\infty.$

Therefore, $x=(x_{kl})\in C_{f_0}(\mathcal{M}+\mathcal{M}',u,\Delta^m,p)$. This completes the proof. □

Theorem 2.9. *Let $\mathcal{M}=(M_{kl})$ and $\mathcal{M}'=(M'_{kl})$ be two Musielak-Orlicz functions, $p=(p_{kl})$ be a bounded sequence of positive real numbers and $u=(u_{kl})$ be a double sequence of strictly positive real numbers. Then,*

$$C_{f_0}(\mathcal{M}',u,\Delta^m,p)\subset C_{f_0}(\mathcal{M}\circ\mathcal{M}',u,\Delta^m,p).$$

Proof. Let $x=(x_{kl})\in C_{f_0}(\mathcal{M}',u,\Delta^m,p)$. Then, we have

$$\lim_{q,r\to\infty}\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}M'_{kl}\Big(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\Big)\Big|^{p_{kl}}=0.$$

Let $\epsilon>0$ and choose $\delta>0$ with $0<\delta<1$ such that $M_{kl}(n)<\epsilon$ for $0\leq n\leq\delta$.

Write $y_{kl} = \left[M'_{kl}\left(\frac{u_{kl}\Delta^m x_{k+s,l+t}}{\varrho}\right)\right]$ and consider

$$\sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{k,l=0,0}^{q,r} M_{kl}(y_{kl})\right|^{p_{kl}}$$

$$= \sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{1} M_{kl}(y_{kl})\right|^{p_{kl}} + \sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{2} M_{kl}(y_{kl})\right|^{p_{kl}}$$

where the first summation is over $y_{kl} \leq \delta$ and second summation is over $y_{kl} > \delta$. Since M_{kl} is continuous, we have

$$\sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{1} M_{kl}(y_{kl})\right|^{p_{kl}} < \epsilon^H \tag{2.3}$$

where $H = \sup p_{kl}$ and for $y_{kl} > \delta$, we use the fact that

$$y_{kl} < \frac{y_{kl}}{\delta} < 1 + \frac{y_{kl}}{\delta}$$

By the definition, we have for $y_{kl} > \delta, M_{kl}(y_{kl}) < 2M_{kl}(1)\frac{y_{kl}}{\delta}$. Hence,

$$\sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{2} M_{kl}(y_{kl})\right|^{p_{kl}}$$

$$\leq \max\left(1, (2F_k(1)\delta^{-1})^H\right) \sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{k,l=0,0}^{q,r} y_{kl}\right|^{p_{kl}} \tag{2.4}$$

Therefore, from equations (2.3) and (2.4), we have

$$C_{f_0}(\mathcal{M}', u, \Delta^m, p) \subset C_{f_0}(\mathcal{M} \circ \mathcal{M}', u, \Delta^m, p).$$

This completes the proof. □

Theorem 2.10. *Let BS be the space of bounded series of double sequences and $C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ be the space of all almost null double sequences. Then the inclusion relation $BS \subset C_{f_0}(\mathcal{M}, u, \Delta^m, p)$ holds.*

Proof. Let $x = (x_{kl}) \in BS$. Then $T = \sup_{s,t\in\mathbb{N}}\left|\sum_{k,l=0,0}^{s,t} x_{kl}\right| < \infty$.

Therefore, for all $q, r, s, t \in \mathbb{N}$, we have

$$\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r} x_{kl}\right| = \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t+r} x_{kl} - \sum_{k=0}^{s-1}\sum_{l=0}^{t+r} x_{kl} - \sum_{k=s}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right|$$

$$\leq \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t+r} x_{kl}\right| + \left|\sum_{k=s}^{s+q}\sum_{l=0}^{t+r} x_{kl}\right| + \left|\sum_{k=s}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right|$$

$$\leq 2T + \left|\sum_{k=s}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right|$$

$$= 2T + \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t-1} x_{kl} - \sum_{k=0}^{s}\sum_{l=0}^{t-1} x_{kl}\right|$$

$$\leq 2T + \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right| + \left|\sum_{k=0}^{s}\sum_{l=0}^{t-1} x_{kl}\right| \leq 4T,$$

which implies that

$$\Big|\frac{1}{(q+1)(r+1)}\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{kl}}{\varrho}\Big)\Big|^{p_{kl}} \leq \Big|\frac{4T}{(q+1)(r+1)}\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{kl}}{\varrho}\Big)\Big|^{p_{kl}}$$

Further, if we take supremum over $s,t\in\mathbb{N}$ in the above relation and also apply the P-limit as $q,r\to\infty$, then, we have

$$P-\lim_{q,r\to\infty}\sup_{s,t>0}\Big|\frac{1}{(q+1)(r+1)}\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}M_{kl}\Big(\frac{u_{kl}\Delta^m x_{kl}}{\varrho}\Big)\Big|^{p_{kl}}=0,$$

therefore, $x\in C_{f_0}(\mathcal{M},u,\Delta^m,p)$. Hence, the result holds. □

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School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
E-mail address: kuldipraj68@gmail.com
E-mail address: renuanand71@gmail.com

ADDITIVE ρ-FUNCTIONAL INEQUALITIES

HARIN LEE, JAE YOUNG CHA, MIN WOO CHO, MYUNGJUN KWON AND CHOONKIL PARK*

Abstract. In this paper, we solve the additive ρ-functional inequalities

$$\|f(2x-y)+f(y-x)-f(x)\| \leq \|\rho\left(f(x+y)-f(x)-f(y)\right)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho|<1$, and

$$\|f(x+y)-f(x)-f(y)\| \leq \|\rho(f(2x-y)+f(y-x)-f(x))\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho|<\frac{1}{2}$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The stability of quadratic functional equation was proved by Skof [16] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [2, 4, 7, 8, 9, 11, 12, 13, 14, 15, 18] for more information on the stability problems of functional equations.

In Section 2, we solve the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in complex Banach spaces.

In Section 3, we solve the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in complex Banach spaces.

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*Corresponding author.

Throughout this paper, let G be a 2-divisible abelian group. Assume that X is a real or complex normed space with norm $\|\cdot\|$ and that Y is a complex Banach space with norm $\|\cdot\|$.

2. Additive ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive ρ-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If a mapping $f : G \to Y$ satisfies*

$$\|f(2x-y)+f(y-x)-f(x)\| \leq \|\rho\,(f(x+y)-f(x)-f(y))\| \tag{2.1}$$

for all $x, y \in G$, then $f : G \to Y$ is additive.

Proof. Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = 0$ and $y = 0$ in (2.1), we get $\|f(0)\| \leq \|\rho\,(f(0))\|$ and so $f(0) = 0$ with $|\rho| < 1$.

Letting $x = 0$ in (2.1), we get $\|f(-y)+f(y)\| \leq 0$ and so f is an *odd mapping*.

Letting $x = z$ and $y = z - w$ in (2.1), we get

$$\|f(z+w)-f(z)-f(w)\| \leq \|\rho\,(f(2z-w)+f(w-z)-f(z))\| \tag{2.2}$$

for all $z, w \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\|f(2x-y)+f(y-x)-f(x)\| &\leq \|\rho\,(f(x+y)-f(x)-f(y))\| \\ &\leq |\rho|^2\|f(2x-y)+f(y-x)-f(x)\|\end{aligned}$$

and so $f(2x-y)+f(y-x) = f(x)$ for all $x, y \in G$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(2x-y)+f(y-x)-f(x)\| \\ &\quad\leq \|\rho\,(f(x+y)-f(x)-f(y))\| + \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \leq \frac{2\theta}{2^r-2}\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$, in (2.3), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = 0$ in (2.3), we get

$$\|f(2x) + f(-x) - f(x)\| \leq \theta\|x\|^r \tag{2.5}$$

for all $x \in X$.

Letting $x = 0$ in (2.3), we get

$$\|f(y) + f(-y)\| \leq \theta\|y\|^r \tag{2.6}$$

for all $y \in X$.

From (2.5) and (2.6), we get

$$\begin{aligned} \|f(2x) - 2f(x)\| &\leq \|f(2x) + f(-x) - f(x)\| + \|f(x) + f(-x)\| \\ &\leq 2\theta\|x\|^r \end{aligned} \tag{2.7}$$

for all $x \in X$. So,

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2}{2^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2}{2^r}\sum_{j=l}^{m-1} \frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} &\|h(2x-y) + h(y-x) - h(x)\| \\ &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{2x-y}{2^n}\right) + f\left(\frac{y-x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 2^n|\rho| \left\|f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\ &= |\rho|\|h(x+y) - h(x) - h(y)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(2x-y) + h(y-x) - h(x)\| \leq |\rho|\|h(x+y) - h(x) - h(y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\le 2^n \left(\left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\le \frac{4 \cdot 2^n}{(2^r - 2)2^{nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.4). □

Theorem 2.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2}f(2x)\right\| \le \theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j}\theta\|x\|^r \end{aligned} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. □

Remark 2.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we solve and investigate the additive ρ-functional inequality (0.2) in complex Banach spaces.

Lemma 3.1. *If a mapping $f : G \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \|\rho(f(2x-y) + f(y-x) - f(x))\| \tag{3.1}$$

for all $x, y \in G$, then $f : G \to Y$ is additive.

Proof. Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.1), we get $\|f(2x) - 2f(x)\| \leq 0$ and so

$$2f(x) = f(2x) \tag{3.2}$$

for all $x \in G$.

Letting $y = 2x$ in (3.1), we get $\|f(3x) - f(x) - f(2x)\| \leq 0$ and from (3.2),

$$3f(x) = f(3x) \tag{3.3}$$

for all $x \in G$.

Letting $y = -x$ in (3.1), we get $\|f(x) + f(-x)\| \leq \|\rho(f(3x)+f(-2x)-f(x))\|$. From (3.2) and (3.3), $f(3x)+f(-2x)-f(x) = 2f(x)+2f(-x)$, so $\|f(x) + f(-x)\| \leq 0$, and we get

$$f(x) + f(-x) = 0 \tag{3.4}$$

for all $x \in G$. So f is an *odd mapping*.

Letting $x = z, y = z - w$ in (3.1), we get

$$\|f(2z-w) - f(z) - f(z-w)\| \leq \|\rho(f(z+w) + f(-w) - f(z))\|$$

and from (3.4),

$$\|f(2z-w) + f(w-z) - f(z)\| \leq \|\rho(f(z+w) - f(z) - f(w))\| \tag{3.5}$$

for all $z, w \in G$.

It follows from (3.1) and (3.5) that

$$\begin{aligned} \|f(x+y) - f(x) - (y)\| &\leq \|\rho(f(2x-y) + f(y-x) - f(x))\| \\ &\leq |\rho|^2 \|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in G$. So f is *additive*. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\quad \leq \|\rho(f(2x-y) + f(y-x) - f(x))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.4), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r \tag{3.8}$$

for all $x \in X$. So

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2}{2^r}\sum_{j=l}^{m-1} \frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.7).

It follows from (3.6) that

$$\begin{aligned} &\|h(x+y) - h(x) - h(y)\| \\ &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 2^n \left\|\rho\left(f\left(\frac{2x-y}{2^n}\right) + f\left(\frac{y-x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right)\right\| + \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\ &= \|\rho(h(2x-y) + h(y-x) - h(x))\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y) - h(x) - h(y)\| \leq \|\rho(h(2x-y) + h(y-x) - h(x))\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.7). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= 2^n \left\|h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\leq 2^n \left(\left\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\leq \frac{2\cdot 2^n \cdot 2}{(2^r - 2)2^{nr}}\theta\|x\|^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.7). □

Theorem 3.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.4)*. Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r \tag{3.10}$$

for all $x \in X$.

Proof. It follows from (3.8) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \\ &\leq \sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\|x\|^r \end{aligned} \tag{3.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.11) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), we get (3.10).

The rest of the proof is similar to the proof of Theorem 3.2. □

Remark 3.4. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

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Harin Lee, Jae Young Cha, Min Woo Cho, Myungjun Kwon
Mathematics Branch, Seoul Science High School, Seoul 03066, Republic of Korea
E-mail address: harinboy@naver.com; ckwodud99@naver.com; whalsdn9903@naver.com

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Dipartimento di Matematica
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Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
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Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang, Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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On generalized Fibonacci k-sequences and Fibonacci k-numbers

Hee Sik Kim, J. Neggers and Choonkil Park*

Abstract. In this paper analogs of Fibonacci sequences and Fibonacci numbers as well as Fibonacci functions (the case $n = 2$) for cases $n = 3, 4, \cdots$ are introduced. It is shown that these analogs are related to each other in a regular manner and that if $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi_k$ for a Fibonacci k-sequence, then $\varphi_1 = 1, \varphi_2 = \frac{1+\sqrt{5}}{2} < \varphi_3 < \cdots < \varphi_n < \cdots < \lim_{n\to\infty} \varphi_n = 2$. Many identities of types similar to those which hold for the case $n = 2$ (i.e., the Fibonacci case) are also established, indicating the existence of a larger theory of which the Fibonacci case is an integrated part.

1. Introduction

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast (see [1, 3, 4, 9]). Han et al. [5] considered several properties of Fibonacci sequences in arbitrary groupoids. Kim et al. [7] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed these in particular for the case where the groupoid contains idempotents and pre-idempotents.

In [6], Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x + 2) = f(x + 1) + f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$.

Kim et al. [7] discussed Fibonacci functions using the (ultimately) periodicity and they also discussed the exponential Fibonacci functions. Especially, given a non-negative real-valued function, they obtained examples of several classes of exponential Fibonacci functions.

In this paper we introduce the family of Fibonacci k-sequences, where $\{F_n\}_{n=0}^{\omega}$ is a Fibonacci k-sequence provided $F(n+k) = F(n+k-1) + \cdots + F(n)$ for all $n \in \mathbf{N} = \{1, 2, \cdots\}$. Thus, if $k = 2$, then $F(n + 2) = F(n + 1) + F(n)$ and $\{F(n)\}_{n=0}^{\omega}$ is an ordinary Fibonacci sequence. Similarly, for $k = 3$, we obtain the formula $F(n+3) = F(n+2)+F(n+1)+F(n)$, where $F(0), F(1), F(2)$ may be taken as arbitrary elements of an abelian group A, usually taken to be the group of integers, rationals, real or complex numbers. If $F(0) = F(1) = F(2) = 1$, then we will consider this special case, i.e., $\{1, 1, 1, 3, 5, 9, 17, 31, \cdots\}$ the sequence of Fibonacci 3-numbers. The properties of this sequence can be expected to be analogous to those of the sequence $\{1, 1, 2, 3, 5, 8, \cdots\}$ of Fibonacci (i.e., Fibonacci 2-) numbers. Thus, e.g., $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi_3 (\doteqdot 1.839)$ is an analogue of $\varphi_2 = \frac{1+\sqrt{5}}{2}$, the golden section. What this number φ_3 may mean in other settings is itself a question of interest as is the question of

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*Correspondence: +82 2 2220 0892(phone), +82 2 2290 0019(fax) (C. Park).

whether and where this number can be observed in nature. It turns out that the corresponding sequence $\varphi_1 = 1, \varphi_2 = \frac{1+\sqrt{5}}{2} < \varphi_3 < \cdots < \varphi_n < \cdots < \lim_{n\to\infty} \varphi_n = 2$ is itself a sequence of interest as we hope to show below as well.

From Fibonacci sequences to go to Fibonacci functions in a natural way has proven to be interesting and it has yields a theory of such functions [6, 8]. The second part of this paper is concerned with the introduction of Fibonacci k-functions, where a real-valued function $f : \mathbf{R} \to \mathbf{R}$ is a Fibonacci k-function if $f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x)$. Again, in analogy with the class of Fibonacci 2-functions (i.e., Fibonacci functions), we are able to construct many examples of such functions for $k = 3, 4, 5, \cdots$ etc., indicating that there is a great deal of material which has yet to be uncovered in this area. Nevertheless, it is clear that there is much work to be done, some of which we will discuss below in further detail.

2. Preliminaries

A function f defined on the real numbers is said to be a *Fibonacci function* ([5]) if it satisfies the formula

$$f(x+2) = f(x+1) + f(x) \tag{1}$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 2.1. ([6]) Let $f(x) := a^x$ be a Fibonacci function on $\mathbf{R}$ where $a > 0$. Then $a^x a^2 = f(x+2) = f(x+1) + f(x) = a^x(a+1)$. Since $a > 0$, we have $a^2 = a+1$ and $a = \frac{1+\sqrt{5}}{2}$. Hence $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, and the unique Fibonacci function of this type on $\mathbf{R}$.

Example 2.2. ([8]) (a). A function $f(x) = (x - \lfloor x \rfloor)(\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function.
(b). A function $f(x)$ defined by

$$f(x) = \begin{cases} (x - \lfloor x \rfloor)(\frac{1+\sqrt{5}}{2})^x & \text{if } x \in Q \\ -(x - \lfloor x \rfloor)(\frac{1+\sqrt{5}}{2})^x & \text{otherwise} \end{cases}$$

is a Fibonacci function.
(c). A function $f(x) = sin(\pi x)(\frac{\sqrt{5}-1}{2})^x$ is a Fibonacci function.

Proposition 2.3. ([5]) *If $f(x)$ is a Fibonacci function, then*

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$$

3. Higher Fibonacci k-sequences

Let α, β, γ be non-zero integers. A sequence $\{F_n\}$ is said to be an (α, β)-*Fibonacci 2-sequence* if $F_1 = \alpha, F_2 = \beta$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, 3, \cdots$. A sequence $\{F_n\}$ is said to be an (α, β, γ)-*Fibonacci 3-sequence* if $F_1 = \alpha, F_2 = \beta, F_3 = \gamma$ and $F_{n+3} = F_{n+2} + F_{n+1} + F_n$ for $n = 1, 2, 3, \cdots$. Especially, if $\alpha = \beta = 1$ or $\alpha = \beta = \gamma = 1$, then we say $\{F_n\}$ a *Fibonacci 2-sequence* or a *Fibonacci 3-sequence* respectively.

Example 3.1. (a). It is well known that $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$ is the Fibonacci 2-sequence. (b). $\{1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, \cdots\}$ is the Fibonacci 3-sequence.

We give some formulas for the Fibonacci 3-sequence as follows.

Proposition 3.2. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_n = \frac{1}{2}[(F_{n+3} - F_{n+1}) - (F_{n+2} - F_n)]. \tag{1}$$

Proof. Since $F_{n+3} = F_{n+2} + F_{n+1} + F_n$, we obtain $F_{n+3} - F_{n+1} = F_{n+2} + F_n$. It follows that

$$\begin{aligned} 2F_n &= (F_{n+2} + F_n) - (F_{n+2} - F_n) \\ &= (F_{n+3} - F_{n+1}) - (F_{n+2} - F_n) \end{aligned}$$

so that we obtain the equality (1). □

Proposition 3.3. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_{n+2} = \frac{1}{2}[(F_{n+3} - F_{n+1}) + (F_{n+2} - F_n)]. \tag{2}$$

Proof. Since $F_{n+3} = F_{n+2} + F_{n+1} + F_n$, we obtain $F_{n+3} - F_{n+1} = F_{n+2} + F_n$. It follows that

$$\begin{aligned} (F_{n+3} - F_{n+1}) + (F_{n+2} - F_n) &= (F_{n+2} + F_n) + (F_{n+2} - F_n) \\ &= 2F_{n+2} \end{aligned}$$

so that we obtain the equality (2). □

Proposition 3.4. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_n F_{n+2} = \frac{1}{4}[(F_{n+3} - F_{n+1})^2 - (F_{n+2} - F_n)^2]. \tag{3}$$

Proof. It follows that

$$\begin{aligned} 4F_n F_{n+2} &= F_{n+2}^2 + 2F_n F_{n+2} + F_n^2 - F_{n+2}^2 + 2F_n F_{n+2} - F_n^2 \\ &= (F_{n+2} + F_n)^2 - (F_{n+2} - F_n)^2 \\ &= (F_{n+3} - F_{n+1})^2 - (F_{n+2} - F_n)^2 \end{aligned}$$

so that we obtain the equality (3). □

Proposition 3.5. *If $\{F_n\}$ is the Fibonacci 3-sequence, then*

$$F_{n+3} = 2(F_1 + F_2 + \cdots + F_n) + F_{n+1}$$

where $F_0 = -1$.

Proof. Since $F_{n+3} = F_{n+2} + F_{n+1} + F_n$, we obtain $F_3 - F_2 = F_1 + F_0, F_4 - F_3 = F_2 + F_1, \cdots, F_{n+3} - F_{n+2} = F_{n+1} + F_n$, which proves the proposition. □

4. Solutions of a Fibonacci polynomial $\xi_n(x)$.

Given a natural number n, we define a polynomial $\xi_n(x)$ by

$$\xi_n(x) := x^n - x^{n-1} - x^{n-2} - \cdots - x - 1.$$

We call such a polynomial $\xi_n(x)$ a *Fibonacci polynomial*. Let φ_n be the largest real root of the equation $\xi_n(x) = 0$. Then $\xi_n(2) = 2^n - (2^{n-1} + 2^{n-2} + \cdots + 2 + 1) = 1$ and $\xi_n(1) = 1 - n < 0$ when $n > 1$. Let $x \geq 2$. Then $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) \geq x^{n-1} + x^{n-2} + \cdots + x + 1$. It follows that $\xi_n(x) = x^n - (x^{n-1} + x^{n-2} + \cdots + x + 1) \geq 1 = \xi_n(2)$. Hence we obtain $1 \leq \varphi_n \leq 2$ for all $n \in \mathbf{N}$. If $n = 1$, then $\xi_1(x) = x - 1$ and $\varphi_1 = 1$, and if $n = 2$, then $\xi_2(x) = x^2 - x - 1$ and $\varphi_2 = \frac{1+\sqrt{5}}{2}$.

Proposition 4.1. *Let $\{\varphi_n\}$ be the sequence of the largest roots of $\xi_n(x) = 0$. Then it is increasing, i.e., $\varphi_n < \varphi_{n+1}$ for all $n \in \boldsymbol{N}$ and $\lim_{n\to\infty} \varphi_n \leq 2$.*

Proof. Since φ_n is the largest real root of $\xi_n(x) = 0$ for all $n \in \mathbf{N}$, $\xi_{n-1}(\varphi_{n-1}) = 0$. It follows that

$$\varphi_{n-1}^{n-1} = \varphi_{n-1}^{n-2} + \varphi_{n-1}^{n-3} + \cdots + \varphi_{n-1} + 1. \tag{4}$$

If we let $x := \varphi_{n-1}$ in $\xi_n(x) = 0$, then by (4) we have

$$\begin{aligned} \xi_n(\varphi_{n-1}) &= \varphi_{n-1}^k - (\varphi_{n-1}^{n-1} + \varphi_{n-1}^{n-2} + \cdots + \varphi_{n-1} + 1 \\ &= \varphi_{n-1}^n - 2\varphi_{n-1}^{n-1} \\ &= \varphi_{n-1}^{n-1}(\varphi_{n-1} - 2) \\ &< 0. \end{aligned}$$

Since φ_n is the largest real root of $\xi_n(x) = 0$, we obtain $\varphi_{n-1} < \varphi_n$ for all $n \in \mathbf{Z}$ with $n \geq 2$. Since $\{\varphi_n\}$ is an increasing sequence and it is bounded above, we have $\lim_{n\to\infty} \varphi_n \leq 2$. □

Given a Fibonacci 3-sequence $\{F_n\}$, we consider the following sequence $\{\frac{F_{n+1}}{F_n}\}$:

$$\{\frac{F_{n+1}}{F_n}\} = \{1, 1, 3, 1.667, 1.8, 1.889, 1.832, 1.839, 1.842, 1.838, 1.839, 1.839, \cdots\}$$

Thus we expect that the limit $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ is approximately 1.839.

Note that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ for the Fibonacci 2-sequence.

A quick computation yields $\xi_3(1.839) = (1.839)^3 - (1.839)^2 - 1.839 - 1 = 0.002$, i.e., $F_3(1.839) = 0.002 \doteqdot 0$.

Since φ_3 is the largest real root of $\xi_3(x) = 0$, we obtain $\xi_3(x) = (x - \varphi_3)(x^2 + \alpha x + \beta)$ for some $\alpha, \beta \in \mathbf{R}$. It follows that

$$\xi_3(x) = x^3 + (\alpha - \varphi_3)x^2 + (\beta - \alpha\varphi_3)x - \varphi_3\beta$$

It follows that $\alpha - \varphi_3 = -1, \beta - \alpha\varphi_3 = -1$ and $-\varphi_3\beta = -1$. Hence $\beta = \alpha\varphi_3 - 1 = (\varphi_3 - 1)\varphi_3 - 1 = \varphi_3^2 - \varphi_3 - 1$ and $\alpha^2 - 4\beta = -3\varphi_3^2 + 2\varphi_3 + 5 \doteqdot -1.468 < 0$ if we let $\varphi_3 = 1.839$. This means that $x^2 + \alpha x + \beta = 0$ has imaginary roots. Let ρ_1, ρ_2 be roots of $x^2 + \alpha x + \beta = 0$. Then $\rho_i = -\frac{\alpha}{2} \pm i\sqrt{\beta - \frac{1}{4}\alpha^2}$ and $\|\rho_i\| = \frac{1}{\sqrt{\varphi_3}} < 1$. If we let $\tau := arg(-\frac{\alpha}{2} \pm i\sqrt{\beta - \frac{1}{4}\alpha^2})$, then $\rho_1 = \frac{1}{\sqrt{\varphi_3}} \exp^{i\tau}$ and $\rho_2 = \frac{1}{\sqrt{\varphi_3}} \exp^{-i\tau}$. If we let $\varphi_3 \doteqdot 1.839$, then $\varphi_3^2 = 3.382$ and hence $\beta = \varphi_3^2 - \varphi_3 - 1 \doteqdot 0.543$ and $\alpha \doteqdot 0.839$. Hence $x^3 - x^2 - x - 1 \doteqdot (x - 1.839)(x^2 + 0.839x + 0.543)$ and $\rho_i \doteqdot -0.420 \pm 0.606i$.

Now, we may assume the polynomial $x^3 - x^2 - x - 1 = 0$ is the characteristic equation of the linear operator ξ_n, the Fibonacci 3-sequence (as in a linear differential equation: $y''' - y'' - y' - y = 0$) which means that the roots φ_3, $\frac{1}{\sqrt{\varphi_3}}e^{\tau i}$, $\frac{1}{\sqrt{\varphi_3}}e^{-\tau i}$ provide the expression:

$$F_n = A\varphi_3^n + B(\frac{1}{\sqrt{\varphi_3}})^n e^{n\tau i} + C(\frac{1}{\sqrt{\varphi_3}})^n e^{-n\tau i} \tag{5}$$

for some $A, B, C \in \mathbf{R}$.

Theorem 4.2. *If F_n is of the form (5), then*

$$A = \frac{1}{\triangle}[\ \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\varphi_3\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3})(e^{-\tau i} - e^{\tau i})\],$$

where $\triangle = \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3^3})(e^{-\tau i} - e^{\tau i})$.

Proof. If we take $n = 1, 2, 3$ in (5) respectively, then we have

$$\begin{aligned}
1 = F_1 &= A\varphi_3 + B\frac{1}{\sqrt{\varphi_3}}e^{\tau i} + C\frac{1}{\sqrt{\varphi_3}}e^{-\tau i},\\
1 = F_2 &= A\varphi_3^2 + B(\frac{1}{\sqrt{\varphi_3}})^2 e^{2\tau i} + C(\frac{1}{\sqrt{\varphi_3}})^2 e^{-2\tau i},\\
1 = F_3 &= A\varphi_3^3 + B(\frac{1}{\sqrt{\varphi_3}})^3 e^{3\tau i} + C(\frac{1}{\sqrt{\varphi_3}})^3 e^{-3\tau i}.
\end{aligned}$$

Using Cramer's rule we obtain

$$\triangle = \begin{vmatrix} \varphi_3 & \frac{1}{\sqrt{\varphi_3}}e^{\tau i} & \frac{1}{\sqrt{\varphi_3}}e^{-\tau i} \\ \varphi_3^2 & (\frac{1}{\sqrt{\varphi_3}})^2 e^{2\tau i} & (\frac{1}{\sqrt{\varphi_3}})^2 e^{-2\tau i} \\ \varphi_3^3 & (\frac{1}{\sqrt{\varphi_3}})^3 e^{3\tau i} & (\frac{1}{\sqrt{\varphi_3}})^3 e^{-3\tau i} \end{vmatrix}$$

which becomes

$$\triangle = \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\sqrt{\varphi_3}}(1 + \frac{1}{\varphi_3^3})(e^{-\tau i} - e^{\tau i})$$

by simple computations. To obtain A, we change the first column into 1's in $\triangle$.

$$A = \frac{1}{\triangle}\begin{vmatrix} 1 & \frac{1}{\sqrt{\varphi_3}}e^{\tau i} & \frac{1}{\sqrt{\varphi_3}}e^{-\tau i} \\ 1 & (\frac{1}{\sqrt{\varphi_3}})^2 e^{2\tau i} & (\frac{1}{\sqrt{\varphi_3}})^2 e^{-2\tau i} \\ 1 & (\frac{1}{\sqrt{\varphi_3}})^3 e^{3\tau i} & (\frac{1}{\sqrt{\varphi_3}})^3 e^{-3\tau i} \end{vmatrix}$$

which becomes

$$A = \frac{1}{\triangle}[\ \frac{1}{\varphi_3^2}(e^{2\tau i} - e^{-2\tau i})\ +\ \frac{1}{\varphi_3\sqrt{\varphi_3}}(1+\frac{1}{\varphi_3})(e^{-\tau i} - e^{\tau i}),\]$$

by simple computations. This proves the theorem. □

Note that we can obtain the coefficients B, C, but it is not necessary to find those. Since $\frac{1}{\sqrt{\varphi_3}} = \frac{1}{\sqrt{1+\alpha}}$, we obtain that $\lim_{n\to\infty}(\frac{1}{\sqrt{\varphi_3}})^n = 0$ and $\lim_{n\to\infty}(\frac{1}{\sqrt{\varphi_3}})^n[Be^{n\tau i} + Ce^{-n\tau i}] = 0$. Hence $\lim_{n\to\infty}\frac{F_n}{\varphi_3^n} = A$. Thus $\lim_{n\to\infty}\frac{F_{n+1}}{\varphi_3 F_n} = \lim_{n\to\infty}\frac{F_{n+1}/\varphi_3^{n+1}}{F_n/\varphi_3^n} = 1$. It follows that $\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \varphi_3$.

Example 4.3. To approximate A we may take an expression $A \doteqdot \frac{F_n}{\varphi_3^n}$ for a "sufficiently large n and a sufficiently accurate value of φ_3". Thus, if $n = 5$, then $F_5 = 9$ and $(1.839)^5 = 21.033$ and hence $A \doteqdot 0.428$. If $n = 7$, then $F_7 = 31, \varphi^7 = 71.132$ and hence $A \doteqdot \frac{F_7}{\varphi_3^7} \doteqdot 0.436$. If $n = 12$, then $F_{12} = 653, (1.836)^{12} = 1496.145$ and hence $A \doteqdot \frac{F_{12}}{(1.839)^{12}} = 0.436$.

5. Higher Fibonacci k-functions

Let $k \geq 2$ be an integer. A function f defined on the real numbers is said to be a *Fibonacci k-function* if it satisfies the formula

$$f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x+1) + f(x) \tag{2}$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 5.1. Let $f(x) := \varphi_3^x$. Then $f(x+3) = \varphi_3^{x+3} = \varphi_3^x\varphi_3^3$ and $f(x+2) + f(x+1) + f(x) = \varphi_3^{x+2} + \varphi_3^{x+1} + \varphi_3^x = \varphi_3^x[\varphi_3^2 + \varphi_3^2 + 1]$. Since $\varphi_3^3 = \varphi_3^2 + \varphi_3^2 + 1$, $f(x) = \varphi_3^x$ is a Fibonacci 3-function.

In a similar manner, we know that $g(x) = \varphi_2^x$ is a Fibonacci 2-function where $\varphi_2 = \frac{1+\sqrt{5}}{2}$. Similarly, $h(x) = \varphi_k^x$ defines a Fibonacci k-function.

Example 5.2. Let $\varphi_k := \omega\theta$ and $f(x) := A\omega^{\lfloor x\rfloor}\theta^x$ where $A \in \mathbf{R}$. Then we have

$$\begin{aligned} f(x+k) &= A\omega^{\lfloor x+k\rfloor}\theta^{x+k} \\ &= A\omega^{\lfloor x\rfloor}\theta^x\omega^k\theta^k \\ &= A\omega^{\lfloor x\rfloor}\theta^x(1 + \omega\theta + \cdots + \omega^{k-1}\theta^{k-1}) \\ &= A\omega^{\lfloor x\rfloor}\theta^x + A\omega^{\lfloor x+1\rfloor}\theta^{x+1} + \cdots + A\omega^{\lfloor x+k-1\rfloor}\theta^{x+k-1} \\ &= f(x) + f(x+1) + \cdots + f(x+k-1). \end{aligned}$$

Hence $f(x) = A\omega^{\lfloor x \rfloor}\theta^x$ is a Fibonacci k-function.

Note that the collection of all Fibonacci k-functions over the real numbers forms a vector space over $\mathbf{R}$.

Proposition 5.3. *Let $f(x)$ be a Fibonacci k-function. If we define $g(x) := f(x+t)$ where $t \in \mathbf{R}$ for any $x \in \mathbf{R}$, then $g(x)$ is also a Fibonacci k-function.*

Proof. Since $f(x)$ is a Fibonacci k-function, we have

$$\begin{aligned} g(x+k) &= f(x+k+t) \\ &= f(x+t+k-1) + f(x+t+k-2) + \cdots + f(x+t) \\ &= g(x+k-1) + g(x+k-2) + \cdots + g(x+1) + g(x), \end{aligned}$$

proving that $g(x)$ is also a Fibonacci k-function. □

For example, a function $g(x) := (\varphi_3)^{x+t}$ is a Fibonacci 3-function.

Theorem 5.4. *Let φ_k be the largest real root of $F_k(x) = 0$ and let $\varphi_k = \omega_1\theta_1 = \omega_2\theta_2 = \cdots = \omega_N\theta_N$, where $\omega_i, \theta_i \in \mathbf{R}$. If we define $F(x) := \sum_{i=1}^{N} A_i\omega_i^{\lfloor x \rfloor}\theta_i^x$, then $F(x)$ is a Fibonacci k-function, where $A_i \in \mathbf{R}$.*

Proof. Given $x \in \mathbf{R}$, we have

$$\begin{aligned} F(x+k) &= \sum_{i=1}^{N} A_i\omega_i^{\lfloor x+k \rfloor}\theta_i^{x+k} \\ &= \sum_{i=1}^{N} A_i\omega_i^{\lfloor x \rfloor}\theta_i^x\omega_i^k\theta_i^k \\ &= \sum_{i=1}^{N} A_i\omega_i^{\lfloor x \rfloor}\theta_i^x\varphi_k^k \\ &= \sum_{i=1}^{N} A_i\omega_i^{\lfloor x \rfloor}\theta_i^x(1 + \omega_i\theta_i + \cdots + \omega_i^{k-1}\theta_i^{k-1}) \\ &= \sum_{i=1}^{N} A_i\omega_i^{\lfloor x \rfloor}\theta_i^x + \sum_{i=1}^{N} A_i\omega_i^{\lfloor x+1 \rfloor}\theta_i^{x+1} + \cdots + \sum_{i=1}^{N} A_i\omega_i^{\lfloor x+k-1 \rfloor}\theta_i^{x+k-1} \\ &= F(x) + F(x+1) + \cdots + F(x+k-1), \end{aligned}$$

proving that $F(x)$ is a Fibonacci k-function. □

Example 5.5. In Example 5.2, if we let $A := 1$ and $\omega := n$ a natural number, then $\varphi_k = n\theta$. Assume $f(x) := n^{\lfloor x \rfloor}\theta^x$. Then $f(x) = n^{\lfloor x \rfloor}\theta^x = n^{\lfloor x \rfloor}(\frac{\varphi_k}{n})^x = n^{\lfloor x \rfloor - x}(\varphi_k)^x$. If we let $x := 2.5$, then $\lfloor 2.5 \rfloor = 2$. Hence $f(2.5) = n^{\lfloor 2.5 \rfloor - 2.5}(\varphi_k)^{2.5} = \frac{1}{\sqrt{n}}(\varphi_k)^2\sqrt{\varphi_k}$. If we let $n := 8$, since $\{\varphi_k\}$ goes to 2, we obtain $f(2.5) \doteqdot \frac{1}{\sqrt{8}}2^2\sqrt{2} = 2$.

Proposition 5.6. *Let $f(x)$ be a Fibonacci k-function and differentiable (integrable, resp.). Then its derivative(integration, resp.) is also a Fibonacci k-function.*

Example 5.7. Consider a Fibonacci k-function $f(x) := \varphi_k^x$. Then $f(x) = e^{x \ln \varphi_k}$ and hence $f'(x) = (e^{x \ln \varphi_k})' = \ln \varphi_k e^{x \ln \varphi_k} = \varphi_k^x \ln \varphi_k = f(x) \ln \varphi_k$. Hence $f'(x)$ is a Fibonacci k-function. Similarly, if we define $g(x) := \int_0^x e^{u \ln \varphi_k} du$, then $g(x) = \frac{1}{\ln \varphi_k}[\varphi_k^x - 1]$ and $g'(x) = \frac{1}{\ln \varphi_k}[\varphi_k^x \ln \varphi_k - 0] = \varphi_k^x$. Hence $g(x)$ is also a Fibonacci k-function.

Example 5.8. In Example 5.2, if $\varphi_k = \omega\theta$, then $f(x) := A\omega^{\lfloor x \rfloor}\theta^x$ is a Fibonacci k-function. Since $\frac{f(x+1)}{f(x)} = (A\omega^{\lfloor x+\rfloor}\theta^{x+1})/(A\omega^{\lfloor x \rfloor}\theta^x) = \omega\theta = \varphi_k$, we obtain $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \varphi_k$.

Conjecture. *If $f(x)$ is a Fibonacci k-function, then*

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \varphi_k$$

It is known that if $f(x)$ is a Fibonacci 2-function, then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \varphi_2$ [5].

6. Fibonacci k-sequences and Fibonacci k-numbers

Proposition 6.1. *Let $f(x)$ be a Fibonacci 3-function, then*

$$f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$$

for all natural numbers n, where $\{\alpha_n\}$ is an $(1,2,4)$-Fibonacci 3-sequence, $\{\beta_n\}$ is an $(1,2,3)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is an $(1,1,2)$-Fibonacci 3-sequence.

Proof. Let $f(x)$ be a Fibonacci 3-function. Then $f(x+3) = f(x+2) + f(x+1) + f(x)$, and $f(x+4) = 2f(x+2) + 2f(x+1) + f(x)$ for all $x \in \mathbf{R}$. In this fashion we obtain

$$\begin{aligned}
f(x+5) &= 4f(x+2) + 2f(x+1) + 2f(x),\\
f(x+6) &= 7f(x+2) + 6f(x+1) + 4f(x),\\
f(x+7) &= 13f(x+2) + 11f(x+1) + 7f(x),\\
f(x+8) &= 24f(x+2) + 20f(x+1) + 13f(x).
\end{aligned}$$

The sequence $\{\alpha_n\}$ of coefficients of $f(x+2)$ is $1, 2, 4, 7, 13, 24, \cdots$, the sequence $\{\beta_n\}$ of the coefficients of $f(x+1)$ is $1, 2, 3, 6, 11, 20, \cdots$ and the sequence $\{\gamma_n\}$ of coefficients of $f(x)$ is $1, 1, 2, 4, 7, 13, \cdots$. This shows that $\{\alpha_n\}$ is the $(1,2,4)$-Fibonacci 3-sequence, $\{\beta_n\}$ is the $(1,2,3)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is the $(1,1,2)$-Fibonacci 3-sequence. This shows that $f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$, proving the theorem. □

Corollary 6.2. *Given a natural number n, we have*

$$\varphi_3^{n+2} = \alpha_n \varphi_3^2 + \beta_n \varphi_3 + \gamma_n$$

where $\{\alpha_n\}$ is the $(1,2,4)$-Fibonacci 3-sequence, $\{\beta_n\}$ is the $(1,2,3)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is the $(1,1,2)$-Fibonacci 3-sequence.

Proof. $f(x) = \varphi_3^x$ is a Fibonacci 3-function. □

Theorem 6.3. *Let a, b, c be non-zero integers. If we define $f(x+3) := af(x+2) + bf(x+1) + cf(x)$ for all $x \in \mathbf{R}$, then*

$$f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$$

for all $x \in \mathbf{R}$ and $n \geq 4$, where $\{\alpha_n\}$ is a $(a, a^2+b, a^3+2ab+c)$-Fibonacci 3-sequence, $\{\beta_n\}$ is a $(b, ab+c, a^2b+ac+b^2)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is a (c, ac, a^2c+bc)-Fibonacci 3-sequence.

Proof. Since $f(x+3) := af(x+2) + bf(x+1) + cf(x)$, we let $(\alpha_1, \beta_1, \gamma_1) := (a, b, c)$. $f(x+4) = af(x+3)+bf(x+2)+cf(x+1) = a[af(x+2)+bf(x+1)+cf(x)] = (a^2+b)f(x+2)+(ab+c)f(x+1)+acf(x)$ leads to $(\alpha_2, \beta_2, \gamma_2) = (a^2+b, ab+c, ac)$. By simple computations, we obtain $f(x+5) = (a^3+2ab+c)f(x+2) + (a^2b+ac+b^2)f(x+1) + (a^2c+bc)f(x)$. Let $(\alpha_3, \beta_3, \gamma_3) := (a^3+2ab+c, a^2b+ac+b^2, a^2c+bc)$. We compute $f(x+6)$ as follows:

$$\begin{aligned} f(x+6) &= af(x+5) + bf(x+4) + cf(x+3) \\ &= a[\alpha_3 f(x+2) + \beta_3 f(x+1) + \gamma_3 f(x)] \\ &+ b[\alpha_2 f(x+2) + \beta_2 f(x+1) + \gamma_2 f(x)] \\ &+ c[\alpha_1 f(x+2) + \beta_1 f(x+1) + \gamma_1 f(x)] \\ &= (a\alpha_3 + b\alpha_2 + c\alpha_1) f(x+2) + (a\beta_3 + b\beta_2 + c\beta_1) f(x+1) \\ &+ (a\gamma_3 + b\gamma_2 + c\gamma_1) f(x), \end{aligned}$$

i.e., $\alpha_4 = a\alpha_3 + b\alpha_2 + c\alpha_1, \beta_4 = a\beta_3 + b\beta_2 + c\beta_1$ and $\gamma_4 = a\gamma_3 + b\gamma_2 + c\gamma_1$. If we let $\alpha_n = a\alpha_{n-1} + b\alpha_{n-2} + c\alpha_{n-3}, \beta_n = a\beta_{n-1} + b\beta_{n-2} + c\beta_{n-3}$ and $\gamma_n = a\gamma_{n-1} + b\gamma_{n-2} + c\gamma_{n-3}$ for $n \geq 4$, then $\{\alpha_n\}$ is an $(a, a^2+b, a^3+2ab+c)$-Fibonacci 3-sequence, $\{\beta_n\}$ is a $(b, ab+c, a^2b+ac+b^2)$-Fibonacci 3-sequence and $\{\gamma_n\}$ is a (c, ac, a^2c+bc)-Fibonacci 3-sequence. Hence $f(x+n+2) = \alpha_n f(x+2) + \beta_n f(x+1) + \gamma_n f(x)$ for all $n \geq 4$. This proves the theorem. □

Note that Proposition 6.1 is a special case of Theorem 6.3 if we let $a = b = c = 1$.

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Hee Sik Kim, Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: heekim@hanyang.ac.kr

J. Neggers, Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A
E-mail address: jneggers@as.ua.edu

Choonkil Park, Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

CONVERGENCE OF SP ITERATIVE SCHEME FOR THREE MULTIVALUED MAPPINGS IN HYPERBOLIC SPACE

BIROL GUNDUZ AND IBRAHIM KARAHAN

Abstract. The present paper aims to deal with multivalued version of SP iterative scheme to approximate a common fixed point of three multivalued nonexpansive mappings in a uniformly convex hyperbolic space and obtain strong and Δ-convergence theorems for the SP process. Our results extend some existing results in the contemporary literature.

1. Introduction

Fixed point theory is one of the most important area of nonlinear analysis and it has applications in different disciplines of science such as in economics, biology, chemistry, engineering and technology, game theory and physics. It has become attractive to many scientists because it directly affects our daily lives. Iterative methods play an important role in calculating the fixed point of nonlinear mappings (see [16, 17, 18, 19]). The oldest known iterative method is Picard iteration which is the pioneer of iterative approximation of fixed point of different class of nonlinear mappings.

W.R. Mann [1], S. Ishikawa [2], M. A. Noor [3] introduced the Mann, Ishikawa, Noor iteration process respectively for a single valued map T defined on nonempty subset of a normed space. Metric space versions of these iterations are following:

(Mann) $$u_{n+1} = W\left(u_n, Tu_n, \alpha_n\right),$$

(Ishikawa) $$\begin{array}{c} u_{n+1} = \left(u_n, Tv_n, \alpha_n\right) \\ v_n = \left(u_n, Tu_n\beta_n\right) \end{array},$$

(Noor) $$\begin{array}{c} u_{n+1} = \left(u_n, Tv_n, \alpha_n\right) \\ v_n = \left(u_n, Tw_n, \beta_n\right) \\ w_n = \left(u_n, Tu_n, \gamma_n\right) \end{array},$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0, 1]$.

In 2008, Thianwan [4] introduced a new two steps iteration. Gunduz et al. [20] modified this iteration process as following and used for computing fixed point of nonexpansive mappings in hyperbolic spaces.

(Thianwan) $$\begin{array}{c} u_{n+1} = \left(v_n, Tv_n, \alpha_n\right) \\ v_n = \left(u_n, Tu_n, \beta_n\right) \end{array},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

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In 2011, Phuengrattana and Suantai [5] gave the SP iteration which defined in metric spaces as follows:

$$\text{(SP)} \qquad \begin{aligned} u_{n+1} &= (v_n, Tv_n, \alpha_n) \\ v_n &= (w_n, Tw_n, \beta_n) \\ w_n &= (u_n, Tu_n, \gamma_n) \end{aligned},$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0, 1]$.

They also showed that SP-iteration is a generalized version of Mann, Ishikawa, Noor iterations and converges faster than all of others for the class of non-decreasing and continuous functions and they supported their analytical results by numerical examples. Chugh and Kumar [6] showed that the SP iterative scheme converges faster than the new two step iterative schemes of Thianwan [4] for increasing functions and decreasing functions.

On the other hand, it is well known that the theory of multivalued maps is more complex than according to the theory of single valued maps. Now we discourse on multivalued maps.

Let E be a metric space and D be a nonempty subset of E. If there exists an element $k \in D$ such that

$$d(x, k) = \inf\{d(x, y) : y \in D\} = d(x, D)$$

for each $x \in E$, then the set D is called proximinal. We shall denote the compact subsets, proximinal bounded subsets, and closed and bounded subsets of K by $C(D)$, $P(D)$, and $CB(D)$, respectively. A Hausdorff metric H induced by the metric d of E is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for every $A, B \in CB(E)$. A multivalued mapping $T : D \to P(D)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that for any $x, y \in D$,

$$H(Tx, Ty) \leq kd(x, y),$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in D$. A point $p \in D$ is called a fixed point of T if $p \in Tp$. Denote the set of all fixed points of T by $F(T)$.

By using the Hausdorff metric, Markin [7, 8] studied the fixed points of multivalued nonexpansive mappings and contractions. Moreover, Lim [33] proved the existence of fixed points for multivalued nonexpansive mappings under suitable conditions in uniformly convex Banach spaces. Later on, since the fixed point theory for this kind of mappings has a lot of application areas such as convex optimization problem, control problem and differential inclusion problem (see, [9] and references cited therein), an interesting fixed point theory was developed. From then on different authors have studied on the convergence of fixed points for this class of mappings in convex metric spaces. For instance, Shimizu and Takahashi [25] generalized the results of Lim [33] to the convex metric spaces. The study of multivalued maps is a rapidly growing area of research (see, [10, 11, 12, 13, 14, 15]).

Shahzad and Zegeye [13] studied strong convergenge of Ishikawa iterative process for quasi-nonexpansive multivalued maps in uniformly convex Banach spaces. They defined Ishikawa iteration as follows:

Let $T : D \to P(D)$ and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. The sequence of Ishikawa iteration is defined by $x_0 \in D$,

$$\begin{cases} y_n = (1 - b_n)x_n + b_n z_n, \\ x_{n+1} = (1 - a_n)x_n + a_n z_n^{'}, \end{cases}$$

where $z_n \in Tx_n$, $z_n^{'} \in Ty_n$, and $\{a_n\}, \{b_n\}$ are sequences of real numbers in $[0, 1]$.

In this paper, we first give multivalued version of the SP iteration scheme in Kohlenbach hyperbolic spaces and then use this scheme to approximate a common fixed point of three multivalued nonexpansive mappings. Our results improve and extend current results in recently literature by via faster, more general iteration and more general space.

2. Hyperbolic Spaces and Lemmas

There are different definitions for hyperbolic space in mathematics. We will study in the hyperbolic space defined by Kohlenbach [22]. We review hyperbolic space and it's features in this section.

Definition 1. [22] *A metric space (E, d) is said to be hyperbolic space if there exists a map $W : E^2 \times [0, 1] \to E$ satisfying:*

W1. $d(u, W(x, y, \alpha)) \leq (1 - \alpha) d(u, x) + \alpha d(u, y)$
W2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
W3. $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
W4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha) d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in E$ and $\alpha, \beta \in [0, 1]$.

A metric space (E, d) is called a convex metric space introduced by Takahashi [23] if it satisfies only W1. Every normed space (and Banach space) is a special convex metric space, but the converse of this statement is not true, in general (see [16]). The class of hyperbolic spaces includes normed spaces, the Hilbert ball (see [24] for a book treatment) and CAT (0)-spaces. The readers can found detailed discussion in [21].

Definition 2. [25] *A hyperbolic space (E, d, W) is said to be uniformly convex if for all $u, x, y \in E$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that*

$$\left.\begin{array}{c} d(x, u) \leq r \\ d(y, u) \leq r \\ d(x, y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1 - \delta) r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A subset D of a hyperbolic space E is convex if $W(x, y, \alpha) \in D$ for all $x, y \in D$ and $\alpha \in [0, 1]$.

Now, we give definition of Δ-convergence which coined by Lim [26] in general metric spaces. To give the definition of Δ-convergence, we first recall the notions of asymptotic radius and asymptotic center. Let $\{x_n\}$ be a bounded sequence in a metric space E. For $x \in E$, define a continuous functional $r(., \{x_n\}) : E \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)$. Then the asymptotic radius $\rho = r(\{x_n\})$

of $\{x_n\}$ is given by $\rho = \inf\{r(x,\{x_n\}) : x \in E\}$ and the asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset D of E is defined as follows:

$$A_D(\{x_n\}) = \{x \in E : r(x,\{x_n\}) \leq r(y,\{x_n\}) \text{ for any } y \in D\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [29] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in E is said to Δ-converge to $x \in E$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [27]. In this case, we write Δ -$\lim_n x_n = x$.

We want to point out that $\Delta-$convergence coincides with weak convergence in Banach spaces with Opial's property [30].

Kirk and Panyanak [27] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [28] continued to work in this direction and studied the Δ-convergence of Picard, Mann and Ishikawa iterates in $CAT(0)$ spaces. Khan et al. [31] was studied this concept in hyperbolic spaces and they gave a couple of helpful lemma as follows.

Lemma 1. [31] *Let (E,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in E$ and $\{\alpha_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty} d(x_n,x) \leq r, \limsup_{n\to\infty} d(y_n,x) \leq r$ and $\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Lemma 2. [31] *Let D be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in D such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in D such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

The following useful first lemma can be found in [14] gives some properties of P_T in metric (and hence hyperbolic) spaces and second can be found in [8].

Lemma 3. [14] *Let $T : D \to P(D)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : d(x,y) = d(x,Tx)\}$. Then the following are equivalent.*
(1) *$x \in F(T)$, that is, $x \in Tx$,*
(2) *$P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$,*
(3) *$x \in F(P_T)$, that is, $x \in P_T(x)$.*
Moreover, $F(T) = F(P_T)$.

Lemma 4. *Let $A, B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \eta$.*

3. Strong and Δ-Convergence Theorems

Before giving our main results, we give the multivalued version of the SP iteration scheme (SP) in hyperbolic spaces. Let E be a hyperbolic space and D be a nonempty convex subset of E. Let $T, S, R : D \to P(D)$ be three multivalued maps

and $P_T(x) = \{y \in Tx : d(x,y) = d(x,Tx)\}$. Choose $x_0 \in D$ and define $\{x_n\}$ as

$$\begin{cases} x_{n+1} = W(u_n, y_n, \alpha_n) \\ y_n = W(v_n, z_n, \beta_n) \\ z_n = W(w_n, x_n, \gamma_n) \end{cases}, \tag{3.1}$$

where $u_n \in P_T(y_n)$, $v_n \in P_S(z_n), w_n \in P_R(x_n)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for all $n \in \mathbb{N}$. The iterative sequence (3.1) is called the *modified SP iterative process* for a multivalued nonexpansive mapping in a Kohlenbach hyperbolic space.

Lemma 5. *Let D be a nonempty closed convex subset of a hyperbolic space X and let $S, T, R : D \to P(D)$ be three multivalued mappings such that P_T, P_S and P_R are nonexpansive mappings with a nonempty common fixed point set F. Then for the modified SP iterative process $\{x_n\}$ in (3.1), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. Then $p \in P_T(p) = \{p\}$, $p \in P_S(p) = \{p\}$ and $p \in P_R(p) = \{p\}$. Using (3.1), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(u_n, y_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(u_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n) d(u_n, P_T(p)) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n) H(P_T(y_n), P_T(p)) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n) d(y_n, p) + \alpha_n d(y_n, p) \\ &= d(y_n, p) = d(W(v_n, z_n, \beta_n), p) \\ &\leq (1-\beta_n) d(v_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n) d(v_n, P_S(p)) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n) H(P_S(z_n), P_S(p)) + \alpha_n d(z_n, p) \\ &\leq (1-\beta_n) d(z_n, p) + \alpha_n d(z_n, p) \\ &= d(z_n, p) = d(W(w_n, x_n, \gamma_n), p) \\ &\leq (1-\gamma_n) d(w_n, p) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) d(w_n, P_R(p)) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) H(P_R(x_n), P_R(p)) + \gamma_n d(x_n, p) \\ &\leq (1-\gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{3.2}$$

That is,

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 6. *Let D be a nonempty closed convex subset of a uniformly convex hyperbolic space X and let $S, T, R : D \to P(D)$ be three multivalued mappings such that P_T, P_S and P_R are nonexpansive mappings with a nonempty common fixed point set F. Then for modified SP iterative process $\{x_n\}$ in (3.1), we have $\lim_{n\to\infty} d(x_n, P_T(y_n)) = \lim_{n\to\infty} d(x_n, P_S(z_n)) = \lim_{n\to\infty} d(x_n, P_R(x_n)) = 0$.*

Proof. By Lemma 5, $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F$. Assume that $\lim\limits_{n\to\infty} d(x_n,p)=c$ for some $c\geq 0$. For $c=0$, the result is trivial. Suppose $c>0$.

Now $\lim_{n\to\infty} d(x_{n+1},p)=c$ can be rewritten as

$$\lim_{n\to\infty} d(W(u_n,y_n,\alpha_n),p)=c. \tag{3.3}$$

Since P_T is nonexpansive, we have

$$\begin{aligned}
d(u_n,p) &= d(u_n,P_T(p))\\
&\leq H(P_T(y_n),P_T(p))\\
&\leq d(y_n,p)=d(W(v_n,z_n,\beta_n),p)\\
&\leq (1-\beta_n)d(v_n,p)+\beta_n d(z_n,p)\\
&\leq (1-\beta_n)H(P_S(z_n),P_S(p))+\beta_n d(z_n,p)\\
&\leq (1-\beta_n)d(z_n,p)+\beta_n d(z_n,p)\\
&= d(z_n,p)=d(W(w_n,x_n,\gamma_n),p)\\
&\leq (1-\gamma_n)d(w_n,p)+\gamma_n d(x_n,p)\\
&\leq (1-\gamma_n)H(P_R(x_n),P_R(p))+\gamma_n d(x_n,p)\\
&\leq (1-\gamma_n)d(x_n,p)+\gamma_n d(x_n,p)\\
&\leq d(x_n,p).
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} d(u_n,p)\leq c, \limsup_{n\to\infty} d(z_n,p)\leq c, \limsup_{n\to\infty} d(y_n,p)\leq c. \tag{3.4}$$

Next

$$\begin{aligned}
d(v_n,p) &= d(v_n,P_S(p))\\
&\leq H(P_S(z_n),P_S(p))\\
&\leq d(z_n,p)
\end{aligned}$$

and so

$$\limsup_{n\to\infty} d(v_n,p)\leq c. \tag{3.5}$$

Further

$$\begin{aligned}
d(w_n,p) &= d(w_n,P_R(p))\\
&\leq H(P_R(x_n),P_R(p))\\
&\leq d(x_n,p)
\end{aligned}$$

and so

$$\limsup_{n\to\infty} d(w_n,p)\leq c. \tag{3.6}$$

On the other hand, since

$$d(W(w_n,x_n,\gamma_n),p)=d(z_n,p)\leq d(x_n,p),$$

we have

$$\limsup_{n\to\infty} d(W(w_n,x_n,\gamma_n),p)\leq c. \tag{3.7}$$

From (3.2), we have

$$c\leq \liminf_{n\to\infty} d(W(w_n,x_n,\gamma_n),p) \tag{3.8}$$

So, it follows from (3.7) and (3.8) that

$$\lim_{n\to\infty} d\left(W\left(w_n, x_n, \gamma_n\right), p\right) = \lim_{n\to\infty} d\left(z_n, p\right) = c. \tag{3.9}$$

Since we assume that $\lim_{n\to\infty} d\left(x_n, p\right) = c$, we can write

$$\limsup_{n\to\infty} d\left(x_n, p\right) \leq c. \tag{3.10}$$

By using the inequalities (3.6), (3.9) and (3.10) and Lemma 1, we obtain that

$$\lim_{n\to\infty} d\left(x_n, w_n\right) = 0. \tag{3.11}$$

On the other hand, it follows from (3.3), (3.4) and Lemma 1 that

$$\lim_{n\to\infty} d\left(u_n, y_n\right) = 0.$$

Nothing that

$$\begin{aligned} d\left(x_{n+1}, p\right) &= d\left(W\left(u_n, y_n, \alpha_n\right), p\right) \\ &\leq \left(1-\alpha_n\right) d\left(u_n, p\right) + \alpha_n d\left(y_n, p\right) \\ &\leq \left(1-\alpha_n\right) d\left(y_n, u_n\right) + d\left(u_n, p\right) \end{aligned}$$

which yields that

$$c \leq \liminf_{n\to\infty} d\left(u_n, p\right). \tag{3.12}$$

Then from (3.4) and (3.12), we have

$$c = \lim_{n\to\infty} d\left(u_n, p\right).$$

Since $d\left(x_{n+1}, p\right) \leq d\left(y_n, p\right)$, this implies that

$$\begin{aligned} c &\leq \lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(W\left(v_n, z_n, \beta_n\right), p\right) \\ &\leq \lim_{n\to\infty} \left[\left(1-\beta_n\right) \limsup_{n\to\infty} d\left(v_n, p\right) + \beta_n \limsup_{n\to\infty} d\left(z_n, p\right)\right] \\ &\leq \lim_{n\to\infty} \left[\left(1-\beta_n\right) c + \beta_n c\right] \\ &= c \end{aligned}$$

Hence, we get

$$c = \lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(W\left(v_n, z_n, \beta_n\right), p\right). \tag{3.13}$$

So, it follows from Lemma 1, (3.4), (3.5) and (3.13) that

$$\lim_{n\to\infty} d\left(z_n, v_n\right) = 0$$

and

$$\begin{aligned} d\left(y_n, p\right) &= d\left(W\left(v_n, z_n, \beta_n\right), p\right) \\ &\leq \left(1-\beta_n\right) d\left(v_n, p\right) + \beta_n d\left(z_n, p\right) \\ &\leq \beta_n d\left(z_n, v_n\right) + d\left(v_n, p\right) \end{aligned}$$

this yields that

$$c \leq \liminf_{n\to\infty} d\left(v_n, p\right),$$

so (3.5) gives that

$$c = \lim_{n\to\infty} d\left(v_n, p\right).$$

On the other hand, since

$$\begin{aligned}\lim_{n\to\infty} d(z_n, x_n) &= \lim_{n\to\infty} d(W(w_n, x_n, \gamma_n), x_n) \\ &\leq \lim_{n\to\infty} \left[(1-\gamma_n) \limsup_{n\to\infty} d(x_n, x_n) + \gamma_n \limsup_{n\to\infty} d(w_n, x_n)\right],\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} d(y_n, z_n) &= \lim_{n\to\infty} d(W(v_n, z_n, \beta_n), z_n) \\ &\leq \lim_{n\to\infty} \left[(1-\beta_n) \limsup_{n\to\infty} d(z_n, z_n) + \beta_n \limsup_{n\to\infty} d(v_n, z_n)\right]\end{aligned}$$

we have

$$\lim_{n\to\infty} d(z_n, x_n) = 0$$

and

$$\lim_{n\to\infty} d(y_n, z_n) = 0.$$

Also,

$$d(u_n, x_n) \leq d(u_n, y_n) + d(y_n, z_n) + d(z_n, x_n),$$

then

$$\lim_{n\to\infty} d(u_n, x_n) = 0.$$

Also,

$$d(v_n, x_n) \leq d(v_n, z_n) + d(z_n, x_n),$$

that is,

$$\lim_{n\to\infty} d(v_n, x_n) = 0.$$

Since

$$d(x, P_R(x)) = \inf_{z\in P_R(x)} d(x, z),$$

therefore

$$d(x_n, P_R(x_n)) \leq d(x_n, w_n) \to 0 \text{ as } n \to \infty.$$

Similarly

$$d(x_n, P_T(y_n)) \leq d(x_n, u_n) \to 0$$

and

$$d(x_n, P_S(z_n)) \leq d(x_n, v_n) \to 0$$

as $n \to \infty$. □

Firstly, we prove that modified SP iterative process defined in (3.1) Δ-converges a common fixed point of S, T and R.

Theorem 1. *Let D be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η. Let S, T, R, P_S, P_T, P_R and F be as in Lemma 6. Then the modified SP iterative process $\{x_n\}$ Δ-converges to a p in F.*

Proof. Let $p \in F(T) = F(P_T)$. By the Lemma 5, $\{x_n\}$ is bounded and so $\lim_{n\to\infty} d(x_n, p)$ exists. This gives that $\{x_n\}$ has a unique asymptotic center. In other words, we have $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. From Lemma 6, we have $\lim_{n\to\infty}(x_n, P_S(x_n)) = 0$. We claim that v is a fixed point of P_S.

To prove this, take another sequence $\{z_m\}$ in $P_S(v)$. Then

$$\begin{aligned} r(z_m, \{v_n\}) &= \lim_{n\to\infty} \sup d(z_m, v_n) \\ &\leq \lim_{n\to\infty} \sup \{d(z_m, P_S(v_n)) + d(P_S(v_n), v_n)\} \\ &\leq \lim_{n\to\infty} \sup H(P_S(v), P_S(v_n)) \\ &\leq \lim_{n\to\infty} \sup d(v, v_n) \\ &= r(v, \{v_n\}). \end{aligned}$$

This gives $|r(z_m, \{v_n\} - r(v, \{v_n\})| \to 0$ $(m \to \infty)$. From Lemma 2, we have $\lim_{m\to\infty} z_m = v$. Note that $Sv \in P(K)$ being proximinal is closed, hence $P_S(v)$ is closed. Moreover, $P_S(v)$ is bounded. Consequently $\lim_{m\to\infty} z_m = v \in P_S(v)$. Similarly, $v \in P_T(v)$ and $v \in P_R(v)$. Hence $v \in F$. From the uniqueness of asymptotic center, we have

$$\begin{aligned} \lim_{n\to\infty} \sup d(v_n, v) &< \lim_{n\to\infty} \sup d(v_n, x) \\ &\leq \lim_{n\to\infty} \sup d(x_n, x) \\ &< \lim_{n\to\infty} \sup d(x_n, v) \\ &= \lim_{n\to\infty} \sup d(v_n, v), \end{aligned}$$

and this is a contradiction. Hence $x = u$. Therefore $A(\{v_n\}) = \{v\}$ for every subsequence $\{v_n\}$ of $\{x_n\}$. Hence $\{x_n\}$ Δ-converges to a p in F. □

We now prove a strong convergence theorem. Then we will apply following theorem to obtain new theorem in a complete and uniformly convex hyperbolic space.

Theorem 2. *Let D be a nonempty closed and convex subset of a hyperbolic space E and, S, T, R, P_S, P_T, P_R and F be as in Lemma 6. Let $\{x_n\}$ be the modified SP iterative process defined in* (3.1)*, then $\{x_n\}$ converges strongly to a p in F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By similar methot in Lemma 5, we get

$$d(x_{n+1}, p) \leq d(x_n, p),$$

which implies

$$d(x_{n+1}, F) \leq d(x_n, F).$$

This gives that $\lim_{n\to\infty} d(x_n, F)$ exists and so by the hypothesis of our theorem, $\liminf_{n\to\infty} d(x_n, F) = 0$. Therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

We now prove that $\{x_n\}$ is a Cauchy sequence in D. Let $m, n \in \mathbb{N}$ and assume $m > n$. Then it follows (along the lines similar to Lemma 5) that

$$d(x_m, p) \leq d(x_n, p)$$

for all $p \in F$. Thus we have

$$d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) \leq 2d(x_n, p).$$

Taking inf on the set F, we have $d(x_m, x_n) \leq d(x_n, F)$. We show that $\{x_n\}$ is a Cauchy sequence in D by taking limit as $m \to \infty,\ n \to \infty$ in the inequality $d(x_m, x_n) \leq d(x_n, F)$. Thus, it converges to a $q \in D$. Now it is left to show that $q \in F(S)$. Indeed, by $d(x_n, F(P_S)) = \inf_{y \in F(P_S)} d(x_n, y)$. So for each $\varepsilon > 0$, there exists $p_n^{(\varepsilon)} \in F(P_S)$ such that,

$$d\left(x_n, p_n^{(\varepsilon)}\right) < d(x_n, F(P_S)) + \frac{\varepsilon}{2}.$$

This implies $\lim_{n\to\infty} d\left(x_n, p_n^{(\varepsilon)}\right) \leq \frac{\varepsilon}{2}$. From $d\left(p_n^{(\varepsilon)}, q\right) \leq d\left(x_n, p_n^{(\varepsilon)}\right) + d(x_n, q)$ it follows that

$$\lim_{n\to\infty} d\left(p_n^{(\varepsilon)}, q\right) \leq \frac{\varepsilon}{2}.$$

Finally,

$$\begin{aligned} d(P_S(q), q) &\leq d\left(q, p_n^{(\varepsilon)}\right) + d\left(p_n^{(\varepsilon)}, P_S(q)\right) \\ &\leq d\left(q, p_n^{(\varepsilon)}\right) + H\left(P_S\left(p_n^{(\varepsilon)}\right), P_S(q)\right) \\ &\leq 2d\left(p_n^{(\varepsilon)}, q\right) \end{aligned}$$

shows $d(P_S(q), q) < \varepsilon$. Therefore $d(P_S(q), q) = 0$. In a similar way, we get $d(P_T(q), q) = 0$ and $d(P_R(q), q) = 0$. Since F is closed, therefore $q \in F$. □

As appropriate our aim, we give definition of multivalued version of condition (I) of Senter and Dotson [32] for three maps and definition of semi-compact map.

Definition 3. *The multivalued nonexpansive mappings $S, T, R : D \to CB(D)$ where D a subset of E, are said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0,\ f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{3}[d(x, Sx) + d(x, Tx) + d(x, Rx)] \geq f(d(x, F))$ for all $x \in D$.*

Definition 4. *A map $T : D \to P(D)$ is called semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

We now give applications of above theorem.

Theorem 3. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and, S, T, R, P_S, P_T, P_R and F be as in Lemma 6. Suppose that P_S, P_T and P_R satisfy condition (I), then the modified SP iterative process $\{x_n\}$ defined in (3.1) converges strongly to $p \in F$.*

Proof. For all $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exists from Lemma 5. We call it c for some $c \geq 0$.

If $c = 0$, there is nothing to prove. Assume $c > 0$. Now $d(x_{n+1}, p) \leq d(x_n, p)$ gives that

$$\inf_{p \in F(T)} d(x_{n+1}, p) \leq \inf_{p \in F(T)} d(x_n, p),$$

which means that $d(x_{n+1},F) \leq d(x_n,F)$.Hence $\lim_{n\to\infty} d(x_n,F)$ exists. By using condition (I) and Lemma 6,we get

$$\lim_{n\to\infty} f(d(x_n,F)) \leq \lim_{n\to\infty} \frac{1}{3}\left[d(x,Sx)+d(x,Tx)+d(x,Rx)\right] = 0.$$

and so

$$\lim_{n\to\infty} f(d(x_n,F)) = 0.$$

By the properties of f, we obtain that $\lim_{n\to\infty} d(x_n,F) = 0$. Finally applying Theorem 2, we get the result. □

Since the proof of following theorem is similar to proof of theorem proved in Banach spaces by various authors, we omit.

Theorem 4. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and, S,T,R,P_S,P_T,P_R and F be as in Lemma 6. Suppose that one of P_S,P_T,P_R is semi-compact, then the modified SP iterative process $\{x_n\}$ defined in* (3.1) *converges strongly to $p \in F$.*

As a corollary of Theorem 2, we have the following theorem which is new in the literature.

Theorem 5. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X and let $T : D \to P(D)$ be a multivalued mapping such that P_T is nonexpansive mapping with a nonempty fixed point set F. Let $\{x_n\}$ defined as*

$$\begin{cases} x_{n+1} = W(u_n, y_n, \alpha_n) \\ y_n = W(v_n, z_n, \beta_n) \\ z_n = W(w_n, x_n, \gamma_n) \end{cases},$$

where $u_n \in P_T(y_n)$, $v_n \in P_T(z_n), w_n \in P_T(x_n)$. Let $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be real sequences such that $0 < a \leq \alpha_n,\beta_n,\gamma_n \leq b < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Remark 1. *(1) Since SP iterative process converges faster than Mann and Ishikawa iterative processes, our theorems are better than results of Fukhar-ud-din et al.* [34].

(2) Since CAT(0)-spaces are uniformly convex hyperbolic spaces with a 'nice' monotone modulus of uniform convexity $\eta(r,\varepsilon) := \frac{\varepsilon^2}{8}$, then our results valid in CAT(0) spaces besides Banach spaces.

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Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey.
E-mail address: birolgndz@gmail.com

Department of Mathematics, Faculty of Science, Erzurum Technical University, Erzurum, 25700, Turkey
E-mail address: ibrahimkarahan@erzurum.edu.tr

ADDITIVE-QUADRATIC ρ-FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN BANACH SPACES

CHOONKIL PARK[1], JUNG RYE LEE[2], AND DONG YUN SHIN[3*]

Abstract. Let

$$\begin{aligned} M_1f(x,y) : &= \frac{3}{4}f(x+y)-\frac{1}{4}f(-x-y) \\ &\quad +\frac{1}{4}f(x-y)+\frac{1}{4}f(y-x)-f(x)-f(y), \\ M_2f(x,y) : &= 2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y). \end{aligned}$$

We solve the additive-quadratic ρ-functional equations

$$M_1f(x,y)=\rho M_2f(x,y), \tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho|<1$, and

$$M_2f(x,y)=\rho M_1f(x,y), \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\le|r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\le\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

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*Corresponding author: Dong Yun Shin (email: dyshin@uos.ac.kr).

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Definition 1.1. ([8]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [18] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 10, 11, 13, 14, 15, 16, 17, 20, 21]).

In Section 2, we solve the additive-quadratic ρ-functional equation (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive-quadratic ρ-functional equation (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Additive-quadratic ρ-functional equation (0.1) in non-Archimedean normed spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive-quadratic ρ-functional equation (0.1) in non-Archimedean normed spaces.

Lemma 2.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$M_1 f(x, y) = \rho M_2 f(x, y) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1)*, then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$f(2x) - 2f(x) = 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$-f(0) = 2\rho f(0).$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\frac{1}{2} f(2x) - 2f(x) = 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}&\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\\&\quad=\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\\&\quad=\rho\left(\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right)\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 2.2. *Let $r<1$ and θ be nonnegative real numbers and let $f: X \to Y$ be an odd mapping such that*

$$\|M_1 f(x,y)-\rho M_2 f(x,y)\| \le \theta(\|x\|^r+\|y\|^r) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A: X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0)=0$.

Letting $y=x$ in (2.4), we get

$$\|f(2x)-2f(x)\| \le 2\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So $\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}&\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| \\&\quad\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right)-2^{l+1}f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|2^{m-1}f\left(\frac{x}{2^{m-1}}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\quad=\max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right)-2f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-2f\left(\frac{x}{2^m}\right)\right\|\right\}\\&\quad\le\max\left\{\frac{|2|^l}{|2|^{rl+r}},\cdots,\frac{|2|^{m-1}}{|2|^{r(m-1)+r}}\right\}2\theta\|x\|^r=\frac{2\theta}{|2|^{(r-1)l+r}}\|x\|^r\end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m>l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A: X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l=0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}\|M_1A(x,y)-\rho M_2A(x,y)\| &= \lim_{n\to\infty}|2|^n\left\|M_1f\left(\frac{x}{2^n},\frac{y}{2^n}\right)-\rho M_2f\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty}\frac{|2|^n\theta}{|2|^{nr}}(\|x\|^r+\|y\|^r)=0\end{aligned}$$

for all $x,y\in X$. So

$$M_1A(x,y)=\rho M_2A(x,y)$$

for all $x,y\in X$. By Lemma 2.1, the mapping $A:X\to Y$ is additive .

Now, let $T:X\to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|2^qA\left(\frac{x}{2^q}\right)-2^qT\left(\frac{x}{2^q}\right)\right\| \\ &\le \max\left\{\left\|2^qA\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|,\left\|2^qT\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|\right\}\le\frac{2\theta}{|2|^{(r-1)q+r}}\|x\|^r,\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in X$. So we can conclude that $A(x)=T(x)$ for all $x\in X$. This proves the uniqueness of h. Thus the mapping $A:X\to Y$ is a unique additive mapping satisfying (2.5). □

Theorem 2.3. *Let $r>1$ and θ be nonnegative real numbers and let $f:X\to Y$ be an odd mapping satisfying* (2.4). *Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\|f(x)-A(x)\|\le\frac{2\theta}{|2|}\|x\|^r \tag{2.8}$$

for all $x\in X$.

Proof. It follows from (2.6) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|\le\frac{2}{|2|}\theta\|x\|^r$$

for all $x\in X$. Hence

$$\begin{aligned}&\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^m}f\left(2^mx\right)\right\| \\ &\le\max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|\right\} \\ &=\max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^mx\right)\right\|\right\} \\ &\le\max\left\{\frac{|2|^{lr}}{|2|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}2\theta\|x\|^r=\frac{2\theta}{|2|^{(1-r)l+1}}\|x\|^r\end{aligned}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 2.4. *Let $r<2$ and θ be nonnegative real numbers and let $f:X\to Y$ be an even mapping satisfying* (2.4). *Then there exists a unique quadratic mapping $Q:X\to Y$ such that*

$$\|f(x)-Q(x)\|\le\frac{|2|}{|2|^r}2\theta\|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $-f(0) = 2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\left\| \frac{1}{2} f(2x) - 2f(x) \right\| \leq 2\theta \|x\|^r \tag{2.10}$$

for all $x \in X$. So $\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{|2|}{|2|^r} 2\theta \|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} &\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \\ &\leq \max\left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \cdots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &= \max\left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \cdots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ &\leq \max\left\{ \frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \frac{|2|}{|2|^r} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l}} \frac{|2|}{|2|^r} \|x\|^r \end{aligned} \tag{2.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.11) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.9).

It follows from (2.4) that

$$\begin{aligned} \|M_1 Q(x,y) - \rho M_2 Q(x,y)\| &= \lim_{n\to\infty} |4|^n \left\| M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - \rho M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$M_1 Q(x,y) = \rho M_2 Q(x,y)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.9). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ &\leq \max\left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \frac{|2|}{|2|^{(r-2)q+r}} 2\theta \|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.9). □

Theorem 2.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (2.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|} \|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.10) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\leq\frac{2\theta}{|2|}\|x\|^r$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.4. □

3. Additive-quadratic ρ-functional equation (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

In this section, we solve the additive-quadratic ρ-functional equation (0.2) in non-Archimedean normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f:X\to Y$ satisfies*

$$M_2f(x,y)=\rho M_1f(x,y) \tag{3.1}$$

for all $x,y\in X$, then $f:X\to Y$ is additive.
(ii) *If an even mapping $f:X\to Y$ satisfies $f(0)=0$ and (3.1), then $f:X\to Y$ is quadratic.*

Proof. (i) Assume that $f:X\to Y$ satisfies (3.1).

Letting $y=0$ in (3.1), we get

$$2f\left(\frac{x}{2}\right)-f(x)=0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right)=\frac{1}{2}f(x)$ for all $x\in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}f(x+y)-f(x)-f(y) &= 2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\\ &= \rho(f(x+y)-f(x)-f(y))\end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in X$.

(ii) Assume that $f:X\to Y$ satisfies (3.1).

Letting $y=0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right)-f(x)=0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right)=\frac{1}{4}f(x)$ for all $x\in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}&\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\\ &\quad=2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\\ &\quad=\rho(\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y))\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces for an odd mapping case.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \leq \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.7)\\
&\quad \leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad = \max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad \leq \max\left\{\frac{|2|^l}{|2|^{rl}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.4)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{|2|^r\theta}{|2|}\|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{|2|^r\theta}{|2|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\| \qquad (3.9)\\
&\le \max\left\{\left\|\frac{1}{2^l}f\left(2^l x\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^m x\right)\right\|\right\}\\
&= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^m x\right)\right\|\right\}\\
&\le \max\left\{\frac{|2|^{rl}}{|2|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r=\frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces for an even mapping case.

Theorem 3.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \le \theta\|x\|^r \qquad (3.10)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $2f(0) = \rho f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\| \le \theta\|x\|^r \qquad (3.11)$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.12)\\
&\le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right)-4^{l+1}f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|4^{m-1}f\left(\frac{x}{2^{m-1}}\right)-4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|4|^l\left\|f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\le \max\left\{\frac{|4|^l}{|2|^{rl}},\cdots,\frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r=\frac{\theta}{|2|^{(r-2)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.12) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r \tag{3.13}$$

for all $x \in X$.

Proof. It follows from (3.11) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \qquad (3.14)\\
&\leq \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\
&= \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\}\\
&\leq \max\left\{\frac{|2|^{rl}}{|4|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(2-r)l+2}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.13).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.4. □

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[1]Research Institute for Natural Sciences,
Hanyang University, Seoul 04763,
Republic of Korea
E-mail address: baak@hanyang.ac.kr

[2]Department of Mathematics,
Daejin University, Kyunggi 11159,
Republic of Korea
E-mail address: jrlee@hdaejin.ac.kr

[3]Department of Mathematics,
University of Seoul, Seoul 02504,
Republic of Korea
E-mail address: dyshin@uos.ac.kr

Complete asymptotic expansions for the genuiune Bernstein-Durrmeyer operator [1]

Chungou Zhang

School of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China
E-mail: 3773@mail.cnu.edu.cn

Abstract In this paper, we discuss properties of asymptotic approximation of the genuiune Bernstein-Durrmeyer operator, for which establish a complete asymptotic expansion formula of approximation and present the saturation theorems as an application.

Key words The genuiune Bernstein-Durrmeyer operator, Jacobi weight, Complete asymptotic expansion, Approximation, Saturation.

1. INTRODUCTION

The Bernstein-Durrmeyer operator with weights, which is one of the objects of interest in approximation theory of operators and play as an important role in learning theory, is defined as follows

$$M_n^{\omega}(f;x) = \sum_{k=0}^{n} \frac{(f, b_{n,k})_{\omega}}{(e_0, b_{n,k})_{\omega}} b_{n,k}(x)$$

where $e_k = e_k(x) = x^k, k = 0, 1, \cdots$, Bernstein basis functions

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad x \in [0,1], k = 0, 1, \cdots, n$$

and inner product weighted ω defined by

$$(f, g)_{\omega} = \int_0^1 f(t) g(t) \omega(t) dt.$$

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A popular case is that the weight ω is taken as the Jacobi weight $\omega(t) = t^{\alpha}(1-t)^{\beta}$, $\alpha, \beta > -1$, at this time we denote this operator $M_n^{\omega}(f)$ by $M_n^{(\alpha,\beta)}(f)$. Because when $\alpha = -1, \beta = -1$, the inner products $(e_0, b_{n,k})_{\omega}$ are no definition at $k = 0$ and n, W. Chen ([1] 1987), T.N.T. Goodman and A. Sharma ([2] 1987) independently modified the operator into

$$M_n^*(f;x) = Lf + (n-1)\sum_{k=1}^{n-1}(f - Lf, b_{n-2,k-1})b_{k,n}(x)$$

where

$$Lf(x) = (1-x)f(0) + xf(1)$$

and

$$(f,g) = \int_0^1 f(x)g(x)dx.$$

The operator $M_n^* f$ is also called as the genuiune Bernstein-Durrmeyer operator(see [3] or [4]) and can be rewritten into the following form

$$M_n^*(f;x) = f(0)(1-x)^n + f(1)x^n + (n-1)\sum_{k=1}^{n-1}(f, b_{n-2,k-1})b_{n,k}(x),$$

which is a particular case of those operators introduced by D. Cárdenas-Morales and V. Gupta in [5]

$$M_{n,\alpha,\beta}(f;x) = \sum_{k\in l_n} b_{n,k}(x)f(\frac{k}{n}) + (n-\alpha+1) \cdot \sum_{k=1}^{n-\alpha+\beta}\int_0^1 f(t)\ b_{n-\alpha,k-\beta}(t)dt\ b_{n,k}(x)$$

where $l_n \subset \{0, 1, \cdots, n\}$. Obviously when $\alpha = 2, \beta = 1$ and $l_n = \{0, n\}$, $M_{n,\alpha,\beta}(f)$ is $M_n^*(f)$, for which there has been extensive research (see [3,4] and [6-10]).

In the next sect, we will establish a complete asymptotic expansion formula for the operator $M_n^*(f)$, and in the section 3, we will present the saturation theorems of approximation as a application of the asymptotic expansion formula.

2. Complete Asymptotic expansions

From [5], one can find that

$$M_n^*(e_0;x) = M_n^*(1,x) = 1; \quad M_n^*(e_1;x) = M_n^*(t;x) = x;$$

$$M_n^*(e_2;x) = M_n^*(t^2,x) = x^2 + \frac{2x(1-x)}{n+1}.$$

For higher order moments of the operator $M_n^*(f)$, we have the following result.

Lemma 2.1 For any natural numbers n, m, there holds that

$$M_n^*(e_m;x) = x(1-x)\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n^{\bar{k}}}\Big[x^{m-1}(1-x)^{k-1}\Big]^{(k)},$$

where the up factorial is defined by

$$n^{\bar{k}} = n(n+1)\cdots(n+k-1), \quad n^{\bar{0}} = 1$$

and the fall factorial, which to be used in the following proof, done by

$$n^{\underline{k}} = n(n-1)\ldots(n-k+1), \quad n^{\underline{0}} = 1.$$

Proof Since when $n, m \le 1$, the conclusion is true obviously, we only need to consider the case $n, m \ge 2$. At this moment, we have

$$\begin{aligned}
M_n^*(e_m;x) &= (n-1)\sum_{i=1}^{n-1}(t^m, b_{n-2,i-1})b_{n,i}(x) + x^n \\
&= \frac{(n-1)!}{(m+n-1)!}\sum_{i=1}^{n-1}\frac{(m+i-1)!}{(i-1)!}\binom{n}{i}x^i(1-x)^{n-i} + x^n \\
&= \frac{(n-1)!}{(m+n-1)!}\Big[\frac{d^m}{dx^m}\sum_{i=1}^{n-1}\binom{n}{i}x^{m+i-1}y^{n-i}\Big]_{y=1-x} + x^n \\
&= \frac{(n-1)!}{(m+n-1)!}\frac{d^m}{dx^m}\Big[x^{m-1}(x+y)^n\Big]_{y=1-x} \\
&= \frac{(n-1)!}{(m+n-1)!}\Big[\sum_{v=1}^{m}\binom{m}{v}\frac{(m-1)!}{(v-1)!}x^{v-1}\frac{n!}{(n-v)!}(x+y)^{n-v}\Big]_{y=1-x} \\
&= \frac{1}{n^{\overline{m}}}\sum_{v=1}^{m}\binom{m}{v}(m-1)^{\underline{m-v}}n^{\underline{v}}x^v.
\end{aligned}$$

Using the Vandermonde formula we get

$$n^{\underline{v}} = \sum_{k=0}^{v}\binom{v}{k}\Big(n-(1-m)\Big)^{\underline{k}}(1-m)^{\underline{v-k}},$$

and notice that when $v=0$, $(m-1)^{\underline{m-v}} = 0$, therefore it follows that

$$M_n^*(e_m;x)$$

$$= \frac{1}{n^{\overline{m}}}\sum_{v=0}^{m}\binom{m}{v}(m-1)^{\underline{m-v}}x^v\sum_{k=0}^{v}\binom{v}{k}\Big(n-(1-m)\Big)^{\underline{k}}(1-m)^{\underline{v-k}}$$
$$= \sum_{k=0}^{m}\frac{1}{(m-k)!}\frac{1}{n^{\overline{k}}}\sum_{v=0}^{k}(-1)^{k-v}\frac{m!}{v!}\frac{(k+m-v-2)^{\underline{k-v}}}{(k-v)!}(m-1)^{\underline{v}}x^{m-v}.$$

Again using the Vandermonde formula we have

$$\frac{(k+m-v-2)^{\underline{k-v}}}{(k-v)!}(m-1)^{\underline{v}} = (m-1)^{\underline{v}}\sum_{i=v}^{k}\binom{k-v}{i-v}(m-1-v)^{\underline{i}}(k-1)^{\underline{k-i}},$$

hence the second sum in the previous equation can be turned into

$$\sum_{v=0}^{k}(-1)^{k-v}\frac{m!}{v!}x^{m-v}\cdot\sum_{i=v}^{k}\frac{(m-1)^{\underline{i}}}{(i-v)!}\frac{(k-1)^{\underline{k-i}}}{(k-i)!}$$
$$= m!\sum_{i=0}^{k}\frac{(m-1)^{\underline{i}}}{i!}\frac{(k-1)^{\underline{k-i}}}{(k-i)!}\cdot\sum_{v=0}^{i}(-1)^{k-v}\binom{i}{v}x^{m-v}$$
$$= m!\sum_{i=0}^{k}\frac{(m-1)^{\underline{i}}}{i!}\frac{(k-1)^{\underline{k-i}}}{(k-i)!}(-1)^{k-i}x^{m-i}(1-x)^i$$
$$= x(1-x)\frac{m!}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{d^i}{dx^i}\Big[x^{m-1}\Big]\frac{d^{k-i}}{dx^{k-i}}\Big[(1-x)^{k-1}\Big]$$
$$= x(1-x)\frac{m!}{k!}\Big[x^{m-1}(1-x)^{k-1}\Big]^{(k)}.$$

That completes the proof of Lemma 2.1.

If denoting $\psi_x^s(t) = (t-x)^s$, then we have the following assertion.

Lemma 2.2 For arbitrary natural number s and $x\in[0,1]$, there holds

$$M_n^*(\psi_x^s;x) == x(1-x)\sum_{k=[\frac{s+1}{2}]}^{s}\frac{s!}{k!}\frac{1}{n^{\overline{k}}}\binom{k}{s-k}\Big[x^{k-1}(1-x)^{k-1}\Big]^{(2k-s)}$$

Proof On account of

$$M_n^*(\psi_x^s;x) = \sum_{m=0}^{s}\binom{s}{m}(-x)^{s-m}M_n^*(e_m;x),$$

by Lemma 2.1 we see easily

$$M_n^*(\psi_x^s;x) = x(1-x)\sum_{m=0}^{s}\binom{s}{m}(-x)^{s-m}\cdot\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n^{\overline{k}}}\Big[x^{m-1}(1-x)^{k-1}\Big]^{(k)}$$
$$= x(1-x)\sum_{k=0}^{s}\frac{\binom{s}{k}}{n^{\overline{k}}}\frac{d^k}{dy^k}\Big[\sum_{m=0}^{s-k}\binom{s-k}{m}(-x)^{s-k-m}y^{m-1+k}(1-y)^{k-1}\Big]\Big|_{y=x}$$
$$= x(1-x)\sum_{k=0}^{s}\frac{\binom{s}{k}}{n^{\overline{k}}}\frac{d^k}{dy^k}\Big[y^{k-1}(1-y)^{k-1}(y-x)^{s-k}\Big]\Big|_{y=x},$$

and recall that

$$\frac{d^k}{dy^k}\Big[y^{k-1}(1-y)^{k-1}(y-x)^{s-k}\Big]_{y=x}$$
$$=\begin{cases}(s-k)!\binom{k}{s-k}\dfrac{d^{2k-s}}{dy^{2k-s}}\Big[y^{k-1}(1-y)^{k-1}\Big]_{y=x}, & s\le 2k\\ 0, & s>2k.\end{cases}$$

The proof is completed.

Remark By Lemma 2.2, we immediately obtain

$$M_n^*(\psi_x^s;x)=O(n^{-[\frac{s+1}{2}]}).$$

For arbitrary natural number q and $x\in I$, let $f\in K[q;x]$ denote the class of functions $f\in B(I)$ (space of bounded functions on I) which q times differentiable at x, then we have the following theorems of approximation.

Lemma 2.3 [11] Let q be arbitrary natural number, $x\in I$ and $A_n: B(I)\to C(I)$(space of continuous functions on I) be a sequence of positive linear operators such that

$$A_n(\psi_x^s;x)=O(n^{-[\frac{s+1}{2}]})\quad (n\to\infty)\quad (s=0,1,\cdots,2q+2),$$

then for arbitrary $f\in K[2q;x]$, there holds

$$A_n(f;x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}A_n(\psi_x^s;x)+o(n^{-q})\quad (n\to\infty).$$

In particular, if $f^{(2q+2)}(x)$ exists, then the $o(n^{-q})$ can be replaced by $O(n^{-q-1})$

Theorem 2.1 For arbitrary natural number q, $x\in[0.1]$ and $f\in K[2q;x]$, there holds that

$$M_n^*(f;x)=\sum_{k=0}^{q}\frac{1}{k!\overline{n^k}}x(1-x)\Big[x^{k-1}(1-x)^{k-1}f^{(k)}(x)\Big]^{(k)}+o(n^{(-q)}$$

Proof By Lemma 2.2 and Lemma 2.3, we have

$$M_n^*(f;x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}x(1-x)$$
$$\cdot\sum_{k=[\frac{s+1}{2}]}^{s}\frac{s!}{k!}\frac{1}{\overline{n^k}}\binom{k}{s-k}\Big[x^{k-1}(1-x)^{k-1}\Big]^{(2k-s)}+o(n^{-q}).$$

Interchanging the two sums and substituting $s+k$ into s, it is followed by

$$\begin{aligned}(M_n^*(f;x) = \quad & x(1-x)\sum_{k=0}^{q}\frac{1}{k!n^{\overline{k}}} \\ & \cdot\sum_{s=0}^{k}\binom{k}{s}f^{(s+k)}(x)\Big[x^{k-1}(1-x)^{k-1}\Big]^{(k-s)}+o(n^{-q}).\end{aligned}$$

With the Leibniz formula, it becomes

$$M_n^*(f;x) = x(1-x)\sum_{k=0}^{q}\frac{1}{k!n^{\overline{k}}}\cdot[x^{k-1}(1-x)^{k-1}f^{(k)}(x)]^{(k)}+o(n^{-q}).$$

That is the proof of Theorem 2.1.

Remark 1 If taking $q=1$, then we can get so-call the Vonorovskaja type asymptotic expansion formula by Theorem 2.1 as below

$$\lim_{n\longrightarrow\infty} n(M_n^*(f;x)-f(x)) = x(1-x)f''(x)$$

Remark 2 Theorem 2.1 shows that the complete asymptotic expansion formula of the operator $M_n^*(f)$ coincides with which of the operator $M_n^{(\alpha,\beta)}(f)$ at $\alpha=-1,\beta=-1$. This seems to be one of reasons why $M_n^*(f)$ is called as the genuiune Bernstein-Durrmeyer operator.

3. Saturations of Approximation

As an application of the asymptotic formula, in this section we will present the saturation theorems of approximation for the operator M_n^*f. Along with those denotations and signs in [12], denote the space of all continuous functions on $[0,1]$ by $C[0,1]$, for $v(x),\omega(x)\in C[0,1]$ and strictly positive, that is $v(x),\omega(x)>0, x\in(0,1)$, let

$$\varphi(x)=\int_0^x v(t)dt,\quad \psi(x)=\int_0^x\omega(t)dt$$
$$\phi(x)=\int_0^x\psi(t)v(t)dt.$$

For a function $f\in C[0,1]$, we define that

$$Df(x)=D_\psi D_\varphi f(x)=\frac{1}{\omega(x)}\Big[\frac{f'(x)}{v(x)}\Big]'$$

where

$$D_\varphi f(t) = \frac{f'(t)}{v(t)}.$$

Suppose that $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operators, $\{\lambda_n\}$ is a sequence of positive numbers such as $\lambda_n \to \propto$ $(n \to \propto)$ and $\rho(x) \in C[0,1]$ is strictly positive. We say that $\{L_n\}$ satisfy with the Voronovskaya condition if and only if for $Df(x), \{\lambda_n\}$ and $\rho(x)$ as above, there holds

$$\lim_{n\longrightarrow\propto} \lambda_n\{L_n(f;x) - f(x)\} = \rho(x)Df(x), \quad x \in [0,1]$$

Lemma 3.1[12] Suppose that $L_n : C[0.1] \to C[0.1]$ is a sequence of positive linear operators and satisfy the Voronovskaya condition and $G \in C[0,1]$. If for all $x \in (0,1)$

$$\lambda_n\Big|L_n(G;x) - G(x)\Big| \le M\rho(x) + o_x(1),$$

for some positive constant M, than there exists $D_\varphi G(x) \in C[0,1]$ and

$$\Big|D_\varphi G(y) - D_\varphi G(x)\Big| \le M\Big|\psi(y) - \psi(x)\Big| \quad x, y \in [0,1],$$

and vice versa.

Remark 1 of Theorem 2.1 shows that the operator $M_n^* f$ satisfies the Voronovskaya condition and

$$v(x) = \omega(x) = 1, \rho(x) = x(1-x), \lambda_n = n$$

$$Df(x) = D_\psi D_\varphi f(x) = f''(x),$$

thus from Lemma 3.1 we can get the following pointwise saturation theorem without any difficult.

Theorem 3.1 (1) Let $f \in C[0,1]$, then for all $x \in [0,1]$, $\lim_{n\longrightarrow\propto} n\{M_n^*(f;x) - f(x)\} = 0$ if and only if $f(x) = A + Bx$;

(2) Let $f \in C[0,1]$, than for all $x \in [0,1]$

$$n\Big|M_n^*(f;x) - f(x)\Big| \le M\rho(x) + o_x(1)$$

if and only if for any $x, y \in [0,1]$,

$$\Big|f'(y) - f'(x)\Big| \le M\Big|y - x\Big|.$$

For the operator $M_n^*(f)$, we also establish the following saturation theorem in L_P. Since it is the same as the approach taken for the Kantorovich operator in [13] and [14], here the process is omitted.

Let $f \in L_p[0,1], (p \geq 1)$ denote $|f|^p$ Lebesgue integrable on $[0,1]$,

$$U_p = \begin{cases} \Big\{h | h \in L_p[0,1], h(0) = h(1) = 0 \ \& \ h' \in L_p[0,1]\}, & p > 1 \\ \{h | h \in L_p[0,1], h(0) = h(1) = 0 \ \& \ h \in BV[0,1]\}, & p = 1. \end{cases}$$

and

$$S_p = \Big\{ f | \quad f \in L_p[0,1], \exists h \in U_p, \xi \in (0,1) \textit{and constants } c, d \textit{ such as} \\ f(x) = c + dx + \int_\xi^x (\int_\xi^u \frac{h(t)}{t(1-t)} dt) du \Big\}.$$

Theorem 3.2 Suppose $f \in L_p[0,1] (p \geq 1)$, then

(i) $\| M_n^*(f) - f \|_p = O(\frac{1}{n})$, if and only if $f \in S_p (p \geq 1)$

(ii) $\| M_n^*(f) - f \|_p = o(\frac{1}{n})$, if and only if $f(x)$ is a linear function.

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IMPULSIVE PERIODIC SOLUTIONS FOR A SINGULAR DAMPED DIFFERENTIAL EQUATION VIA VARIATIONAL METHODS

SHENGJUN LI [1,2],XIANHUA TANG [2], HUXIAO LUO [2]

Abstract. In this paper, we study impulsive periodic solutions for second order non-autonomous singular damped differential equations. The proof of the main result relies on a variational approach on mountain pass theorem, together with a truncation technique.

1. Introduction

In this work, we are concerned with the existence of periodic solutions for the following second order non-autonomous singular damped problems

$$\begin{cases} u'' + a(t)u' - \frac{b(t)}{u^\alpha} = g(t), \text{ a.e. } t \in (0,T), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1.1}$$

under the impulse conditions

$$\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)),\ j = 1, 2, \ldots, p-1, \tag{1.2}$$

where $u'(t_j^\pm) = \lim_{t \to t_j^\pm} u'(t)$, $\alpha > 1$, $a, g \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\int_0^T a(t)dt = 0$, and $b \in C(\mathbb{R}/T\mathbb{Z}, (0,\infty))$, t_j for $j = 1, 2, \ldots, p-1$, are the instants when the impulses occur and $0 = t_0 < t_1 < t_2 < \ldots < t_{p-1} < t_p = T, I_j : \mathbb{R} \to \mathbb{R} (j = 1, 2, \ldots, p-1)$ are continuous.

Impulsive effects occur widely in many evolution processes in which their states are changed abruptly at certain moments of time. In recent years, second-order differential boundary value problems with impulses have been studied extensively in the literature [1, 3, 11, 12, 14, 15, 16, 17, 18]. In [18], Tian and Ge studied the existence of solutions for impulsive differential equations:

$$\begin{cases} -(\rho(t)\phi_p(u'(t)))' + s(t)\phi_p(u(t)) = f(t, u(t)),\ t \neq t_j \text{ a.e. } t \in [a,b], \\ -\Delta(\rho(t_j)\phi_p(u'(t_j)))' = I_j(u(t_j)),\ j = 1, 2, \ldots, l, \\ \alpha u'(a) + \beta u(a) = A,\ \gamma u(b) + \sigma u'(b) = B, \end{cases}$$

by using a variational method. Later, Nieto and O'Regan [12] further developed the variational framework for impulsive problems and established existence results for the following impulsive differential equations with Dirichlet boundary conditions:

$$\begin{cases} -u'' + \lambda u(t) = f(t, u(t)),\ t \neq t_j \text{ a.e. } t \in (0,T), \\ \Delta(u'(t_j)) = I_j(u(t_j)),\ j = 1, 2, \ldots, l, \\ u(0) = u(T) = 0, \end{cases}$$

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From then on, the variational method has been a powerful tool in the study of impulsive differential equations.

On the other hand, singular problems without impulse effects have been investigated extensively in the literature. Usually, in the literature, the proof is based on variational methods [10], or topological methods[2, 4, 6, 7, 8, 9], which were started with the pioneering paper of Lazer and Solimini [5].

In 1987, Lazer and Solimini [5] considered a the second order singular problem

$$u''(t) - \frac{1}{u^{\alpha}(t)} = e(t), \quad t \in (0,T). \tag{1.3}$$

By using the method of upper and lower solutions, they obtained a famous sufficient and necessary condition on positive T-periodic solution for Problem (1.3) as follows

Theorem 1.1[5] Assume that $e \in L^1([0,T],\mathbb{R})$ is T-periodic. Then Problem (1.3) has a positive T-periodic weak solution if and only if $\int_0^T e(t)dt < 0$.

Motivated by the above fact, in the present paper we shall consider Problem (1.1) with impulsive effects, In general cases, it is impossible to apply variational methods to Eq.(1.1) when $\int_0^T a(t)dt > 0$. In this paper, using a variant of the mountain pass theorem, we consider the case $\int_0^T a(t)dt = 0$, on an appropriate Sobolev space, we establish the corresponding variational framework of periodic solutions to guarantee the existence of at least one nontrivial solution of Eq.(1.1).

In order to state our main result, we need the following assumptions:

(H_1) $a \in C(\mathbb{R}/T\mathbb{Z})$ with $\int_0^T a(t)dt = 0$;
(H_2) $b \in C(\mathbb{R}/T\mathbb{Z},(0,\infty))$ is $T-$periodic and $b'(t) \geq 0$ for all $t \in [0,T]$;
(H_3) $g \in L^2([0,T],\mathbb{R})$ is $T-$periodic and $\int_0^T g(t)dt < 0$;
(H_4) There exist two constants m, M such that for any $t \in \mathbb{R}$,

$$m \leq I_j(t) \leq M, \ j = 1,2,\ldots,p-1,$$

where $m < 0$ and $0 \leq M < -\frac{1}{p-1}\int_0^T g(t)dt$;
(H_5) For any $t \in \mathbb{R}$,

$$\int_0^t I_j(s)ds \geq 0, \ j = 1,2,\ldots,p-1.$$

Theorem 1.1 Assume that $(\mathrm{H}_1)-(\mathrm{H}_5)$ are satisfied. Then problem (1.1)-(1.2) has at least one solution.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of variational method, we will state and prove the main results.

2. PRELIMINARIES

In this section, we present some results which will be applied in Sections 3.

Let

$$H_T^1 = \{u : [0,T] \to \mathbb{R} \mid u \text{ is absolutely continuous, } u(0) = u(T) \text{ and } u' \in L^2([0,T],\mathbb{R})\}$$

with the inner product

$$(u,v) = \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u,v \in H_T^1.$$

The corresponding norm is defined by

$$\|u\|_{H_T^1}=\left(\int_0^T|u(t)|^2dt+\int_0^T|u'(t)|^2dt\right)^{\frac{1}{2}},\ \forall u\in H_T^1.$$

Then H_1^T is a Banach space.

If $u\in H_T^1$, then u is absolutely continuous and $u'\in L^2([0,T],\mathbb{R})$. In this case, $\Delta u'(t)=u'(t^+)-u'(t^-)=0$ is not necessarily valid for every $t\in(0,T)$ and the derivative u' may exist some discontinuities. It may lead to impulse effects.

From (1.1), we get

$$-\left[e^{A(t)}u'(t)\right]'+e^{A(t)}\left[\frac{b(t)}{u^\alpha}+g(t)\right]=0, \tag{2.1}$$

where $A(t)=\int_0^t a(s)ds$. Following the ideas of [12], take $v\in H_T^1$ and multiply the two sides of (2.1) by v and integrate from 0 to T, so we have

$$\int_0^T-\left[e^{A(t)}u'(t)\right]'v(t)dt+\int_0^T e^{A(t)}\left[\frac{b(t)}{u^\alpha}+g(t)\right]v(t)dt=0 \tag{2.2}$$

Note that, since $u'(0)=u'(T)$, one has

$$\begin{aligned}
&\int_0^T\left[e^{A(t)}u'(t)\right]'v(t)dt\\
&=\sum_{j=0}^{p-1}\int_{t_j}^{t_{j+1}}\left[e^{A(t)}u'(t)\right]'v(t)dt\\
&=\sum_{j=0}^{p-1}e^{A(t)}\left[u'(t_{j+1}^-)v(t_{j+1}^-)-u'(t_{j+1}^+)v(t_{j+1}^+)\right]-\sum_{j=0}^{p-1}\int_{t_j}^{t_{j+1}}e^{A(t)}u'(t)v'(t)dt\\
&=e^{A(T)}u'(T)v(T)-e^{A(0)}u'(0)v(0)-\sum_{j=0}^{p-1}e^{A(t)}\Delta u'(t_j)v(t_j)-\int_0^T e^{A(t)}u'(t)v'(t)dt\\
&=-e^{A(t)}\sum_{j=1}^{p-1}I_j(u(t_j))v(t_j)-\int_0^T e^{A(t)}u'(t)v'(t)dt.
\end{aligned}$$

Combining with (2.2), we get

$$\int_0^T e^{A(t)}\left[u'(t)v'(t)+\frac{b(t)}{u^\alpha}v(t)dt+g(t)v(t)\right]dt+e^{A(t)}\sum_{j=1}^{p-1}I_j(u(t_j))v(t_j)=0.$$

As a result we introduce the following concept of a weak solution for problem (1.1)-(1.2).

Definition 2.1 We say that a function $u\in H_T^1$ is a weak solution of problem (1.1)-(1.2) if

$$\int_0^T e^{A(t)}\left[u'(t)v'(t)+\frac{b(t)}{u^\alpha}v(t)dt+g(t)v(t)\right]dt+e^{A(t)}\sum_{j=1}^{p-1}I_j(u(t_j))v(t_j)=0$$

holds for any $v\in H_T^1$.

Define the functional $\Phi : H_T^1 \to \mathbb{R}$ by

$$\Phi(u) := \int_0^T e^{A(t)} \left[\frac{1}{2}|u'(t)|^2 + b(t) \int_1^{u(t)} \frac{1}{s^\alpha} ds + g(t)u(t) \right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s) ds$$

for every $u \in H_T^1$. Under the conditions of Theorem 1.2, it is easy to verify that Φ is well defined on H_T^1, continuously differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ are the weak solutions of problem (1.1)-(1.2).

In next section, the following version of the mountain pass theorem will be used in our argument.

Theorem 2.2 [13] Let X be a Banach space and let $\varphi \in C(X, \mathbb{R})$. Assume that there exist $x_0, x_1 \in X$ and a bounded open neighborhood Ω of x_0 such that $x_1 \in X/\bar{\Omega}$ and

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x \in \partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \in C([0,1], X) : h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} \varphi(h(s)).$$

If φ satisfies the (PS)-condition, i.e., a sequence $\{u_n\}$ in X satisfying $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence, then c is a critical value of φ and $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

3. Main results

In order to study problem (1.1)-(1.2), for any $\lambda \in (0,1)$ we consider the following modified problem

$$\begin{cases} u'' + a(t)u' + b(t)f_\lambda(u(t)) = g(t), \ a.e.\ t \in (0,1), \\ \Delta u'(t_j) = I_j(u(t_j)), \ j = 1, 2, \ldots, p-1, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{3.1}$$

where $f_\lambda : [0,T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\lambda(u) = \begin{cases} -\frac{1}{u^\alpha}, \ u \geq \lambda, \\ -\frac{1}{\lambda^\alpha}, \ u < \lambda. \end{cases}$$

Let $F_\lambda(u) = \int_1^u f_\lambda(s) ds$

$$\Phi_\lambda : H_T^1 \to \mathbb{R}$$

defined by

$$\Phi_\lambda(u) := \int_0^T e^{A(t)} \left[\frac{1}{2}|u'(t)|^2 - b(t)F_\lambda(u(t))dt + g(t)u(t) \right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s) ds.$$

Clearly, Φ_λ is well defined on H_T^1, continuously differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ_λ are the weak solutions of problem (3.1).

Proof. The proof will be divided into four steps.

Step 1. Φ_λ satisfies the Palais-Smale condition.

Let a sequence $\{u_n\}$ in H_T^1 satisfy $\Phi_\lambda(u_n)$ is bounded and $\Phi'_\lambda(u_n) \to 0$ as $n \to +\infty$. That is, there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that, for all n,

$$|\int_0^T e^{A(t)}\left[\frac{1}{2}|u_n'(t)|^2 - b(t)F_\lambda(u_n(t))dt + g(t)u_n(t)\right]dt + e^{A(t)}\sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds\,| \le c_1, \tag{3.2}$$

and for every $v \in H_T^1$,

$$|\int_0^T e^{A(t)}\left[u_n'(t)v'(t) - b(t)f_\lambda(u_n(t))v(t) + g(t)v(t)\right]dt + e^{A(t)}\sum_{j=1}^{p-1} I_j(u_n(t_j))v(t_j)\,| \le \epsilon_n\|v\|_{H_T^1}. \tag{3.3}$$

Using a standard argument, it is suffices to show that $\{u_n\}$ is bounded when verifying the (PS)-condition.

Taking $v(t) \equiv -1$ in (3.3), one has

$$\left|\int_0^T e^{A(t)}\left[b(t)f_\lambda(u_n(t)) - g(t)\right]dt - e^{A(t)}\sum_{j=1}^{p-1} I_j(u_n(t_j))\right| \le \epsilon_n\sqrt{T} \text{ for all } n.$$

By (H_3), we have

$$\begin{aligned}\left|\int_0^T e^{A(t)}b(t)f_\lambda(u_n(t))\right| &\le \epsilon_n\sqrt{T} + \left|\int_0^T e^{A(t)}g(t)dt+\right| + e^{A(t)}\sum_{j=1}^{p-1}|I_j(u_n(t_j))| \\ &\le \epsilon_n\sqrt{T} + e^{\|a\|_{L^1}}\left|\int_0^T g(t)\right| + e^{\|a\|_{L^1}}(p-1)M := c_2.\end{aligned}$$

Note that for any $t \in [0,T], b(t)f_\lambda(u_n(t)) < 0$. Thus

$$\int_0^T \left|e^{A(t)}b(t)f_\lambda(u_n(t))\right|dt = \left|\int_0^T e^{A(t)}b(t)f_\lambda(u_n(t))\right| \le c_2.$$

On the other hand, if we take, in (3.3), $v(t) \equiv w_n(t) := u_n(t) - \bar{u}_n$, where $\bar{u}_n$ is the average of u_n over the interval $[0,T]$, we have

$$\begin{aligned}c_3\|w\|_{H_T^1} &\ge \left|\int_0^T e^{A(t)}\left[w_n'(t)^2 - b(t)f_\lambda(u_n(t))w_n(t) + g(t)w_n(t)\right]dt + e^{A(t)}\sum_{j=1}^{p-1} I_j(u_n(t_j))w_n(t_j)\right| \\ &\ge e^{-\|a\|_{L^1}}\|w_n'\|_{L^2} - (c_2 + e^{\|a\|_{L^1}}\|g\|_{L^1})\|w_n\|_{L^\infty} + e^{-\|a\|_{L^1}}(p-1)m\|w_n\|_{L^\infty} \\ &\ge e^{-\|a\|_{L^1}}\|w_n'\|_{L^2} - (c_2 + e^{\|a\|_{L^1}}\|g\|_{L^1} - e^{-\|a\|_{L^1}}(p-1)m)\|w_n\|_{L^\infty} \\ &\ge e^{-\|a\|_{L^1}}\|w_n'\|_{L^2} - c_4\|w_n\|_{H_T^1},\end{aligned}$$

where c_3 and c_4 are two positive constants. Consequently, using the Wirtinger inequality for zero mean functions in the Sobolev space H_T^1, there exists $c_5 > 0$ such that

$$\|u_n'\|_{L^2}^2 \le \|w_n\|_{H_T^1} \le c_5. \tag{3.4}$$

Now, suppose that

$$\|u_n\|_{H_T^1} \to +\infty \text{ as } n \to +\infty.$$

Since (3.4) holds, we have, passing to subsequence if necessary, that either

$$M_n := \max u_n \to +\infty \text{ as } n \to +\infty, \text{ or}$$
$$m_n := \min u_n \to -\infty \text{ as } n \to +\infty,$$

(i) Assume that the first possibility occurs. By (H_3) and the fact that $f_\lambda < 0$, one has

$$\begin{aligned}
&\int_0^T e^{A(t)} \left[b(t)F_\lambda(u_n(t)) - g(t)u_n(t)\right] dt - e^{A(t)} \sum_{j=1}^{p-1} \int_0^T I_j(s)ds \\
\geq &\int_0^T e^{A(t)} \left[\int_1^{u_n(t)} b(t)f_\lambda(s)ds - g(t)u_n(t)\right] dt - e^{\|a\|_{L^1}}(p-1)MM_n \\
= &\int_0^T e^{A(t)} \left[\int_1^{M_n(t)} b(t)f_\lambda(s)ds - \int_{u_n(t)}^{M_n} b(t)f_\lambda(s)ds - g(t)u_n(t)\right] dt \\
&-e^{\|a\|_{L^1}}(p-1)MM_n \\
= &\int_0^T e^{A(t)} b(t)F_\lambda(M_n)dt - \int_0^T e^{A(t)} M_n g(t)dt - \int_0^T \left[\int_{u_n(t)}^{M_n} e^{A(t)} \left(b(t)f_\lambda(s) - g(t)\right) ds\right] dt \\
&-e^{\|a\|_{L^1}}(p-1)MM_n \\
\geq &\int_0^T e^{A(t)} \left[b(t)F_\lambda(M_n) - M_n g(t)\right] dt + \int_0^T e^{A(t)}(M_n - u_n(t))g(t)dt - e^{\|a\|_{L^1}}(p-1)MM_n \\
\geq &\int_0^T e^{A(t)} \left[b(t)F_\lambda(M_n) - M_n g(t)\right] dt - e^{\|a\|_{L^1}} \|M_n - u_n\| \|g\|_{L^1} - e^{\|a\|_{L^1}}(p-1)MM_n.
\end{aligned}$$

Thus, using Sobolev and Poincare's inequalities, one has

$$\begin{aligned}
&-e^{\|a\|_{L^1}} \left((p-1)M + \int_0^T e^{A(t)} g(t)dt\right) M_n \\
\leq &\int_0^T e^{A(t)} [b(t)F_\lambda(u_n(t)) - g(t)u_n(t)]dt - e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u_n(t_j)} I_j(s)ds \\
&+\sqrt{T} e^{\|a\|_{L^1}} \|g\|_{L^1} \|u_n'\|_{L^2} - F_\lambda(M_n) e^{\|a\|_{L^1}} \int_0^T b(t)dt \\
\leq &\int_0^T e^{A(t)} [b(t)F_\lambda(u_n(t)) - g(t)u_n(t)]dt - e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u_n(t_j)} I_j(s)ds \\
&+\sqrt{T} e^{\|a\|_{L^1}} \|g\|_{L^1} \|u_n'\|_{L^2} - \frac{\int_0^T b(t)dt}{\alpha - 1} \left(\frac{1}{M_n^{\alpha-1}} - 1\right) e^{\|a\|_{L^1}},
\end{aligned}$$

From (3.2),(3.4) and the fact that $\frac{1}{M_n^{\alpha-1}} \to 0$ as $n \to +\infty$, we see that the right hand side of the above inequality is bounded, which is contradiction.

(ii) Assume the second possibility occurs, i.e., $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by m_n in the preceding arguments, and we also get a contradictions.

Therefore, Φ_λ satisfies the Palais-Smale condition. This completes the proof of the claim.

Step 2. Let

$$\Omega = \left\{ u \in H_T^1 | \min_{t\in[0,T]} u(t) > 1 \right\},$$

and

$$\partial\Omega = \left\{ u \in H_T^1 | \min_{t\in[0,T]} u(t) \geq 1 \text{ for all } t \in (0,T), \exists t_u \in (0,T) \text{ such that } u(t_n) = 1 \right\}.$$

We show that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) \geq -d$ whenever $\lambda \in (0,1)$.

For any $u \in \partial\Omega$, there exists some $t_u \in (0,T)$ such that $\min_{t\in[0,T]} u(t) = u(t_n) = 1$. By (H_4) and extending the functions by T-periodicity, we obtain that

$$\begin{aligned}
\Phi_\lambda(u) &= \int_{t_u}^{t_u+T} e^{A(t)} \left[\frac{1}{2}|u'(t)|^2 - b(t)F_\lambda(u(t))dt + g(t)u(t) \right] dt + e^{A(t)} \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)ds \\
&\geq \frac{1}{2} \int_{t_u}^{t_u+T} e^{A(t)} u'(t)^2 dt + \frac{1}{\alpha-1} \int_{t_u}^{t_u+T} e^{A(t)} b(t) \left(1 - \frac{1}{u(t)^{\alpha-1}} \right) dt \\
&\quad + \int_{t_u}^{t_u+T} e^{A(t)} g(t)(u(t)-1)dt + \int_{t_u}^{t_u+T} e^{A(t)} g(t)dt \\
&\geq \frac{1}{2} \int_{t_u}^{t_u+T} e^{A(t)} u'(t)^2 dt + \int_{t_u}^{t_u+T} e^{A(t)} g(t)(u(t)-1)dt + \int_{t_u}^{t_u+T} e^{A(t)} g(t)dt.
\end{aligned}$$

By the Schwarz inequality and the fact that $u'(t) = (u(\cdot)-1)'(t)$, one has

$$\Phi_\lambda(u) \geq \frac{e^{-\|a\|_{L^1}}}{2} \|(u(\cdot)-1)'\|_{L^2} - e^{\|a\|_{L^1}} \|g\|_{L^2} \|u(\cdot)-1\|_{L^2} - e^{\|a\|_{L^1}} \|g\|_{L^1}.$$

Applying Poincare's inequality to $u(\cdot) - 1$, we get

$$\Phi_\lambda(u) \geq \frac{e^{-\|a\|_{L^1}}}{2} \|(u(\cdot)-1)'\|_{L^2} - e^{\|a\|_{L^1}} \gamma \|g\|_{L^2} \|u'\|_{L^2} - e^{\|a\|_{L^1}} \|g\|_{L^1},$$

where $\gamma = \gamma(t_u)$. The above inequality shows that

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u'\|_{L^2} \to +\infty.$$

Since $\min_{t\in[0,T]} u(t) = 1$, we have that $\|u(\cdot)-1\|_{H_T^1} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$. Hence

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u\|_{H_T^1} \to +\infty, \forall u \in \partial\Omega,$$

which shows that Φ_λ is coercive. Thus it has a minimizing sequence. The weak lower semi-continuity of Φ_λ yields

$$\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -d$ for all $\lambda \in (0,1)$.

Step 3. We show that there exists $\lambda_0 \in (0,1)$ with the property that, for every $\lambda \in (0,\lambda_0)$, any solution u of problem (3.1) satisfying $\Phi_\lambda(u) > -d$ such that $\min_{u\in[0,T]} u(t) \geq \lambda_0$, and hence u is a solution of problem (1.1)-(1.2).

Assume on the contrary that there are sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that
(i) $\lambda_n \leq \frac{1}{n}$;
(ii) u_n is a solution of (3.1) with $\lambda = \lambda_n$;

(iii) $\Phi_{\lambda_n}(u_n) \geq -d$;
(iv) $\min_{t\in[0,T]} u_n(t) < \frac{1}{n}$.

Since $f_{\lambda_n} < 0$ and

$$\int_0^T e^{A(t)}[b(t)f_{\lambda_n}(u_n(t)) - g(t)]dt = 0, \tag{3.5}$$

one has

$$\|e^{A(t)}b(\cdot)f_{\lambda_n}(u_n(\cdot))\|_{L^1} \leq c_6, \quad \text{for some constant } c_6 > 0.$$

On the other hand, since $u_n(0) = u_n(T)$, there exists $\tau_n \in (0,T)$ such that

$$u_n'(\tau_n) = 0.$$

Therefore, we obtain that

$$e^{A(t)}u_n'(t) - e^{A(\tau_n)}u_n'(\tau_n) = \int_{\tau_n}^t e^{A(s)}[f_\lambda(u_n(s)) - g(s)]ds,$$

which, from (3.5), yields that

$$\|u_n'\|_{L^\infty} \leq c_7 \quad \text{for some constant } c_7 > 0. \tag{3.6}$$

$$\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -\infty.$$

From $\Phi_{\lambda_n}(u_n) \geq -d$, it follows that there must exist two constants R_1 and R_2, with $0 < R_1 < R_2$ such that

$$\max\{u_n(t); t \in [0,T]\} \subset [R_1, R_2].$$

If not, u_n would tend uniformly to 0 or $+\infty$. In both cases, by $(H_2) - (H_3)$ and (3.6), we have

$$\Phi_{\lambda_n}(u_n) \to -\infty \text{ as } n \to +\infty,$$

which contradicts $\Phi_{\lambda_n}(u_n) \geq -d$.

Let τ_n^1, τ_n^2 be such that, for n large enough

$$u_n(\tau_n^1) = \frac{1}{n} < R_1 = u_n(\tau_n^2).$$

Multiplying the differential equation in (3.1) by u_n' and integrating the equation on $[\tau_n^1, \tau_n^2]$, (or $[\tau_n^2, \tau_n^1]$), we get

$$\begin{aligned}\Psi &:= \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t)dt + \int_{\tau_n^1}^{\tau_n^2} a(t)u_n'(t)^2(t)dt + \int_{\tau_n^1}^{\tau_n^2} b(t)f_{\lambda_n}(u_n(t))u_n'(t)dt \\ &= \int_{\tau_n^1}^{\tau_n^2} g(t)u_n'(t)dt.\end{aligned}$$

It is easy to verify that

$$\Psi = \Psi_1 + \frac{1}{2}\left[u_n'(\tau_n^2) - u_n'(\tau_n^1)\right] + \int_{\tau_n^2}^{\tau_n^1} a(t)u_n'^2(t)dt,$$

where

$$\Psi_1 = \int_{\tau_n^1}^{\tau_n^2} b(t)f_{\lambda_n}(u_n(t))u_n'(t)dt.$$

From (H_2) and (3.6), it follows that Ψ is bounded, and consequently Ψ_1 is bounded.

On the other hand, it is easy to see that

$$b(t)f_{\lambda_n}(u_n(t))u_n'(t) = \frac{d}{dt}[b(t)F_{\lambda_n}(u_n(t))] - b'(t)F_{\lambda_n}(u_n(t)).$$

Thus, by (H_1) we have

$$\begin{aligned}\Psi_1 &= b(\tau_n^2)F_{\lambda_n}(R_1) - b(\tau_n^1)F_{\lambda_n}\left(\frac{1}{n}\right) - \int_{\tau_n^1}^{\tau_n^2} b'(t)F_{\lambda_n}(u_n(t))dt \\ &\leq b(\tau_n^2)F_{\lambda_n}(R_1) - b(\tau_n^1)F_{\lambda_n}\left(\frac{1}{n}\right) - \frac{1}{\alpha-1}\int_{\tau_n^1}^{\tau_n^2} b'(t)\left(\frac{1}{R_2^{\alpha-1}} - 1\right)dt.\end{aligned}$$

From the fact that $F_{\lambda_n}(\frac{1}{n}) \to +\infty$ as $n \to +\infty$, we obtain $\Psi_1 \to -\infty$, i.e., Ψ_1 is unbounded. This is a contradiction.

Step 4. We prove that Φ_λ has a mountain-pass geometry for $\lambda \leq \lambda_0$.

Fix $\lambda \in (0, \lambda_0]$, one has

$$\begin{aligned}F_\lambda(0) &= \int_1^0 f_\lambda(s)ds = -\int_0^1 f_\lambda(s)ds \\ &= -\int_0^\lambda f_\lambda(s)ds - \int_\lambda^1 f_\lambda(s)ds \\ &= \frac{1}{\lambda^{\alpha-1}} - \int_\lambda^1 f_\lambda(s)ds.\end{aligned}$$

This implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s)ds = \int_1^\lambda f_\lambda(s)ds = F_\lambda(\lambda).$$

Hence

$$\begin{aligned}\Phi_\lambda(0) &= -F_\lambda(0)\int_0^T e^{A(t)}b(t)dt < -F_\lambda(\lambda)\int_0^T e^{A(t)}b(t)dt \\ &\leq -\frac{\int_0^T e^{A(t)}b(t)dt}{\alpha-1}\left(\frac{1}{\lambda^{\alpha-1}} - 1\right).\end{aligned} \tag{3.7}$$

Consider $\lambda \in (0, \lambda_0]$ such that

$$\frac{1}{\lambda^{\alpha-1}} > 1 + \frac{d(\alpha-1)}{\int_0^T e^{A(t)}b(t)dt}.$$

Thus it follows from (3.7) that $\Phi_\lambda(0) < -d$.

Also, using (H_3) we can choose $R > 1$ enough large such that

$$-e^{\|a\|_{L^1}}\left(M(p-1) + \int_0^T g(t)dt\right)R - \frac{e^{\|a\|_{L^1}}\int_0^T b(t)dt}{\alpha-1}\left(1 - \frac{1}{R^{\alpha-1}}\right) > d.$$

Thus,

$$\begin{aligned}\Phi_\lambda(R) &= e^{A(t)}\sum_{j=1}^{p-1}\int_0^R I_j(s)ds - F_\lambda(R)\int_0^T e^{A(t)}b(t)dt + R\int_0^T g(t)dt\\ &\le e^{\|a\|_{L^1}}M(p-1)R + \frac{1}{\alpha-1}e^{\|a\|_{L^1}}\left(1-\frac{1}{R^{\alpha-1}}\right)\int_0^T b(t)dt\\ &\quad + Re^{\|a\|_{L^1}}\int_0^T g(t)dt\\ &= e^{\|a\|_{L^1}}\left(M(p-1)+\int_0^T g(t)dt\right)R + e^{\|a\|_{L^1}}\frac{\int_0^T b(t)dt}{\alpha-1}\left(1-\frac{1}{R^{\alpha-1}}\right)\\ &< -d.\end{aligned}$$

Since Ω is a neighborhood of $R, 0 \notin \Omega$ and

$$\max\{\Phi_\lambda(0), \Phi_\lambda(R)\} < \inf_{x\in\partial\Omega}\Phi_\lambda(u).$$

Step 1 and Step 2 imply that Φ_λ has a critical point u_λ such taht

$$\Phi_\lambda(u_\lambda) = \inf_{h\in\Gamma}\max_{s\in[0,1]}\Phi_\lambda(h(s)) \ge \inf_{x\in\partial\Omega}\Phi_\lambda(u),$$

where

$$\Gamma = \{h \in C([0,1], H_T^1) : h(0) = 0, h(1) = R\}.$$

Since $\inf_{u\in\partial\Omega}\Phi_\lambda(u_\lambda) \ge -d$, it follows from Step 3 that u_λ is a solution of problem (1.1)-(1.2). The proof of the main result is complete. □

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[1] College of Information Sciences and Technology, Hainan University, Haikou, 570228, China

[2] School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China

E-mail address: shjli626@126.com (S. Li)
E-mail address: tangxh@mail.csu.edu (X. Tang)
E-mail address: wshrm7@126.com (H. Luo)

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Accelerated SNS and accelerated SSS iteration methods for non-Hermitian linear systems

Min-Li Zeng and Guo-Feng Zhang

Abstract. Recently, Bai proposed the skew-normal splitting (SNS) and skew-scaling splitting (SSS) iteration methods for large sparse non-Hermitian positive definite systems. Compared with the Hermitian and skew-Hermitian splitting (HSS) iteration method, both of the SNS and SSS methods are making more use of the skew-Hermitian parts than the HSS method. In this paper, we introduce an accelerated skew-normal splitting (ASNS) iteration method and an accelerated skew-scaling splitting (ASSS) iteration method for solving large sparse non-Hermitian positive definite system of linear equations. We study the convergence properties of the the new iteration methods and the quasi-optimal parameters. Moreover, the inexact forms of the new methods are proposed by employing some subspace methods as the inner iteration processes at each step of the outer iterations. Numerical experiments are given to verify the correctness of the theoretical results and the effectiveness of the new methods.

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1. Introduction

Consider the numerical solution of the large sparse system of linear equations of the form

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \quad \text{and} \quad x, b \in \mathbb{C}^n, \tag{1.1}$$

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where A is a non-Hermitian positive definite matrix. The linear system (1.1) arises from many scientific computing areas, such as diffuse optical tomography [1], lattice quantum chromo dynamics [16], structural dynamics [15], eddy current problems [11] and so on. See [2] and references therein for more applications of the linear system of the form (1.1).

Based on the Hermitian and skew-Hermitian (HS) splitting of the coefficient matrix A: $A = H + S$, with

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*),$$

Bai, Golub and Ng [2] present and studied an efficient Hermitian and skew-Hermitian splitting (HSS) iteration method. Because of the unconditionally convergent property and effectiveness, the HSS iteration method has captured a lot of researchers' attention. A multitude of researchers focused on the HSS method and proposed varieties of variants based on the Hermitian and skew-Hermitian splitting, such as the HSS-like method [9], the modified HSS method [10], the accelerated HSS method [5] and the preconditioned HSS method [6, 3] and so on, see [12, 20, 22, 17, 4, 18]. As is shown in [7] that the HSS method is more effective when S dominates H than vice versa. Furthermore, when S is very small compared to H, the subsystem

$$(\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b$$

in the HSS method contributes little to convergence and the inner iterations must be designed to terminate properly since

$$(I - \frac{1}{\alpha}S)^{-1} = I + \frac{1}{\alpha}S + \frac{1}{\alpha^2}S^2 + \cdots .$$

As is known that the iterative matrix of HSS

$$(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$$

is similar to $W(\alpha)Q(\alpha)$, where

$$W(\alpha) = (I - \frac{1}{\alpha}H)(I + \frac{1}{\alpha}H)^{-1}$$

is Hermitian, so forth $\|W(\alpha)\|_2 < 1$, and

$$Q(\alpha) = (I - \frac{1}{\alpha}S)(I + \frac{1}{\alpha}S)^{-1}$$

is unitary for all $\alpha > 0$. Therefore, according to [7], when S is small compared to H, different iterations combing H and S^*S are more naturally.

Suppose that S is invertible, then we can multiply (1.1) on the left by $-S$ to obtain the following equivalent form:

$$-SHx - S^2x = -Sb, \tag{1.2}$$

i.e.,

$$-SHx = S^2x - Sb \quad \text{or} \quad -S^2x = SHx - Sb.$$

Adding αH to each side will lead to the following two fixed-point equations

$$(\alpha H - SH)x \equiv (\alpha I - S)Hx = (\alpha H + S^2)x - Sb, \tag{1.3}$$

$$(\alpha H - S^2)x = (\alpha H + SH)x - Sb \equiv (\alpha I + S)Hx - Sb. \tag{1.4}$$

Based on the above fixed-point equations, Bai [7] established the skew-normal splitting (SNS) methods as the following algorithm.

Algorithm 1.1. *(SNS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - Sb, \\ (\alpha H - S^2)x^{(k+1)} = (\alpha I + S)x^{(k+\frac{1}{2})} - Sb, \end{cases}$$

where α is a given positive constant.

Another way to use the skew-Hermitian matrix S is to employ it to scale the linear system (1.1). By first adding and then subtracting $\frac{1}{\alpha}S^2x$ to the fixed-point equations

$$-Sx = Hx - b \quad \text{and} \quad Hx = -Sx + b,$$

respectively, it follows,

$$(I - \frac{1}{\alpha}S)(-Sx) = (H + \frac{1}{\alpha}S^2)x - b \quad \text{and} \quad (H - \frac{1}{\alpha}S^2)x = (I + \frac{1}{\alpha}S)(-Sx) + b.$$

In analogy to the SNS method, Bai further present the skew-scaling splitting (SSS) method in [7] as the following algorithm.

Algorithm 1.2. *(SSS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$, until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - \alpha b, \\ (\alpha H - S^2)x^{(k+1)} = (\alpha I + S)x^{(k+\frac{1}{2})} + \alpha b, \end{cases}$$

where α is a given positive constant.

For the SNS and SSS methods, it has been shown in [7] that when S is small compared to H, the Corollary 2.3 in [2] makes clear how to choose a good iterative parameter. When S dominates H, it is not clear. However, the SNS and SSS methods give an exact way to choose the optimal iterative parameter no matter S dominates H or not.

In this paper, we will first accelerate the SNS and SSS methods by adding another parameter in the second iterate step of each iterations and then we obtain two new methods, which are named as the ASNS method and the ASSS method. The new methods can be seen as generalized forms of the original SNS and SSS methods. Futher, the iterative parameters can be chosen in a more extensive range.

The outline of this paper is arranged as follows. In Section 2, we present the accelerated SNS (ASNS) and the accelerated SSS (ASSS) methods. Then we analyze the convergence properties of both methods. In Section 3, we determine the quasi-optimal parameters by minimizing the upper bound of the spectral radius of the iteration matrix and then show the case about the new methods superiority to the SNS and SSS methods. Section 4 is devoted to

the inexact variant form of the new methods and the asymptotically convergent rate property of the new methods. Numerical experiments are given in Section 5 to illustrate the correctness of the theoretical results obtained in this paper. Section 6 draws some conclusions and remarks to end this paper.

Throughout this paper, the skew-Hermitian part S of the coefficient matrix A is assumed to be invertible and the Hermitian part H to be positive definite. We use $A \sim B$ to denote that the matrix A is similar to the matrix B.

2. The ASNS and ASSS methods

In this section, we will propose the accelerated SNS (ASNS) and the accelerated SSS (ASSS) methods. Firstly, we add αH to each side of the equation $-SHx = S^2x - Sb$ and then add βH to each side of the equation $-S^2x = SHx - Sb$. Then we obtain the fixed-point equations

$$(\alpha H - SH)x \equiv (\alpha I - S)Hx = (\alpha H + S^2)x - Sb \tag{2.1}$$

and

$$(\beta H - S^2)x = (\beta H + SH)x - Sb \equiv (\beta I + S)Hx - Sb. \tag{2.2}$$

Subsequent algorithm is the ASNS iteration method.

Algorithm 2.1. *(ASNS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - Sb, \\ (\beta H - S^2)x^{(k+1)} = (\beta I + S)x^{(k+\frac{1}{2})} - Sb, \end{cases}$$

where α and β are given positive constants.

For the fixed-point equations

$$-Sx = Hx - b \quad \text{and} \quad Hx = -Sx + b,$$

we first add $\frac{1}{\alpha}S^2x$ and then subtract $\frac{1}{\beta}S^2x$ on both sides to obtain

$$(I - \frac{1}{\alpha}S)(-Sx) = (H + \frac{1}{\alpha}S^2)x - b$$

and

$$(H - \frac{1}{\beta}S^2)x = (I + \frac{1}{\beta}S)(-Sx) + b.$$

After rearranging these equations and choosing $x^{(0)}$ wisely, we can straightforwardly get the accelerated skew-scaling splitting (ASSS) method as the next algorithm.

Algorithm 2.2. *(ASSS method) Given an initial approximate solution $x^{(0)}$, for $k = 0, 1, 2, \cdots$, until convergence, solve*

$$\begin{cases} (\alpha I - S)x^{(k+\frac{1}{2})} = (\alpha H + S^2)x^{(k)} - \alpha b, \\ (\beta H - S^2)x^{(k+1)} = (\beta I + S)x^{(k+\frac{1}{2})} + \beta b, \end{cases}$$

where α and β are given positive constants.

Obviously, when $\alpha = \beta$, the ASNS method and the ASSS method reduce to the SNS method and the SSS method, respectively. Comparing the ASSS method with the ASNS method, we find that the coefficient matrices of the ASNS and ASSS method are exactly the same. Therefore, they have the same iteration matrix $M(\alpha, \beta)$, where

$$M(\alpha, \beta) := (\beta H - S^2)^{-1}(\beta I + S)(\alpha I - S)^{-1}(\alpha H + S^2).$$

It can be seen that the ASSS method is much cheaper than the ASNS method, because of the constant vector terms αb, βb instead of Sb.

It is seen that S^{-1} is skew-Hermitian, then $-S^{-1}HS^{-1} = (S^{-1})^* HS^{-1}$ is Hermitian positive definite. Denote

$$\xi_{\max} = \max\{|\xi_j| \big| \mathbf{i}\xi_j \in \sigma(S)\}, \xi_{\min} = \min\{|\xi_j| \big| \mathbf{i}\xi_j \in \sigma(S)\},$$

$$\lambda_{\max} = \max\{\lambda_j \big| \lambda_j \in \sigma((S^{-1})^* HS^{-1})\}, \lambda_{\min} = \min\{\lambda_j \big| \lambda_j \in \sigma((S^{-1})^* HS^{-1})\},$$

the following theorem concentrates on the convergence property of the ASNS method.

Theorem 2.1. *Given a non-Hermitian matrix $A \in \mathbb{C}^{n\times n}$. Let $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$. If H is positive definite , S is invertible, then the spectral radius $\rho(M(\alpha, \beta))$ of the ASNS and ASSS iteration matrix is bounded by $\delta(\alpha, \beta)$, where*

$$\delta(\alpha, \beta) = \max_{\mathbf{i}\xi_j \in \sigma(S)} \sqrt{\frac{\beta^2 + \xi_j^2}{\alpha^2 + \xi_j^2}} \cdot \max_{\lambda_k \in \sigma((S^{-1})^* HS^{-1})} \Big| \frac{\alpha\lambda_k - 1}{\beta\lambda_k + 1} \Big|.$$

*Further, if α and β satisfy $\delta(\alpha, \beta) < 1$, then the ASNS method and the ASSS method are convergent to the unique solution of the linear system (*1.1*).*

Proof. As

$$\begin{aligned} M(\alpha, \beta) &= (\beta H - S^2)^{-1}(\beta I + S)(\alpha I - S)^{-1}(\alpha H + S^2) \\ &\sim S^{-1}(\beta I + S)(\alpha I - S)^{-1}(\alpha H + S^2)(\beta H - S^2)^{-1} S \\ &= (\beta I + S)S^{-1}(\alpha I - S)^{-1} S(-S^{-1})(\alpha H - S^* S)S^{-1} S(\beta H + S^* S)^{-1}(-S) \\ &= (\beta I + S)(\alpha I - S)^{-1}(\alpha (S^{-1})^* HS^{-1} - I)(\beta (S^{-1})^* HS^{-1} + I)^{-1} \\ &:= \overline{M}(\alpha, \beta), \end{aligned} \tag{2.3}$$

then it follows

$$\begin{aligned} &\rho(M(\alpha, \beta)) \\ &\le \|(\beta I + S)(\alpha I - S)^{-1}\| \cdot \|(\alpha (S^{-1})^* HS^{-1} - I)(\beta (S^{-1})^* HS^{-1} + I)^{-1}\| \\ &\le \max_{\mathbf{i}\xi_j \in \sigma(S^{-1})} \sqrt{\frac{\beta^2 + \xi_j^2}{\alpha^2 + \xi_j^2}} \cdot \max_{\lambda_k \in \sigma((S))^* HS^{-1})} \Big| \frac{\alpha\lambda_k - 1}{\beta\lambda_k + 1} \Big| \\ &= \delta(\alpha, \beta) \end{aligned}$$

If $\delta(\alpha, \beta) < 1$, then $\rho(M(\alpha, \beta)) \le \delta(\alpha, \beta) < 1$, i.e., the ASNS and the ASSS methods are convergent. □

Theorem 2.2. *The ASNS and ASSS iteration methods are convergent if either of the following conditions holds:*
(1) $\alpha \geq \beta$, $(\lambda_{\min}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\min})$ *and* $(\lambda_{\max}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\max})$*;*
(2) $\alpha < \beta$, $(\lambda_{\min}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\min})$ *and* $(\lambda_{\max}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\max})$.

Proof. $\delta(\alpha, \beta) < 1$ leads to

$$\max_{\mathbf{i}\xi_j \in \sigma(S)} \sqrt{\frac{\beta^2 + \xi_j^2}{\alpha^2 + \xi_j^2}} \cdot \max_{\lambda_k \in \sigma((S^{-1})^* H S^{-1})} |\frac{\alpha\lambda_k - 1}{\beta\lambda_k + 1}| < 1.$$

Or equivalently,

$$\max_{\lambda_k \in \sigma((S^{-1})^* H S^{-1})} |\frac{\alpha\lambda_k - 1}{\beta\lambda_k + 1}| < \min_{\mathbf{i}\xi_j \in \sigma(S)} \sqrt{\frac{\alpha^2 + \xi_j^2}{\beta^2 + \xi_j^2}}. \tag{2.4}$$

(1) If $\alpha \geq \beta$, then

$$\min_{\mathbf{i}\xi_j \in \sigma(S)} \sqrt{\frac{\alpha^2 + \xi_j^2}{\beta^2 + \xi_j^2}} = \sqrt{\frac{\alpha^2 + \xi_{\max}^2}{\beta^2 + \xi_{\max}^2}}.$$

Since

$$\max_{\lambda_k \in \sigma((S^{-1})^* H S^{-1})} |\frac{\alpha\lambda_k - 1}{\beta\lambda_k + 1}| = \max\{|\frac{\alpha\lambda_{\min} - 1}{\beta\lambda_{\min} + 1}|, |\frac{\alpha\lambda_{\max} - 1}{\beta\lambda_{\max} + 1}|\},$$

then (2.4) is equivalent to

$$\begin{cases} (\frac{\alpha\lambda_{\min} - 1}{\beta\lambda_{\min} + 1})^2 < \frac{\alpha^2 + \xi_{\max}^2}{\beta^2 + \xi_{\max}^2}, \\ (\frac{\alpha\lambda_{\max} - 1}{\beta\lambda_{\max} + 1})^2 < \frac{\alpha^2 + \xi_{\max}^2}{\beta^2 + \xi_{\max}^2}. \end{cases}$$

After some simple computations, it follows

$$\begin{cases} (\alpha\lambda_{\min} - 1)^2(\beta^2 + \xi_{\max}^2) < (\alpha^2 + \xi_{\max}^2)(\beta\lambda_{\min} + 1)^2, \\ (\alpha\lambda_{\max} - 1)^2(\beta^2 + \xi_{\max}^2) < (\alpha^2 + \xi_{\max}^2)(\beta\lambda_{\max} + 1)^2. \end{cases} \tag{2.5}$$

Because α and β are positive constants, the first equation of (2.5) leads to

$$(\lambda_{\min}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\min}).$$

The second equation of (2.5) leads to

$$(\lambda_{\max}^2\xi_{\max}^2 - 1)\alpha - 2\xi_{\max}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\max}^2 - 1 + 2\alpha\lambda_{\max}).$$

(2) If $\alpha < \beta$, then

$$\min_{\mathbf{i}\xi_j \in \sigma(S)} \sqrt{\frac{\alpha^2 + \xi_j^2}{\beta^2 + \xi_j^2}} = \sqrt{\frac{\alpha^2 + \xi_{\min}^2}{\beta^2 + \xi_{\min}^2}}.$$

Using the same strategy as $\alpha \geq \beta$, we can easily obtain

$$(\lambda_{\min}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\min} < \beta(\lambda_{\min}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\min})$$

and

$$(\lambda_{\max}^2\xi_{\min}^2 - 1)\alpha - 2\xi_{\min}^2\lambda_{\max} < \beta(\lambda_{\max}^2\xi_{\min}^2 - 1 + 2\alpha\lambda_{\max}).$$

□

Remark 2.3. *If $\alpha = \beta$, from the result (1) of Theorem 2.2, we obtain that when $\alpha^2+\xi_{\min}^2 > 0$, the ASNS and ASSS iteration methods are convergent. Because $\alpha^2 + \xi_{\min}^2 > 0$ invariably holds, then we know the ASNS and ASSS iteration methods are convergent unconditionally, which agrees with the results in* [7].

3. The quasi-optimal iterative parameters

In this section, we will give the quasi-optimal parameters of the ASNS iteration method and the ASSS iteration method. Then we will analyze the optimal parameters by minimizing the upper bound $\delta(\alpha, \beta)$. According to the analysis in the previous section, we have

$$\delta(\alpha,\beta) = \begin{cases} \sqrt{\dfrac{\beta^2+\xi_{\min}^2}{\alpha^2+\xi_{\min}^2}} \cdot \max\left\{|\dfrac{\alpha\lambda_{\min}-1}{\beta\lambda_{\min}+1}|, |\dfrac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}|\right\}, & \alpha \geq \beta; \\ \sqrt{\dfrac{\beta^2+\xi_{\max}^2}{\alpha^2+\xi_{\max}^2}} \cdot \max\left\{|\dfrac{\alpha\lambda_{\min}-1}{\beta\lambda_{\min}+1}|, |\dfrac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}|\right\}, & \alpha < \beta. \end{cases}$$

We rewrite the matrix $\overline{M}(\alpha, \beta)$ in (2.3) as

$$\begin{aligned} &\overline{M}(\alpha,\beta) \\ &= (\frac{1}{\beta}I + S^{-1})(\frac{1}{\alpha}I - S^{-1})^{-1}(\frac{1}{\alpha}I - (S^{-1})^*HS^{-1})(\frac{1}{\beta}I + (S^{-1})^*HS^{-1})^{-1} \\ &\sim (\frac{1}{\beta}I + (S^{-1})^*HS^{-1})^{-1}(\frac{1}{\alpha}I - (S^{-1})^*HS^{-1})(\frac{1}{\alpha}I - S^{-1})^{-1}(\frac{1}{\beta}I + S^{-1}). \end{aligned}$$

By making use of the same strategy of Theorem 4.2 in [19], we can obtain the optimal iterative parameters and the corresponding upper bound in the following theorem.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian matrix. If the ASNS and ASSS iteration methods are convergent, then*
(1) when $\sqrt{\lambda_{\min}\lambda_{\max}} > \frac{1}{\xi_{\min}}$ or $\sqrt{\lambda_{\min}\lambda_{\max}} < \frac{1}{\xi_{\max}}$, the quasi-optimal parameters are given by

$$\alpha_* = \frac{\sqrt{(c^2+\lambda_{\min}^2)(c^2+\lambda_{\max}^2)} + c^2 - \lambda_{\min}\lambda_{\max}}{(\lambda_{\max}+\lambda_{\min})c^2},$$

$$\beta_* = \frac{\sqrt{(c^2+\lambda_{\min}^2)(c^2+\lambda_{\max}^2)} - c^2 + \lambda_{\min}\lambda_{\max}}{(\lambda_{\max}+\lambda_{\min})c^2},$$

and the corresponding optimal upper bound $\delta(\alpha^*, \beta^*)$ of $\rho(M(\alpha, \beta))$ is

$$\delta(\alpha^*,\beta^*) = \frac{\sqrt{(c^2+\lambda_{\min}^2)(c^2+\lambda_{\max}^2)} - (c^2+\lambda_{\min}\lambda_{\max})}{(\lambda_{\max}-\lambda_{\min})c},$$

where the constant c is given by

$$c = \begin{cases} \dfrac{1}{\xi_{\max}}, & \text{if} \quad \sqrt{\lambda_{\max}\lambda_{\min}} > \dfrac{1}{\xi_{\min}}; \\ \dfrac{1}{\xi_{\min}}, & \text{if} \quad \sqrt{\lambda_{\max}\lambda_{\min}} < \dfrac{1}{\xi_{\max}}. \end{cases}$$

(2) when $\frac{1}{\xi_{\max}} \le \sqrt{\lambda_{\min}\lambda_{\max}} \le \frac{1}{\xi_{\min}}$, the quasi-optimal parameters α_* and β_* are

$$\alpha_* = \beta_* = \frac{1}{\sqrt{\lambda_{\min}\lambda_{\max}}}$$

and the corresponding optimal upper bound is

$$\delta(\alpha_*, \beta_*) = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}.$$

Remark 3.2. *When $e_{\min} \le \sqrt{\lambda_{\min}\lambda_{\max}} \le e_{\max}$, the optimal upper bound $\delta(\alpha, \beta)$ of the ASNS and GSSS iteration methods reduces to the optimal bound of the SNS and SSS iteration methods, respectively, i.e.,*

$$\alpha_* = \beta_* = \frac{1}{\sqrt{\lambda_{\min}\lambda_{\max}}}$$

and the corresponding optimal bound is

$$\nu_* = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}.$$

Theorem 3.3. *When $\sqrt{\lambda_{\min}\lambda_{\max}} > \frac{1}{\xi_{\min}}$ or $\sqrt{\lambda_{\min}\lambda_{\max}} < \frac{1}{\xi_{\max}}$, then we have $\delta(\alpha_*, \beta_*) < \nu_*$, where $\delta(\alpha_*, \beta_*)$ and ν_* are defined in Theorem* 3.1 *and Remark* 3.2*, respectively.*

Proof. We rewrite ν_* as

$$\nu_* = \frac{(\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}})^2}{\lambda_{\max} - \lambda_{\min}}.$$

Then $\delta(\alpha_*, \beta_*) < \nu_*$ if and only if

$$(\lambda_{\min}^2 + c^2)(\lambda_{\max}^2 + c^2) < [c(\sqrt{\lambda_{\min}} - \sqrt{\lambda_{\min}})^2 + (\lambda_{\min}\lambda_{\min} + c^2)]^2.$$

Denote $k_1 = \lambda_{\max} + \lambda_{\min}$ and $k_2 = \lambda_{\max}\lambda_{\min}$, then the above inequality can be simplified as

$$4c^2k_2 - 2c^2k_1\sqrt{k_2} + ck_1k_2 - 2c^2c^3\sqrt{k_2} - 2ck_2\sqrt{k_2} + c^3k_1 > 0.$$

Or equivalently,

$$(k_1 - 2\sqrt{k_2})(\sqrt{k_2} - c)^2 > 0.$$

It follows

$$(\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}})^2(\sqrt{\lambda_{\min}\lambda_{\max}} - c)^2 > 0,$$

i.e., $(\sqrt{\lambda_{\min}\lambda_{\max}} - c)^2 > 0$. That is, $\sqrt{\lambda_{\min}\lambda_{\max}} > \frac{1}{\xi_{\min}}$ or $\sqrt{\lambda_{\min}\lambda_{\max}} < \frac{1}{\xi_{\max}}$. □

4. The inexact ASNS and ASSS methods

The two half-steps at each step of the ASNS and ASSS methods require finding solutions with the coefficient matrices $\alpha I - S$ and $\beta H - S^2$. Because $\beta H - S^2$ is Hermitian positive definite, then we may use the CG method to solve the linear system with coefficient matrix $\beta H - S^2$. For solving the linear system with coefficient matrix $\alpha I - S$, we may use some Krylov subspace method [21]. The inexact ASNS (IASNS) method can be described as follows.

Algorithm 4.1. *(IASNS method) Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \cdots$, until $\{\bar{x}^{(k)}\}$ converges, solve $\{\bar{x}^{(k+\frac{1}{2})}\}$ approximately from*

$$(\alpha I - S)\bar{x}^{(k+\frac{1}{2})} = (\alpha H + S^2)\bar{x}^{(k)} - Sb$$

by employing an inner iteration (e.g., some Krylov subspace method) with $\bar{x}^{(k)}$ as the initial guess, then solve $\bar{x}^{(k+1)}$ approximately from

$$(\beta H - S^2)\bar{x}^{(k+1)} = (\beta I + S)\bar{x}^{(k+\frac{1}{2})} - Sb$$

by employing an inner iteration (e.g., the CG method) with $\bar{x}^{(k+\frac{1}{2})}$ as the initial guess, where α and β are given positive constants.

The inexact ASSS (IASSS) method can be described as follows.

Algorithm 4.2. *(IASSS method) Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \cdots$, until $\{\bar{x}^{(k)}\}$ converges, solve $\{\bar{x}^{(k+\frac{1}{2})}\}$ approximately from*

$$(\alpha I - S)\bar{x}^{(k+\frac{1}{2})} = (\alpha H + S^2)\bar{x}^{(k)} - \alpha b$$

by employing an inner iteration (e.g., some Krylov subspace method) with $\bar{x}^{(k)}$ as the initial guess, then solve $\bar{x}^{(k+1)}$ approximately from

$$(\beta H - S^2)\bar{x}^{(k+1)} = (\beta I + S)\bar{x}^{(k+\frac{1}{2})} + \beta b$$

by employing an inner iteration (e.g., the CG method) with $\bar{x}^{(k+\frac{1}{2})}$ as the initial guess, where α and β are given positive constants.

Subsequently, we will concentrate on the IASNS iteration method. The results about the IASSS iteration method can be obtained in the similar way. To simplify numerical implementation and convergence analysis, the IASNS iteration method can be rewritten as the following equivalent scheme.

Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \cdots$, compute Step 1 and Step 2 until $\bar{x}^{(k)}$ converges:

Step 1. approximate the solution of

$$(\alpha I - S)\bar{z}^{(k)} = -S\bar{r}^{(k)}, (\bar{r}^{(k)} = b - A\bar{x}^{(k)})$$

by iterating until $\bar{z}^{(k)}$ is such that the residual

$$\bar{p}^{(k)} = \bar{r}^{(k)} - (\alpha I - S)\bar{z}^{(k)}$$

satisfies

$$\|\bar{p}^{(k)}\| \leq \varepsilon_k \|\bar{r}^{(k)}\|,$$

and then compute $\bar{x}^{(k+\frac{1}{2})} = \bar{x}^{(k)} + \bar{z}^{(k)}$;

Step 2. approximate the solution of

$$(\beta H - S^2)\bar{z}^{(k+\frac{1}{2})} = -S\bar{r}^{(k+\frac{1}{2})}, (\bar{r}^{(k+\frac{1}{2})} = b - A\bar{x}^{(k+\frac{1}{2})})$$

by iterating until $\bar{z}^{(k+\frac{1}{2})}$ is such that the residual

$$\bar{q}^{(k+\frac{1}{2})} = \bar{r}^{(k+\frac{1}{2})} - (\beta H - S^2)\bar{z}^{(k+\frac{1}{2})}$$

satisfies

$$\|\bar{p}^{(k+\frac{1}{2})}\| \leq \eta_k \|\bar{r}^{(k+\frac{1}{2})}\|,$$

and then compute $\bar{x}^{(k+1)} = \bar{x}^{(k+\frac{1}{2})} + \bar{z}^{(k+\frac{1}{2})}$. Here $\|\cdot\|$ is a norm of a vector.

If the two inner systems are solved inexactly with corresponding quantities $\{\varepsilon_k\}$ and $\{\eta_k\}$. Denote $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$ and $\eta_{\max} = \max_k\{\eta_k\}$. Let $|||\cdot|||_M$ denote

$$|||X|||_M = \|MXM^{-1}\|$$

for all $X \in \mathbb{C}^{n\times n}$. Then the following theorem concentrates on the convergent results of the IASNS and IASSS iteration methods. According to Theorem 3.1 in [8] and by specializing the splitting as

$$\begin{aligned}(-S)A = M_1 - N_1 &:= (\alpha H - SH) - (\alpha H + S^2)\\ &= M_2 - N_2 := (\beta H - S^2) - (\beta H + SH),\end{aligned}$$

we can immediately obtain the following theorem.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian positive definite matrix. $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, and let α and β be positive constants. If S is invertible, $\{\bar{x}^{(k)}\}$ is an iterative sequence generated by the IASNS iteration method and $x^* \in \mathbb{C}^n$ is the exact solution of the linear system (1.1), then it holds that*

$$\begin{aligned}&|||\bar{x}^{(k+1)} - x^*|||_{M_2}\\ &\quad\leq (\sigma(\alpha,\beta) + \mu(\alpha,\beta)\theta(\beta)\varepsilon_k + \theta(\beta)(\rho(\alpha,\beta) + \theta(\beta)\nu(\alpha,\beta)\varepsilon_k)\eta_k)\\ &\qquad\cdot |||\bar{x}^{(k)} - x^*|||_{M_2},\end{aligned}$$

where

$$\begin{aligned}\sigma(\alpha,\beta) &= \|(\beta I + S)(\alpha I - S)^{-1}(\alpha H + S^2)(\beta H - S^2)^{-1}\|,\\ \rho(\alpha,\beta) &= \|(\beta H - S^2)H^{-1}(\alpha I - S)^{-1}(\alpha H + S^2)(\beta H - S^2)^{-1}\|,\\ \mu(\alpha,\beta) &= \|(\beta I + S)(\alpha I - S)^{-1}\|,\\ \theta(\beta) &= \|(-SA)(\beta H - S^2)^{-1}\|,\\ \nu(\alpha,\beta) &= \|(\beta H - S^2)H^{-1}(\alpha I - S)^{-1}\|.\end{aligned}$$

In particularly, if

$$\sigma(\alpha,\beta) + \mu(\alpha,\beta)\theta(\beta)\varepsilon_{\max} + \theta(\beta)(\rho(\alpha,\beta) + \theta(\beta)\nu(\alpha,\beta)\varepsilon_{\max})\eta_{\max} < 1,$$

then the iterative sequence $\{\bar{x}^{(k)}\}$ converges to $x^* \in \mathbb{C}^n$, where

$$\varepsilon_{\max} = \max_k\{\varepsilon_k\} \quad \textit{and} \quad \eta_{\max} = \max_k\{\eta_k\}.$$

Theorem 4.2. *Let the assumption in Theorem* 4.1 *be satisfied. Suppose that both* $\{\tau_1(k)\}$ *and* $\{\tau_2(k)\}$ *are nondecreasing and positive sequences satisfying* $\tau_1(k) \geq 1$ *and* $\tau_2(k) \geq 1$*, and*

$$\lim_{k\to\infty} \sup \tau_1(k) = \lim_{k\to\infty} \sup \tau_2(k) = +\infty,$$

and that both δ_1 *and* δ_2 *are real constants in the interval* $[0, 1]$ *satisfying*

$$\varepsilon_k \leq c_1 \delta_1^{\tau_1(k)} \quad \text{and} \quad \eta_k \leq c_2 \delta_2^{\tau_2(k)},$$

for $k = 0, 1, 2, \cdots$*, where* c_1 *and* c_2 *are nonnegative constants. Then we have*

$$|||\bar{x}^{(k+1)} - x^*|||_{M_2} \leq (\sqrt{\sigma(\alpha,\beta)} + \omega(\alpha,\beta)\theta(\beta)\delta^{\tau(k)})^2 \cdot |||\bar{x}^{(k)} - x^*|||_{M_2},$$

where $k = 0, 1, 2, \cdots$, $\tau(k) = \min\{\tau_1(k), \tau_2(k)\}, \delta = \max\{\delta_1, \delta_2\}$ *and*

$$\omega = \max\{\sqrt{c_1 c_2 \nu(\alpha,\beta)}, \frac{1}{2\sqrt{\sigma(\alpha,\beta)}}(c_1\mu(\alpha,\beta) + c_2\rho(\alpha,\beta))\}.$$

In particular, we have

$$\lim_{k\to\infty} \sup \frac{|||\bar{x}^{(k+1)} - x^*|||_{M_2}}{|||\bar{x}^{(k)} - x^*|||_{M_2}} = \sigma(\alpha,\beta),$$

i.e., the convergence rate of the IASNS method is asymptotically the same as that of the exact two-step iterative scheme ASNS.

5. Numerical results

In this section, we will consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z) \tag{5.1}$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, with constant coefficient q and subject to Dirichlet-type boundary conditions. When the seven-point finite difference discretization, for example, the centered differences to diffusive terms, and the centered differences or the first order upwind approximations to the convective terms are applied to the above model convection-diffusion equation, we get the system of linear equations (1.1) with the coefficient matrix

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z,$$

where the equidistant step-size $h = \frac{1}{n+1}$ is used in the distretization on all of the three directions and the natural lexicographic ordering is employed to the unknowns. In addition, $\otimes$ denotes the Kronecker product, and T_x, T_y, and T_z are tri-diagonal matrices given by

$$T_x = tridiag(t_2, t_1, t_3), \quad T_y = tridiag(t_2, 0, t_3), \quad T_z = tridiag(t_2, 0, t_3),$$

with

$$t_1 = 6, t_2 = -1 - r, t_3 = -1 + r$$

if the first order derivatives are approximated by the centered difference scheme and with

$$t_1 = 6 + 6r, t_2 = -1 - 2r, t_3 = -1$$

if the first order derivatives are approximated by the upwind difference scheme. Here $r = \frac{qh}{2}$ is the mesh Reynolds number. For details, we refer to [13, 14, 2].

Figure 1 plots $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ for the centered difference scheme with $n = 8$ and $q = 1, 10$ when α varies in $[0, 1]$. When α is fixed, according to [19], we can compute β as

$$\frac{1-\alpha\lambda_{\min}}{\beta\lambda_{\min}+1} = \frac{\alpha\lambda_{\max}-1}{\beta\lambda_{\max}+1}.$$

That is,

$$\beta = \frac{\alpha(\lambda_{\max}+\lambda_{\min})-2}{\lambda_{\max}+\lambda_{\min}-2\alpha\lambda_{\max}\lambda_{\min}}. \tag{5.2}$$

Besides, Figure 2 plots $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ for the upwind difference scheme with $n = 8$ and $q = 1, 10$ when α varies in $[0, 1]$. β is also computed according to (5.2).

From Figures 1-2, we find that, when β is chosen according to (5.2) and let α vary in $[0, 1]$, the point such that the value of $\rho(M(\alpha,\beta))$ reaches the minimum is extremely close to the point such that the value of $\delta(\alpha,\beta)$ reaches the minimum. Therefore, the theoretical optimal parameters in Theorem 3.1 would be intensely close to the real optimal parameters of the methods.

In Figure 3 and Figure 4, we plot the distributions of the eigenvalues of the iterative matrices. Here, we choose the experimental optimal parameters $\alpha = \beta = \frac{qh}{2}$ in the SNS method and we plot the eigenvalues distribution in Figure 3. We replace α in the ASNS method by $\frac{h}{2}$ and we plot the eigenvalues distributions in Figure 4.

It can be seen from Figures 3-4 that the eigenvalues of the ASNS iterative matrix are more cluster around 0 than the eigenvalues of the SNS iterative matrix when the iterative parameters are chosen appropriately. Further, if we can find a cheaper way to choose the optimal parameters, the accelerated methods would be more efficient, which will be our next work.

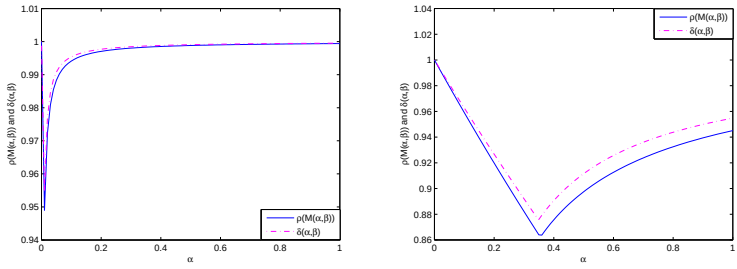


FIGURE 1. The comparison between $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ when $q = 1$ (left) and $q = 10$ (right) (centered).

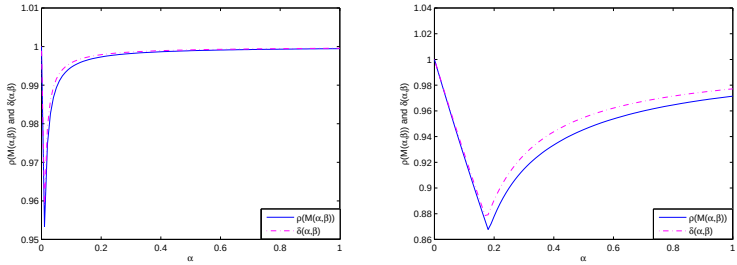


FIGURE 2. The comparison between $\rho(M(\alpha,\beta))$ and $\delta(\alpha,\beta)$ when $q = 1$ (left) and $q = 10$ (right) (upwind).

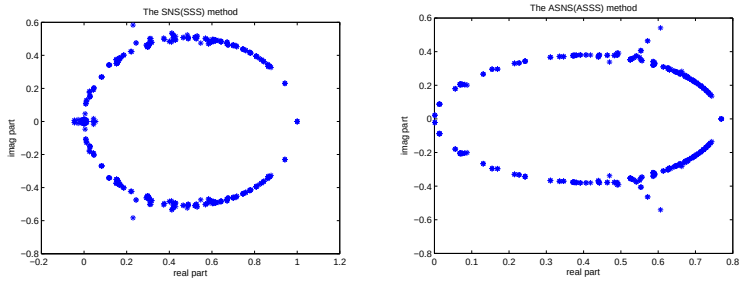


FIGURE 3. The distribution of the eigenvalues of the iterative matrix for the SNS method (left) and the ASNS method (right).

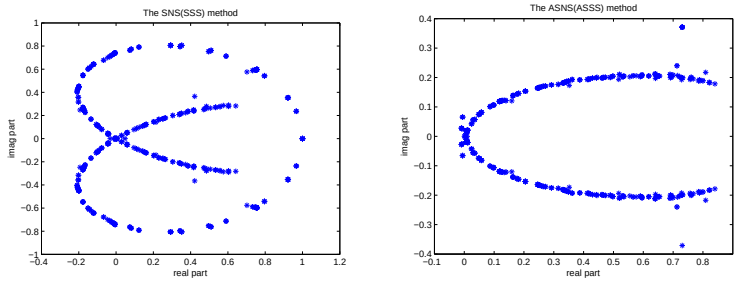


FIGURE 4. The distribution of the eigenvalues of the iterative matrix for the SNS method (left) and the ASNS method (right).

6. Conclusions

For solving the non-Hermitian positive definite with invertible skew-Hermitian parts, we proposed an accelerated SNS and an accelerated SSS iteration method in this paper. Compared with the HSS iteration method, the new

methods not only concentrate more on the balance between the Hermitian parts and the skew-Hermitian parts, but also accelerate the original SNS and SSS methods. The convergence properties and the quasi-optimal parameters are analyzed. In actual implementations, we give the inexact forms of the new methods. Numerical results demonstrate that the point such that $\delta(\alpha, \beta)$ reaches the minimum is exact the same as the point such that $\rho(M(\alpha, \beta))$ reaches the minimum. Therefore, the theoretical results obtained in this paper is correct. Meanwhile, by choosing appropriate parameters, the new methods are more efficient than the SNS and SSS methods.

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Min-Li Zeng
School of Mathematics, Putian University, Putian 351100, P.R. China
e-mail: zengml12@lzu.edu.cn

Guo-Feng Zhang
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P.R. China

Aggregating of Interval-valued Intuitionistic Uncertain Linguistic Variables based on Archimedean t-norm and It Applications in Group Decision Makings[†]

Juan Li[a,*], Xiao-Lei Zhang[b], Zeng-Tai Gong[c]
[a]*Department of Mathematics, Baoji University of Arts and Sciences, Baoji, Shaanxi, 721013, P.R. China*
[b]*Group of Mathematics, Dongfanghong Middle School of Dingxi, Dingxi, Gansu 743000, China*
[b]*College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730070 China*

Abstract With respect to multiple attribute group decision making (MAGDM) problems in which the attribute weights and the expert weights take the form of real numbers and the attribute values take the form of interval-valued intuitionistic uncertain linguistic variable, we propose group decision making methods of interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm and Choquet integral. First, we introduce some concepts of fuzzy measure and interval-valued intuitionistic uncertain lingistic variables based on Archimedean t-norm. Then, interval-valued intuitionistic uncertain linguistic weighted average(geometric) and interval-valued intuitionistic uncertain lingistic ordered weighted average operator based on Archimedean t-norm are developed. Furthermore, some desirable properties of these operators, such as commutativity, idempotency and monotonicity have been studied, and interval-valued intuitionistic uncertain linguistic hybrid average operator based on Archimedean t-norm are developed. Based on these operators, two methods for multiple attribute group decision making problems with intuitionistic uncertain linguistic information have been proposed. Finally, an illustrative example is given to verify the developed approaches and demonstrate their practicality and effectiveness.
Keywords: Interval-valued intuitionistic fuzzy sets; aggregation operators; Archimedean t-norm; group decision making.

1. Introduction

Multiple attribute decision making (MADM) problems are an important research topic in decision theory. Because the objects are fuzzy and uncertain, the attributes involved in decision problems are not always expressed as real numbers, and some better suited to be denoted by fuzzy numbers, such as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic numbers on uncertain linguistic variables, and intuitionistic fuzzy numbers. Because Zadeh initially proposed the basic model of fuzzy decision making based on the theory of fuzzy mathematics, fuzzy MADM has been receiving more and more attention. The fuzzy set (FS) theory proposed by Zadeh [1] was a very good tool to research the fuzzy MADM problems, the fuzzy set is used to character the fuzziness just by membership degree. Different from fuzzy set,there is another parameter: non-membership degree in intuitionistic fuzzy set (IFS) which is proposed by Atanassov [2,3]. Clearly, the IFS can describe and character the fuzzy essence of the objective world more accurately [2] than the fuzzy set, and has received more and more attention since its appearance. Later, Atanassov and Gargov [4,5] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are interval numbers rather real numbers.

On the other hand, in the real decision-making, there are many qualitative attributes which are difficult to give attribute values by quantitative measurement. While, they are easy to give linguistic assessment values. However, for a linguistic assessment value, it is usually implied that the membership

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[*]Corresponding Author: Juan Li. Tel.: +8613992795738. Email Address: lijuanbjasu@163.com

degree is one, and the non-membership degree and hesitation degree of decision makers cannot be expressed. On the basis of the intuitionistic fuzzy set and the linguistic assessment set, Wang and Li [6] proposed the concept of intuitionistic linguistic set, the intuitionistic linguistic number, the intuitionistic two-semantic and the Hamming distance between two intuitionistic two-semantics and ranked the alternatives by calculating the comprehensive membership degree to the ideal solution for each alternative.

As an aggregation function, the Choquet integral [7] with respect to fuzzy measures has performed successfully in multicriteria decision making (MCDM). There are many works on the Choquet integral of single-valued functions. It is of interest to combine the Choquet integral and the IFS theory or MCDM under intuitionistic fuzzy environment, because, by doing this, we cannot only deals with the imprecise and uncertain decision information but also efficiently take into account the various interactions among the decision criteria.

Base on Archimedean t-conorm and t-norm [8-11], and the aggregation functions for the classical fuzzy sets (FSs), Beliakov et al. gave some operations about intuitionistic fuzzy sets, proposed two general concepts for constructing other types of aggregation operators for intuitionistic fuzzy sets (IFSs) extending the existing methods and showed that the operators obtained by using the Lukasiewicz t-norm are consistent with the ones on ordinary FSs. We can find above aggregation operators are all based on different relationships of the aggregated arguments, which can provide more choices for the decision makers.

In summary, based on intuitionistic linguistic set proposed by Wang and Li [6] , combining interval-valued uncertain linguistic variables, Archimedean t-norm and Choquet integral, in this paper, we propose the interval-valued uncertain linguistic variables based on Archimedean t-norm and investigate the MAGDM problems. First, we introduced some concepts of fuzzy measure and interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm. Then, interval-valued intuitionistic uncertain linguistic weighted average(geometric) operator based on Archimedean t-norm ,interval-valued intuitionistic uncertain linguistic ordered weighted average(geometric) operator based on Archimedean t-norm are developed. Furthermore, some desirable properties of these operators, such as commutativity, idempotency and monotonicity have been studied, and an intuitionistic uncertain linguistic hybrid average(geometric) operator based on Archimedean t-norm was developed. Based on these operators, two methods for multiple attribute group decision making problems with intuitionistic uncertain linguistic information have been proposed.

1. Preliminaries

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions (ref. to [8,9]):

1) $T(1,x) = x$, for all x.

2) $T(x,y) = T(y,x)$, for all x and y.

3) $T(x,T(y,z)) = T(T(x,y),z)$, for all x,y and z.

4) $x \leqslant x^{'}, y \leqslant y^{'}$ implies $T(x,y) \leqslant T(x^{'},y^{'}),\ x,y,x^{'},y^{'} \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions (ref. to [8,9]):

1) $S(0,x) = x$, for all x.

2) $S(x,y) = S(y,x)$, for all x and y.

3) $S(x,S(y,z)) = S(S(x,y),z)$, for all x,y and z.

4) $x \leqslant x^{'}, y \leqslant y^{'}$ implies $S(x,y) \leqslant S(x^{'},y^{'}),\ x,y,x^{'},y^{'} \in [0,1]$.

A t-norm function $T(x,y)$ is called Archimedean t-norm if it is continuous and $T(x,x) < x$ for all $x \in [0,1]$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x,y \in (0,1)$, (ref. to [8,9]).

A t-conorm function $S(x,y)$ is called Archimedean t-conorm if it is continuous and $S(x,x) > x$ for all $x \in [0,1]$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x,y \in (0,1)$, (ref. to [8,9]).

A mapping $N : [0,1] \to [0,1]$ is called negation operator， if N is decreasing and $N(0) = 1$, $N(1) = 0$.

Suppose that $S = (s_0, s_1, \ldots, s_{l-1})$ is a finite and fully ordered discrete term set, where l is an odd number. In real situations, l would be equal to 3,5,7,9,etc. For example, when $l = 7$, a set S can be given

as follows:

$$S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6) = \{very\ poor, poor, slightly\ poor, fair, slightly\ good, good, very\ good\}.$$

For any linguistic set $S = (s_0, s_1, \ldots, s_{l-1})$, the relationship between the element s_i and its subscript i is strictly monotonically increasing [12,13,14], so the function can be defined as follows: $f : s_i = f(i)$. Clearly, the function $f(i)$ is a strictly monotonically increasing function about a subscript i. To preserve all of the given information, the discrete linguistic label $S = (s_0, s_1, \ldots, s_{l-1})$ is extended to a continuous linguistic label $\overline{S} = \{s_\alpha \mid \alpha \in R\}$, which satisfies the above characteristics.

Suppose $\widetilde{s} = [s_a, s_b]$, $s_a, s_b \in \overline{S}$ and $a \leqslant b$, s_a and s_b are the lower limit and the upper limit of $\widetilde{s}$, respectively. Then, $\widetilde{s}$ is called an uncertain linguistic variable [15].

For each x, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are, respectively, denoted by $\mu_A^L(x)$, $\mu_A^U(x)$,$\nu_A^L(x)$,$\nu_A^U(x)$. We can denote by

$$A = \{\langle x[[s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\},$$

where $s_{\theta(x)}, s_{\tau(x)} \in \overline{S}$, $0 \leqslant \mu_A^U(x) + \nu_A^U(x) \leqslant 1$, $x \in X$, $\mu_A^L(x) \geqslant 0$ and $\nu_A^L(x) \geqslant 0$.

For each element x, we can compute its hesitation interval of x to uncertain linguistic variable $[s_{\theta(x)}, s_{\tau(x)}]$ as:

$$\pi_A(x) = [\pi_A^L(x), \pi_A^U(x)] = [1 - \nu_A^U(x) - \mu_A^U(x), 1 - \nu_A^L(x) - \mu_A^L(x)].$$

Definition 1.1. Let $A = \{\langle x[[s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\}$ be $IVIULS$, 6-Tuple $< [s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)] >$ is called an interval-valued intuitionistic uncertain linguistic number ($IVIULN$), and A can also be viewed as a collection of the interval-valued intuitionistic uncertain linguistic variables. So it can also be expressed as $A = \{\langle [s_{\theta(x)}, s_{\tau(x)}], [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]\rangle | x \in X\}$.

2. Interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm

Definition 2.1. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$ be two interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm and $\lambda \geqslant 0$, we can define the operational rules about $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ as follows

(1) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [s_{\theta(\alpha_1)+\theta(\alpha_2)}, s_{\tau(\alpha_1)+\tau(\alpha_2)}], [S(\mu^L(\alpha_1), \mu^L(\alpha_2)), S(\mu^U(\alpha_1), \mu^U(\alpha_2))],$
$[T(\nu^L(\alpha_1), \nu^L(\alpha_2)), T(\nu^U(\alpha_1), \nu^U(\alpha_2))]\rangle$
$= \langle [s_{\theta(\alpha_1)+\theta(\alpha_2)}, s_{\tau(\alpha_1)+\tau(\alpha_2)}], [h^{-1}(h(\mu^L(\alpha_1)) + h(\mu^L(\alpha_2))), h^{-1}(h(\mu^U(\alpha_1)) + h(\mu^U(\alpha_2)))],$
$[g^{-1}(g(\nu^L(\alpha_1)) + g(\nu^L(\alpha_2))), g^{-1}(g(\nu^U(\alpha_1)) + g(\nu^U(\alpha_2)))]\rangle$;

(2) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [s_{\theta(\alpha_1)\times\theta(\alpha_2)}, s_{\tau(\alpha_1)\times\tau(\alpha_2)}], [T(\mu^L(\alpha_1), \mu^L(\alpha_2)), T(\mu^U(\alpha_1), \mu^U(\alpha_2))],$
$[S(\nu^L(\alpha_1), \nu^L(\alpha_2)), S(\nu^U(\alpha_1), \nu^U(\alpha_2))]\rangle$
$= \langle [s_{\theta(\alpha_1)\times\theta(\alpha_2)}, s_{\tau(\alpha_1)\times\tau(\alpha_2)}], [g^{-1}(g(\mu^L(\alpha_1)) + g(\mu^L(\alpha_2))), g^{-1}(g(\mu^U(\alpha_1)) + g(\mu^U(\alpha_2)))],$
$[h^{-1}(h(\nu^L(\alpha_1)) + h(\nu^L(\alpha_2))), h^{-1}(h(\nu^U(\alpha_1)) + h(\nu^U(\alpha_2)))]\rangle$;

(3) $\lambda\widetilde{\alpha}_1 = \langle [s_{\lambda\times\theta(\alpha_1)}, s_{\lambda\times\tau(\alpha_1)}], [h^{-1}(\lambda h(\mu^L(\alpha_1))), h^{-1}(\lambda h(\mu^U(\alpha_1)))], [g^{-1}(\lambda g(\nu^L(\alpha_1))), g^{-1}(\lambda g(\nu^U(\alpha_1)))]\rangle$;

(4) $\widetilde{\alpha}_1^\lambda = \langle [s_{(\theta(\alpha_1))^\lambda}, s_{(\tau(\alpha_1))^\lambda}], [g^{-1}(\lambda g(\mu^L(\alpha_1))), g^{-1}(\lambda g(\mu^U(\alpha_1)))], [h^{-1}(\lambda h(\nu^L(\alpha_1))), h^{-1}(\lambda h(\nu^U(\alpha_1)))]\rangle$.

Theorem 2.1. For any two interval-valued intuitionistic uncertain linguistic numbers based on Archimedean t-norm $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$, it can be proved the calculation rules shown as follows:

(a) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \oplus \widetilde{\alpha}_1$;
(b) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \otimes \widetilde{\alpha}_1$;
(c) $\lambda(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2) = \lambda\widetilde{\alpha}_1 \oplus \lambda\widetilde{\alpha}_2$, $\lambda > 0$;
(d) $\lambda_1\widetilde{\alpha}_1 \oplus \lambda_2\widetilde{\alpha}_1 = (\lambda_1 + \lambda_2)\widetilde{\alpha}_1$, $\lambda_1, \lambda_2 > 0$;
(e) $\widetilde{\alpha}_1^{\lambda_1} \otimes \widetilde{\alpha}_1^{\lambda_2} = (\widetilde{\alpha}_1)^{\lambda_1+\lambda_2}$, $\lambda_1, \lambda_2 > 0$;
(f) $\widetilde{\alpha}_1^\lambda \otimes \widetilde{\alpha}_2^\lambda = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2)^\lambda$, $\lambda > 0$.

Definition 2.2 [16]. Let $\widetilde{\alpha}_1 = \langle [s_{\theta(\alpha_1)}, s_{\tau(\alpha_1)}], [\mu^L(\alpha_1), \mu^U(\alpha_1)], [\nu^L(\alpha_1), \nu^U(\alpha_1)]\rangle$ be an interval-valued intuitionistic uncertain linguistic number based on Archimedean t-norm, an expected value $E(\widetilde{\alpha}_1)$ of $\widetilde{\alpha}_1$ can be represented as follows

$$E(\widetilde{\alpha}_1) = \frac{1}{2} \times (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + 1 - \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2}) \times s_{(\theta(\alpha_1)+\tau(\alpha_1))/2}$$
$$= s_{((\theta(\alpha_1)+\tau(\alpha_1))\times(\mu^L(\alpha_1)+\mu^U(\alpha_1)+2-\nu^L(\alpha_1)-\nu^U(\alpha_1))/8}.$$

Definition 2.3 [16]. Let $\widetilde{\alpha}_1 = \langle [s_{\theta(\alpha_1)}, s_{\tau(\alpha_1)}], [\mu^L(\alpha_1), \mu^U(\alpha_1)], [\nu^L(\alpha_1), \nu^U(\alpha_1)] \rangle$ be an interval-valued intuitionistic uncertain linguistic number based on Archimedean t-norm, an accuracy function $H(\widetilde{\alpha}_1)$ can be represented as follows

$$H(\widetilde{\alpha}_1) = (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2}) \times s_{(\theta(\alpha_1)+\tau(\alpha_1))/2}$$
$$= s_{((\theta(\alpha_1)+\tau(\alpha_1))\times(\mu^L(\alpha_1)+\mu^U(\alpha_1)+\nu^L(\alpha_1)+\nu^U(\alpha_1))/4}.$$

Definition 2.4 [16]. If $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2)$ are any two interval-valued intuitionistic uncertain linguistic numbers based on Archimedean t-norm, then

(1) If $E(\widetilde{\alpha}_1) > E(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

(2) If $E(\widetilde{\alpha}_1) = E(\widetilde{\alpha}_2)$, then:

If $H(\widetilde{\alpha}_1) > H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

If $H(\widetilde{\alpha}_1) = H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

3. Aggregating of the interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm aggregating operator and fuzzy measure

A fuzzy measure on X is a set function $\mu : P(X) \to [0, 1]$ such that

(i) $\mu(\emptyset) = 0, \mu(X) = 1$;

(ii) $A, B \subseteq X$, $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$.

Let $A, B \in P(X), A \cap B = \varnothing$. If fuzzy measure g satisfies the following conditions:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B)$$

and $\lambda \in (-1, \infty)$.

If $\lambda = 0$, then g is an additive measure, which means there is no interaction between coalitions A and B .

If $\lambda > 0$, then g is called a supperadditive measure, which reflects there exists complementary interaction between coalitions A and B.

If $-1 < \lambda < 0$, then g is said to be a subadditive measure, which shows there exists redundancy interaction between coalitions A and B.

Let $X = \{x_1, x_2, ...x_n\}$ be a attribute index set, if $i, j = 1, 2, ..., n$ and $i \neq j, x_i \cap x_j = \varnothing, \bigcup_{i=1}^n x_i = X$, then

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i=1}^n [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i=1}^n g(x_i) & \lambda = 0, \end{cases}$$

where x_i, $g(x_i)$ is called a fuzzy measure function, and it indicates the importance degree of x_i.

From $g(X) = 1$, we know λ is determined by $\lambda + 1 = \prod_{i=1}^n (1 + \lambda g(x_i))$.

Based on the the above operational rules, we propose weighted average (geometric) operator, ordered weighted average (geometric) operator and hybrid average (geometric) operator for interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm in this part.

Definition 3.1. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $ATS - IVIULWA (or\ ATS - IVIULWGA) : \Omega^n \to \Omega$, if

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \sum_{j=1}^n \mu_j \widetilde{\alpha}_j,$$

$$(or\ \ ATS - IVIULWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \prod_{j=1}^n (\widetilde{\alpha}_j)^{\mu_j},)$$

where Ω is the set of all interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$, then $ATS - IVIULWA$ ($ATS - IVIULWGA$) is called the interval-valued intuitionistic uncertain linguistic weighted average (weighted geometric average) operator based on Archimedean t-norm.

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $ATS - IVIULWA$ operator is interval-valued intuitionistic uncertain linguistic arithmetic average operator based on Archimedean t-norm ($ATS - IVIULAA$):

$$ATS - IVIULAA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \frac{1}{n}(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 \oplus \ldots \oplus \widetilde{\alpha}_n).$$

Theorem 3.1. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, then, the result aggregated by Definition 3.1 is still an interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm, and

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\sum_{j=1}^{n}(\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j \times \tau(\alpha_j))}],$$
$$[h^{-1}(\textstyle\sum_{j=1}^{n} \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^U(\alpha_j)))] \rangle.$$

$$(or \quad ATS - IVIULWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\prod_{j=1}^{n}((\theta(\alpha_j))^{\mu_j})}, s_{\prod_{j=1}^{n}((\tau(\alpha_j))^{\mu_j})}],$$
$$[g^{-1}(\textstyle\sum_{j=1}^{n} \mu_j g(\mu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^U(\alpha_j)))], [h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^U(\alpha_j)))] \rangle,)$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, the parentheses used for indices represent a permutation on X such that $\widetilde{\alpha}_1 \leqslant \widetilde{\alpha}_2 \leqslant \cdots \leqslant \widetilde{\alpha}_n$, $A_{(j)} = (j, ..., n), A_{(n+1)} = \varnothing$.

Theorem 3.1 can be proven by mathematical induction. The steps in the proof are shown as follows:

Proof. We only prove the case of $ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n)$.

(1) When $n = 1$, obviously, it is right.

(2) When $n = 2$,

$$\mu_1\widetilde{\alpha}_1 = \langle [s_{\mu_1 \times \theta(\alpha_1)}, s_{\mu_1 \times \tau(\alpha_1)}], [h^{-1}(\mu_1 h(\mu^L(\alpha_1))), h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],$$
$$[g^{-1}(\mu_1 g(\nu^L(\alpha_1))), g^{-1}(\nu_1 g(\nu^U(\alpha_1)))] \rangle.$$
$$\mu_2\widetilde{\alpha}_2 = \langle [s_{\mu_2 \times \theta(\alpha_2)}, s_{\mu_2 \times \tau(\alpha_2)}], [h^{-1}(\mu_2 h(\mu^L(\alpha_2))), h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],$$
$$[g^{-1}(\mu_2 g(\nu^L(\alpha_2))), g^{-1}(\nu_2 g(\nu^U(\alpha_2)))] \rangle.$$
$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2) = \mu_1\widetilde{\alpha}_1 \oplus \mu_2\widetilde{\alpha}_2$$
$$= \langle [s_{\mu_1 \times \theta(\alpha_1)}, s_{\mu_1 \times \tau(\alpha_1)}], [h^{-1}(\mu_1 h(\mu^L(\alpha_1))), h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],$$
$$[g^{-1}(\mu_1 g(\nu^L(\alpha_1))), g^{-1}(\nu_1 g(\nu^U(\alpha_1)))] \rangle$$
$$\oplus \langle [s_{\mu_2 \times \theta(\alpha_2)}, s_{\mu_2 \times \tau(\alpha_2)}], [h^{-1}(\mu_2 h(\mu^L(\alpha_2))), h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],$$
$$[g^{-1}(\mu_2 g(\nu^L(\alpha_2))), g^{-1}(\nu_2 g(\nu^U(\alpha_2)))] \rangle$$
$$= \langle [s_{\mu_1 \times \theta(\alpha_1) + \mu_2 \times \theta(\alpha_2)}, s_{\mu_1 \times \tau(\alpha_1) + s_{\mu_2 \times \tau(\alpha_2)}}],$$
$$[h^{-1}(h(h^{-1}(\mu_1 h(\mu^L(\alpha_1)))) + h(h^{-1}(\mu_2 h(\mu^L(\alpha_2))))),$$
$$h^{-1}(h(h^{-1}(\mu_1 h(\mu^U(\alpha_1)))) + h(h^{-1}(\mu_2 h(\mu^U(\alpha_2))))),$$
$$[g^{-1}(g(g^{-1}(\mu_1 g(\nu^L(\alpha_1)))) + g(g^{-1}(\mu_2 g(\nu^L(\alpha_2))))),$$
$$g^{-1}(g(g^{-1}(\mu_1 g(\nu^U(\alpha_1)))) + g(g^{-1}(\mu_2 g(\nu^U(\alpha_2)))))] \rangle$$
$$= \langle [s_{\sum_{j=1}^{2}(\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^{2}(\mu_j \times \tau(\alpha_j))}], [h^{-1}(\textstyle\sum_{j=1}^{2} \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{2} \mu_j h(\mu^U(\alpha_j)))],$$
$$[g^{-1}(\textstyle\sum_{j=1}^{2} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{2} \mu_j g(\nu^U(\alpha_j)))] \rangle.$$

Therefore, when $n = 2$, the conclusion is right.

(3) Suppose when $n = k$, the conclusion is right, i.e.

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k) = \langle [s_{\sum_{j=1}^{k}(\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^{k}(\mu_j \times \tau(\alpha_j))}], [h^{-1}(\textstyle\sum_{j=1}^{k} \mu_j h(\mu^L(\alpha_j))),$$
$$h^{-1}(\textstyle\sum_{j=1}^{k} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{k} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{k} \mu_j g(\nu^U(\alpha_j)))] \rangle.$$

Then, when $n = k + 1$,

$$ATS - IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k, \widetilde{\alpha}_{k+1}) = \langle [s_{\sum_{j=1}^{k}(\mu_j \times \theta(\alpha_j))}, s_{\sum_{j=1}^{k}(\mu_j \times \tau(\alpha_j))}], [h^{-1}(\textstyle\sum_{j=1}^{k} \mu_j h(\mu^L(\alpha_j))),$$
$$h^{-1}(\textstyle\sum_{j=1}^{k} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{k} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{k} \mu_j g(\nu^U(\alpha_j)))] \rangle$$
$$\oplus \langle [s_{\mu_{k+1} \times \theta(\alpha_{k+1})}, s_{\mu_{k+1} \times \tau(\alpha_{k+1})}], [h^{-1}(\mu_{k+1} h(\mu^L(\alpha_{k+1}))), h^{-1}(\mu_{k+1} h(\mu^U(\alpha_{k+1})))],$$
$$[g^{-1}(\mu_{k+1} g(\nu^L(\alpha_{k+1}))), g^{-1}(\nu_{k+1} g(\nu^U(\alpha_{k+1})))] \rangle$$
$$= \langle [s_{\sum_{j=1}^{k}(\mu_j \times \theta(\alpha_j)) + \mu_{k+1} \times \theta(\alpha_{k+1})}, s_{\sum_{j=1}^{k}(\mu_j \times \tau(\alpha_j)) + \mu_{k+1} \times \tau(\alpha_{k+1})}],$$

$$
\begin{aligned}
&[h^{-1}(h(h^{-1}(\textstyle\sum_{j=1}^{k}\mu_j h(\mu^L(\alpha_j)))) + h(h^{-1}(\mu_{k+1}h(\mu^L(\alpha_{k+1}))))),\\
&h^{-1}(h(h^{-1}(\textstyle\sum_{j=1}^{k}\mu_j h(\mu^U(\alpha_j)))) + h(h^{-1}(\mu_{k+1}h(\mu^U(\alpha_{k+1}))))),\\
&[g^{-1}(g(g^{-1}(\textstyle\sum_{j=1}^{k}\mu_j g(\nu^L(\alpha_j)))) + g(g^{-1}(\mu_{k+1}g(\nu^L(\alpha_{k+1}))))),\\
&g^{-1}(g(g^{-1}(\textstyle\sum_{j=1}^{k}\mu_j g(\nu^U(\alpha_j)))) + g(g^{-1}(\mu_{k+1}g(\nu^U(\alpha_{k+1})))))]\rangle\\
&= \langle[s_{\sum_{j=1}^{k+1}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{k+1}(\mu_j\times\tau(\alpha_j))}], [h^{-1}(\textstyle\sum_{j=1}^{k+1}\mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{k+1}\mu_j h(\mu^U(\alpha_j)))],\\
&[g^{-1}(\textstyle\sum_{j=1}^{k+1}\mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^U(\alpha_j)))]\rangle.
\end{aligned}
$$

So, when $n = k+1$, the conclusion is right, too.

According to steps (1), (2) and (3), we can conclude the conclusion is right for all n.

Example 3.1. If $N(x) = 1-x,\ g(t) = -\log t$, and Algebraic t-conorm and t-norm [11] defined by $T^A(x,y) = x\cdot y,\ S^A(x,y) = x+y-xy$, then

$IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}]$,

$[1-\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}, 1-\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}]$,

$[\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}, \prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}]\rangle$,

which is the interval-valued intuitionistic uncertain linguistic weighted average(IVIULWA) operator.

Example 3.2. If $N(x) = 1-x,\ g(t) = \log(\frac{2-t}{t})$, and Einstein t-conorm and t-norm [11] defined by $T^E(x,y) = \frac{xy}{1+(1-x)(1-y)}\ ,\ S^E(x,y) = \frac{x+y}{1+xy}$, then

$EIVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}]$,

$[\frac{\prod_{j=1}^{n}(1+\mu^L(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+\mu^L(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}, \frac{\prod_{j=1}^{n}(1+\mu^U(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+\mu^U(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}]$,

$[\frac{2\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(2-\nu^L(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}, \frac{2\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(2-\nu^U(\alpha_j))^{\mu_j}+\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}]\rangle$,

which is called the Einstein interval-valued intuitionistic uncertain linguistic weighted average(EIVIUL WA) operator.

Example 3.3. If $N(x) = 1-x,\ g(t) = \log(\frac{\gamma+(1-\gamma)t}{t})\ ,\ \gamma>0$, and Hamacher t-conorm and t-norm [11] defined by $T_\gamma^H(x,y) = \frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,\ S_\gamma^H(x,y) = \frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}\ ,\ \gamma>0$, then

$HIVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}]$,

$[\frac{\prod_{j=1}^{n}(1+(\gamma-1)\mu^L(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\mu^L(\alpha_j))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(1-\mu^L(\alpha_j))^{\mu_j}}, \frac{\prod_{j=1}^{n}(1+(\gamma-1)\mu^U(\alpha_j))^{\mu_j}-\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\mu^U(\alpha_j))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(1-\mu^U(\alpha_j))^{\mu_j}}]$,

$[\frac{\gamma\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-\nu^L(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}}, \frac{\gamma\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-\nu^U(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}}]\rangle$,

which is called the Hammer interval-valued intuitionistic uncertain linguistic weighted average(HIVIUL WA) operator.

Especially, if $\gamma = 1$, then the $HIVIULWA$ operator reduces to the $IVIULWA$ operator; if $\gamma = 2$, then the $HIVIULWA$ operator reduces to the $EIVIULWA$ operator.

Example 3.4. If $N(x) = 1-x,\ g(t) = \log(\frac{\gamma-1}{\gamma^t-1})\ ,\ \gamma>1$, and Frank t-conorm and t-norm [11] defined by $T_\gamma^F(x,y) = \log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,\ S_\gamma^F(x,y) = 1-\log_\gamma(1+\frac{(\gamma^{1-x}-1)(\gamma^{1-y}-1)}{\gamma-1})\ ,\ \gamma>1$, then

$FIVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}]$,

$[1-\log_\gamma(1+\prod_{j=1}^{n}(\gamma^{1-\mu^L(\alpha_j)}-1)^{\mu_j}), 1-\log_\gamma(1+\prod_{j=1}^{n}(\gamma^{1-\mu^U(\alpha_j)}-1)^{\mu_j})]$,

$[\log_\gamma(1+\prod_{j=1}^{n}(\gamma^{\nu^L(\alpha_j)}-1)^{\mu_j}), \log_\gamma(1+\prod_{j=1}^{n}(\gamma^{\nu^U(\alpha_j)}-1)^{\mu_j})]\rangle$,

which is called the Frank interval-valued intuitionistic uncertain linguistic weighted average(FIVIULWA) operator.

Especially, if $\gamma\to 1$, then the $FIVIULWA$ operator reduces to the $IVIULWA$ operator.

Example 3.5. If $N(x) = 1-x^2,\ g(t) = -\log t$, and Algebraic t-conorm and t-norm [11] defined by $T_2^A(x,y) = xy\ ,\ S_2^A(x,y) = \sqrt{1-(1-x^2)(1-y^2)}$, then

$IVIULWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle[s_{\sum_{j=1}^{n}(\mu_j\times\theta(\alpha_j))}, s_{\sum_{j=1}^{n}(\mu_j\times\tau(\alpha_j))}]$,

$[\sqrt{1-\prod_{j=1}^{n}(1-(\mu^L(\alpha_j))^2)^{\mu_j}}, \sqrt{1-\prod_{j=1}^{n}(1-(\mu^U(\alpha_j))^2)^{\mu_j}}]$,

$[\prod_{j=1}^{n}(\nu^L(\alpha_j))^{\mu_j}, \prod_{j=1}^{n}(\nu^U(\alpha_j))^{\mu_j}]\rangle$,

which is the interval-valued intuitionistic uncertain linguistic weighted average(IVIULWA) operator.

Example 3.6. If $N(x)=1-x^2,\ g(t)=\log(\frac{2-t}{t})$, and Einstein t-conorm and t-norm [11] defined by $T_2^E(x,y)=\frac{xy}{1+(1-x)(1-y)}$, $S_2^E(x,y)=\sqrt{\frac{x^2+y^2}{1+x^2y^2}}$, then

$EIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\langle[s_{\sum_{j=1}^n(\mu_j\times\theta(\alpha_j))},s_{\sum_{j=1}^n(\mu_j\times\tau(\alpha_j))}],$

$[\sqrt{\frac{\prod_{j=1}^n(1+(\mu^L(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\mu^L(\alpha_j))^2)^{\mu_j}+\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}},\sqrt{\frac{\prod_{j=1}^n(1+(\mu^U(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\mu^U(\alpha_j))^2)^{\mu_j}+\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}}],$

$[\frac{2\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(2-\nu^L(\alpha_j))^{\mu_j}+\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}},\frac{2\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(2-\nu^U(\alpha_j))^{\mu_j}+\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}]\rangle,$

which is called the Einstein interval-valued intuitionistic uncertain linguistic weighted average(EIVIUL WA) operator.

Example 3.7. If $N(x)=1-x^2,\ g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, and Hamacher t-conorm and t-norm defined by $T_{2\gamma}^H(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$, $\gamma>0$, $S_{2\gamma}^H(x,y)=\sqrt{\frac{x^2+y^2-x^2y^2-(1-\gamma)x^2y^2}{1-(1-\gamma)x^2y^2}}$, $\gamma>0$, then

$HIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\langle[s_{\sum_{j=1}^n(\mu_j\times\theta(\alpha_j))},s_{\sum_{j=1}^n(\mu_j\times\tau(\alpha_j))}],$

$[\sqrt{\frac{\prod_{j=1}^n(1+(\gamma-1)(\mu^L(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(\mu^L(\alpha_j))^2)^{\mu_j}+(\gamma-1)\prod_{j=1}^n(1-(\mu^L(\alpha_j))^2)^{\mu_j}}},\sqrt{\frac{\prod_{j=1}^n(1+(\gamma-1)(\mu^U(\alpha_j))^2)^{\mu_j}-\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(\mu^U(\alpha_j))^2)^{\mu_j}+(\gamma-1)\prod_{j=1}^n(1-(\mu^U(\alpha_j))^2)^{\mu_j}}}],$

$[\frac{\gamma\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(1-\nu^L(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^n(\nu^L(\alpha_j))^{\mu_j}},\frac{\gamma\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}{\prod_{j=1}^n(1+(\gamma-1)(1-\nu^U(\alpha_j)))^{\mu_j}+(\gamma-1)\prod_{j=1}^n(\nu^U(\alpha_j))^{\mu_j}}]\rangle,$

which is called the Hammer interval-valued intuitionistic uncertain linguistic weighted average(HIVIUL WA) operator.

Especially, if $\gamma=1$, then the $HIVIULWA$ operator reduces to the $IVIULWA$ operator; if $\gamma=2$, then the $HIVIULWA$ operator reduces to the $EIVIULWA$ operator.

Example 3.8. If $N(x)=1-x^2,\ g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, and Frank t-conorm and t-norm defined by $T_{2\gamma}^F(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})$, $\gamma>1$, $S_{2\gamma}^F(x,y)=\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-x^2}-1)(\gamma^{1-y^2}-1)}{\gamma-1})}$, $\gamma>1$, then

$FIVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\langle[s_{\sum_{j=1}^n(\mu_j\times\theta(\alpha_j))},s_{\sum_{j=1}^n(\mu_j\times\tau(\alpha_j))}],$

$[\sqrt{1-\log_\gamma(1+\prod_{j=1}^n(\gamma^{1-(\mu^L(\alpha_j))^2}-1)^{\mu_j})},\sqrt{1-\log_\gamma(1+\prod_{j=1}^n(\gamma^{1-(\mu^U(\alpha_j))^2}-1)^{\mu_j})}],$

$[\log_\gamma(1+\prod_{j=1}^n(\gamma^{\nu^L(\alpha_j)}-1)^{\mu_j}),\log_\gamma(1+\prod_{j=1}^n(\gamma^{\nu^U(\alpha_j)}-1)^{\mu_j})]\rangle,$

which is called the Frank interval-valued intuitionistic uncertain linguistic weighted average(FIVIULWA) operator.

Especially, if $\gamma\to1$, then the $FIVIULWA$ operator reduces to the $IVIULWA$ operator.

The $ATS-IVIULWA$ operator has the following properties, such as idempotency, monotonicity, bounded, and so on.

In the following section, we propose the interval-valued intuitionistic uncertain linguistic ordered weighted average operator based on Archimedean t-norm.

Definition 3.2. Let $\widetilde{\alpha}_i=\langle[s_{\theta(\alpha_i)},s_{\tau(\alpha_i)}],[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle\ (i=1,2,\ldots,n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $ATS-IVIULOWA\ (ATS-IVIULOWG):\Omega^n\to\Omega$, if

$$ATS-IVIULOWA_w(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\sum_{j=1}^n w_j\widetilde{\alpha}_{\sigma_j},$$

$$(or\quad ATS-IVIULOWG_w(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\prod_{j=1}^n\widetilde{\alpha}_{\sigma_j}^{w_j},)$$

where Ω is the set of all interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $w=(w_1,w_2,\ldots,w_n)^T$ is an associated weighted vector with $ATS-IVIULOWA\ (ATS-IVIULOWG)$ and $w_j\in[0,1]$, $\sum_{j=1}^n w_j=1$. If $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is any permutation of $(1,2,\ldots,n)$, such that $\widetilde{\alpha}_{\sigma_{j-1}}\succeq\widetilde{\alpha}_{\sigma_j}$ for all $j=1,2,\ldots,n$, then $ATS-IVIULOWA\ (ATS-IVIULOWG)$ operator is called the interval-valued intuitionistic uncertain linguistic ordered weighted average (weighted geometric) operator based on Archimedean t-norm. w_j is decided only by the jth position in the aggregation process. Therefore, w can also be called the position-weighted vector.

The position-weighted vector w can be determined according to actual needs, or it can be determined based on the fuzzy semantic quantitative operator[17] or the combination number[18].

Theorem 3.2. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, the result aggregated from the Definition 3.2 is still an interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm, and

$ATS - IVIULOWA_w(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\sum_{j=1}^n (w_j \times \theta(\alpha_{\sigma_j}))}, s_{\sum_{j=1}^n (\mu_j \times \tau(\alpha_{\sigma_j}))}], [h^{-1}(\sum_{j=1}^n w_j h(\mu^L(\alpha_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\mu^U(\alpha_{\sigma_j})))], [g^{-1}(\sum_{j=1}^n w_j g(\nu^L(\alpha_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\nu^U(\alpha_{\sigma_j})))] \rangle$.

$(or\ \ ATS - IVIULOWG_w(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\prod_{j=1}^n ((\theta(\alpha_{\sigma_j}))^{w_j})}, s_{\prod_{j=1}^n ((\tau(\alpha_{\sigma_j}))^{w_j})}], [g^{-1}(\sum_{j=1}^n w_j g(\mu^L(\alpha_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\mu^U(\alpha_{\sigma_j})))], [h^{-1}(\sum_{j=1}^n w_j h(\nu^L(\alpha_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\nu^U(\alpha_{\sigma_j})))] \rangle,)$

where $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighted vector with $ATS - IVIULOWA$ $(ATS - IVIULOWG)$ and $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is any permutation of $(1, 2, \ldots, n)$, such that $\widetilde{\alpha}_{\sigma_{j-1}} \succeq \widetilde{\alpha}_{\sigma_j}$ for all $j = 1, 2, \ldots, n$.

In a similar way to the proof of Theorem 3.1, Theorem 3.2 can be proven by mathematical induction, and the proof steps are therefore omitted.

The $ATS - IVIULOWA$ operator has some similar properties to the $ATS - IVIULWA$ operator, such as idempotency, monotonicity, bounded, commutativity and so on.).

Definition 3.3. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $ATS - IVIULHA : \Omega^n \to \Omega$, if

$$ATS - IVIULHA_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \sum_{j=1}^n w_j \widetilde{\beta}_{\sigma_j},$$

$$(or\ \ ATS - IVIULHG_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \prod_{j=1}^n (\widetilde{\beta}_{\sigma_j})^{w_j},)$$

where Ω is the set of all interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, and $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighted vector with $ATS - IVIULHA$ $(ATS - IVIULHG)$ and $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. If $\widetilde{\beta}_{\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{\beta}_k (\widetilde{\beta}_k = n\mu_k \widetilde{\alpha}_k, k = 1, 2, \ldots, n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, $\sum_{j=1}^n \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$ and n is the balancing coefficient, then $ATS - IVIULHA$ $(ATS - IVIULHG)$ is called the interval-valued intuitionistic uncertain linguistic hybrid average (hybrid geometric) operator based on Archimedean t-norm.

Theorem 3.3. Let $\widetilde{\alpha}_i = \langle [s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)] \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm, then, the result aggregated from Definition 3.3 is still an interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm, and

$ATS - IVIULHA_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\sum_{j=1}^n (w_j \times \theta(\beta_{\sigma_j}))}, s_{\sum_{j=1}^n (w_j \times \tau(\beta_{\sigma_j}))}], [h^{-1}(\sum_{j=1}^n w_j h(\mu^L(\beta_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\mu^U(\beta_{\sigma_j})))], [g^{-1}(\sum_{j=1}^n w_j g(\nu^L(\beta_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\nu^U(\beta_{\sigma_j})))] \rangle$.

$(or\ ATS - IVIULHG_{\mu,w}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \langle [s_{\prod_{j=1}^n ((\theta(\beta_{\sigma_j}))^{w_j})}, s_{\prod_{j=1}^n ((\tau(\beta_{\sigma_j}))^{w_j})}], [g^{-1}(\sum_{j=1}^n w_j g(\mu^L(\beta_{\sigma_j}))), g^{-1}(\sum_{j=1}^n w_j g(\mu^U(\beta_{\sigma_j})))], [h^{-1}(\sum_{j=1}^n w_j h(\nu^L(\beta_{\sigma_j}))), h^{-1}(\sum_{j=1}^n w_j h(\nu^U(\beta_{\sigma_j})))] \rangle),$

where $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighted vector with $ATS - IVIULHA$ $(ATS - IVIULHG)$ and $w_j \in [0, 1], \sum_{j=1}^n w_j = 1$. If $\widetilde{\beta}_{\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{\beta}_k (\widetilde{\beta}_k = n\mu_k \widetilde{\alpha}_k, k = 1, 2, \ldots, n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, $\sum_{j=1}^n \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$ and n is the balancing coefficient.

4. Aggregating of the interval-valued intuitionistic uncertain linguistic variables in group decision making based on Archimedean t-norm

Consider a multiple attribute group decision making problem with intuitionistic linguistic information. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be the

set of attributes. $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighting vector of the attribute $C_j (j = 1, 2, \ldots, n)$, where $\mu_j \geqslant 0$ $(j = 1, 2, \ldots . n)$, $\sum_{j=1}^{n} \mu_j = 1$. Let $D = \{D_1, D_2, \ldots, D_p\}$ be the set of decision makers and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_P)^T$ be the weighted vector of the decision makers, where $\lambda_k \geqslant 0$ $(k = 1, 2, \ldots, p)$, $\sum_{k=1}^{p} \lambda_k = 1$. Suppose that $\widetilde{R}^k = [\widetilde{r}_{ij}^k]_{m\times n}$ is the decision matrix, where $\widetilde{r}_{ij}^k = \langle [\alpha_{ijk}^L, \alpha_{ijk}^U], [\mu_{ijk}^L(\alpha_i), \mu_{ijk}^U(\alpha_i)], [\nu_{ijk}^L(\alpha_i), \nu_{ijk}^U(\alpha_i)] \rangle$ takes the form of the intuitionistic uncertain linguistic variable given by the decision maker D_k for an alternative A_i with respect to an attribute C_j, and $0 \leqslant \mu_{ijk}^L \leqslant 1, 0 \leqslant \nu_{ijk}^L \leqslant 1, \mu_{ijk}^L \leqslant \mu_{ijk}^U, \nu_{ijk}^L \leqslant \nu_{ijk}^U, \mu_{ijk}^U + \nu_{ijk}^U \leqslant 1, \alpha_{ijk}^L, \alpha_{ijk}^U \in S$. Then, the ranking of alternatives is required.

In the following section, we will apply the above operators to solve multiple group decision making problems based on intuitionistic uncertain linguistic information. There are two methods, which are as follows:

4.1 The method of aggregating the attribute values first

Step 1. We can utilize the $ATS - IVIULWA$ operator(or $ATS - IVIULWGA$ operator) to aggregate the attribute values $(\widetilde{r}_{i1}^k, \widetilde{r}_{i2}^k, \ldots, \widetilde{r}_{in}^k)$ given by each decision maker with respect to each alternative into a comprehensive attribute value $\widetilde{r}_i^k$,

$$\widetilde{r}_i^k = ATS - IVIULWA_\mu(\widetilde{r}_{i1}^k, \widetilde{r}_{i2}^k, \ldots, \widetilde{r}_{in}^k) = \sum_{j=1}^{n} \mu_j \widetilde{r}_{ij}^k$$

where $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p, \mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)}) \in [0, 1]$ is the attribute weight vector, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$.

Step 2. We can utilize the $ATS - IVIULHA$ operator(or $ATS - IVIULHG$ operator) operator to aggregate the information $(\widetilde{r}_i^1, \widetilde{r}_i^2, \ldots, \widetilde{r}_i^p)$ of each decision maker into a collective value $\widetilde{r}_i$ for each alternative,

$$\widetilde{r}_i = ATS - IVIULHA_{\lambda,w}(\widetilde{r}_i^1, \widetilde{r}_i^2, \ldots, \widetilde{r}_i^p) = \sum_{k=1}^{p} w_k \widetilde{b}_i^{\sigma_k}$$

$i = 1, 2, \ldots, m$, where $w = (w_1, w_2, \ldots, w_p)^T$ is an associated weighted vector with $ATS - IVIULHA$ (or $ATS - IVIULHG$) and $w_k \in [0, 1]$, $\sum_{k=1}^{p} w_k = 1$. $\widetilde{b}_i^{\sigma_k}$is the kth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{b}_i^l (\widetilde{b}_i^l = p\lambda_l \widetilde{r}_i^l$ or $\widetilde{b}_i^l = \widetilde{r}_i^{lp\lambda_l}, l = 1, 2, \ldots, p)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ is the weighted vector of the decision makers, $\lambda_k \in [0, 1]$, $\sum_{k=1}^{n} \lambda_k = 1$ and p is the balancing coefficient.

Step 3. By Definition 2.2, we can calculate the expected value $E(\widetilde{r}_i)(i = 1, 2, \ldots, m)$ of the collective value $\widetilde{r}_i (i = 1, 2, \ldots, m)$, rank all of the alternatives $A_i (i = 1, 2, \ldots, m)$ and then select the best one(s). If there is no difference between two expected values $E(\widetilde{r}_i)$ and $E(\widetilde{r}_j)$, then by Definition 2.3, we must calculate the accuracy degrees $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$ of the collective overall preference values $\widetilde{r}_i$ and $\widetilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy function values $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$.

Step 4. Rank all of the alternatives $A_i (i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with $E(\widetilde{r}_i)$ and $H(\widetilde{r}_i)$ $(i = 1, 2, \ldots, m)$.

4.2 The method of first aggregating the information from different decision makers

Step 1. We can utilize the $ATS - IVIULWA$ operator(or $ATS - IVIULWGA$ operator) to aggregate all of the decision matrices $\widetilde{R}^k (k = 1, 2, \ldots, p)$ into a collective decision matrix $\widetilde{R} = [\widetilde{r}_{ij}]_{m\times n}$,

$$\widetilde{r}_{ij} = ATS - IVIULWA_\lambda(\widetilde{r}_{ij}^1, \widetilde{r}_{ij}^2, \ldots, \widetilde{r}_{ij}^p) = \sum_{k=1}^{p} \lambda_k \widetilde{r}_{ij}^k$$

$i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ is the weighted vector of the decision makers.

Step 2. We can utilize the $ATS-IVIULHA$ operator(or $ATS-IVIULHG$ operator) to derive the collective overall preference value $\widetilde{r}_i(i=1,2,\ldots,m)$ of the alternative A_i,

$$\widetilde{r}_i = ATS-IVIULHA_{\mu,w} = (\widetilde{r}_{i1},\widetilde{r}_{i2},\ldots,\widetilde{r}_{in}) = \sum_{j=1}^{n} w_j \widetilde{b}_{i\sigma_j}$$

$i=1,2,\ldots,m$, where $w=(w_1,w_2,\ldots,w_n)^T$ is an associated weighted vector with $ATS-IVIULHA$ (or $ATS-IVIULHG$) and $w_j\in[0,1]$, $\sum_{j=1}^{n} w_j=1$. $\widetilde{b}_{i\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{b}_{ik}$ ($\widetilde{b}_{ik}=n\mu_k\widetilde{r}_{ik}$ or $\widetilde{b}_{ik}=(\widetilde{r}_{ik})^{n\mu_k}, k=1,2,\ldots,n$), $\mu=(\mu_1,\mu_2,\ldots,\mu_n)^T$ is the weight vector of $\widetilde{r}_{ij}(j=1,2,\ldots,n), \mu_j\in[0,1]$, $\sum_{j=1}^{n}\mu_j=1$, and n is the balancing coefficient.

Step 3. By Definition 2.2, we can calculate the expected value $E(\widetilde{r}_i)(i=1,2,\ldots,m)$ of the collective value $\widetilde{r}_i(i=1,2,\ldots,m)$, rank all of the alternatives $A_i(i=1,2,\ldots,m)$ and then select the best one(s). If there is no difference between two expected values $E(\widetilde{r}_i)$ and $E(\widetilde{r}_j)$, then by Definition 2.3, we must calculate the accuracy degrees $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$ of the collective overall preference values $\widetilde{r}_i$ and $\widetilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy function values $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$.

Step 4. Rank all of the alternatives $A_i(i=1,2,\ldots,m)$ and select the best one(s) in accordance with $E(\widetilde{r}_i)$ and $H(\widetilde{r}_i)$ $(i=1,2,\ldots,m)$.

5. An application example

Let us consider an investment company that wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money:

(1) A_1 is a car company.
(2) A_2 is a computer company.
(3) A_3 is a TV company.
(4) A_4 is a food company.

The investment company must make a decision according to the following four attributes:

(1) C_1 is the risk index.
(2) C_2 is the growth index.
(3) C_3 is the social-political impact index.
(4) C_4 is the environment impact index.

The fuzzy measure of each attributes: $g(x_1)=0.4,\ g(x_2)=0.25,\ g(x_3)=0.37,\ g(x_4)=0.2$. Since

$$\lambda+1=\prod_{i=1}^{n}(1+\lambda g(x_i)), i=1,2,3,4,$$

we have $\lambda=-0.44$, then $g(x_1,x_2)=0.60,\ g(x_1,x_3)=0.70,\ g(x_1,x_4)=0.56,\ g(x_2,x_3)=0.68,\ g(x_2,x_4)=0.43,\ g(x_3,x_4)=0.54,\ g(x_1,x_2,x_4)=0.88,\ g(x_2,x_3,x_4)=0.75,\ g(x_1,x_3,x_4)=0.73,\ g(x_1,x_2,x_3,x_4)=1$.

The four possible alternatives A_1,A_2,A_3,A_4 are evaluated by three decision makers $D_k(k=1,2,3)$ (whose weight vector is $\lambda=(0.4,0.32,0.28)^T$) using the linguistic term set $S=(s_0,s_1,s_2,s_3,s_4,s_5,s_6)$ about the interval-valued intuitionistic uncertain linguistic variables based on Archimedean t-norm $\widetilde{r}_{ij}^k=\langle[\alpha_{ijk}^L,\alpha_{ijk}^U],[\mu_{ijk}^L,\mu_{ijk}^U],[\nu_{ijk}^L,\nu_{ijk}^U]\rangle$ under the above four attributes. The decision matrices $\widetilde{R}^k=[\widetilde{r}_{ij}^k]_{4\times4}$ $(k=1,2,3))$ are listed in Tables 1-3.

To obtain the best alternative(s), we can use the two methods introduced to obtain the selection results.

Let $N(x)=1-x,\ g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma=2$, i.e. $g(t)=\log(\frac{2-t}{t})$.

5.1 The method of aggregating the attribute values first

Step 1. We can utilize the $ATS-IVIULWGA$ operator to aggregate the attribute values $(\widetilde{r}_{i1}^k,\widetilde{r}_{i2}^k,\widetilde{r}_{i3}^k,\widetilde{r}_{i4}^k)$ into a comprehensive attribute value $\widetilde{r}_i^k$, we can obtain the aggregating results shown in Table 4.

Step 2. We can utilize the $ATS-IVIULHG$ operator to aggregate the information $(\widetilde{r}_i^1, \widetilde{r}_i^2, \widetilde{r}_i^3)$ of each decision maker into a collective value $\widetilde{r}_i$ for each alternative.

Suppose the position weight is $w = (0.25, 0.5, 0.25)^T$. We can obtain

$\widetilde{r}_1 = \langle [s_{3.97}, s_{4.85}], [0.73, 0.78], [0.12, 0.16] \rangle$, $\widetilde{r}_2 = \langle [s_{4.12}, s_{5.03}], [0.61, 0.75], [0.17, 0.14] \rangle$,

$\widetilde{r}_3 = \langle [s_{4.28}, s_{4.40}], [0.72, 0.75], [0.10, 0.18] \rangle$, $\widetilde{r}_4 = \langle [s_{3.28}, s_{3.97}], [0.63, 0.70], [0.16, 0.22] \rangle$.

Step 3. We can calculate the expected value $E(\widetilde{r}_i)(i = 1, 2, 3, 4)$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $\widetilde{r}_i(i = 1, 2, 3, 4)$ as

$E(\widetilde{r}_1) = s_{3.976}$, $E(\widetilde{r}_2) = s_{3.701}$, $E(\widetilde{r}_3) = s_{3.499}$, $E(\widetilde{r}_4) = s_{2.984}$.

Step 4. By definition 4.2, we can rank all of the alternatives (A_1, A_2, A_3, A_4) in accordance with the expected values $(E(\widetilde{r}_1), E(\widetilde{r}_2), E(\widetilde{r}_3), E(\widetilde{r}_4))$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $(\widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3, \widetilde{r}_4)$. We can obtain $A_1 \succ A_2 \succ A_3 \succ A_4$, and thus, the most desirable alternatives is A_1.

5.2 The method of first aggregating the information from different decision makers

Step 1. We can utilize the $ATS-IVIULWGA$ operator to aggregate all of the decision matrices $\widetilde{R}^k(k = 1, 2, 3)$ into a collective decision matrix $\widetilde{R} = [\widetilde{r}_{ij}]_{4\times 4}$. We can obtain the aggregation result shown in Table 5.

Step 2. We can utilize

$$\widetilde{r}_i = ATS-IVIULHG_{\mu,w} = (\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{i4}) = \prod_{j=1}^{4} (\widetilde{b}_{i\sigma_j})^{w_j} \ (i = 1, 2, 3, 4)$$

to derive the collective overall values $\widetilde{r}_i$ of alterative A_i, where $\widetilde{b}_{i\sigma_j}$ is the jth the largest of the interval-valued intuitionistic uncertain linguistic weighted argument $\widetilde{b}_{ik}(\widetilde{b}_{ik} = (\widetilde{r}_{ik})^{4\mu_k}, k = 1, 2, 3, 4)$. Suppose the position weight is $w = (0.15, 022., 0.35, 0.28)^T$. We can obtain

$\widetilde{r}_1 = \langle [s_{4.65}, s_{5.11}], [(0.72, 0.78], [0.10, 0.15] \rangle$, $\widetilde{r}_2 = \langle [s_{4.74}, s_{5.17}], [(0.63, 0.70], [0.15, 0.20] \rangle$,

$\widetilde{r}_3 = \langle [s_{4.34}, s_{4.45}], [(0.70, 0.76], [0.11, 0.17] \rangle$, $\widetilde{r}_4 = \langle [s_{3.71}, s_{4.08}], [(0.66, 0.73], [0.13, 0.23] \rangle$.

Step 3. We can calculate the expected value $E(\widetilde{r}_i)(i = 1, 2, 3, 4)$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $\widetilde{r}_i(i = 1, 2, 3, 4)$ as

$E(\widetilde{r}_1) = s_{3.971}$, $E(\widetilde{r}_2) = s_{3.694}$, $E(\widetilde{r}_3) = s_{3.494}$, $E(\widetilde{r}_4) = s_{2.952}$.

Step 4. By definition 4.2, we can rank all of the alternatives (A_1, A_2, A_3, A_4) in accordance with the expected values $(E(\widetilde{r}_1), E(\widetilde{r}_2), E(\widetilde{r}_3), E(\widetilde{r}_4))$ of the collective interval-valued intuitionistic uncertain linguistic variables values based on Archimedean t-norm $(\widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3, \widetilde{r}_4)$. We can obtain $A_1 \succ A_2 \succ A_3 \succ A_4$, and thus, the most desirable alternatives is A_1.

Table 1: Decision matrix $\widetilde{R}^1$

	C_1	C_2	C_3	C_4
A_1	$\langle [s_4, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.6, 0.6], [0.1, 0.3] \rangle$	$\langle [s_5, s_6], [0.8, 0.8], [0.1, 0.1] \rangle$	$\langle [s_4, s_4], [0.8, 0.8], [0.1, 0.1] \rangle$
A_2	$\langle [s_5, s_5], [0.6, 0.6], [0.1, 0.2] \rangle$	$\langle [s_5, s_6], [0.7, 0.7], [0.2, 0.2] \rangle$	$\langle [s_4, s_5], [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [s_4, s_5], [0.5, 0.6], [0.1, 0.3] \rangle$
A_3	$\langle [s_4, s_4], [0.7, 0.7], [0.2, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.7, 0.7], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.3] \rangle$
A_4	$\langle [s_3, s_4], [0.6, 0.7], [0.2, 0.3] \rangle$	$\langle [s_3, s_3], [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [s_4, s_4], [0.6, 0.7], [0.2, 0.3] \rangle$	$\langle [s_3, s_4], [0.7, 0.7], [0.2, 0.2] \rangle$

Table 2: Decision matrix $\widetilde{R}^2$

	C_1	C_2	C_3	C_4
A_1	$\langle [s_5, s_6], [0.6, 0.7], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.7, 0.7], [0.1, 0.1] \rangle$	$\langle [s_4, s_5], [0.7, 0.9], [0.2, 0.1] \rangle$	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$
A_2	$\langle [s_5, s_5], [0.5, 0.7], [0.2, 0.2] \rangle$	$\langle [s_4, s_5], [0.6, 0.7], [0.2, 0.2] \rangle$	$\langle [s_5, s_4], [0.7, 0.7], [0.1, 0.2] \rangle$	$\langle [s_6, s_6], [0.6, 0.7], [0.1, 0.1] \rangle$
A_3	$\langle [s_5, s_5], [0.6, 0.7], [0.0, 0.2] \rangle$	$\langle [s_4, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_4, s_4], [0.6, 0.6], [0.2, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.7], [0.2, 0.2] \rangle$
A_4	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_4, s_4], [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [s_3, s_3], [0.9, 0.9], [0.0, 0.1] \rangle$	$\langle [s_3, s_4], [0.8, 0.8], [0.1, 0.2] \rangle$

Table 3: Decision matrix $\widetilde{R}^3$

	C_1	C_2	C_3	C_4
A_1	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_5, s_6], [0.7, 0.8], [0.2, 0.2] \rangle$
A_2	$\langle [s_5, s_6], [0.6, 0.7], [0.1, 0.2] \rangle$	$\langle [s_5, s_6], [0.7, 0.7], [0.1, 0.2] \rangle$	$\langle [s_5, s_5], [0.8, 0.8], [0.1, 0.1] \rangle$	$\langle [s_5, s_5], [0.9, 0.9], [0.1, 0.1] \rangle$
A_3	$\langle [s_5, s_5], [0.8, 0.8], [0.0, 0.1] \rangle$	$\langle [s_5, s_5], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [s_4, s_4], [0.7, 0.8], [0.1, 0.1] \rangle$
A_4	$\langle [s_4, s_5], [0.8, 0.9], [0.1, 0.1] \rangle$	$\langle [s_4, s_4], [0.8, 0.8], [0.0, 0.2] \rangle$	$\langle [s_4, s_5], [0.8, 0.8], [0.0, 0.1] \rangle$	$\langle [s_4, s_5], [0.7, 0.7], [0.1, 0.2] \rangle$

Table 4: The comprehensive attribute value $\widetilde{r}_i^k$

	D_1	D_2	D_3
A_1	$\langle[s_{4.07}, s_{4.89}], [0.58, 0.67], [0.12, 0.16]\rangle$	$\langle[s_{4.21}, s_{4.97}], [0.64, 0.80], [0.11, 0.18]\rangle$	$\langle[s_{5.21}, s_{5.43}], [0.56, 0.73], [0.14, 0.19]\rangle$
A_2	$\langle[s_{4.46}, s_{5.10}], [0.61, 0.75], [0.1, 0.2]\rangle$	$\langle[s_{4.21}, s_{4.86}], [0.67, 0.78], [0.13, 0.17]\rangle$	$\langle[s_{4.60}, s_{4.96}], [0.68, 0.80], [0.13, 0.20]\rangle$
A_3	$\langle[s_{3.82}, s_{4.19}], [0.68, 0.79], [0.07, 0.10]\rangle$	$\langle[s_{4.30}, s_{4.81}], [0.61, 0.74], [0.12, 0.22]\rangle$	$\langle[s_{4.33}, s_{4.62}], [0.71, 0.84], [0.11, 0.23]\rangle$
A_4	$\langle[s_{4.37}, s_{4.94}], [0.72, 0.80], [0.10, 0.16]\rangle$	$\langle[s_{4.79}, s_{5.04}], [0.61, 0.74], [0.08, 0.20]\rangle$	$\langle[s_{4.57}, s_{5.08}], [0.64, 0.76], [0.12, 0.19]\rangle$

Table 5: The collective decision matrix $\widetilde{R} = (C_1, C_2, C_3, C_4)$.

	C_1	C_2
A_1	$\langle[s_{4.57}, s_{5.28}], [0.67, 0.77], [0.10, 0.14]\rangle$	$\langle[s_{5.00}, s_{5.00}], [0.69, 0.71], [0.10, 0.19]\rangle$
A_2	$\langle[s_{5.00}, s_{5.28}], [0.57, 0.66], [0.13, 0.20]\rangle$	$\langle[s_{4.68}, s_{5.68}], [0.67, 0.70], [0.17, 0.20]\rangle$
A_3	$\langle[s_{4.57}, s_{4.57}], [0.70, 0.73], [0.09, 0.17]\rangle$	$\langle[s_{4.28}, s_{4.57}], [0.73, 0.83], [0.10, 0.17]\rangle$
A_4	$\langle[s_{3.81}, s_{4.57}], [0.69, 0.79], [0.14, 0.21]\rangle$	$\langle[s_{3.57}, s_{3.57}], [0.58, 0.65], [0.14, 0.27]\rangle$

	C_3	C_4
A_1	$\langle[s_{4.68}, s_{5.38}], [0.83, 0.86], [0.07, 0.10]\rangle$	$\langle[s_{4.57}, s_{4.83}], [0.74, 0.80], [0.13, 0.16]\rangle$
A_2	$\langle[s_{4.57}, s_{4.68}], [0.64, 0.69], [0.14, 0.21]\rangle$	$\langle[s_{4.83}, s_{5.28}], [0.63, 0.71], [0.19, 0.19]\rangle$
A_3	$\langle[s_{4.37}, s_{4.37}], [0.67, 0.70], [0.13, 0.20]\rangle$	$\langle[s_{4.37}, s_{4.37}], [0.70, 0.77], [0.13, 0.17]\rangle$
A_4	$\langle[s_{3.67}, s_{3.92}], [0.74, 0.79], [0.09, 0.19]\rangle$	$\langle[s_{3.50}, s_{4.28}], [0.73, 0.73], [0.14, 0.20]\rangle$

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Non-integer variable order dynamic equations on time scales involving Caputo-Fabrizio type differential operator

Dumitru Baleanu[1]*, Mehdi Nategh[2]†,

Abstract

This work deals with the conecept of a Caputo-Fabrizio type non-integer variable order differential opertor on time scales that involves a non-singular kernel. A measure theoretic discussion on the limit cases for the order of differentiation is presented. Then, corresponding to the fractional derivative, we discuss on an integral for constant and variable orders. Beside the obtaining solutions to some dynamic problems on time scales involving the proposed derivative, a fractional folrmulation for the viscoelastic oscillation problem is studied and its conversion into a third order dynamic equation is presented.

keywords: Time scales, Fractional calculus, Caputo-Fabrizio derivative, Non-integer variable order derivative and integral , Dirac delta functional, Viscoelastic Oscillation.

MSC 2010: 34N05, 26A33.

1 Introduction

This work deals with two theories, namely non-integer order calculus (or as what it is called, fractional calculus) and Δ-calculus (calculus on time scales). The first one, originally is as old as the classical calculus in the sense of Leibnitz and the latter which was started by an effort in 80's, was aimed to unify the difference and the differential. For an overview on the trends and achievements in Δ-calculus, see [5].

In the recent years, to propose a non-integer order counterpart of Δ-calculus, a number of efforts have been made [1, 3]. One of the main challenges for such proposals was to overcome the limitations caused by the nature of the time scales, since a typical cluster of points may appear in a variety of scattered or dense patterns.

In view of the real world applications of the non-local fractional calculus, one presumption is to find a suitable kernel for the purpose of the better description to a class of phenomena.

*1. Corresponding author. Department of Mathematics and Computer Sciences, Çankaya University, Ankara, Turkey. E-mail: dumitru@cankaya.edu.tr, Tel: +903122331420, Fax: +903122331025.

†2. Department of Mathematics, University of Mazandran, Babolsar, Iran. E-mail: m.nateghp@yahoo.com, Tel: +989216246609, Fax: +981135302460.

Recently, a Caputo-like non-integer order derivative with non-singular kernel is proposed by Caputo and Fabrizio [7],

$$\mathfrak{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[- \frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau, \tag{1.1}$$

in which, $M(\alpha)$ is called, the normalization function and it satisfies $M(0) = M(1) = 1$. As is is graphically illustrated in [7], this new definition of the non-integer order derivative seems to be more appropriate in describing a process which is affected by its past. Indeed, compare to Caputo derivative, the new derivative with an exponential kernel, shows rapid stabilization with respect to the memory effect.

A substitution of the exponential kernel with Mittag-Leffler type, to define another non-local derivative, is suggested by Atangana and Baleanu with an application to the non-integer order heat transfer model [2].

Fractional integral of variable order with singular kernel first was introduced in 1993 by Samko and Ross in [14] using direct and Fourier based approaches. The direct appraoch formulation for the derivative reads

$$D_{a^+}^{\alpha(t)} = \frac{1}{\Gamma(1-\alpha(t))} . \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^{-\alpha(\tau)}} \tag{1.2}$$

with the assumtion that $\Re(\alpha(t)) \in (0,1)$. The corresponded integral was defined in a similar manner. Having no idea about any prospect application at that time, it has been noticed by the authors in the introduction that, "the work is stimulated by intelectual curiosity". However, later on, the theory found its applicability in viscoelastic dynamics and anomalous diffusion as well. (see [13] for a variable order model of a harmonically forced oscillator with viscoelastic damping).

Lorenzo & Hartley [11], in 2002, proposed the following formulation for the integral

$${}_0 d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau)(1-(dq(\tau)/d\tau))-1}}{\gamma(q(t,\tau)(1-\frac{dq(\tau)}{d\tau}))} f(\tau) d\tau, \tag{1.3}$$

then a simplified version of (1.3), that is

$${}_0 d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau))-1}}{\Gamma(q(t,\tau))} f(\tau) d\tau, \tag{1.4}$$

is considered. A slightly different approach is introduced by Coimbra non-integer variable order derivative ([6, 13]) which is defined by

$$D^{\alpha(t)} f(t) = \frac{1}{1-\alpha(t)} \left\{ \int_a^t \left[\frac{df(\tau)}{d\tau} \right] \frac{1}{(t-\tau)^{\alpha(t)}} d\tau + f(0) t^{-\alpha(t)} . \right\}. \tag{1.5}$$

From the definition, it is clear that, (1.3) and (1.5) are of the Riemann-Liouville and Caputo type. However, according to the other well-known singular non-integer (constant)

order integrals and derivatives, the variable order counterparts are reported (see [16], section 1.1.5, and [12] for a comprehensive list of singular and non-singular fractional derivatives).

All the above mentioned works was dealt with the continuous time rather than considering lattices or in its more general form the time scales approach. The appearance of variable order fractional differnce operator of Caputo type, i.e., constant pace time scale of the form $(h\mathbb{N})_{a_0} = \{a_0, a_0 + h, \dots\}$, was in [15], in which, the order $\alpha(t)$ involved a chaos,

$$\begin{aligned} {}^{C}_{a_0}\Delta_t^{\alpha(i)} f\Big[t + (1-\alpha(N-1))h\Big] = \\ \frac{1}{\Gamma(1-\alpha(N-1))} \sum_{i=0}^{N-1} \frac{\Gamma(N-i-\alpha(i))}{\Gamma(n-i)h^{\alpha(i)}}, \Delta f(a_0+ih), \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} \alpha(i) &= \beta + \omega(0.5 - X(i)), \\ X(i) &= 4X(i)(1-X(i)), \quad i = 1, 2, \dots. \end{aligned} \tag{1.7}$$

In this work, relaxing the order α to be a function of a time domain, we propose the non-integer and non-singular (in the sense of Caputo-Fabrizio) variable order differential operator which is based on the calculus on time scales.

Definition 1.1. *Suppose $\mathbb{T}_0 \subset [0, +\infty)$ is a time scale with $0 \in \mathbb{T}$ and let $\alpha : \mathbb{T}_0 \to [0,1]$ be such that $\mu(t) \neq \frac{1-\alpha(t)}{\alpha(t)}$ ($\frac{1-\alpha}{\alpha} \in \mathcal{R}$). Let $M : [0,1] \to \mathbb{R}$ be a function which satisfies $M(0) = M(1) = 1$. The non-integer derivative of f at t of order $\alpha(s)$, denoted by the operational notation $\mathfrak{D}\{f\}(t,s)$, is defined by*

$$\mathfrak{D}_{0^+}^{\alpha(s)} f(t) = \frac{M(\alpha(s))}{1-\alpha(s)} \int_0^t f^{\Delta}(\tau) e_{\ominus \frac{\alpha(s)}{1-\alpha(s)}}(t, \sigma(\tau)) \Delta\tau. \quad s \in \mathbb{T}_0 \tag{1.8}$$

An specific case, which is defined in [11] as a variable order derivative is $t = s$, i.e., pertaining to the values of $\mathfrak{D}\{f\}$ on the diagonal line in $\mathbb{T}_0 \times \mathbb{T}_0$.

Remark 1.1. *Interpreting the order α as the rate of anomalousity of a diffusion process, a suitable plausible candidate for α may be regarded below*

$$\alpha(t) = \sum_{j \in \mathbb{N}} a_j . \chi_{A_j}(t), \tag{1.9}$$

where $\sum_{j\in\mathbb{N}} a_j = 1$ ($a_j > 0$), $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{T}_0$ and A_j can be voided for infinitely many $j \in \mathbb{N}$. Indeed, (1.9) is pertaining to a process for which, the order of proportionality between average of displacement and time is subject to a sudden change caused by some external forces or an unknown mechanism. The special case $t = s$ reflects a synchronization between the process and its anomalousity variable order.

2 An Investigation on the Limit Cases

One of the aims of this section is to clarify that, in the original CF definition, the denominator, i.e., $\frac{1}{1-\alpha}$ will be restrictive (in the case $\alpha \to 1^-$) when we extend the theory (with loyalty) to include the time scales. However, not being confined by the original CF, it will be possible to modify the multiplier $\frac{M(\alpha)}{1-\alpha}$ in such a way that, it complies with the expectations in the sense of limit cases.

It is noticeable that, the limit case $\alpha \to 0^+$ gives $f(t) - f(0)$. The limit is understood in the sense of existing a sequence in $\mathbb{T}_0$ for which, $\alpha(t)$ tends to 0 for a subsequence. In this section, we put aside variable order and only the constant order will be considered.

Investigating the other limit case, namley $\alpha \to 1^-$, in the case t is right dense, demands some time sclae type measure theoretical background (see [9] for measure theory on time scales).

To this end, let t_0 be right-scattered. The (Hilger) Dirac delta function δ_{t_0} is defined by $\frac{1}{\mu(t_0)}$ if $t = t_0$ and 0 if $t \neq t_0$. We define $L_{t_0} \in (C_c^\infty(\mathbb{T}_0, \mathbb{R}))^*$ by ([8], section 3)

$$\langle L_{t_0}, f\rangle = \begin{cases} \int_0^\infty \delta_{t_0}(\tau) f(\tau)\Delta\tau, & \mu(t_0) > 0 \\ \lim_{\epsilon\to 0} \int_0^\infty \frac{1}{\epsilon} h(\frac{\tau}{\epsilon}) f(\sigma(\tau))\Delta\tau. & \mu(t_0) = 0 \end{cases} \tag{2.1}$$

where $h : \mathbb{T}_0 \to \mathbb{R}$ has the property $\int_0^\infty h(\tau)\Delta\tau = 1$. It can be observed that, for a function f with $f^\sigma \in C_c^\infty(\mathbb{T}_0, \mathbb{R})$, we have [8]:

$$\langle L_{t_0}, f\rangle = f(t_0). \tag{2.2}$$

For the time scale version of the exponential funation to be taken into account and be substitited with h, wee need to modify the above mentioned theory in this manner:

Let $t \in \mathbb{T}_0$ be a left scattered then, the corresponding Hilger Dirac delta function is defined by

$$\delta_t(\tau) = \begin{cases} \frac{1}{\mu(\rho(t))}, & \rho(t) = \tau \\ 0, & t \neq \tau \end{cases} \tag{2.3}$$

Now supose $t \in \mathbb{T}_0$ is left dense and define

$$g(\epsilon, t, \sigma(\tau)) = \begin{cases} \frac{1}{\epsilon}, & t - \epsilon \leq \sigma(\tau) \leq t \\ 0, & elsewhere \end{cases} \tag{2.4}$$

we infer that

$$\lim_{\epsilon\to 0^+} g(\epsilon, t, \sigma(\tau)) = \begin{cases} +\infty, & t = \sigma(\tau) \\ 0. & t \neq \sigma(\tau) \end{cases} \tag{2.5}$$

The definition (2.1) with a slight change based on the left density property (instead of right density property) is

$$\langle L_t, f\rangle = \begin{cases} \int_0^\infty \delta_t(\tau) f(\sigma(\tau))\Delta\tau, & \mu(\rho(t)) > 0 \\ \lim_{\epsilon\to 0} \int_0^\infty g(\epsilon, t, \sigma(\tau)) f(\tau)\Delta\tau. & \mu(\rho(t)) = 0 \end{cases} \tag{2.6}$$

To obtain the expected result, i.e., $\langle L_t, f\rangle = f(t)$, first we assume that $\rho(t) < t$ and we observe

$$\begin{aligned}\langle L_t, f\rangle &= \int_0^{\infty} \delta_t(\tau) f(\sigma(\tau))\Delta\tau \\ &= \int_{\rho(t)}^{t} \delta_t(\tau) f(\sigma(\tau))\Delta\tau \\ &= f(t). \end{aligned} \tag{2.7}$$

For the case $\rho(t) = t$, we have

$$\begin{aligned}\langle L_t, f\rangle &= \lim_{\epsilon\to 0^+} \int_{t-\epsilon}^{t} g(\epsilon, t, \sigma(\tau)) f(\tau)\Delta\tau \\ &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_{t-\epsilon}^{t} f(\tau)\Delta\tau \\ &= F^{\Delta}(t) = f(t), \end{aligned} \tag{2.8}$$

where $F(t) = \int_a^t f(\tau)\Delta\tau$ $(a \in \mathbb{T}_0)$.

Now suppose $\{\epsilon_n\}_n$ be such that $\frac{1}{\epsilon_n} \in \mathcal{R}(\mathbb{T}_0)$ (for $n \in \mathbb{N}$) and let $\epsilon_n \to 0$. Since in the case $\mu(s) > 0$, we have $1 \ominus \frac{1}{\epsilon_n}\mu(s) \in [0, 1]$ for $s \in \mathbb{T}$, then,

$$\lim_{n\to+\infty} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)) = \begin{cases} +\infty, & \sigma(\tau) = t \\ 0, & \sigma(\tau) < t \end{cases} \tag{2.9}$$

and in fact

$$\lim_{\epsilon\to 0^+} g(\epsilon, t, \sigma(\tau)) = \lim_{n\to+\infty} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)). \tag{2.10}$$

Suppose $\{\alpha_n\}_n \subset (0, 1)$ be such that $\alpha_n \to 1^-$ and satisfying $\frac{\alpha_n}{1-\alpha_n} \in \mathcal{R}(\mathbb{T}_0)$. Making use of the auxillary variable $\epsilon_n = \frac{1-\alpha_n}{\alpha_n}$, we have $\frac{M(\alpha_n)}{1-\alpha_n} = \frac{N(\epsilon_n)}{\epsilon_n}$, where $N(\epsilon_n) = (1 + \epsilon_n)M(\frac{1}{1+\epsilon_n})$. For ϵ_n small enough, we have

$$\int_0^t \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau))\Delta\tau \cong \int_{t-\epsilon_n}^{t} \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau))\Delta\tau \cong 1, \tag{2.11}$$

and we infer that

$$\begin{aligned}\lim_{\alpha\to 1^-} \mathfrak{D}^{\alpha}_{0^+} f(t) &= \lim_{n\to+\infty} N(\epsilon_n) \int_0^t \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)) f^{\Delta}(\tau)\Delta\tau \\ &= \lim_{n\to+\infty} N(\epsilon_n) \int_{t-\epsilon_n}^t \frac{1}{\epsilon_n} e_{\ominus\frac{1}{\epsilon_n}}(t, \sigma(\tau)) f^{\Delta}(\tau)\Delta\tau \\ &= \lim_{n\to+\infty} N(\epsilon_n) \int_0^t g(\epsilon_n, t, \sigma(\tau)) f^{\Delta}(\tau)\Delta\tau \\ &= \langle L_t, f^{\Delta}\rangle = f^{\Delta}(t). \end{aligned} \tag{2.12}$$

In the right scattered case, i.e., $\rho(t) < t$, the continuation may gives rise to a divergence. Let $\mathbb{T}_0 = \mathbb{N}_0$. Since $e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau)) = (1-\alpha)^{t-\sigma(\tau)}$, then

$$\begin{aligned}\frac{M(\alpha)}{1-\alpha}\int_0^t f^{\Delta}(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau))\Delta\tau &= \frac{M(\alpha)}{1-\alpha}\sum_{\tau=0}^{\rho(t)} f^{\Delta}(\tau)(1-\alpha)^{t-\sigma(\tau)}\\ &= M(\alpha)\sum_{\tau=0}^{\rho(t)} f^{\Delta}(\tau)(1-\alpha)^{t-\sigma(\tau)-1}\\ &= M(\alpha)\sum_{\tau=0}^{\rho^2(t)} f^{\Delta}(\tau)(1-\alpha)^{t-\sigma(\tau)-1}\\ &\quad + M(\alpha)\frac{f^{\Delta}(\rho(t))}{1-\alpha}, \end{aligned} \tag{2.13}$$

and in the case $f^{\Delta}(\rho(t) \neq 0$, (2.13) gives rise to $\lim_{\alpha\to 1^-}\mathfrak{D}_{0^+}^{\alpha}f(t) = \infty$. Assuming $\mathbb{T}_0$ arbitraryily and a left scattered $t \in \mathbb{T}_0$, we obtain

$$\begin{aligned}\lim_{\alpha\to 1^-}\mathfrak{D}_{0^+}^{\alpha}f(t) &= \lim_{n\to+\infty}\frac{N(\epsilon_n)}{\epsilon_n}\int_0^t e_{\ominus\frac{1}{\epsilon_n}}(t,\sigma(\tau))f^{\Delta}(\tau)\Delta\tau\\ &= \lim_{n\to+\infty}\frac{N(\epsilon_n)}{\epsilon_n}\int_{\rho(t)}^t e_{\ominus\frac{1}{\epsilon_n}}(t,\sigma(\tau))f^{\Delta}(\tau)\Delta\tau\\ &= \Big(f^{\Delta}(\rho(t))\mu(\rho(t))\Big)\lim_{n\to+\infty}\frac{N(\epsilon_n)}{\epsilon_n}\\ &= \begin{cases}\infty, & f^{\Delta}(\rho(t)) \neq 0\\ 0, & f^{\Delta}(\rho(t)) = 0.\end{cases}\end{aligned} \tag{2.14}$$

The above discussion is summarized below:

Proposition 2.1. *Let $f : \mathbb{T}_0 \to \mathbb{R}$ be such that $\mathfrak{D}_{0^+}^{\alpha_n}f(t)$ exists for $\alpha_n \in (0,1)$. The*

a) For $t \in \mathbb{T}_0$, we have

$$\lim_{\alpha\to 0^+}\mathfrak{D}_{0^+}^{\alpha}f(t) = f(t). \tag{2.15}$$

b) If $t \in \mathbb{T}_0$ is left dense then

$$\lim_{\alpha\to 1^-}\mathfrak{D}_{0^+}^{\alpha}f(t) = f^{\Delta}(t). \tag{2.16}$$

c) If $t \in \mathbb{T}_0$ is left scattered, then

$$\lim_{\alpha\to 1^-}\mathfrak{D}_{0^+}^{\alpha}f(t) = \begin{cases}\infty, & f^{\Delta}(\rho(t)) \neq 0\\ 0, & f^{\Delta}(\rho(t)) = 0.\end{cases} \tag{2.17}$$

Remark 2.1. *As it is illustrated, the restriction in the limit case $\alpha \to 1^-$ for a left scattered point, is originated from the definition 1.1. In the special case $\mathbb{T}_0 = [0,+\infty)$, we have the equality below*

$$\lim_{\alpha\to 1^-}\int_0^t \frac{1}{1-\alpha}f(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau)) = f(t). \tag{2.18}$$

Unlike (2.18), in the time scale counterpart theory of the Dirac functional, generally, there is no consistency between the formal definition of L_t and the exponential based one, i.e.,

$$\lim_{\alpha\to 1^-}\int_0^t \frac{1}{1-\alpha} f(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau)) \neq \langle L_t, f\rangle. \tag{2.19}$$

An alternative multiplier based on the original CF definition, to overcome this pitfall, is proposed below:

Suppose $T : \mathbb{T}_0 \times [0,1) \to \mathbb{R}$ satisfies the following asymptotic condintions:

$$\begin{cases} \lim_{\alpha\to 1^-} T(t,\alpha) = 1, & \rho(t) < t \\ T(t,\alpha) \sim \frac{1}{1-\alpha}, \quad as \ \ \alpha \to 1^- & \rho(t) = t \\ \lim_{\alpha\to 0^+} T(t,\alpha) = T(t,0) = 1. & \forall t \in \mathbb{T}_0 \end{cases} \tag{2.20}$$

It is clear that, for the CF derivative (with constant order) on time scales introduced by

$$\mathfrak{D}^{\alpha}_{0^+} f(t) = T(t,\alpha)\int_0^t f^{\Delta}(\tau)e_{\ominus\frac{\alpha}{1-\alpha}}(t,\sigma(\tau))\Delta\tau, \qquad s \in \mathbb{T}_0 \tag{2.21}$$

we have

$$\begin{cases} \lim_{\alpha\to 0^+} \mathfrak{D}^{\alpha}_{0^+} f(t) = f(t) - f(0), & \forall t \in \mathbb{T}_0 \\ \lim_{\alpha\to 0^+} \mathfrak{D}^{\alpha}_{0^+} f(t) = f^{\Delta}(t), & \rho(t) = t \\ \lim_{\alpha\to 0^+} \mathfrak{D}^{\alpha}_{0^+} f(t) = f^{\Delta}(\rho(t))\mu(\rho(t)). & \rho(t) = t \end{cases} \tag{2.22}$$

For the sake of loyalty to the Caputo and Fabrizio approach, in the rest of this paper, only the Definition 1.1 will be considered.

3 Laplace Transform

Let $\mathbb{T}_0$ be as in the Definition 1.1. we need the following background for the Laplace transform on time scales. For the details and more discussion, see [4].

Definition 3.1. *Assume that $f : \mathbb{T}_0 \to \mathbb{R}$ is regulated. Then the Laplace transform of f is defined by*

$$\mathcal{L}\{f\}(z) := \int_0^{\infty} f(t)e^{\sigma}_{\ominus z}(t,0)\Delta t, \tag{3.1}$$

for $z \in \mathcal{D}(f)$, where $\mathcal{D}(f)$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper integral exists.

As in the classical case, the time scale counterpart satisfies almost all expected properties. However, one should be careful about the considering sufficient conditions for the convergence issues. An important one, which will be needed in this sequel, is given below (see [4], Theorems 3.87 & 3.89)

Theorem 3.1. *Supposez $\in \mathbb{C}$ is regressive. 1) Assume $f : \mathbb{T}_0 \to \mathbb{R}$ is such that, f^{Δ} is regulated. Then*

$$\mathcal{L}\{f^{\Delta}\}(z) = z\mathcal{L}\{f\}(z) - f(0), \tag{3.2}$$

provided

$$\lim_{t\to\infty}\{f(t)e_{\ominus z}(t,0)\}=0. \tag{3.3}$$

@) Suupose f satisfies the the limit condition

$$\lim_{t\to+\infty}\Big\{e_{\ominus z}(t,0)\int_0^t f(\tau)\Delta\tau\Big\}=0, \tag{3.4}$$

then

$$\mathcal{L}\Big\{\int_0^t f(\tau)\Delta\tau\Big\}(z)=\frac{\mathcal{L}\{f\}(z)}{z}. \tag{3.5}$$

Remark 3.1. *The limit condition (3.3) is satisfied provided f is of exponentioal type II, i.e.,*

$$|f(t)|\leq ce_{c_0}(t,0), \tag{3.6}$$

for some positive constants c, c_0 and $\Re_\mu(z)>\Re_\mu(c_0)$ ([8], Theorem 1.1). Also, this is sufficient to the convergence of (3.1) [8].

Note that, the limit condition (3.3) implies

$$\begin{aligned}\lim_{t\to\infty}\{tf(t)e_{\ominus z}(t,0)\} &= 0,\\ \lim_{t\to\infty}\{\|f\|_{[0,t]}e_{\ominus z}(t,0)\} &= 0.\end{aligned} \tag{3.7}$$

($\|.\|$ is the maximum norm.) The first limit is obvious by the definition of $e_{\ominus z}$ and for the second limit we have

$$\max_{[0,t]}\{|f(\tau)|e_{\ominus z}(t,0)\}\leq\max_{[0,t]}\{|f(\tau(t))|e_{\ominus z}(\tau(t),0)\}\to 0, \tag{3.8}$$

as $t\to+\infty$. In view of the estimation $\Big|\int_0^t u(\tau)\Delta\tau\Big|\leq t\|u\|_{[0,t]}$, we conclude that, (3.3) gives rise to (3.4).

An special case of the convolution integral is defined here

Definition 3.2. *For a regulated function $f:\mathbb{T}_0\to\mathbb{R}$, and $\alpha\in\mathcal{R}$, the convolution of f and $e_\alpha(t,\tau)$ is defined by*

$$(e_\alpha * f)(t):=\int_0^t f(\tau)e_\alpha(t,\sigma(\tau))\Delta\tau. \tag{3.9}$$

Likewise the classical case, the convolution theorem (see [4], Theorem 3.106) is stated:

Theorem 3.2. *(Convolution Theorem) For a regulated function $f:\mathbb{T}_0\to\mathbb{R}$, we have*

$$\mathcal{L}\{e_\alpha * f\}(z)=\frac{1}{z-\alpha}.\mathcal{L}\{f\}(z), \tag{3.10}$$

where $\frac{1}{z-\alpha}$ is $\mathcal{L}\{e_\alpha(.,0)\}(z)$.

In view of Remark 2.1, The statement below is immediate.

Corollary 3.1. *Let $\alpha(s) \equiv \alpha$, i.e., equivalent to the case $s \in \mathbb{T}_0$ is fixed. Assume $f : \mathbb{T}_0 \to \mathbb{R}$ be such that, $\mathfrak{D}^{\alpha}_{0^+}(f(t))$ is regulated, continuous and satisfies (3.3) (substituting f with $\mathfrak{D}^{\alpha}_{0^+}$). Then, one can simply observe the following relation*

$$\mathcal{L}\Big\{\mathfrak{D}^{\alpha}_{0^+}(f(t))\Big\}(z) = \frac{M(\alpha)(1-\alpha+\alpha\mu(t))\Big\{z\mathcal{L}\{f\}(z) - f(0)\Big\}}{(1-\alpha)\Big[z(1-\alpha+\alpha\mu(t)) + \alpha\Big]}. \tag{3.11}$$

4 Non-integer order integral

In this section, following the same method as in [10] section 2, we tilize the Laplace transform given in Definition 2.1, to obtain an associated integral for the derivative introduced in definition 1.1.

Suppose $\alpha : \mathbb{T}_0 \to \mathbb{R}$ be as in definition 1.1. we assume two cases:

Case 1. Assume that α is constant, i.e., $\alpha(s) \equiv \alpha$ and consider

$$\mathfrak{D}^{\alpha(s)}_{0^+} f(t) = u(t), \tag{4.1}$$

for some $u : \mathbb{T}_0 \to \mathbb{R}$. Applying $\mathcal{L}$ on the both sides of (4.1) and in view of (3.11), we solve the equation for $\mathcal{L}\{f\}(z)$ to obtain

$$\mathcal{L}\{f\}(z) = \frac{1-\alpha}{M(\alpha)}\mathcal{L}\{u\}(z) + \frac{\alpha(1-\alpha)}{zM(\alpha)(1-\alpha+\alpha\mu(t))}\mathcal{L}\{u\}(z) + \frac{f(0)}{z}, \tag{4.2}$$

that is

$$f(t) = \frac{1-\alpha}{M(\alpha)}u(t) + \frac{\alpha(1-\alpha)}{M(\alpha)(1-\alpha+\alpha\mu(t))}\int_0^t u(s)\Delta s + f(0). \tag{4.3}$$

In order to introduce an integral (which is denoted by $\mathfrak{I}^{\alpha(t)}_{0^+} f(t)$), we expect that, it satisfies

$$\mathfrak{D}^{\alpha(t)}_{0^+}\mathfrak{I}^{\alpha}_{0^+} f(t) = f(t). \tag{4.4}$$

In view of (4.3) that is the solution of (4.1), we arrive at the definition below.

Definition 4.1. *The non-singular and constant non-integer order integral of a regulated function $f : \mathbb{T}_0 \to \mathbb{R}$ is defined by*

$$\mathfrak{I}^{\alpha}_{0^+} f(t) = \frac{1-\alpha}{M(\alpha)}f(t) + \frac{\alpha(1-\alpha)}{M(\alpha)(1-\alpha+\alpha\mu(t))}\int_0^t f(s)\Delta s. \tag{4.5}$$

Remark 4.1. *Substituting $\frac{M(\alpha)}{1-\alpha}$ in Definition 1.1 with $\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$, as Losada and Nieto did in [10] to define ${}^{CF}\mathfrak{D}^{\alpha}$, the associated integral ${}^{CF}\mathfrak{I}^{\alpha}$ ([10], Definition 1) takes the form*

$${}^{CF}\mathfrak{I}^{\alpha} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds. \quad t \geq 0 \tag{4.6}$$

In comparison with (4.5) with the presence of the graininess function $\mu(t)$, the Losada & Nieto's integral introduced by (4.6) is an espcial case of (4.5) up to the multiplier, while assuming $\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$ as the multiplier and $\mu(t) = 0$ in (4.5), we obtain (4.6).

Case 2. Suppose $s = t$. Let α be non-constant and Δ-differentiable. Further assume that, $\alpha(\mathbb{T}_0)$, (i.e., the image of α) is closed and $M : \alpha(\mathbb{T}_0) \to \mathbb{R}$ is Δ-differentiable. Making use of the properties of exponential function and considering (4.1) as follows

$$\int_0^t f^{\Delta}(\tau) e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(\tau), 0)\Delta\tau = \frac{1-\alpha(t)}{M(\alpha(t))} e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).u(t), \tag{4.7}$$

then Δ-differentiating, we obtain

$$\begin{aligned} f^{\Delta}(t).e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), 0) &= \frac{\alpha(t)}{1-\alpha(t)}.e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).\frac{1-\alpha(t)}{M(\alpha(t))}.u(t) \\ &+ e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).\Big(\frac{1-\alpha(t)}{M(\alpha(t))}\Big)^{\Delta}.u(t) + e_{\frac{\alpha(t)}{1-\alpha(t)}}(t, 0).\frac{1-\alpha(t)}{M(\alpha(t))}.u^{\Delta}(t). \end{aligned} \tag{4.8}$$

Multiplying the both sides by $e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), 0)$

$$f^{\Delta}(t) = e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t).\left\{\Big(\frac{\alpha(t)}{M(\alpha(t))} + \Big(\frac{1-\alpha(t)}{M(\alpha(t))}\Big)^{\Delta}\Big).u(t) + \frac{1-\alpha(t)}{M(\alpha(t))}.u^{\Delta}(t)\right\}, \tag{4.9}$$

then Δ-Integrating (4.9), we get

$$f(t) = \int_0^t e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t).\Big(p(\tau)u(\tau) + q(\tau)u^{\Delta}(\tau)\Big)\Delta\tau + f(0), \tag{4.10}$$

where

$$\begin{cases} p(t) = \frac{\alpha(t)}{M(\alpha(t))} + \frac{\alpha^{\Delta}(t)}{M(\alpha(t)).M(\alpha^{\sigma}(t))}.\Big(\alpha(t).M^{\Delta}(\alpha(t)) - 2M(\alpha(t))\Big), \\ q(t) = \frac{1-\alpha(t)}{M(\alpha(t))}. \end{cases} \tag{4.11}$$

Denoting The corresponded integral to the variable order $\alpha(t)$ by $\mathfrak{I}_{0^+}^{\alpha(t)}$, the discussion above, together with having the expectation similar to (4.4), is summarized in the definition below

Definition 4.2. *The non-singular and non-integer variable order integral of a Δ-differentiable function $f : \mathbb{T}_0 \to \mathbb{R}$ is introduced by*

$$\mathfrak{I}_{0^+}^{\alpha(t)} f(t) = \int_0^t e_{\ominus\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t).\Big(p(\tau)f(\tau) + q(\tau)f^{\Delta}(\tau)\Big)\Delta\tau, \tag{4.12}$$

where p and q are defined by (4.11).

Example 4.1. *Suppose the dynamic problem below*

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha} f(t) = \lambda f(t), \quad \lambda \in \mathbb{R} \\ \\ f(0) = f_0. \end{cases} \tag{4.13}$$

with constant $\alpha \in [0, 1]$. If we assume $\mathbb{T}_0 = [0, 1]$ then, it will be easily observed that, the exact solution is

$$f(t) = \frac{f_0 M(\alpha)}{M(\alpha)\lambda(1-\alpha)} \exp\Big[\frac{\lambda\alpha t}{M(\alpha) - \lambda(1-\alpha)}\Big]. \tag{4.14}$$

With the same assumption on α and $\mathbb{T}_0$, considering the IVP

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+} f(t) = a(t)f(t), \quad \lambda \in \mathbb{R} \\ \\ f(0) = f_0. \end{cases} \tag{4.15}$$

from (4.5), it is obvious that, the differentiability of $a(t)$ as asufficient condition, supposrts the existence issue. Then, (4.15) is converted into the ODE

$$f'(t) + \frac{M(\alpha)a(t)}{a(t)(1-\alpha) - M(\alpha)} f(t) = \frac{M(\alpha)f_0}{M(\alpha) - a(t)(1-\alpha)}, \tag{4.16}$$

which has a unique solution provided the denominator is non-vanishing.

Example 4.2. *Suppose* $\mathbb{T}_0 = (h\mathbb{N})_0 \cap [0,1]$. *Let* $\alpha(t) = t$, *then*

$$\begin{aligned} e_{\ominus \frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t), t) &= \exp \int_t^{\sigma(t)} \frac{\log(1 + \mu(\tau). \ominus \frac{\alpha(t)}{1-\alpha(t)})\Delta\tau}{\mu(\tau)} \\ &= \frac{\alpha(t) - 1}{\alpha(t)(1-\mu(t)) - 1} \\ &= \frac{t-1}{t-1-th}. \end{aligned} \tag{4.17}$$

and by $e_{\frac{\alpha}{1-\alpha}}(\sigma(t), t) = \frac{1}{e_{\ominus \frac{\alpha}{1-\alpha}}(\sigma(t),t)}$, *we have* $e_{\frac{\alpha}{1-\alpha}}(\sigma(t), t) = \frac{1-t+th}{1-t}$

Assume $M(\alpha) \equiv 1$ *and consider the IVP*

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+} f(t) = \lambda f(t), \\ \\ f(0) = f_0. \end{cases} \tag{4.18}$$

In equation (4.10) we have $p(t) = t - 2$, $q(t) = 1 - t$, *so*

$$\Big(\frac{\lambda(1-t+th)}{1-t}\Big) f(t) = \int_0^t (\tau - 2) f(\tau)\Delta\tau + \int_0^t (1-\tau) f^{\Delta}(\tau)\Delta\tau. \tag{4.19}$$

If we Δ*-differentiate of the both sides in (4.19), then we have*

$$f^{\Delta}(t) = \beta(t) f(t), \tag{4.20}$$

where

$$\beta(t) = \frac{t(1-t+th) - \lambda(1-t)^2(t-2)}{(1-t)\Big[\lambda(1-t)^2 - th + t + 1\Big]}. \tag{4.21}$$

Therefore, the unique solution to the problem (4.18) is

$$f(t) = f_0 e_{\beta(t)}(t, 0), \tag{4.22}$$

provided $\beta \in \mathcal{R}$. *This condition holds true if the roots of the polynomial* $Q(t)$ *(denoted by* $t(\lambda, h)$*) defined by*

$$Q(t) = \lambda(1+h)t^3 + (h^2 - 4\lambda h - 3\lambda + 1)t^2 + (5\lambda h + 2h + 3\lambda - 2)t + 2\lambda h - \lambda + 1, \tag{4.23}$$

satisfy $t(\lambda, h) \notin \mathbb{T}_0$. *Indeed, a sufficient condition is* $t(\lambda, h) > 1$.

5 Dynamic Equations on Viscoelastic Oscillations

The governing equation of motion for the viscoelastic oscillator in a dimensionless form is given by

$$x^{(\alpha)}(t) = \lambda(x(t) + x''(t)), \tag{5.1}$$

in which, $x^{(\alpha)}$ means a fractional order derivative in any of the existing senses and λ simplifies the conastants such as mass, charactristic length and natural frequency. Note that, compare to the classic form, the damping term appears as a fractional order derivative, rather than first order, that is interpreted as a viscoelasticity. The variable order derivative in the sense of Coimbra, (which involves non-singular kernel and continuous time) has been studied in [13].

Now consider the IVP

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha(t)} x(t) = \lambda(x(t) + x^{\Delta\Delta}(t)), \\ \\ f(0) = 0. \end{cases} \tag{5.2}$$

where $\mathfrak{D}_{0^+}^{\alpha(t)}$ is the differential operator introduced by Definition 1.1. Let $\mathbb{T}_0$ be a time scale of the desired type, that is introduced in Definition 1.1 with $\mathbb{T}_0 \subset [0,1]$. Moreover, assume that $\alpha : \mathbb{T}_0 \to \mathbb{T}_0$. By (4.10), we obtain

$$e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t),t).x(t) = \lambda \int_0^t \Big\{ p(\tau)\Big(x(\tau) + x^{\Delta\Delta}(\tau)\Big) + q(\tau)\Big(x^{\Delta}(\tau) + x^{\Delta\Delta\Delta}(\tau)\Big)\Big\}\Delta\tau. \tag{5.3}$$

Δ-differentiating gives rise to the third order dynamic equation on $\mathbb{T}_0$

$$\begin{aligned} \lambda q(t)x^{\Delta\Delta\Delta}(t) \quad &+ \quad \lambda p(t)x^{\Delta\Delta}(t) + \Big(\lambda q(t) - e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t),t)\Big)x^{\Delta}(t) \\ &+ \quad \Big(\lambda p(t) - \frac{\alpha(t)}{1-\alpha(t)} e_{\frac{\alpha(t)}{1-\alpha(t)}}(\sigma(t),t)\Big)x(t) = 0. \end{aligned} \tag{5.4}$$

If we further assume that $M : \mathbb{T}_0 \to \mathbb{T}_0$ satisfies

$$\alpha(t)M^{\Delta}(\alpha(t)) = 2M(\alpha(t)), \tag{5.5}$$

then in (4.11), we infer that $p(t) = \frac{\alpha(t)}{1-\alpha(t)}$. Indeed, $M(s) = e_{\frac{2}{s}}(s, s_0)$ $(s_0 \in \mathbb{T}_1 = \alpha(\mathbb{T}_0))$ is well defined and satisfies (5.5), since $\frac{2}{\alpha(t)} \in \mathcal{R}(\mathbb{T}_1)$ for $t \in \mathbb{T}_0$. Thefore, (5.4) takes the form below

$$\frac{\lambda(1-\alpha(t))}{M(\alpha(t))}x^{\Delta^3}(t) + \frac{\lambda\alpha(t)}{M(\alpha(t))}x^{\Delta^2}(t) + \beta(t)x^{\Delta}(t) + \frac{\alpha(t)}{1-\alpha(t)}\beta(t)x(t) = 0. \tag{5.6}$$

in which

$$\beta(t) = \frac{\lambda(1-\alpha(t))^2 - M(\alpha(t))(1-\alpha(t)+\mu(t)\alpha(t))}{M(\alpha(t))(1-\alpha(t))}. \tag{5.7}$$

6 Conclusion

In the present paper, having a look at the anomalous diffusion phenomena, by extending the concept of fractional differential operator of Caputo-Fabrizio type, to include a class of variable orders, a time-scale counterpart of the non-integer order differential operator is introduced. Implementing the measure theory on time scales and introducing Dirac delta functional based on the left density property of a given point, it has been deduced that, the both limit cases, namely $\alpha \to 1^{-}$or 0^{+}, give the well-known Δ-derivative and a shifted zeroth derivative, i.e., $f^{\Delta}(t)$ and $f(t) - f(0)$ respectively provided t is left dense. Moreover, through the discussion, the inconsistency between the formal definition of Dirac delta functional and the exponential based definition with the given multiplier in the original CF definition, is illustrated. By making use of the Laplace transform and direct Δ-calculus based approach, the associated integral for constant and variable orders are discussed. To illustrate the theory, some dynamic equations on time scales are studied and a dynamic problem, which governs a class of viscoelastic oscillation phenomena and involving the new introduced derivative is studied.

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FIXED POINT THEOREMS ON CONE METRIC SPACES WITH $c-$DISTANCE

YOUNG-OH YANG* AND HONG JOON CHOI

ABSTRACT. In this paper, we obtain a new sufficient conditions for existence of a unique coincidence point and a common fixed point for a pair of self mappings satisfying the contractive condition in a cone metric space by using $c-$distance.

1. Introduction

Since Huang and Zhang([5]) introduced the cone metric space which is more general than the concept of a metric space, many fixed point theorems have been proved in normal or non-normal cone metric spaces by some authors ([1], [3], [5], [6], [9] ,[12], [3]). Note that Cho et al.([3]) introduced the $c-$distance in a cone metric space which is a cone version of the $w-$distance of Kada et al.([7]).

In this paper, we obtain a new sufficient conditions for existence of a unique coincidence point and a common fixed point for a pair of self mappings satisfying contractive conditions in a cone metric space by using $c-$distance.

Let E be a real Banach space and θ denote the zero element in E. A *cone* P is a subset of E such that

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \;\Rightarrow\; ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called *normal* if there exists a number K such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1.1}$$

Equivalently, the cone P is normal if

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x \tag{1.1.2}$$

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*The corresponding author: yangyo@jejunu.ac.kr (Young-Oh Yang).

The least positive number K satisfying condition (1.1.1) is called the *normal constant* of P.

Definition 1.1. Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subseteq E$. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

Definition 1.2. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in E$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in E$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Lemma 1.3. *([10]) Let E be a real Banach space with a cone P. Then*

(1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

Lemma 1.4. *([10]) Let E be a real Banach space with cone P. Then*

(1) *If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*
(2) *If $\{a_n\}, \{b_n\}$ are sequences in E such that $a_n \to a, b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Lemma 1.5. *([5]) Let (X, d) be a cone metric space, P a normal cone, $x \in X$ and $\{x_n\}$ a sequence in X. Then*

(1) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$.*
(2) *The limit point of every sequence is unique.*
(3) *Every convergent sequence is a Cauchy sequence.*
(4) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.*
(5) *If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

Definition 1.6. Let (X, d) be a cone metric space. Then a mapping $q : X \times X \to E$ is called a $c-$*distance* on X if the following are satisfied :

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$.
(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$.
(q3) for all $x \in X$ and all $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$.
(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that
$$q(z, x) \ll e \text{ and } q(z, y) \ll e \text{ imply } d(x, y) \ll c.$$

Example 1.7. ([3]) Let (X, d) be a cone metric space and let P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a $c-$distance.

Example 1.8. ([3]) Let (X, d) be a cone metric space and let P be a normal cone. Put $q(x, y) = d(u, y)$ for all $x, y \in X$, where $u \in X$ is constant. Then q is a $c-$distance.

Example 1.9. ([3]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a $c-$distance.

Remark 1.10. (1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Lemma 1.11. *([3]) Let (X, d)be a cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following facts hold:*
(1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.
(2) If $q(x_n, y_n) \preceq u_n$and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.
(3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
(4) If $q(y, x_n) \preceq u_n$, then $\{x_n\}$is a Cauchy sequence in X.

Definition 1.12. Let T and S be self mappings of a set X.

(1) If $y = Tx = Sx$ for some $x \in X$, then x is called a *coincidence point* of T and S and y is called a *point of coincidence* of T and S.
(2) $T, S : X \to X$ are *weakly compatible* if for every $x \in X$, the following holds:
$$T(Sx) = S(Tx) \quad \text{whenever} \quad Sx = Tx.$$
(2) $T : X \to X$ is *continuous* if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$.

2. **Main results**

In [12], Wang and Guo proved some common fixed point theorem of two mappings (in a normal cone metric space) satisfying the contractive condition (2.2.1). We state Theorem 2.1 in [12] as follows.

Theorem 2.1. *([12]) Let (X,d) be a cone metric space. Let P be a normal cone with normal constant K and let q be a $c-$distance on X. Let $a_i \in [0,1)\,(i=1,2,3,4)$ be constants with $a_1+a_2+a_3+2a_4<1$ and $f,g: X\to X$ be two mappings satisfying the condition*

$$q(fx,fy)\preceq a_1q(gx,gy)+a_2q(gx,fx)+a_3q(gy,fy)+a_4q(gx,fy) \tag{2.2.1}$$

for all $x,y\in X$. Suppose that $f(X)\subseteq g(X)$ and $g(X)$ is a complete subset of X. If f and g satisfy

$$\inf\{\|q(gx,y)\|+\|q(fx,y)\|+\|q(gx,fx)\|: x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a common fixed point in X.

In a normal cone metric space we prove a new common fixed point theorem of two mappings satisfying the contractive condition independent of the condition (2.2.1).

Theorem 2.2. *Let (X,d) be a cone metric space, P be a normal cone with normal constant K and q be a $c-$distance on X. Let $a_i\in[0,1)\,(i=1,2,3,4,5)$ be constants with $a_1+2a_2+2a_3+3a_4+a_5<1$ and $f,g:X\to X$ be two mappings satisfying the condition*

$$q(fx,fy)\preceq a_1q(gx,gy)+a_2q(gx,fx)+a_3q(gy,fy)+a_4q(gx,fy)+a_5q(gy,fx)$$

for all $x,y\in X$. Suppose that $f(X)\subseteq g(X)$ and $g(X)$ be a complete subset of X. If f and g satisfy

$$\inf\{\|q(gx,y)\|+\|q(fx,y)\|+\|q(gx,fx)\| \;:\; x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. Let $x_0, x_1 \in X$. Using the fact that $f(X) \subseteq g(X)$, construct $\{x_{2n}\}, \{x_{2n+1}\}$ such that $gx_{2n} = fx_{2n-2}$ and $gx_{2n+1} = fx_{2n-1} \quad (n \in \mathbb{N})$. Then we have

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &= q(fx_{2n-2}, fx_{2n-1}) \\ &\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, fx_{2n-2}) + a_3 q(gx_{2n-1}, fx_{2n-1}) \\ &+ a_4 q(gx_{2n-2}, fx_{2n-1}) + a_5 q(gx_{2n-1}, fx_{2n-2}) \\ &= a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, gx_{2n}) \\ &+ a_3 q(gx_{2n-1}, gx_{2n+1}) + a_4 q(gx_{2n-2}, gx_{2n+1}) + a_5 q(gx_{2n-1}, gx_{2n}) \\ &\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2\{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n})\} \\ &+ a_3\{q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\ &+ a_4\{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\ &+ a_5 q(gx_{2n-1}, gx_{2n}). \end{aligned}$$

Hence

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-1}, gx_{2n}) \\ &+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}). \end{aligned} \tag{2.2.2}$$

Similarly,

$$\begin{aligned} q(gx_{2n-1}, gx_{2n}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}) \\ &+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-3}, gx_{2n-2}). \end{aligned} \tag{2.2.3}$$

Clearly $0 \leq \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} < 1$ and $0 \leq \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$. Set

$$b_1 = \alpha = \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} \quad \text{and} \quad c_1 = \beta = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}.$$

Applying (2.2.2) and (2.2.3) and putting $b_2 = c_1 + \alpha b_1 = \beta + \alpha b_1, c_2 = \beta b_1$,

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &\preceq b_1 q(gx_{2n-1}, gx_{2n}) + c_1 q(gx_{2n-2}, gx_{2n-1}) \\ &\preceq b_2 q(gx_{2n-2}, gx_{2n-1}) + c_2 q(gx_{2n-3}, gx_{2n-2}) \\ &\vdots \\ &\preceq b_{2n-1} q(gx_1, gx_2) + c_{2n-1} q(gx_0, gx_1), \end{aligned} \tag{2.2.4}$$

where $b_{2n-1} = \beta b_{2n-3} + \alpha b_{2n-2}$ and $c_{2n-1} = \beta b_{2n-2}$. Similarly

$$q(gx_{2n-1}, gx_{2n}) \preceq b_{2n-2} q(gx_1, gx_2) + c_{2n-2} q(gx_0, gx_1) \tag{2.2.5}$$

where $b_{2n-2} = \beta b_{2n-4} + \alpha b_{2n-3}$ and $c_{2n-2} = \beta b_{2n-3}$. From (2.2.4) and (2.2.5),

$$q(gx_{n+1}, gx_{n+2}) \preceq b_n q(gx_1, gx_2) + c_n q(gx_0, gx_1)$$

where $b_n = \beta b_{n-2} + \alpha b_{n-1}$ and $c_n = \beta b_{n-1}$.

Consider

$$b_{n+2} = \alpha b_{n+1} + \beta b_n \quad (0 \le \alpha, \beta < 1, b_1, b_2 \ge 0).$$

Then $b_n \ge 0$ for all $n \in \mathbb{N}$. Its characteristic equation is that $t^2 - \alpha t - \beta = 0$. If $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$ then it has two roots t_1, t_2 such that $-1 < t_1 \le 0 \le t_2 < 1$. Also the hypothesis $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ implies $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$. For such t_1, t_2, $b_n = k_1(t_1)^n + k_2(t_2)^n$ for some $k_1, k_2 \in \mathbb{R}$.

Let $m > n \ge 1$. It follows that

$$\begin{aligned}
q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\
&\preceq (b_{n-1} + b_n + \cdots + b_{m-2}) q(gx_1, gx_2) + (c_{n-1} + c_n + \cdots + c_{m-2}) q(gx_0, gx_1) \\
&\preceq \{k_1(t_1^{n-1} + t_1^n + \cdots + t_1^{m-2}) + k_2(t_2^{n-1} + \cdots + t_2^{m-2})\} q(gx_1, gx_2) \\
&+ \beta\{k_1(t_1^{n-2} + \cdots + t_1^{m-3}) + k_2(t_2^{n-2} + \cdots + t_2^{m-3})\} q(gx_0, gx_1) \\
&\preceq (\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2}) q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2}) q(gx_0, gx_1) \\
&\to \theta
\end{aligned}$$

as $n \to \infty$. Therefore $\{gx_n\}$ is a Cauchy sequence in $g(X)$ by Lemma 1.11 (3). Since $g(X)$ is complete, there exists $x' \in g(X)$ such that $gx_m \to x'$ as $m \to \infty$. By definition 1.6(q3)

$$q(gx_n, x') \preceq (\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2}) q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2}) q(gx_0, gx_1)$$

Since P is a normal cone with normal constant K, we have

$$\begin{aligned}
\|q(gx_n, gx_m)\| &\le K\|(\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2}) q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2}) q(gx_0, gx_1)\| \\
&\le K(\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2})\|q(gx_1, gx_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2})\|q(gx_0, gx_1)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Also

$$\begin{aligned}
\|q(gx_n, x')\| &\le K\|(\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2}) q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2}) q(gx_0, gx_1)\| \\
&\le K(\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2})\|q(gx_1, gx_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2})\|q(gx_0, gx_1)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

Suppose that x' is not a point of coincidence of f and g. Then by assumption,

$$\begin{aligned} 0 &< \inf\{\|q(gx,x')\| + \|q(fx,x')\| + \|q(gx,fx)\| \,:\, x \in X\} \\ &\leq \inf\{\|q(gx_n,x')\| + \|q(fx_n,x')\| + \|q(gx_n,fx_n)\| \,:\, n \in \mathbb{N}\} \\ &= \inf\{\|q(gx_n,x')\| + \|q(gx_{n+2},x')\| + \|q(gx_n,gx_{n+2})\| \,:\, x \in \mathbb{N}\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore x' is a point of coincidence of f and g. So there exists $x \in X$ such that $fx = gx = x'$. If there exists $w \in X$ such that $fy = gy = w$ for some $y \in X$,

$$\begin{aligned} q(x',x') &= q(fx,fx) \\ &\preceq a_1q(gx,gx) + a_2q(gx,fx) + a_3q(gx,fx) + a_4q(gx,fx) + a_5q(gx,fx) \\ &= (a_1 + a_2 + a_3 + a_4 + a_5)q(x',x'). \end{aligned}$$

Hence

$$q(x',x') = \theta. \tag{2.2.6}$$

Similarly

$$q(w,w) = \theta. \tag{2.2.7}$$

Now by (2.2.6) and (2.2.7)

$$\begin{aligned} q(x',w) &= q(fx,fy) \\ &\preceq a_1q(gx,gy) + a_2q(gx,fx) + a_3q(gy,fy) + a_4q(gx,fy) + a_5q(gy,fx) \\ &= a_1q(x',w) + a_2q(x',x') + a_3q(w,w) + a_4q(x',w) + a_5q(w,x') \\ &= (a_1 + a_4)q(x',w) + a_5q(w,x'). \end{aligned}$$

Similarly $q(w,x') \preceq (a_1 + a_4)q(w,x') + a_5q(x',w)$. Thus

$$q(x',w) + q(w,x') \preceq (a_1 + a_4 + a_5)\{q(x',w) + q(w,x')\}.$$

Therefore $q(x',w) + q(w,x') = \theta$ which implies

$$q(x',w) = q(w,x') = \theta. \tag{2.2.8}$$

By (2.2.7),(2.2.8) and Lemma 1.11(1), $x' = w$. Consequently x' is a unique point of coincidence of f and g.

Moreover if f and g are weakly compatible,

$$gx' = ggx = gfx = fgx = fx'$$

which implies gx' is a point of coincidence of f and g. By uniqueness of the point of coincidence , $fx' = gx' = x'$. In other words, x' is the unique common fixed point of f and g. □

Corollary 2.3. *Let (X, d) be a complete cone metric space and let P be a normal cone with normal constant K and q be a c-distance on X. Let $a_i \in [0,1)$, $i = 1,2,3,4,5$ be constants with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ and $f : X \to X$ be a mapping satisfying the condition*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy) + a_5 q(y, fx)$$

for all $x, y \in X$. If f satisfies the condition

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \,:\, x \in X\} > 0$$

if $fy \neq y$, then f has a unique fixed point in X.

Proof. Take $g = I$ in the above theorem. □

Corollary 2.4. *Let (X, d) be a complete cone metric space and let P be a normal cone with normal constant K and q be a c-distance on X. Let $a_i \in [0,1)$, $i = 1,2,3,4$ be constants with $a_1 + 2a_2 + 2a_3 + 3a_4 < 1$. If $f : X \to X$ is a continuous mapping satisfying the condition*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy),$$

then f has a unique fixed point in X.

Proof. Assume there exists $y \in X$ such that $fy \neq y$ and

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \,:\, x \in X\} = 0.$$

Then we can construct $\{x_n\}$ in X such that

$$\inf\{\|q(x_n,y)\| + \|q(fx_n,y)\| + \|q(x_n,fx_n)\| \,:\, n \in \mathbb{N}\} = 0.$$

Hence

$$q(x_n, y) \to \theta, \quad q(fx_n, y) \to \theta, \quad q(x_n, fx_n) \to \theta.$$

By Lemma 1.11(2), $fx_n \to y$. By the contractive condition, we have

$$\begin{aligned} q(fx_n, f^2x_n) &\preceq a_1 q(x_n, fx_n) + a_2 q(x_n, fx_n) + a_3 q(fx_n, f^2x_n) + a_4 q(x_n, f^2x_n) \\ &\preceq a_1 q(x_n, fx_n) + a_2 q(x_n, fx_n) + a_3 q(fx_n, f^2x_n) \\ &+ a_4 q(x_n, fx_n) + a_4 q(fx_n, f^2x_n). \end{aligned}$$

Therefore $q(fx_n, f^2x_n) \preceq \frac{a_1+a_2+a_4}{1-a_3-a_4} q(x_n, fx_n)$. Hence

$$\begin{aligned} q(x_n, f^2x_n) &\preceq q(x_n, fx_n) + q(fx_n, f^2x_n) \\ &\preceq q(x_n, fx_n) + \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_n, fx_n) \to \theta \end{aligned}$$

as $n \to \infty$. This implies $q(x_n, f^2x_n) \to \theta$. Consequently, $f^2x_n \to y$ by Lemma 1.11(2) . Since f is continuous, we have

$$fy = f(\lim_{n\to\infty} fx_n) = \lim_{n\to\infty} f^2x_n = y$$

which is a contradiction. Therefore if $fy \neq y$, then

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \, : \, x \in X\} > 0.$$

By Corollary 2.3, the proof is complete. □

Example 2.5. Let $X = \{0,1,2,3\}$, $E = \mathbb{R}$ and $P = \{x \in \mathbb{R} : x \geq 0\}$. Define $d : X \times X \to E$ by $d(x,y) = |x-y|$. Then (X,d) is a complete cone metric space. Define $q : X \times X \to E$ by the following :

$$\begin{aligned}
q(0,0) &= 0, \quad q(0,1) = 1, \quad q(0,2) = 1.1, \quad q(0,3) = 0.5,\\
q(1,0) &= 1, \quad q(1,1) = 0, \quad q(1,2) = 0.1, \quad q(1,3) = 0.5,\\
q(2,0) &= 1, \quad q(2,1) = 1, \quad q(2,2) = 0, \quad q(2,3) = 0.5,\\
q(3,0) &= 1, \quad q(3,1) = 0.5, \; q(3,2) = 0.6, \quad q(3,3) = 0.
\end{aligned}$$

Then q is a $c-$distance. In fact, Definition 1.6 (q1),(q3) are obvious. If we put $e = 0.01$, (q4) is also clear. From the direct calculation, (q2) is satisfied.

Define $f : X \to X$ by $f0 = 1, f1 = 2, f2 = 2, f3 = 2$ and define $g : X \to X$ by $gx = x$. Then $f(X) \subseteq g(X)$ and $g(X)$ is complete. If we take $x = 2, y = 0$, then $q(f2, f0) = q(2,1) = 1$ and

$$\begin{aligned}
a_1 q(g2, g0) &+ a_2 q(g2, f2) + a_3 q(g0, f0) + a_4 q(g2, f0)\\
&= a_1 q(2,0) + a_2 q(2,2) + a_3 q(0,1) + a_4 q(2,1)\\
&= a_1 + a_3 + a_4 \leq a_1 + a_3 + 2a_4 < 1
\end{aligned}$$

for any real numbers $a_i \in [0,1)\,(i = 1,2,3,4)$ with $a_1 + a_2 + a_3 + 2a_4 < 1$. Hence the contractive condition (2.2.1) of Theorem 2.1 is not satisfied and so Theorem 2.1 can not be applied to this example.

But Theorem 2.2 can be applied to this example. In fact we take $a_1 = 0.14, a_2 = a_3 = a_4 = 0$ and $a_5 = 0.85$. Then for any $x, y \in X$, the contractive condition of Theorem 2.2 is satisfied. Also

$$\inf\{\|q(gx,y)\| + \|q(fx,y)\| + \|q(gx,fx)\| \, : \, x \in X\} > 0$$

for $y = 0, 1, 3$. Hence the hypotheses are satisfied and so by Theorem 2.2 f and g have a unique point of coincidence. Since $f2 = 2$ and $g2 = 2$, 2 is a unique point of coincidence. Since $2 = gf2 = fg2$, f and g are weakly compatible. Hence 2 is the unique common fixed point of f and g.

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Young-Oh Yang,
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
E-mail address: yangyo@jejunu.ac.kr

Hong Joon Choi,
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
E-mail address: next-xy@hanmail.net

L-FUZZY CLOSURE OPERATORS, L-FUZZY TOPOLOGIES AND L-FUZZY QUASI-UNIFORMITIES

JU-MOK OH, YONG CHAN KIM*

Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea

Abstract. We investigate L-fuzzy closure operators and L-(fuzzy) topologies in a complete residuated lattice. Also, we study the relationship among L-fuzzy closure operator and L-fuzzy quasi-uniform space. Finally, we give their examples.

Keywords:Complete residuated lattice, L-fuzzy closure space L-fuzzy topological space and L-fuzzy quasi-uniform space.

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1. Introduction

Hájek [6] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules in complete residuated lattices. Höhle [7,8]introduced L-fuzzy topological structure with algebraic structure L(cqm, quantales, MV-algebra). Uniformities in fuzzy sets, have the following approach of Lowen [14] based on powersets of the form $L^{X\times X}$ as a viewpoint of the enourage approach, the uniform covering approach of Kotzé [12], the uniform operator approach of Rodabaugh [20] as a generalization of Hutton [9] based on powersets of the form $(L^X)^{(L^X)}$, the unification approach of Gutiérrez García[3]. Many researchers studied the different approach as powerset [12] or the uniform covering [9].

We investigate L-fuzzy closure operators and L-(fuzzy) topologies in a complete residuated lattice. Also, we study the relationship among L-fuzzy closure operator and L-fuzzy quasi-uniform space. Finally, we give their examples.

2. Preliminaries

Definition 2.1. [2,6] An algebra $(L, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

* Corresponding author

(L1) $(L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(L2) $(L, \odot, \top)$ is a commutative monoid;

(L3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $L = (L, \vee, \wedge, \odot, \to, \bot, \top)$ be a complete residuated lattice with with an order reversing involution * which is defined by $x \oplus y = (x^* \odot y^*)^*,\ x^* = x \to 0$.

Lemma 2.2. [2,6] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$ and $x \to 0 = x^*$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigwedge_i y_i) \leq \bigwedge_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$, $x \oplus (\bigvee_i y_i) = \bigvee_i (x \oplus y_i)$,

(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(13) $x \odot (x^* \oplus y^*) \leq y^*$,

(14) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(15) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,

(16) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(17) $x \odot y \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$.

For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \to \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$.

Definition 2.3.[2,4] Let X be a set. A mapping $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions:

(E1) reflexive if $R(x, x) = \top$ for all $x \in X$,

(E2) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$,

(E3) antisymmetric if $R(x, y) = R(y, x) = \top$, then $x = y$.

Lemma 2.4. [2,4] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda, \mu, \rho, \nu \in L^X$, and $\alpha \in L$, the following properties hold.

(1) S is an L-partial order on L^X.

(2) $\lambda \leq \mu$ iff $S(\lambda, \mu) \geq \top$,

(3) If $\lambda \leq \mu$, then $S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$,

(4) $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$ and $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \oplus \nu, \mu \oplus \rho)$,

(5) $S(\mu, \rho) \leq S(\lambda, \mu) \to S(\lambda, \rho)$ and $S(\mu, \rho) \leq S(\rho, \lambda) \to S(\mu, \lambda)$,

(6) $\bigvee_{\mu \in L^X} (S(\mu, \rho) \odot S(\lambda, \mu)) = S(\lambda, \rho)$.

(7) If $\phi : X \to Y$ is a map, then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \leq S(\phi^{\rightarrow}(\lambda), \phi^{\rightarrow}(\mu)),$$

$$S(\rho, \nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.5. [8,22] A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions:

(T1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,

(T2) $\mathcal{T}(\lambda \odot \mu) \geq \mathcal{T}(\lambda) \odot \mathcal{T}(\mu),\ \forall\ \lambda, \mu \in L^X$,

(T3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i),\ \forall\ \{\lambda_i\}_{i \in \Gamma} \subseteq L^X$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space. An L-fuzzy topological space is called enriched if

(R) $\mathcal{T}(\alpha \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be L-fuzzy continuous iff for each $\lambda \in L^Y$, $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda))$.

Remark 2.6. A set $\tau \subset L^X$ is called an L-topology on X if (t1) $\bot_X, \top_X \in \tau$, (t2) $(\lambda \odot \mu) \in \tau$, for each $\lambda, \mu \in \tau$, (t3) $\bigvee_i \lambda_i \in \tau$, for all $\lambda_i \in \tau$. An L-topology τ is called enriched if $\alpha \odot \lambda \in \tau$, for all $\lambda \in \tau$ and $\alpha \in L$.

Definition 2.7. [11] A map $\mathcal{U} : L^{X \times X} \to L$ is called an L-fuzzy quasi-uniformity on X iff the following conditions hold.

(QU1) There exists $u \in L^{X \times X}$ such that $\mathcal{U}(u) = \top$.

(QU2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$.

(QU3) For every $u, v \in L^{X\times X}$,$\mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$.

(QU4) If $\mathcal{U}(u) \neq \perp$ then $\top_{\triangle} \leq u$ where

$$\top_{\triangle}(x, y) = \begin{cases} \top, & \text{if x = y} \\ \perp, & \text{if } x \neq y, \end{cases}$$

(QU5) $\mathcal{U}(u) \leq \bigvee\{\mathcal{U}(v) \mid v \circ v \leq u\},\ \forall\ u \in L^{X\times X}$, where

$$v \circ v(x, y) = \bigvee_{z\in X} v(x, z) \odot v(z, y),\ \forall\ x, y \in X.$$

An L-fuzzy quasi-uniformity $\mathcal{U}$ on X is said to be stratified if

(R) $\mathcal{U}(\alpha \odot u) \geq \alpha \odot \mathcal{U}(u),\ \forall\ u \in L^{X\times X}$.

The pair $(X,\mathcal{U})$ is called an L-fuzzy quasi-uniform space.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be L-fuzzy quasi-uniform spaces, and $\phi : X \to Y$ be a mapping. Then ϕ is said to be L-fuzzy uniformly continuous if

$\mathcal{V}(v) \leq \mathcal{U}((\phi \times \phi)^{\leftarrow}(v))$, for every $v \in L^{Y\times Y}$.

Remark 2.8. Let $(X,\mathcal{U})$ be an L-fuzzy quasi- uniform space. By (QU1) and (QU2), we have $\mathcal{U}(\top_{X\times X}) = \top$ because $u \leq \top_{X\times X}$ for all $u \in L^{X\times X}$.

3. L-fuzzy closure spaces and L-fuzzy topological spaces

Definition 3.1. A map $\mathcal{C} : L^X \to L^X$ is called an L-fuzzy closure operator if it satisfies the following conditions:

(C1) $\mathcal{C}(\perp_X) = \perp_X$,

(C2) for $\lambda \in L^X$, $\lambda \leq \mathcal{C}(\lambda)$,

(C3) if $\lambda \leq \mu$, $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$,

(C4) for all $\lambda, \mu \in L^X$, $\mathcal{C}(\lambda \oplus \mu) \leq \mathcal{C}(\lambda) \oplus \mathcal{C}(\mu)$.

The pair $(X, \mathcal{C})$ is called an L-fuzzy closure space.

An L-fuzzy closure space is called stratified if

(R) $\mathcal{C}(\alpha \to \lambda) \leq \alpha \to \mathcal{C}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two L-fuzzy closure spaces. A mapping $\phi : X \to Y$ is said to be C-map if $\phi^{\rightarrow}(\mathcal{C}_1(\lambda)) \leq \mathcal{C}_2(\phi^{\rightarrow}(\lambda))$ for each $\lambda \in L^X$.

Lemma 3.2. Let $\mathcal{C} : L^X \to L^X$ a map. The following statement are equivalent.

(1) For all $\lambda, \mu \in L^X$, $S(\lambda, \mu) \leq S(\mathcal{C}(\lambda), \mathcal{C}(\mu))$.

(2) If $\lambda \leq \mu$, then $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$ and $\mathcal{C}(\alpha \odot \rho) \geq \alpha \odot \mathcal{C}(\rho)$ for all $\lambda \in L^X$ and $\alpha \in L$.

(3) If $\lambda \leq \mu$, then $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$ and $\mathcal{C}(\alpha \to \rho) \leq \alpha \to \mathcal{C}(\rho)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Proof. (1) ($\Rightarrow$) (2). If $\lambda \leq \mu$, then $\top = S(\lambda, \mu) \leq S(\mathcal{C}(\lambda), \mathcal{C}(\mu))$. Hence $\mathcal{C}(\lambda) \leq \mathcal{C}(\mu)$. Put $\mu = \alpha \odot \lambda$. Then $\alpha \leq S(\lambda, \alpha \odot \lambda) \leq S(\mathcal{C}(\lambda), \mathcal{C}(\alpha \odot \lambda))$. Hence $\alpha \odot \mathcal{C}(\lambda) \leq \mathcal{C}(\alpha \odot \lambda)$.

(2) ($\Rightarrow$) (3). Since $\alpha \odot \mathcal{C}(\alpha \to \lambda) \leq \mathcal{C}(\alpha \odot (\alpha \to \lambda)) \leq \mathcal{C}(\lambda)$, $\mathcal{C}(\alpha \to \lambda) \leq \alpha \to \mathcal{C}(\lambda)$.

(3) ($\Rightarrow$) (1). Since $S(\lambda, \mu) \odot \lambda \leq \mu$ iff $\lambda \leq S(\lambda, \mu) \to \mu$, $\mathcal{C}(\lambda) \leq \mathcal{C}(S(\lambda, \mu) \to \mu) \leq S(\lambda, \mu) \to \mathcal{C}(\mu)$. Hence $S(\lambda, \mu) \leq S(\mathcal{C}(\lambda), \mathcal{C}(\mu))$.

Theorem 3.3. Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Define a map $\mathcal{C}_{\mathcal{T}} : L^X \longrightarrow L^X$ as follows:

$$\mathcal{C}_{\mathcal{T}}(\lambda) = \bigwedge_{\mu \in L^X} (\mathcal{T}(\mu) \to (S(\lambda, \mu^*) \to \mu^*))$$

Then $(X, \mathcal{C}_{\mathcal{T}})$ is a stratified L-fuzzy closure space.

Proof. (C1) $\mathcal{C}_{\mathcal{T}}(\bot_X) = \bigwedge_{\mu \in L^X} (\mathcal{T}(\mu) \to (S(\bot_X, \mu^*) \to \mu^*)) = (\bigvee (\mathcal{T}(\mu) \odot \mu))^* = \bot_X$.

(C2), we have $\mathcal{C}_{\mathcal{T}}(\lambda) \geq \lambda$ for each $\lambda \in L^X$ from:

$$\begin{aligned}
S(\lambda, \mathcal{C}_{\mathcal{T}}(\lambda)) &= \bigwedge_{x \in X} (\lambda(x) \to \mathcal{C}_{\mathcal{T}}(\lambda)(x)) \\
&= \bigwedge_{x \in X} \Big(\lambda(x) \to \bigwedge_{\mu \in L^X} (\mathcal{T}(\mu) \to (S(\lambda, \mu^*) \to \mu^*(x)))\Big) \\
&= \bigwedge_{x \in X} \bigwedge_{\mu \in L^X} \Big((\mathcal{T}(\mu) \odot S(\lambda, \mu^*) \odot \lambda(x)) \longrightarrow \mu^*(x))\Big) \\
&= \bigwedge_{x \in X} \bigwedge_{\mu \in L^X} \Big((\mathcal{T}(\mu) \odot S(\lambda, \mu^*) \longrightarrow (\lambda(x) \longrightarrow \mu^*(x))\Big) \\
&= \bigwedge_{\mu \in L^X} \Big((\mathcal{T}(\mu) \odot S(\lambda, \mu^*)) \to \bigwedge_{x \in X} (\lambda(x) \longrightarrow \mu^*(x))\Big) \\
&= \bigwedge_{\mu \in L^X} \Big((\mathcal{T}(\mu) \odot S(\lambda, \mu^*)) \to S(\lambda, \mu^*)\Big) \geq \top.
\end{aligned}$$

(C3) and, by Lemma 3.2, $\mathcal{C}_{\mathcal{T}}$ is stratified from

$$
\begin{aligned}
&S(\mathcal{C}_{\mathcal{T}}(\lambda),\mathcal{C}_{\mathcal{T}}(\mu)) = \bigwedge_{x\in X}(\mathcal{C}_{\mathcal{T}}(\lambda)(x)\to\mathcal{C}_{\mathcal{T}}(\mu)(x))\\
&= \bigwedge_{x\in X}\Big(\bigwedge_{\rho\in L^X}(\mathcal{T}(\rho)\odot S(\lambda,\rho^*)\to\rho^*(x))\to\bigwedge_{\nu\in L^X}(\mathcal{T}(\nu)\odot S(\mu,\nu^*)\to\nu^*(x))\Big)\\
&\geq \bigwedge_{x\in X}\bigwedge_{\rho\in L^X}\Big((\mathcal{T}(\rho)\odot S(\lambda,\rho^*)\to\rho^*(x))\to((\mathcal{T}(\rho)\odot S(\mu,\rho^*)\to\rho^*(x))\Big)\\
&\geq \bigwedge_{x\in X}\bigwedge_{\rho\in L^X}\Big(\mathcal{T}(\rho)\odot S(\mu,\rho^*)\to(\mathcal{T}(\rho)\odot S(\lambda,\rho^*)\Big) \text{ (by Lemma 2.2 (16))}\\
&\geq \bigwedge_{x\in X}\bigwedge_{\rho\in L^X}\Big(S(\mu,\rho^*)\to S(\lambda,\rho^*)\Big)\geq S(\lambda,\mu).\text{(by Lemma 2.4(5))}
\end{aligned}
$$

(C4)

$$
\begin{aligned}
&(a\odot c)\odot(b\oplus d)^* = (a\odot b^*)\odot(c\odot d^*)\\
&\Leftrightarrow (a\odot c)\to(b\oplus d) = (a\to b)\oplus(c\to d),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{C}_{\mathcal{T}}(\lambda)\oplus\mathcal{C}_{\mathcal{T}}(\mu)\\
&= \bigwedge_{\rho\in L^X}(\mathcal{T}(\rho)\to(S(\lambda,\rho^*)\to\rho^*))\oplus\bigwedge_{\nu\in L^X}(\mathcal{T}(\nu)\to(S(\mu,\nu^*)\to\nu^*))\\
&\text{(by Lemma 2.2(6))}\\
&= \bigwedge_{\rho\in L^X}\bigwedge_{\nu\in L^X}\Big((\mathcal{T}(\rho)\odot S(\lambda,\rho^*)\to\rho^*)\oplus(\mathcal{T}(\nu)\odot S(\mu,\nu^*)\to\nu^*)\Big)\\
&= \bigwedge_{\rho,\nu\in L^X}\Big(\mathcal{T}(\rho)\odot\mathcal{T}(\nu)\odot S(\lambda,\rho^*)\odot S(\mu,\nu^*)\to(\rho^*\oplus\nu^*)\Big)\\
&\geq \bigwedge_{\rho,\nu\in L^X}\Big(\mathcal{T}(\rho\odot\nu)\odot S(\lambda\oplus\mu,(\rho^*\oplus\nu^*))\to(\rho^*\oplus\nu^*)\Big)\\
&\text{(by Lemma 2.4(4))}\\
&\geq \mathcal{C}_{\mathcal{T}}(\lambda\oplus\mu).
\end{aligned}
$$

Remark 3.4. Let (X,τ) be an L-topological space. Define a map $\mathcal{C}_\tau : L^X\to L^X$ as follows:

$$\mathcal{C}_\tau(\lambda) = \bigwedge\{\mu\in L^X \mid \lambda\leq\mu,\mu^*\in\tau\}.$$

Then $(X,\mathcal{C}_\tau)$ is an L-fuzzy closure space. Moreover, if (X,τ) is enriched, $(X,\mathcal{C}_\tau)$ is stratified.

Theorem 3.5. Let $(X, \mathcal{C})$ be an L-fuzzy closure space. Define a map $\mathcal{T}_{\mathcal{C}} : L^X \to L$ by:

$$\mathcal{T}_{\mathcal{C}}(\lambda) = S(\mathcal{C}(\lambda^*), \lambda^*).$$

Then, $\mathcal{T}_{\mathcal{C}}$ is an L-fuzzy topology on X. If $\mathcal{C}$ is stratified, then $\mathcal{T}_{\mathcal{C}}$ is an enriched L-fuzzy topology.

Proof. (T1)

$$\mathcal{T}_{\mathcal{C}}(\top_X) = \bigwedge_{x\in X} (\mathcal{C}(\top_X^*) \to \top_X^*(x)) = \bigwedge_{x\in X} (\bot_X(x) \to \bot_X(x)) = \top,$$

$$\mathcal{T}_{\mathcal{C}}(\bot_X) = \bigwedge_{x\in X} (\mathcal{C}(\bot_X^*) \to \bot_X^*(x)) = \bigwedge_{x\in X} (\top_X(x) \to \top_X(x)) = \top.$$

(T2) By Lemma 2.2(12), we havre

$$\begin{aligned}
\mathcal{T}_{\mathcal{C}}(\lambda \odot \mu) &= \bigwedge_{x\in X} (\mathcal{C}((\lambda \odot \mu)^*)(x) \to (\lambda \odot \mu)^*(x)) \\
&\geq \bigwedge_{x\in X} (\mathcal{C}(\lambda^*)(x) \oplus \mathcal{C}(\mu^*)(x) \to \lambda^*(x) \oplus \mu^*(x)) \\
&\geq \bigwedge_{x\in X} (\mathcal{C}(\lambda^*)(x) \to \lambda^*(x)) \odot \bigwedge_{x\in X} (\mathcal{C}(\mu^*)(x) \to \mu^*(x)) \\
&= \mathcal{T}_{\mathcal{C}}(\lambda) \odot \mathcal{T}_{\mathcal{C}}(\mu).
\end{aligned}$$

(T3) By Lemma 2.2(8),we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{C}}(\bigvee_i \lambda_i) &= \bigwedge_{x\in X} (\mathcal{C}((\bigvee_i \lambda_i)^*)(x) \to (\bigvee_i \lambda_i)^*(x)) \\
&\geq \bigwedge_{x\in X} (\mathcal{C}(\bigwedge_i \lambda_i^*)(x) \to \bigwedge_i \lambda_i^*(x)) \\
&\geq \bigwedge_{x\in X} (\bigwedge_i \mathcal{C}(\lambda_i^*)(x) \to \bigwedge_i \lambda_i^*(x)) \\
&\geq \bigwedge_i \bigwedge_{x\in X} (\mathcal{C}(\lambda_i^*)(x) \to \lambda_i^*(x)) = \bigwedge_i \mathcal{T}_{\mathcal{C}}(\lambda_i).
\end{aligned}$$

(R) By Lemma 2.2 (12), we have

$$\begin{aligned}\mathcal{T}_{\mathcal{C}}(\alpha \odot \lambda) &= \bigwedge_{x\in X} (\mathcal{C}((\alpha \odot \lambda)^*)(x) \to (\alpha \odot \lambda)^*(x)) \\ &= \bigwedge_{x\in X} (\mathcal{C}(\alpha \to \lambda^*)(x) \to (\alpha \to \lambda^*(x)) \\ &\geq \bigwedge_{x\in X} ((\alpha \to \mathcal{C}(\lambda^*)(x)) \to (\alpha \to \lambda^*(x))) \\ &\geq \bigwedge_{x\in X} (\mathcal{C}(\lambda^*)(x) \to \lambda^*(x)) = \mathcal{T}_{\mathcal{C}}(\lambda).\end{aligned}$$

Remark 3.6. Let $(X, \mathcal{C})$ be an L-fuzzy closure space. Define a subset $\tau_{\mathcal{C}} \subset L^X$ by:

$$\tau_{\mathcal{C}} = \{\lambda \in L^X \mid \mathcal{C}(\lambda^*) = \lambda^*\}.$$

Then, $\tau_{\mathcal{C}}$ is an L-topology on X. If $\mathcal{C}$ is stratified, then $\tau_{\mathcal{C}}$ is an enriched L-topology.

Theorem 3.7. (1) If $\mathcal{C}$ is an L-fuzzy closure operator on X, then $\mathcal{C}_{\mathcal{T}_{\mathcal{C}}} \geq \mathcal{C}$ and $\mathcal{C}_{\tau_{\mathcal{C}}} \geq \mathcal{C}$.
(2) If $\mathcal{T}$ is an L-fuzzy topology on X, then $\mathcal{T}_{\mathcal{C}_{\mathcal{T}}} \geq \mathcal{T}$ and $\tau_{\mathcal{C}_{\mathcal{T}}} = \{\lambda \in L^X \mid \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda^*) = \top\}$.
Proof. (1)

$$\begin{aligned}\mathcal{C}_{\mathcal{T}_{\mathcal{C}}}(\lambda)(x) &= \bigwedge_{\mu\in L^X} (\mathcal{T}_{\mathcal{C}}(\mu) \to (S(\lambda, \mu^*) \to \mu^*(x))) \\ &= \bigwedge_{\mu\in L^X} ((\mathcal{T}_{\mathcal{C}}(\mu) \odot S(\lambda, \mu^*)) \to \mu^*(x))) \\ &\geq \bigwedge_{\mu\in L^X} (S(\mathcal{C}(\mu^*), \mu^*) \odot S((\mathcal{C}(\lambda), (\mathcal{C}(\mu^*))) \to \mu^*(x))) \\ &= S(\mathcal{C}(\lambda), \mu^*) \to \mu^*(x) \geq (\mathcal{C}(\lambda)(x) \to \mu^*(x)) \to \mu^*(x) \geq \mathcal{C}(\lambda)(x).\end{aligned}$$

$$\mathcal{C}_{\tau_{\mathcal{C}}}(\lambda) = \bigwedge\{\mu \mid \lambda \leq \mu, \mathcal{C}(\mu) = \mu\} \geq \mathcal{C}(\lambda).$$

(2)

$$\begin{aligned}\mathcal{T}_{\mathcal{C}_\mathcal{T}}(\lambda) &= S\Big(\mathcal{C}_\mathcal{T}(\lambda^*), \lambda^*\Big)\\ &= \bigwedge_{x\in X}\Big(\bigwedge_{\mu\in L^X}(\mathcal{T}(\mu)\to(S(\lambda^*,\mu^*)\to\mu^*))\to\lambda^*(x)\Big)\\ &\geq \bigwedge_{x\in X}\Big((\mathcal{T}(\lambda)\to(S(\lambda^*,\lambda^*)\to\lambda^*))\to\lambda^*(x)\Big)\\ &= \bigwedge_{x\in X}\Big((\mathcal{T}(\lambda)\to\lambda^*(x))\to\lambda^*(x)\Big)\\ &\geq \mathcal{T}(\lambda).\end{aligned}$$

$$\tau_{\mathcal{C}_\mathcal{T}} = \{\lambda\in L^X \mid \mathcal{C}_\mathcal{T}(\lambda^*) = \lambda^*\} = \{\lambda\in L^X \mid \mathcal{T}_{\mathcal{C}_\mathcal{T}}(\lambda^*) = \top\}.$$

Theorem 3.8 Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fuzzy topological spaces and $\phi: X\to Y$ be a map. Then

(1) For each $\lambda\in L^X$,

$$\bigwedge_{\nu\in L^Y}(\mathcal{T}_Y(\nu)\to(\mathcal{T}_X(\phi^{\leftarrow}(\nu))\leq S(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda)),\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda)))$$

(2) If a mapping $\phi:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is continuous, then $\phi:(X,\mathcal{C}_{\mathcal{T}_X})\to(Y,\mathcal{C}_{\mathcal{T}_Y})$ is a C-map.

Proof. (1)

$$\begin{aligned}&S(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda)),\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda)))\\ &= \bigwedge_{y\in Y}(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda))(y)\to\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))(y))\\ &= \bigwedge_{x\in X}(\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda))(\phi(x))\to\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))(\phi(x)))\\ &= \bigwedge_{x\in X}(\mathcal{C}_{\mathcal{T}_X}(\lambda)(x)\to\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))(\phi(x)))\\ &= \bigwedge_{x\in X}(\bigwedge_{\rho\in L^X}((\mathcal{T}_X(\rho)\odot S(\lambda,\rho^*)\to\rho^*(x))\\ &\to \bigwedge_{\nu\in L^Y}((\mathcal{T}_Y(\nu)\odot S(\phi^{\rightarrow}(\lambda),\nu^*)\to\nu^*(\phi(x)))))\\ &= \bigwedge_{x\in X}(\bigwedge_{\nu\in L^Y}((\mathcal{T}_X(\phi^{\leftarrow}(\nu))\odot S(\lambda,\phi^{\leftarrow}(\nu)^*)\to\phi^{\leftarrow}(\nu)^*(x))\\ &\to \bigwedge_{\nu\in L^Y}((\mathcal{T}_Y(\nu)\odot S(\phi^{\rightarrow}(\lambda),\nu^*)\to\nu^*(\phi(x)))))\end{aligned}$$

$$= \bigwedge_{x\in X}\Big(\bigwedge_{\nu\in L^Y}\Big((\mathcal{T}_X(\phi^{\leftarrow}(\nu))\odot S(\lambda,\phi^{\leftarrow}(\nu)^*)\to\phi^{\leftarrow}(\nu)^*(x))$$

$$\to((\mathcal{T}_Y(\nu)\odot S(\phi^{\rightarrow}(\lambda),\nu^*)\to\nu^*(\phi(x))))\Big)$$

$$= \bigwedge_{x\in X}\Big(\bigwedge_{\nu\in L^Y}\Big((\mathcal{T}_Y(\nu)\odot S(\phi^{\rightarrow}(\lambda),\nu^*)\to(\mathcal{T}_X(\phi^{\leftarrow}(\nu))\odot S(\lambda,\phi^{\leftarrow}(\nu)^*)\Big)$$

$$= \bigwedge_{\nu\in L^Y}(\mathcal{T}_Y(\nu)\to\mathcal{T}_X(\phi^{\leftarrow}(\nu)))$$

(2) Since $\mathcal{T}_Y(\nu)\le\mathcal{T}_X(\phi^{\leftarrow}(\nu))$, by (1), $\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_X}(\lambda))\le\mathcal{C}_{\mathcal{T}_Y}(\phi^{\rightarrow}(\lambda))$.

Theorem 3.9 Let $(X,\mathcal{C}_X)$ and $(Y,\mathcal{C}_Y)$ be fuzzy closure spaces and $\phi:X\to Y$ be a map. Then

(1) $S(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*)),\phi^{\leftarrow}(\mathcal{C}_Y(\lambda^*)))\le\mathcal{T}_{\mathcal{C}_Y}(\lambda)\to\mathcal{T}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda))$ for each $\lambda\in L^Y$.

(2) If a mapping $\phi:(X,\mathcal{C}_X)\to(Y,\mathcal{C}_Y)$ is an C-map, then $\phi:(X,\mathcal{T}_{\mathcal{C}_X})\to(Y,\mathcal{T}_{\mathcal{C}_Y})$ is continuous.

Proof. (1) By Lemma 2.2, we have

$$\mathcal{T}_{\mathcal{C}_Y}(\lambda)\to\mathcal{T}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda))$$

$$= \bigwedge_{y\in Y}\Big(\mathcal{C}_Y(\lambda^*)(y)\to\lambda^*(y)\Big)\to\bigwedge_{x\in X}\Big(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*))(x)\to\phi^{\leftarrow}(\lambda^*)(x)\Big)$$

$$\ge \bigwedge_{x\in X}\Big(\phi^{\leftarrow}(\mathcal{C}_Y(\lambda^*))(x)\to\phi^{\leftarrow}(\lambda^*)(x)\Big)\to\bigwedge_{x\in X}\Big(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*))(x)\to\phi^{\leftarrow}(\lambda^*)(x)\Big)$$

$$\ge \bigwedge_{x\in X}\Big(\mathcal{C}_X(\phi^{\leftarrow}(\lambda^*))(x)\to\phi^{\leftarrow}(\mathcal{C}_Y(\lambda^*))(x)\Big)$$

(2) Let $\phi^{\rightarrow}(\mathcal{C}_X(\lambda))\le\mathcal{C}_Y(\phi^{\rightarrow}(\lambda))$. Then, put $\lambda=\phi^{\leftarrow}(\mu)$,

$$\mathcal{C}_X(\phi^{\leftarrow}(\mu))\le\phi^{\leftarrow}(\phi^{\rightarrow}(\mathcal{C}_X(\phi^{\leftarrow}(\mu))))\le\phi^{\leftarrow}(\mathcal{C}_Y(\phi^{\rightarrow}(\phi^{\leftarrow}(\mu))))\le\phi^{\leftarrow}(\mathcal{C}_Y(\mu)).$$

Thus, by (1), if $\mathcal{C}_X(\phi^{\leftarrow}(\lambda))\le\phi^{\leftarrow}(\mathcal{C}_Y(\lambda))$, then $\mathcal{T}_{\mathcal{C}_Y}(\lambda)\le\mathcal{T}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda))$.

Example 3.10. Let $(L=[0,1],\odot,\to)$ be a complete residuated lattice defined by

$$x\odot y=(x+y-1)\vee 0,\quad x\to y=(1-x+y)\wedge 1.$$

$$x\oplus y=(x+y)\wedge 1,\quad x^*=1-x.$$

Let $X=\{x,y,z\}$ be a set and $\rho,\rho\odot\rho\in L^X$ such that

$$\rho(x)=0.1,\rho(y)=0.8,\rho(z)=0.7,$$

$$\rho\odot\rho(x)=0,\rho\odot\rho(y)=0.6,\rho\odot\rho(z)=0.4.$$

(1) We define an L-fuzzy topology $\mathcal{T} : L^X \to L$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ \lambda = 0_X, \\ 0.6, & \text{if } \lambda = \rho, \\ 0.3, & \text{if } \lambda = \rho \odot \rho, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 3.3, we obtain an L-fuzzy closure operator $\mathcal{C}_{\mathcal{T}} : L^X \to L^X$ as follows

$$\begin{aligned} &\mathcal{C}_{\mathcal{T}}(\lambda) = \textstyle\bigwedge_{\mu \in L^X} (\mathcal{T}(\mu) \to (S(\lambda, \mu^*) \to \mu^*)) \\ &= (S(\lambda, 0_X) \to 0_X) \wedge (0.6 \to (S(\lambda, \rho^*) \to \rho^*)) \wedge (0.3 \to (S(\lambda, \rho^* \oplus \rho^*) \to \rho^* \oplus \rho^*)). \end{aligned}$$

For $\lambda_1 = (0.9, 0.4, 0.2)$ and $\lambda_2 = (1, 0, 0)$,

$$\begin{aligned} \mathcal{C}_{\mathcal{T}}(\lambda_1) &= (S(\lambda, 0_X) \to 0_X) \wedge (0.6 \to (S(\lambda, \rho^*) \to \rho^*)) \\ &\wedge (0.3 \to (S(\lambda, \rho^* \oplus \rho^*) \to \rho^* \oplus \rho^*)) = (0.9, 0.8, 0.9) \\ \mathcal{C}_{\mathcal{T}}(\lambda_2) &= 0.6 \to (0.7 \to \rho^*) = (1, 0.7, 0.8), \end{aligned}$$

$$\begin{aligned} \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda_1^*) &= S(\mathcal{C}_{\mathcal{T}}(\lambda_1), \lambda_1) = 0.3 \geq \mathcal{T}(\lambda_1^*) = 0 \\ \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}(\lambda_2^*) &= S(\mathcal{C}_{\mathcal{T}}(\lambda_2), \lambda_2) = 0 = \mathcal{T}(\lambda_2^*). \end{aligned}$$

(2) We define an L-fuzzy closure operator $\mathcal{C} : L^X \to L^X$ as follows

$$\mathcal{C}(\lambda) = \begin{cases} 0_X, & \text{if } \lambda = 0_X, \\ \rho, & \text{if } 0_X \neq \lambda \leq \rho, \\ \rho \oplus \rho, & \text{if } \rho \not\geq \lambda \leq \rho \oplus \rho, \\ 1_X, & \text{otherwise.} \end{cases}$$

$\mathcal{C}$ is not stratified because

$$\mathcal{C}(0.9 \to \rho) = \mathcal{C}((0.2, 0.9, 0.8)) = (1, 1, 1) \not\leq 0.9 \to \mathcal{C}(\rho) = (0.2, 0.9, 0.8).$$

From Theorem 3.5, we obtain an L-fuzzy topology $\mathcal{T}_{\mathcal{C}} : L^X \to L$ as follows

$$\mathcal{T}_{\mathcal{C}}(\lambda) = \begin{cases} 1_X, & \text{if } \lambda = 1_X, \\ S(\rho, \lambda^*), & \text{if } \lambda \geq \rho^*, \\ S(\rho \oplus \rho, \lambda^*), & \text{if } \rho^* \not\leq \lambda \geq \rho^* \odot \rho^*, \\ S(1_X, \lambda^*), & \text{otherwise.} \end{cases}$$

Moreover,

$$\begin{aligned} &\mathcal{C}_{\mathcal{T}_{\mathcal{C}}}(\lambda) = (S(\lambda, 0_X) \to 0_X) \wedge (\wedge_{\mu \geq \rho^*} S(\rho, \mu^*) \to (S(\lambda, \mu^*) \to \mu^*) \\ &\wedge (\wedge_{\rho^* \not\leq \mu \geq \rho^* \odot \rho^*} S(\rho \oplus \rho, \mu^*) \to (S(\lambda, \mu^*) \to \mu^*) \\ &\wedge (\wedge_{otherwise} S(1_X, \mu^*) \to (S(\lambda, \mu^*) \to \mu^*) \end{aligned}$$

$$\mathcal{C}_{\mathcal{T}_{\mathcal{C}}}(\lambda_1) = (S(\lambda_1, 0_X) \to 0_X) \wedge (S(\rho,\rho) \to (S(\lambda,\rho) \to \rho)$$
$$\wedge (S(\rho\oplus\rho, \rho\oplus\rho) \to (S(\lambda, \rho\oplus\rho) \to \rho\oplus\rho)$$
$$\wedge (S(1_X, \rho\oplus\rho) \to (S(\lambda, \rho\oplus\rho) \to \rho\oplus\rho)$$
$$= (0.9, 0.9, 0.9) \wedge (0.9, 1, 1) \wedge (0.9, 1, 1) \wedge (1, 1, 1) = (0.9, 0.9, 0.9)$$
$$\leq \mathcal{C}(\lambda_1) = (1, 1, 1).$$
$$\mathcal{C}_{\mathcal{T}_{\mathcal{C}}}(\lambda_2) = (1, 0.9, 0.9) \leq \mathcal{C}(\lambda_2) = (1, 1, 1).$$

4. L-fuzzy uniformities induced by L-fuzzy closure spaces

Lemma 4.1. For every $\rho \in L^X$, we define $u_\rho, u_\rho^{-1} : L^{X\times X} \to L$ by

$$u_\rho(x, y) = \rho(x) \to \rho(y)$$

$$u_\rho^{-1}(x, y) = u_\rho(y, x).$$

then we have the following statements

(1) $1_{X\times X} = u_{0_X} = u_{1_X}$,

(2) $1_\triangle \leq u_\rho$,

(3) For every $u_\rho \in L^{X\times X}$, $u_\rho \circ u_\rho = u_\rho$.

(4) $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\odot\rho_2}$,

(5) $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\oplus\rho_2}$

(6) $u_\rho^{-1} = u_{\rho^*}$

(7) $u^{-1}_{\rho_1\odot\rho_2} = u_{\rho_1^*\oplus\rho_2^*}$

Proof.

(1) Since $u_{0_X}(x, y) = 0_X(x) \to 0_X(y) = 1$ and $u_{1_X}(x, y) = 1_X(x) \to 1_X(y) = 1$, we have

$$1_{X\times X} = u_{0_X} = u_{1_X}.$$

(2) Since $u\rho(x, x) = \rho(x) \to \rho(x) = 1$, then $1_\triangle \leq u_\rho$.

(3)

$$u_\rho \circ u_\rho(x, z) = \bigvee_{y\in X}(u_\rho(x, y) \odot u_\rho(y, z))$$
$$= \bigvee_{y\in X}((\rho(x) \to \rho(y)) \odot (\rho(y) \to \rho(z)))$$
$$= \rho(x) \to \rho(z) = u_\rho(x, z)$$

(4) Since $(\rho_1(x) \to \rho_1(y)) \odot (\rho_2(x) \to \rho_2(y)) \leq (\rho_1(x) \odot \rho_2(x) \to \rho_1(y) \odot \rho_2(y))$, we have $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\odot\rho_2}$.

(5) Since $(\rho_1(x) \to \rho_1(y)) \odot (\rho_2(x) \to \rho_2(y)) = (\rho_1^*(y) \to \rho_1^*(x)) \odot (\rho_2^*(y) \to \rho_2^*(x)) \leq (\rho_1^*(y) \odot \rho_2^*(y) \to \rho_1^*(x) \odot \rho_2^*(x)) = (\rho_1^*(x) \odot \rho_2^*(x))^* \to (\rho_1^*(y) \odot \rho_2^*(y))^*$, we have $u_{\rho_1} \odot u_{\rho_2} \leq u_{\rho_1\oplus\rho_2}$.

(6) $u_{\rho}^{-1}(x,y) = \rho(y) \to \rho(x) = \rho^*(x) \to r^*(y) = u_{\rho^*}(x,y)$.

(7) $u^{-1}_{\rho_1 \odot \rho_2}(x,y) = (\rho_1 \odot \rho_2)(y) \to (\rho_1 \odot \rho_2)(x) = (\rho_1 \odot \rho_2)^*(x) \to (\rho_1 \odot \rho_2)^*(y) = u_{\rho_1^* \oplus \rho_2^*}$.

In the following theorem, we obtain an L-fuzzy quasi-uniform structure from an L-fuzzy closure operator.

Theorem 4.2 Let $(X, \mathcal{C})$ be an L-fuzzy closure space. Define $\mathcal{U}_{\mathcal{C}} : L^{X\times X} \to L$ by

$$\mathcal{U}_{\mathcal{C}}(u) = \bigvee \{ \bigvee_{x\in X} \odot_{i=1}^{n} \mathcal{C}^*(\lambda_i)(x) \mid \odot_{i=1}^{n} u_{\lambda_i} \le u \},$$

where $\bigvee$ is taken over every finite family $\{u_{\lambda_i} \mid i = 1, 2, 3, ..., n\}$. Then $\mathcal{U}_{\mathcal{C}}$ is an L-fuzzy quasi-uniformity on X.

Proof. (QU1) Since $\mathcal{C}(0_X) = 0_X$, there exists $1_{X\times X} = u_{0_X} \in L^{X\times X}$ such that $\mathcal{U}_{\mathcal{C}}(1_{X\times X}) = 1$.

(QU2) It is trivial from the definition of $\mathcal{U}_{\mathcal{C}}$.

(QU3) Suppose there exist $u, v \in L^{X\times X}$ such that

$$\mathcal{U}_{\mathcal{C}}(u \odot v) \not\ge \mathcal{U}_{\mathcal{C}}(u) \odot \mathcal{U}_{\mathcal{C}}(v).$$

There exist two finite families $\{\lambda_i \in L^X \mid \odot_{i=1}^{m} u_{\lambda_i} \le u\}$ and $\{\rho_j \in L^X \mid \odot_{j=1}^{n} u_{\rho_j} \le v\}$ such that

$$\mathcal{U}_{\mathcal{C}}(u \odot v) \not\ge \bigvee_{x\in X} (\odot_{i=1}^{m} \mathcal{C}^*(\lambda_i)(x)) \odot \bigvee_{x\in X} (\odot_{j=1}^{n} \mathcal{C}^*(\rho_i)(x)).$$

On the other hand, since $u \odot v \ge (\odot_{i=1}^{m} u_{\lambda_i}) \odot (\odot_{j=1}^{n} u_{\rho_j})$, we have

$$\mathcal{U}_{\mathcal{C}}(u \odot v) \ge \bigvee_{x\in X} (\odot_{i=1}^{m} \mathcal{C}^*(\lambda_i)(x)) \odot \bigvee_{x\in X} (\odot_{j=1}^{n} \mathcal{C}^*(\rho_i)(x)).$$

It is a contradiction.

(QU4) Let $\mathcal{U}_{\mathcal{C}}(u) \ne 0$. Then there exists a finite family $\{\lambda_i \in L^X \mid \odot_{i=1}^{m} u_{\lambda_i} \le u\}$ such that

$$\mathcal{U}_{\mathcal{C}}(u) = \bigvee_{x\in X} (\odot_{i=1}^{m} \mathcal{C}^*(\lambda_i)(x)) \ne 0.$$

Since $u_{\lambda_i} \ge \top_{\triangle}$ from Lemma 4.1(2),

$$\top_{\triangle} \le \odot_{i=1}^{m} u_{\lambda_i} \le u.$$

(QU5) Suppose there exists $u \in L^{X\times X}$ such that

$$\bigvee \{\mathcal{U}_{\mathcal{C}}(v) \mid v \circ v \le u\} \not\ge \mathcal{U}_{\mathcal{C}}(u).$$

There exists a finite family $\{\rho_i \in L^{X\times X} \mid \odot_{i=1}^{m} u_{\rho_i} \le u\}$ such that

$$\bigvee \{\mathcal{U}_{\mathcal{C}}(v) \mid v \circ v \le u\} \not\ge \bigvee_{x\in X} (\odot_{i=1}^{m} \mathcal{C}^*(\rho_i)(x)).$$

On the other hand, since $u_{\rho_i} \circ u_{\rho_i} = u_{\rho_i}$ for each $i \in \{1, ..., m\}$ from Lemma 4.1(3), we have $(\odot_{i=1}^m u_{\rho_i}) \circ (\odot_{i=1}^m u_{\rho_i}) \leq \odot_{i=1}^m u_{\rho_i}$ from

$$\begin{aligned}
&\bigvee_{y\in X}((\odot_{i=1}^m u_{\rho_i}(x,y)) \odot (\odot_{i=1}^m u_{\rho_i}(y,z)))\\
&= \bigvee_{y\in X}((\odot_{i=1}^m (\rho_i(x) \to \rho_i(y)) \odot (\odot_{i=1}^m (\rho_i(y) \to \rho_i(z))))\\
&= \bigvee_{y\in X}((\odot_{i=1}^m (\rho_i(x) \to \rho_i(y)) \odot (\rho_i(y) \to \rho_i(z))))\\
&\leq \odot_{i=1}^m (\rho_i(x) \to \rho_i(z)).
\end{aligned}$$

Put $v = \odot_{i=1}^m u_{\rho_i}$. Since $\odot_{i=1}^m u_{\rho_i} \leq v$, $v \circ v \leq u$ and

$$\bigvee\{\mathcal{U}_{\mathcal{C}}(w) \mid w \circ w \leq u\} \geq \mathcal{U}_{\mathcal{C}}(v) \geq \bigvee_{x\in X}(\odot_{i=1}^m \mathcal{C}^*(\rho_i)(x)).$$

It is a contradiction. Thus $\bigvee\{\mathcal{U}_{\mathcal{C}}(v) \mid v \circ v \leq u\} \geq \mathcal{U}_{\mathcal{C}}(u)$. Hence $\mathcal{U}_{\mathcal{C}}$ is an L-fuzzy quasi-uniformity on X.

Theorem 4.3. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-fuzzy closure spaces. Let $\phi : X \to Y$ be a surjective map. Then the following properties hold.

(1) $\mathcal{U}_{\mathcal{C}_Y}(v) \to \mathcal{U}_{\mathcal{C}_X}((\phi \times \phi)^{\leftarrow}(v)) \geq \bigwedge\{\odot_{i=1}^m S(\mathcal{C}_X(\phi^{\leftarrow}(\rho_i)), \phi^{\leftarrow}(\mathcal{C}_Y(\rho_i))) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}$ for each $v \in L^{X\times X}$.

(2) Let $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ be a surjective C-map. Then $\phi : (X, \mathcal{U}_{\mathcal{C}_X}) \to (Y, \mathcal{U}_{\mathcal{C}_Y})$ is fuzzy uniformly continuous.

Proof. Since $v(\phi(x), \phi(y)) = (\phi \times \phi)^{\leftarrow}(v)(x, y)$ and $(\phi \times \phi)^{\leftarrow}(v_\lambda)(x, y) = v_{\phi^{\leftarrow}(\lambda)}(x, y)$, we have

$$\begin{aligned}
&\mathcal{U}_{\mathcal{C}_Y}(v) \to \mathcal{U}_{\mathcal{C}_X}((\phi \times \phi)^{\leftarrow}(v))\\
&= \bigvee\{\odot_{i=1}^m \bigvee_{y\in Y} \mathcal{C}_Y^*(\rho_i)(y) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}\\
&\to \bigvee\{\odot_{j=1}^n \bigvee_{x\in X} \mathcal{C}_X^*(\mu_i)(x) \mid \odot_{j=1}^n u_{\mu_j} \leq (\phi \times \phi)^{\leftarrow}(v)\}\\
&\geq \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_Y^*(\rho_i)(\phi(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}\\
&\to \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x) \mid \odot_{j=1}^n (\phi \times \phi)^{\leftarrow}(\rho_i) \leq (\phi \times \phi)^{\leftarrow}(v)\}\\
&\geq \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_Y^*(\rho_i)(\phi(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}\\
&\to \bigvee\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x) \mid \odot_{j=1}^n v_{\phi^{\leftarrow}(\rho_i)} \leq (\phi \times \phi)^{\leftarrow}(v)\}\\
&\geq \bigwedge\{\odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_Y^*(\rho_i)(\phi(x)) \to \odot_{i=1}^m \bigvee_{x\in X} \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}\\
&\geq \bigwedge\{\odot_{i=1}^m \bigwedge_{x\in X} (\mathcal{C}_Y^*(\rho_i)(\phi(x)) \to \mathcal{C}_X^*(\phi^{\leftarrow}(\rho_i))(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}\\
&= \bigwedge\{\odot_{i=1}^m \bigwedge_{x\in X} (\mathcal{C}_X(\phi^{\leftarrow}(\rho_i))(x) \to \mathcal{C}_Y(\rho_i)(\phi(x)) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}.\\
&= \bigwedge\{\odot_{i=1}^m S(\mathcal{C}_X(\phi^{\leftarrow}(\rho_i)), \phi^{\leftarrow}(\mathcal{C}_Y(\rho_i))) \mid \odot_{i=1}^m v_{\rho_i} \leq v\}
\end{aligned}$$

(2) Since $\mathcal{C}_X(\phi^{\leftarrow}(\rho_i)) \leq \phi^{\leftarrow}(\mathcal{C}_Y(\rho_i))$, by (1), $\mathcal{U}_{\mathcal{C}_Y}(v) \leq \mathcal{U}_{\mathcal{C}_X}((\phi \times \phi)^{\leftarrow}(v))$.

Theorem 4.4 Let $(X,\mathcal{U})$ be an L-fuzzy quasi-uniform space. Define $\mathcal{C}_{\mathcal{U}} : L^X \to L^X$ by

$$\mathcal{C}_{\mathcal{U}}(\lambda)(x) = \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big).$$

Then $\mathcal{C}_{\mathcal{U}}$ is a stratified L-fuzzy closure operator on X.

Proof. (C1)

$$\mathcal{C}_{\mathcal{U}}(\perp_X) = \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \perp_X(y))\Big) = (\bigvee_{u} \mathcal{U}(u))^* = \perp_X.$$

(C2)

$$\begin{aligned} S(\lambda, \mathcal{C}_{\mathcal{U}}(\lambda)) &= \bigwedge_{x\in X} (\lambda(x) \to \mathcal{C}_{\mathcal{U}}(\lambda)(x)) \\ &= \bigwedge_{x\in X} \Big(\lambda(x) \to \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big)\Big) \\ &= \bigwedge_{x\in X} \bigwedge \Big((\lambda(x) \odot \mathcal{U}(u)) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big)\Big) \\ &\geq \bigwedge_{x\in X} \bigwedge \Big((\lambda(x) \odot \mathcal{U}(u)) \to (u(x,x) \odot \lambda(x))\Big)\Big) \\ &\geq \bigwedge_{x\in X} \bigwedge \Big(\mathcal{U}(u) \to u(x,x)\Big) = \top \end{aligned}$$

That is, $\mathcal{C}_{\mathcal{U}}(\lambda) \geq \lambda$ for each $\lambda \in L^X$.

(C3) and, by Lemma 3.2, $\mathcal{C}_{\mathcal{U}}$ is stratified from

$$\begin{aligned} &S(\mathcal{C}_{\mathcal{U}}(\lambda), \mathcal{C}_{\mathcal{U}}(\mu)) = \bigwedge_{x\in X} \mathcal{C}_{\mathcal{U}}(\lambda)(x) \to \mathcal{C}_{\mathcal{U}}(\mu)(x) \\ &= \bigwedge_{x\in X} \Big(\bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big) \to \bigwedge \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \mu(y))\Big) \\ &\geq \bigwedge_{x\in X} \bigwedge \bigwedge_{y\in X} \Big(u(x,y) \odot \lambda(y) \to u(x,y) \odot \mu(y)\Big) \\ &\geq \bigwedge_{y\in X} (\lambda(y) \to \mu(y)) = S(\lambda,\mu) \end{aligned}$$

(C4), Since

$$\begin{aligned} &(a \odot c) \odot (b \oplus d)^* = (a \odot b^*) \odot (c \odot d^*) \\ &\Leftrightarrow (a \odot c) \to (b \oplus d) = (a \to b) \oplus (c \to d), \end{aligned}$$

$$
\begin{aligned}
&\mathcal{C}_{\mathcal{U}}(\lambda)(x) \oplus \mathcal{C}_{\mathcal{U}}(\mu)(x) \\
&= \bigwedge_{u} \Big(\mathcal{U}(u) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big) \oplus \bigwedge_{v} \Big(\mathcal{U}(v) \to \bigvee_{y\in X} (v(x,y) \odot \mu(y))\Big) \\
&= \bigwedge_{u,v} \Big(\mathcal{U}(u) \odot \mathcal{U}(v) \to \bigvee_{y\in X} (u(x,y) \odot \lambda(y))\Big) \oplus \bigvee_{y\in X} (v(x,y) \odot \mu(y))\Big) \\
&= \bigwedge_{u,v} \Big(\mathcal{U}(u \odot v) \to \bigvee_{y\in X} ((u \odot v)(x,y) \odot (\lambda \oplus \mu)(y))\Big) \\
&\geq \mathcal{C}_{\mathcal{U}}(\lambda \oplus \mu).
\end{aligned}
$$

Theorem 4.5. Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be L-fuzzy quasi-uniform spaces and t $\phi : X \to Y$ be a surjective map. Then we have the following properties.

(1) $\bigwedge \Big(\mathcal{U}_Y(v) \to \mathcal{U}_X((\phi \times \phi)^{\leftarrow}(v))\Big) \leq S(\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho)), \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho)))$.

(2) If $\phi : (X,\mathcal{U}_X) \to (Y,\mathcal{U}_Y)$ is fuzzy uniformly continuous, then $\phi : (X,\mathcal{C}_{\mathcal{U}_X}) \to (Y,\mathcal{C}_{\mathcal{U}_Y})$ is a C-map.

Proof. (1)

$$
\begin{aligned}
&S(\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho)), \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho))) \\
&= \textstyle\bigwedge_{x\in X} \Big(\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho))(x) \to \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho))(x)\Big) \\
&= \textstyle\bigwedge_{x\in X} \Big(\bigwedge \Big(\mathcal{U}_X(u) \to \bigvee_{z\in X}(u(x,z) \odot \phi^{\leftarrow}(\rho)(z))\Big) \\
&\to \textstyle\bigwedge \Big(\mathcal{U}_Y(v) \to \bigvee_{y\in Y}(v(\phi(x),y) \odot \rho(y))\Big)\Big) \\
&\geq \textstyle\bigwedge_{x\in X} \Big(\bigwedge \Big(\mathcal{U}_X((\phi \times \phi)^{\leftarrow}(v)) \to \bigvee_{z\in X}((\phi \times \phi)^{\leftarrow}(v)(x,z) \odot \phi^{\leftarrow}(\rho)(z))\Big) \\
&\to \textstyle\bigwedge \Big(\mathcal{U}_Y(v) \to \bigvee_{y\in Y}(v(\phi(x),y) \odot \rho(y))\Big)\Big) \\
&\geq \textstyle\bigwedge_{x\in X} \bigwedge_{v} \Big(\Big(\mathcal{U}_X((\phi \times \phi)^{\leftarrow}(v)) \to \bigvee_{z\in X}((\phi \times \phi)^{\leftarrow}(v)(x,z) \odot \phi^{\leftarrow}(\rho)(z))\Big) \\
&\to \Big(\mathcal{U}_Y(v) \to \bigvee_{z\in X}(v(\phi(x),\phi(z)) \odot \rho(\phi(z)))\Big)\Big) \\
&= \textstyle\bigwedge \Big(\mathcal{U}_Y(v) \to \mathcal{U}_X((\phi \times \phi)^{\leftarrow}(v))\Big).
\end{aligned}
$$

(2) Let $\mathcal{U}_Y(v) \leq \mathcal{U}_X((\phi \times \phi)^{\leftarrow}(v))$ for $v \in L^{X\times X}$. By (1), $\mathcal{C}_{\mathcal{U}_X}(\phi^{\leftarrow}(\rho)) \leq \phi^{\leftarrow}(\mathcal{C}_{\mathcal{U}_Y}(\rho))$ for all $\rho \in L^Y$.

Example 4.6. Let $(L = [0,1], \odot, \to)$ be a complete residuated lattice as in Example 3.10. Let $X = \{x,y,z\}$ be a set and $\rho, \rho \odot \rho \in L^X$ such that

$$\rho(x) = 0.4, \rho(y) = 0.6, \rho(z) = 0.7,$$

$$\rho \oplus \rho(x) = 0.8, \rho \oplus \rho(y) = 1, \rho \oplus \rho(z) = 1.$$

We define an L-fuzzy closure operator $\mathcal{C} : L^X \to L^X$ as follows

$$\mathcal{C}(\lambda) = \begin{cases} 0_X, & \text{if } \lambda = 0_X, \\ \rho, & \text{if } 0_X \neq \lambda \leq \rho, \\ \rho \oplus \rho, & \text{if } \rho \not\geq \lambda \leq \rho \oplus \rho, \\ 1_X, & \text{otherwise.} \end{cases}$$

From Lemma 4.1, we obtain

$$u_\rho = \begin{pmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 1 \\ 0.7 & 0.9 & 1 \end{pmatrix} u_{\rho\oplus\rho} = \begin{pmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 1 \\ 0.8 & 1 & 1 \end{pmatrix}.$$

$$u_\rho \odot u_\rho = \begin{pmatrix} 1 & 1 & 1 \\ 0.6 & 1 & 1 \\ 0.4 & 0.8 & 1 \end{pmatrix}.$$

From Theorem 4.2, we obtain an L-fuzzy quasi-uniformity $\mathcal{U}_\mathcal{C} : L^{X\times X} \to L$ as follows

$$\mathcal{U}_\mathcal{C}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X},\ \lambda = 0_X, \\ 0.6, & \text{if } u_\rho \leq u \neq 1_{X\times X}, \\ 0.2, & \text{if } u_{\rho\odot\rho} \leq u \not\geq u_\rho, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, from Theorem 4.4, we obtain a stratified L-fuzzy closure operator $\mathcal{C}_{\mathcal{U}_\mathcal{C}} : L^X \to L^X$ as follows

$$\begin{aligned} \mathcal{C}_{\mathcal{U}_\mathcal{C}}(\lambda)(x) &= (\textstyle\bigvee_{y\in X} \lambda(y)) \wedge (0.6 \to (\lambda(x) \vee \lambda(y) \vee \lambda(z))) \\ &\wedge (0.2 \to (\lambda(x) \vee \lambda(y) \vee \lambda(z))) \\ \mathcal{C}_{\mathcal{U}_\mathcal{C}}(\lambda)(y) &= (\textstyle\bigvee_{y\in X} \lambda(y)) \wedge (0.6 \to ((0.8 \odot \lambda(x)) \vee \lambda(y) \vee \lambda(z))) \\ &\wedge (0.2 \to ((0.6 \odot \lambda(x)) \vee \lambda(y) \vee \lambda(z))) \\ \mathcal{C}_{\mathcal{U}_\mathcal{C}}(\lambda)(z) &= (\textstyle\bigvee_{y\in X} \lambda(y)) \wedge (0.6 \to ((0.7 \odot \lambda(x)) \vee (0.9 \odot \lambda(y)) \vee \lambda(z))) \\ &\wedge (0.2 \to ((0.4 \odot \lambda(x)) \vee (0.8 \odot \lambda(y)) \vee \lambda(z))). \end{aligned}$$

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SOFT L-NEIGHBORHOOD SYSTEMS AND SOFT L-FUZZY QUASI-UNIFORM SPACES

JUNG MI KO, YONG CHAN KIM*

Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea

Abstract. In this paper, we introduce soft L-neighborhood spaces and investigate the topological properties of soft L-fuzzy quasi-uniformity in complete residuated lattices. We obtain soft L-fuzzy topologies and soft L-neighborhood spaces induced by soft L-fuzzy quasi-uniformity. Moreover, we study the relations among soft L-fuzzy topology, soft L-neighborhood system and soft L-fuzzy quasi-uniformity.

Keywords: Complete residuated lattice, soft L-neighborhood systems, soft L-fuzzy topologies, soft L-fuzzy quasi-uniformities, (uniformly) continuous soft maps

2000 AMS Subject Classification: 54A40; 03E72; 03G10; 06A15

1. Introduction

Molodtsov [19] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,3,6,11,15,16,28]. Pawlak's rough set [20,21] can be viewed as a special case of soft rough sets [6]. The topological structures of soft sets have been developed by many researchers [3,25,26,30,31].

On the other hand, Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [2,8,12-14,23,24,27].

Kim [13,14] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where L is a complete residuated lattice. He introduced soft L-fuzzy topologies, soft L-fuzzy quasi-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices.

In this paper, we introduce soft L-neighborhood spaces and investigate the topological properties of soft L-fuzzy quasi-uniformity in complete residuated lattices. We obtain soft L-fuzzy topologies and soft L-neighborhood spaces induced by soft L-fuzzy quasi-uniformity. Moreover, we study

* Corresponding author

the relations among soft L-fuzzy topology, soft L-neighborhood system and soft L-fuzzy quasi-uniformity. We give their examples.

2. Preliminaries

Definition 2.1. [2,8,9,30] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \rightarrow, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution $x^* = x \rightarrow 0$ which is defined by $x \oplus y = (x^* \odot y^*)^*$ unless otherwise specified and we denote $L_0 = L - \{0\}$.

Lemma 2.2. [2,8,9,30] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \rightarrow x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$,

(7) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,

(8) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,

(9) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$,

(10) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,

(11) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(12) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(13) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,

(14) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)$,

(15) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,

(16) $(x \oplus z) \odot y \leq x \oplus (y \odot z)$,

(17) $x \rightarrow y = y^* \rightarrow x^*$.

Definition 2.3. [13,14] Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F, A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \to L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 2.4. [13,14] Let (F, A) and (G, A) be two fuzzy soft sets over a common universe X.

(1) (F, A) is a fuzzy soft subset of (G, A), denoted by $(F, A) \leq (G, A)$ if $F(a) \leq G(a)$, for each $a \in A$.

(2) $(F, A) \wedge (G, A) = (F \wedge G, A)$ if $(F \wedge G)(a) = F(a) \wedge G(a)$ for each $a \in A$.

(3) $(F, A) \vee (G, A) = (F \vee G, A)$ if $(F \vee G)(a) = F(a) \vee G(a)$ for each $a \in A$.

(4) $(F, A) \odot (G, A) = (F \odot G, A)$ if $(F \odot G)(a) = F(a) \odot G(a)$ for each $a \in A$.

(5) $(F, A)^* = (F^*, A)$ if $F^*(a) = (F(a))^*$ for each $a \in A$.

(6) $(F, A) \oplus (G, A) = (F \oplus G, A)$ if $(F \oplus G)(a) = (F^*(a) \odot G^*(a))^*$ for each $a \in A$.

(7) $\alpha \odot (F, A) = (\alpha \odot F, A)$ for each $\alpha \in L$.

Definition 2.5. [13,14] Let $S(X, A)$ and $S(Y, B)$ be the families of all fuzzy soft sets over X and Y, respectively. The mapping $f_\phi : S(X, A) \to S(Y, B)$ is a soft mapping where $f : X \to Y$ and $\phi : A \to B$ are mappings.

(1) The image of $(F, A) \in S(X, A)$ under the mapping f_ϕ is denoted by $f_\phi((F, A)) = (f_\phi(F), B)$ where

$$f_\phi(F)(b)(y) = \begin{cases} \bigvee_{a \in \phi^{-1}(\{b\})} f^{\rightarrow}(F(a))(y), & \text{if } \phi^{-1}(\{b\}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The inverse image of $(G, B) \in S(Y, B)$ under the mapping f_ϕ is denoted by $f_\phi^{-1}((G, B)) = (f_\phi^{-1}(G), A)$ where

$$f_\phi^{-1}(G)(a)(x) = f^{\leftarrow}(G(\phi(a)))(x), \ \forall a \in A, x \in X.$$

(3) The soft mapping $f_\phi : S(X, A) \to S(Y, B)$ is called injective (resp. surjective, bijective) if f and ϕ are both injective (resp. surjective, bijective).

Lemma 2.6. [13,14] Let $f_\phi : S(X, A) \to S(Y, B)$ be a soft mapping. Then we have the following properties. For $(F, A), (F_i, A) \in S(X, A)$ and $(G, B), (G_i, B) \in S(Y, B)$,

(1) $(G, B) \geq f_\phi(f_\phi^{-1}((G, B)))$ with equality if f is surjective,

(2) $(F, A) \leq f_\phi^{-1}(f_\phi((F, A)))$ with equality if f is injective,

(3) $f_\phi^{-1}((G, B)^*) = (f_\phi^{-1}((G, B)))^*$,

(4) $f_\phi^{-1}(\bigvee_{i \in I}(G_i, B)) = \bigvee_{i \in I} f_\phi^{-1}((G_i, B))$,

(5) $f_\phi^{-1}(\bigwedge_{i \in I}(G_i, B)) = \bigwedge_{i \in I} f_\phi^{-1}((G_i, B))$,

(6) $f_\phi(\bigvee_{i \in I}(F_i, A)) = \bigvee_{i \in I} f_\phi((F_i, A))$,

(7) $f_\phi(\bigwedge_{i \in I}(F_i, A)) \leq \bigwedge_{i \in I} f_\phi((F_i, A))$ with equality if f is injective,

(8) $f_\phi^{-1}((G_1,B)\odot(G_2,B)) = f_\phi^{-1}((G_1,B))\odot f_\phi^{-1}((G_2,B))$,

(9) $f_\phi^{-1}((G_1,B)\oplus(G_2,B)) = f_\phi^{-1}((G_1,B))\oplus f_\phi^{-1}((G_2,B))$,

(10) $f_\phi((F_1,A)\odot(F_2,A)) \le f_\phi((F_1,A))\odot f_\phi((F_2,A))$ with equality if f is injective,

(11) $f_\phi((F_1,A)\oplus(F_2,A)) \le f_\phi((F_1,A))\oplus f_\phi((F_2,A))$ with equality if f is injective.

Definition 2.7. [13,14] A map $\mathcal{T} : S(X,A) \to L$ is called a soft L-fuzzy topology on X if it satisfies the following conditions.

(ST1) $\mathcal{T}((0_X,A)) = \mathcal{T}((1_X,A)) = 1$, where $0_X(a)(x) = 0, 1_X(a)(x) = 1$ for all $a \in A, x \in X$,

(ST2) $\mathcal{T}((F,A)\odot(G,A)) \ge \mathcal{T}((F,A))\odot\mathcal{T}((F,A))$,

(ST3) $\mathcal{T}(\bigvee_{i\in I}(F_i,A)) \ge \bigwedge_{i\in I}\mathcal{T}((F_i,A))$.

The triple $(X,A,\mathcal{T})$ is called a soft L-fuzzy topological space.

A soft L-fuzzy topology is called enriched if $\mathcal{T}(\alpha\odot(F,A)) \ge \mathcal{T}((F,A))$ for all $\alpha \in L$ and $(F,A) \in S(X,A)$.

Let $(X,A,\mathcal{T}_1)$ and $(X,A,\mathcal{T}_2)$ be soft L-fuzzy topological spaces. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $\mathcal{T}_2((F,A)) \le \mathcal{T}_1((F,A))$, for all $(F,A) \in S(X,A)$.

Let $(X,A,\mathcal{T}_X)$ and $(Y,B,\mathcal{T}_Y)$ be soft L-fuzzy topological spaces and $f_\phi : S(X,A) \to S(Y,B)$ be a soft map. Then f_ϕ is called a continuous soft map if

$$\mathcal{T}_Y((G,B)) \le \mathcal{T}_X(f_\phi^{-1}((G,B)),\ \forall (G,B) \in S(Y,B).$$

Definition 2.8. [13,14] A mapping $\mathcal{U} : S(X\times X,A) \to L$ is called a soft L-fuzzy quasi-uniformity on X iff it satisfies the properties.

(SU1) There exists $(U,A) \in S(X\times X,A)$ such that $\mathcal{U}((U,A)) = 1$,

(SU2) If $(V,A) \le (U,A)$, then $\mathcal{U}((V,A)) \le \mathcal{U}((U,A))$,

(SU3) For every $(U,A),(V,A) \in S(X\times X,A)$,

$$\mathcal{U}((U,A)\odot(U,A)) \ge \mathcal{U}((U,A))\odot\mathcal{U}((V,A))$$

(SU4) If $\mathcal{U}((U,A)) \ne 0$, then $(1_\triangle,A) \le (U,A)$, where, for all $a \in A$,

$$1_\Delta(a)(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \ne y, \end{cases}$$

(SU5) $\bigvee\{\mathcal{U}((V,A)) \mid (V,A)\circ(V,A) \le (U,A)\} \ge \mathcal{U}((U,A))$

$$\begin{aligned}((V,A)\circ(V,A))(a)(x,y) &= (V(a)\circ V(a))(x,y)\\ &= \textstyle\bigvee_{z\in X}(V(a)(z,x)\odot V(a)(x,y)),\ \forall\ x,y \in X, a \in A.\end{aligned}$$

The triple $(X,A,\mathcal{U})$ is called a soft L-fuzzy quasi-uniform space.

A soft L-fuzzy quasi-uniformity $\mathcal{U}$ is called stratified if $\mathcal{U}(\alpha\odot(U,A)) \ge \alpha\odot\mathcal{U}((U,A))$ for all $\alpha \in L$ and $(U,A) \in S(X\times X,A)$.

Let $(X, A, \mathcal{U}_1)$ and $(Y, B, \mathcal{U}_2)$ be soft L-fuzzy quasi-uniform spaces and $(f \times f)_\phi$ be a soft map. Then $(f \times f)_\phi$ is called an uniformly continuous soft map if, for all $(V, B) \in S(Y \times Y, B)$,

$$\mathcal{U}_2((V, B)) \leq \mathcal{U}_1((f \times f)_\phi^{-1}((V, B))).$$

Definition 2.9. [14] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e_X satisfies (E1) and (E2), e_X is a fuzzy preorder on X. If e_X satisfies (E1), (E2) and (E3), e_X is a fuzzy partially order on X.

Lemma 2.10. [14] For a given set X, define a binary mapping $e_X : S(X, A) \times S(X, A) \to L$ by

$$e_X((F, A), (G, A)) = \bigwedge_{a \in A} \bigwedge_{x \in X} (F(a)(x) \to G(a)(x)).$$

Then, for each $(F, A), (G, A), (H, A), (K, A) \in S(X, A)$ and $\alpha \in L$ the following properties hold.

(1) $(F, A) \leq (G, A)$ iff $e_X((F, A), (G, A)) = 1$.

(2) e_X is a fuzzy partially order on $S(X, A)$,

(3) If $(F, A) \leq (G, A)$, then

$$e_X((H, A), (F, A)) \leq e_X((H, A), (G, A)),$$
$$e_X((F, A), (H, A)) \geq e_X((G, A), (H, A)).$$

(4) $e_X((F, A), (G, A)) \odot e_X((K, A), (H, A)) \leq e_X((F, A) \odot (K, A), (G, A) \odot (H, A))$.

(5) $e_X((F, A), (G, A)) \odot e_X((K, A), (H, A)) \leq e_X((F, A) \oplus (K, A), (G, A) \oplus (H, A))$.

(6) $e_X((F, A), \alpha \to (G, A)) = e_X(\alpha \odot (F, A), (G, A)) = \alpha \to e_X((F, A), (G, A))$ and $\alpha \odot e_X((F, A), (G, A)) \leq e_X((F, A), \alpha \odot (G, A))$.

(7) $(G, A) \odot e_X((G, A), (F, A)) \leq (F, A)$ and $(G, A) \leq e_X((G, A), (F, A)) \to (F, A)$.

(8) $e_X((G, A), (H, A)) \leq e_X((F, A), (G, A)) \to e_X((F, A), (H, A))$.

(9) $e_X((F, A), (G, A)) \leq e_X((G, A), (H, A)) \to e_X((F, A), (H, A))$.

(10) if $x^* = x \to 0$, then $e_X((F, A), (G, A)) = e_X((G, A)^*, (F, A)^*)$.

(11) Let $f_\phi : (X, A) \to (Y, B)$ be a soft map. Then for $(F, A), (G, A) \in S(X, A)$ and $(H, A), (K, A) \in S(Y, B)$,

$$e_X((F, A), (G, A)) \leq e_Y(f_\phi((F, A)), f_\phi((G, A))),$$

$$e_Y((H, A), (K, A)) \leq e_X(f_\phi^{-1}((H, A)), f_\phi^{-1}((K, A))),$$

and the equalities hold if f_ϕ is bijective.

3. Soft L-neighborhood systems and soft L-fuzzy quasi-uniform spaces

Definition 3.1. [13,14] A map $N : X \to L^{S(X,A)}$ is called a soft L-neighborhood system on X if $N = \{N_x = N(x) \mid x \in X\}$ satisfies the following conditions

(SN1) $N_x((1_X, A)) = 1$ and $N_x((0_X, A)) = 0$,

(SN2) $N_x((F, A) \odot (G, A)) \geq N_x((F, A)) \odot N_x((G, A))$ for each $(F, A), (G, A) \in S(X, A)$,

(SN3) If $(F, A) \leq (G, A)$, then $N_x((F, A)) \leq N_x((G, A))$,

(SN4) $N_x((F, A)) \leq (F, A)(x)$ for all $(F, A) \in S(X, A)$ where $(F, A)(x) = F(-)(x)$.

A soft L-neighborhood system is called stratified if

(S) $N_x(\alpha \odot (F, A)) \geq \alpha \odot N_x((F, A))$ for all $(F, A) \in S(X, A)$ and $\alpha \in L$.

The triple (X, A, N) is called a soft L-neighborhood space.

Let (X, A, N) and (Y, B, M) be soft L-neighborhood spaces. A mapping $f_\phi : (X, A, N) \to (Y, B, M)$ is said to be a continuous soft map iff $M_{f(x)}((G, B)) \leq N_x(f_\phi^{-1}((G, B)))$ for each $x \in X, (G, B) \in S(Y, B)$.

Theorem 3.2. *Let $(X, A, \mathcal{U})$ be a soft L-fuzzy quasi-uniform space. Define a map $rN^{\mathcal{U}} : X \to L^{S(X,A)}$ by, $\forall\ (F, A) \in S(X, A),\ x \in X$,*

$$rN^{\mathcal{U}}_x((F, A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U, A)) \odot e_X((U[x], A), (F, A)),$$

where $(U[x], A)(y) = U(-)(y, x)$. Then the following properties hold.

(1) *$(X, A, rN^{\mathcal{U}})$ is a soft L-neighborhood space.*

(2) *If $\mathcal{U}$ is stratified, then $rN^{\mathcal{U}}$ is also stratified.*

Proof. (1) (SN1) For $\mathcal{U}((U, A)) \neq 0$, by (SU4), $(1_\triangle, A) \leq (U, A)$. Then

$$\begin{aligned} &rN^{\mathcal{U}}_x((0_X, A)) = \textstyle\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U, A)) \odot e_X((U[x], A), (0_X, A)) \\ &\leq \textstyle\bigvee_{(U,A)\in S(X\times X,A)} \big(\mathcal{U}((U, A)) \odot \bigwedge_{a\in A, x\in X}(U(a)(x, x) \to 0(a)(x))\big) \\ &= \textstyle\bigvee_{(U,A)\in S(X\times X,A)} \big(\mathcal{U}((U, A)) \odot (1_\triangle(a)(x, x) \to 0)\big) = 0. \end{aligned}$$

Hence $rN^{\mathcal{U}}_x((0_X, A)) = 0$. Also, $rN^{\mathcal{U}}_x((1_X, A)) = 1$, because

$$rN^{\mathcal{U}}_x((1_X, A)) \geq \mathcal{U}((1_{X\times X}, A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} (1_{X\times X}(a)(x, y) \to 1_X(a)(y)) = 1.$$

(SN2) By Lemma 2.4 (4) , we have

$$\begin{aligned}
&rN_x^{\mathcal{U}}((F,A)) \odot rN_x^{\mathcal{U}}((G,A)) = \big(\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\big)\\
&\odot\big(\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}(V,A) \odot e_X((V[x],A),(G,A))\big)\\
&= \bigvee_{(U,A),(V,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot \mathcal{U}(V,A)\\
&\odot e_X((U[x],A),(F,A)) \odot e_X((V[x],A),(G,A)) \text{(by Lemma 2.2 (13))}\\
&\le \bigvee_{(U,A),(V,A)\in S(X\times X,A)} \mathcal{U}((U\odot V),A) \odot e_X(((U\odot V)[x],A),(F,A)\odot(G,A))\\
&\le \bigvee_{(W,A)\in S(X\times X,A)} \mathcal{U}((W,A)) \odot e_X((W[x],A),(F,A)\odot(G,A))\\
&= rN_x^{\mathcal{U}}((F,A)\odot(G,A)).
\end{aligned}$$

(SN3) It follows from the definition of $rN_x^{\mathcal{U}}$ and Lemma 2.10(3).

(SN4) For $\mathcal{U}((U,A)) \neq 0,\ 1_\triangle \le U$.

$$\begin{aligned}
&rN_x^{\mathcal{U}}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot \bigwedge_{a\in A}\bigwedge_{y\in X}(U(a)(y,x)\to F(a)(y))\\
&\le \bigvee_{(U,A)\in S(X\times X,A)} \{\mathcal{U}((U,A)) \odot (U(-)(x,x)\to (F,A)(x))\} \le (F,A)(x).
\end{aligned}$$

Hence $(X,A,rN^{\mathcal{U}})$ is a soft L-neighborhood space.

(2)

$$\begin{aligned}
&\alpha\odot rN_x^{\mathcal{U}}((F,A)) = \alpha\odot\bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= \bigvee_{(U,A)\in S(X\times X,A)} \alpha\odot\mathcal{U}((U,A)) \odot e_X(\alpha,\alpha) \odot e_X((U[x],A),(F,A))\\
&\le \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}(\alpha\odot(U,A)) \odot e_X(\alpha\odot(U[x],A),\alpha\odot(F,A))\\
&\le rN_x^{\mathcal{U}}(\alpha\odot(F,A)).
\end{aligned}$$

Corollary 3.3. *Let $(X,A,\mathcal{U})$ be a soft L-fuzzy quasi-uniform space. Define a map $lN^{\mathcal{U}} : X \to L^{S(X,A)}$ by, $\forall\ (F,A)\in S(X,A),\ x\in X$,*

$$lN_x^{\mathcal{U}}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[[x]],A),(F,A)),$$

where $(U[[x]],A)(y) = U(-)(x,y)$. Then the following properties hold.

(1) *$(X,A,lN^{\mathcal{U}})$ is a soft L-neighborhood space.*

(2) *If $\mathcal{U}$ is stratified, then $lN^{\mathcal{U}}$ is also stratified.*

Theorem 3.4. (1) *The soft L-neighborhood system $rN^{\mathcal{U}} = \{rN_x^{\mathcal{U}} \mid x\in X\}$ can be constructed from the cuts $\{(U,A)\in S(X\times X,A) \mid \mathcal{U}((U,A))\ge\alpha\}$ as follows:*

$$rN_x^{\mathcal{U}}((F,A)) = \bigvee_{\alpha\in L} \alpha\odot rN_x^{\mathcal{U}}((F,A),\alpha),$$

where

$$rN_x^{\mathcal{U}}((F,A),\alpha) = \bigvee_v\{e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A))\ge\alpha\}.$$

(2)

$$rN_x^{\mathcal{U}}((F,A)) \leq \bigvee_{(V,A)\in S(X\times X,A)} \{rN_x^{\mathcal{U}}((G,A)) \mid (G,A)(z) \leq rN_z^{\mathcal{U}}((F,A),\mathcal{U}(V,A))\}.$$

Proof. (1) For $F \in L^X$ with $F(y) \geq \alpha$, we have $F(y) \odot G(y) \geq \alpha \odot G(y)$ and

$$\bigvee\{F(x) \odot G(x) \mid F(x) \geq \alpha\} \geq \bigvee\{\alpha \odot G(y) \mid F(x) \geq \alpha\}.$$

Suppose

$$\bigvee\{F(x) \odot G(x) \mid x \in X\} \not\leq \bigvee_{\alpha\in L} \bigvee\{\alpha \odot G(x) \mid F(x) \geq \alpha\}.$$

There exists $x_0 \in X$ such that

$$F(x_0) \odot G(x_0) \not\leq \bigvee_{\alpha\in L} \bigvee\{\alpha \odot G(x) \mid F(x) \geq \alpha\}.$$

It is a contradiction. Hence

$$\bigvee\{F(x) \odot G(x) \mid x \in X\} = \bigvee_{\alpha\in L} \bigvee\{\alpha \odot G(x) \mid F(x) \geq \alpha\}.$$

By the above equality, we obtain

$$\begin{aligned} rN_x^{\mathcal{U}}((F,A)) &= \bigvee_{(U,A)\in S(X\times X,A)} (\mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))) \\ &= \bigvee_{\alpha\in L} \bigvee\{\alpha \odot e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \geq \alpha\} \\ &= \bigvee_{\alpha\in L} \{\alpha \odot \bigvee\{e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \geq \alpha\}\} \\ &= \bigvee_{\alpha\in L} \{\alpha \odot rN_x^{\mathcal{U}}((F,A),\alpha)\}. \end{aligned}$$

(2) For $(U,A) \in S(X \times X, A)$ and $(F,A) \in S(X,A)$, we have

$$\begin{aligned} &rN_x^{\mathcal{U}}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A)) \\ &= \bigvee_{(U,A)\in S(X\times X,A)} \{\mathcal{U}((U,A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} (U(a)(y,x) \to F(a)(y))\} \text{ (by SU(5))} \\ &\leq \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}((V,A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} ((V \circ V)(a)(y,x) \to F(a)(y))\} \\ &= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}((V,A)) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} ((\bigvee_{z\in X} V(a)(z,x) \odot V(a)(y,z)) \to F(a)(y))\} \\ &= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} \bigwedge_{z\in X} (V(a)(z,x) \odot V(a)(y,z)) \to F(a)(y))\} \\ &= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A} \bigwedge_{y\in X} \bigwedge_{z\in X} (V(a)(z,x) \to (V(a)(y,z) \to (F,A)(y))\} \\ &\text{(by Lemma 2.2 (12))} \\ &= \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A} \bigwedge_{z\in X} (V(a)(z,x) \to \bigwedge_{y\in X} (V(a)(y,z) \to (F,A)(y))\}. \end{aligned}$$

Put $(G,A)(z) = \bigwedge_{a\in A} \bigwedge_{y\in X} (V(a)(y,z) \to (F,A)(y))$. Then

$$\begin{aligned} &rN_z^{\mathcal{U}}((F,A),\mathcal{U}(V,A)) = \bigvee_{(V,A)\in S(X\times X,A)} \{e_X((U[x],A),(F,A)) \mid \mathcal{U}((U,A)) \geq \mathcal{U}(V,A)\} \\ &\geq e_X((V[z],A),(F,A)) = \bigwedge_{a\in A} \bigwedge_{y\in X} (V(a)(y,z) \to (F,A)(y)) = (G,A)(z). \end{aligned}$$

Thus,

$$\begin{aligned}&rN_x^{\mathcal{U}}((F,A)) \leq \bigvee_{(V,A)\in S(X\times X,A)} \{\mathcal{U}(V,A) \odot \bigwedge_{a\in A}\bigwedge_{z\in X}(V(a)(z,x) \to (G,A)(z)) \\ &\mid (G,A)(z) \leq rN_z^{\mathcal{U}}((F,A),\mathcal{U}(V,A))\} \\ &\leq \bigvee_{(V,A)\in S(X\times X,A)} \{rN_x^{\mathcal{U}}((G,A)) \mid (G,A)(z) \leq rN_z^{\mathcal{U}}((F,A),\mathcal{U}(V,A))\}.\end{aligned}$$

Theorem 3.5. *If $f_\phi : (X,A,\mathcal{U}) \to (Y,B,\mathcal{V})$ is a uniformly continuous soft map, then $f_\phi : (X,A,rN^{\mathcal{U}}) \to (Y,B,rN^{\mathcal{V}})$ and $f_\phi : (X,A,lN^{\mathcal{U}}) \to (Y,B,lN^{\mathcal{V}})$ are continuous soft maps.*

Proof. First we show that $f_\phi^{-1}((V[f(x)],B)) = ((f\times f)_\phi^{-1}(V)[x],A)$ from

$$\begin{aligned}&f_\phi^{-1}((V[f(x)])(a)(z) = f^{\leftarrow}(V[f(x)](\phi(a))(z) = V(\phi(a))[f(x)](f(z)) \\ &= V(\phi(a))(f(z),f(x)) = (f\times f)_\phi^{-1}(V)(a)(z,x) = (f\times f)_\phi^{-1}(V)[x](a)(z).\end{aligned}$$

Thus, by Lemma 2.10(11), we have

$$\begin{aligned}&e_Y((V[f(x)],B),(F,B)) \leq e_X(f_\phi^{-1}((V[f(x)]),A), f_\phi^{-1}((F,B))) \\ &= e_X(((f\times f)_\phi^{-1}(V)[x],A), f_\phi^{-1}((F,B))).\end{aligned}$$

$f_\phi : (X,A,rN^{\mathcal{U}}) \to (Y,B,rN^{\mathcal{V}})$ is a continuous soft map from:

$$\begin{aligned}&rN_{f(x)}^{\mathcal{V}}((F,B)) = \bigvee_{(V,A)\in S(X\times X,A)} \mathcal{V}(V,A) \odot e_Y((V[f(x)],B),(F,B)) \\ &\leq \bigvee_{(V,A)\in S(X\times X,A)} \mathcal{V}(V,A) \odot e_X(((f\times f)_\phi^{-1}(V)[x],A), f_\phi^{-1}((F,B))) \\ &\leq \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((f\times f)_\phi^{-1}(V,A)) \odot e_X(((f\times f)_\phi^{-1}(V)[x],A), f_\phi^{-1}((F,B))) \\ &\leq rN_x^{\mathcal{U}}(f_\phi^{-1}((F,B))).\end{aligned}$$

Similarly, $f_\phi : (X,A,lN^{\mathcal{U}}) \to (Y,B,lN^{\mathcal{V}})$ is a continuous soft map.

Theorem 3.6. *Let $f_\phi : (X,A) \to (Y,B)$ be a soft mapping and (Y,B,M) be a soft L-neighborhood space. We define $N : X \to L^{S(X,A)}$ as follows*

$$N_x((F,A)) = \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \leq (F,A)\}.$$

Then (X,A,N) is the coarsest soft L-neighborhood space for which $f_\phi : (X,A,N) \to (Y,B,N)$ be a continuous soft map. Moreover, if M is stratified, then N is also stratified.

Proof. (SN1) and (SN3) are clearly true.

(SN2) Let $(F_1,A),(F_2,A) \in S(X,A)$, $(G_1,B),(G_2,B) \in S(Y,B)$ and $x \in X$, then we have

$$
\begin{aligned}
&N_x((F_1,A)) \odot N_x((F_2,A))\\
&= \bigvee\{M_{f(x)}((G_1,B)) \mid f_\phi^{-1}((G_1,B)) \le (F_1,A)\} \odot \bigvee\{M_{f(x)}((G_2,B)) \mid f_\phi^{-1}((G_2,B)) \le (F_2,A)\}\\
&= \bigvee\{M_{f(x)}((G_1,B)) \odot M_{f(x)}((G_2,B)) \mid f_\phi^{-1}((G_1,B)) \le (F_1,A),\ f_\phi^{-1}((G_2,B)) \le (F_2,A)\}\\
&\le \bigvee\{M_{f(x)}((G_1,B) \odot (G_2,B)) \mid f_\phi^{-1}((G_1,B)) \odot f_\phi^{-1}((G_2,B)) \le (F_1,A) \odot (F_2,A)\}\\
&= \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \le (F_1,A) \odot (F_2,A)\}\\
&= N_x((F_1,A) \odot (F_2,A)).
\end{aligned}
$$

(SN4) For any $(F,A) \in S(X,A), (G,B) \in S(Y,B)$ and $x \in X$, we have

$$
\begin{aligned}
&N_x((F,A)) = \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \le (F,A)\} \le \bigvee\{(G,B)(f(x)) \mid f_\phi^{-1}((G,B)) \le (F,A)\}\\
&= \bigvee\{f_\phi^{-1}((G,B))(x) \mid f_\phi^{-1}((G,B)) \le (F,A)\} \le (F,A)(x).
\end{aligned}
$$

The a map $f_\phi : (X,N) \to (Y,M)$ is a continuous soft map because, $x \in X, (G,B) \in S(Y,B)$,

$$
\begin{aligned}
&N_x(f_\phi^{-1}((G,B))) = \bigvee\{M_{f(x)}((F,B)) \mid f_\phi^{-1}((F,B)) \le f_\phi^{-1}((G,B))\}\\
&\ge \bigvee\{M_{f(x)}((F,B)) \mid (F,B) \le (G,B)\} \ge M_{f(x)}((G,B)).
\end{aligned}
$$

Let $f_\phi : (X,N') \to (Y,M)$ be a continuous soft map. Suppose there exists $x \in X, (F,A) \in S(X,A)$ such that

$$N_x((F,A)) \not\le N'_x((F,A)).$$

By the definition of N_x, there exists $(G,B) \in S(Y,B)$ with $f_\phi^{-1}((G,B)) \le (F,A)$ such that

$$M_{f(x)}((G,B)) \not\le N'((F,A)).$$

On the other hand,

$$M_{f(x)}((G,B)) \le N'(f_\phi^{-1}((G,B))) \le N'((F,A)).$$

It is a contradiction. Hence $N \le N'$.

Finally, if M is stratified, then N is also stratified. In fact, for any $\alpha \in L$ and $(F,A) \in S(X,A)$, we have

$$
\begin{aligned}
&\alpha \odot N_x((F,A)) = \alpha \odot \bigvee\{M_{f(x)}((G,B)) \mid f_\phi^{-1}((G,B)) \le (F,A)\}\\
&\le \bigvee\{\alpha \odot M_{f(x)}((G,B)) \mid \alpha \odot f_\phi^{-1}((G,B)) \le \alpha \odot (F,A)\}\\
&\le \bigvee\{M_{f(x)}(\alpha \odot (G,B)) \mid f_\phi^{-1}(\alpha \odot (G,B)) \le \alpha \odot (F,A)\} = N_x(\alpha \odot (F,A)).
\end{aligned}
$$

Theorem 3.7. *Let (X,A,N) be a soft L-neighborhood space. Define a map $\mathcal{T}_N : S(X,A) \to L$ by*

$$\mathcal{T}_N((F,A)) = \bigwedge_{a\in A} \bigwedge_{x\in X} (F(a)(x) \to N_x((F,A))).$$

Then (1) *$\mathcal{T}_N$ is a soft L-fuzzy topology on X,*

(2) *If N_x is stratified, then $\mathcal{T}_N$ is an enriched soft L-fuzzy topology.*

Proof. (1) (ST1)

$$\mathcal{T}_N((1_X, A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(1_X(a)(x) \to N_x((1_X, A)) = 1,$$
$$\mathcal{T}_N((0_X, A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(0_X(a)(x) \to N_x((0_X, A))) = 0.$$

(ST2)

$$\begin{aligned}
&\mathcal{T}_N((F, A) \odot (G, A))\\
&= \bigwedge_{a\in A}\bigwedge_{x\in X}((F(a)(x) \odot G(a)x) \to N_x((F, A) \odot (G, A)))\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((F(a)(x) \odot G(a)x) \to (N_x((F, A)) \odot N_x((G, A))))\\
&\quad \text{(by Lemma 2.2 (12))}\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x) \to N_x((F, A))) \odot \bigwedge_{a\in A}\bigwedge_{x\in X}(G(a)(x) \to N_x((G, A)))\\
&= \mathcal{T}_N((F, A)) \odot \mathcal{T}_N((G, A)).
\end{aligned}$$

(ST3)

$$\begin{aligned}
&\mathcal{T}_N(\bigvee_i(F_i, A) = \bigwedge_{a\in A}\bigwedge_{x\in X}((\bigvee_i F_i(a)(x) \to N_x(\bigvee_i(F_i, A))\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((\bigvee_i(F_i(a)(x) \to \bigvee_i N_x((F_i, A)) \quad \text{(by Lemma 2.2 (8))}\\
&\geq \bigwedge_i\bigwedge_{a\in A}\bigwedge_{x\in X}((F_i(a)(x) \to N_x((F_i, A)) = \bigwedge_i \mathcal{T}_N((F_i, A).
\end{aligned}$$

(2) By Lemma 2.2 (12), we have

$$\begin{aligned}
&\mathcal{T}_N(\alpha \odot (F, A)) = \bigwedge_{x\in X}((\alpha \odot (F, A))(x) \to N_x(\alpha \odot (F, A)))\\
&\geq \bigwedge_{x\in X}((\alpha \odot (F, A))(x) \to (\alpha \odot N_x((F, A))))\\
&\geq \bigwedge_{x\in X}((F, A)(x) \to N_x((F, A))) = \mathcal{T}_N((F, A)).
\end{aligned}$$

Corollary 3.8. *Let Let $(X,\mathcal{U})$ be a soft L-fuzzy quasi-uniform space, $\{rN_x^{\mathcal{U}} \mid x \in X\}$ and $\{lN_x^{\mathcal{U}} \mid x \in X\}$ be soft L-neighorhood systems on X. Define maps $\mathcal{T}_{rN^{\mathcal{U}}}, \mathcal{T}_{lN^{\mathcal{U}}} : S(X, A) \to L$ by*

$$\mathcal{T}_{rN^{\mathcal{U}}}((F, A)) = \bigwedge_{x\in X}((F, A)(x) \to rN_x^{\mathcal{U}}((F, A))),$$

$$\mathcal{T}_{lN^{\mathcal{U}}}((F, A)) = \bigwedge_{x\in X}((F, A)(x) \to lN_x^{\mathcal{U}}((F, A))).$$

Then,

(1) *$\mathcal{T}_{rN^{\mathcal{U}}}$ and $\mathcal{T}_{lN^{\mathcal{U}}}$ are soft L-fuzzy topologies on X,*

(2) *If $rN^{\mathcal{U}}$ (resp. $rN^{\mathcal{U}}$) is stratified, then $\mathcal{T}_{rN^{\mathcal{U}}}$ (resp. $\mathcal{T}_{lN^{\mathcal{U}}}$) is an enriched soft L-fuzzy topology.*

Theorem 3.9. *Let $(X, A,\mathcal{U})$ and $(Y, B, \mathcal{V})$ be soft L-fuzzy quasi-uniform spaces. If a map $f_\phi : (X, A,\mathcal{U}) \to (Y, B, \mathcal{V})$ is an uniformly continuous soft map, then maps $f_\phi : (X, A, \mathcal{T}_{rN^{\mathcal{U}}}) \to (Y, B, \mathcal{T}_{rN^{\mathcal{V}}})$ and $f_\phi : (X, A, \mathcal{T}_{lN^{\mathcal{U}}}) \to (Y, B, \mathcal{T}_{lN^{\mathcal{V}}})$ are continuous soft maps.*

Proof.

$$\begin{aligned}
&\mathcal{T}_{rN^{\mathcal{V}}}((G,B)) \to \mathcal{T}_{rN^{\mathcal{V}}}(f_{\phi}^{-1}((G,B)))\\
&= \bigwedge_{b\in B}\bigwedge_{y\in Y}((G(b)(y) \to rN_y^{\mathcal{V}}((G,B))) \to \bigwedge_{a\in A}\bigwedge_{x\in X}(f_{\phi}^{-1}(G)(a)(x) \to rN_x^{\mathcal{U}}(f_{\phi}^{-1}((G,B))))\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((G(\phi(a))(f(x)) \to rN_{f(x)}^{\mathcal{V}}((G,B)))\\
&\to \bigwedge_{a\in A}\bigwedge_{x\in X}(G(\phi(a))(f(x)) \to rN_x^{\mathcal{U}}(f_{\phi}^{-1}((G,B))))\\
&\geq \bigwedge_{a\in A}\bigwedge_{x\in X}((G(\phi(a))(f(x)) \to rN_{f(x)}^{\mathcal{V}}((G,B)))\\
&\to (G(\phi(a))(f(x)) \to rN_x^{\mathcal{U}}(f_{\phi}^{-1}((G,B)))) \quad \text{(by Lemma 2.2 (8))}\\
&\geq \bigwedge_{x\in X}\left(rN_{f(x)}^{\mathcal{V}}((G,B)) \to rN_x^{\mathcal{U}}(f_{\phi}^{-1}((G,B)))\right) \text{ (by Lemma 2.2 (10)).}
\end{aligned}$$

Thus, if $rN_{f(x)}^{\mathcal{V}}((G,B)) \leq rN_x^{\mathcal{U}}(f_{\phi}^{-1}((G,B)))$, then $\mathcal{T}_{\mathcal{V}}((G,B)) \leq \mathcal{T}_{\mathcal{U}}(f_{\phi}^{-1}((G,B)))$. So, f_{ϕ} is a continuous soft map.

Example 3.10. Let $H = \{h_i \mid i = \{1,...,6\}\}$ with h_i=house and $E = \{e, b, w, c, i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings.

Define a binary operation $\odot$ on $[0,1]$ by

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}$$

$$x \oplus y = \min\{1, x+y\},\ x^* = 1-x$$

Then $([0,1], \odot, \to, 0, 1)$ is a complete residuated lattice (ref.[2,8,27]). Let $A = \{b, c\} \subset E$ and $X = \{h^1, h^4, h^5\}$. Put a fuzzy soft set (U, A) as follow:

$$(U,\{b\}) = \begin{pmatrix} b & h^1 & h^4 & h^5 \\ h^1 & 1 & 0.6 & 0.7 \\ h^4 & 0.1 & 1 & 0.5 \\ h^5 & 0.4 & 0.6 & 1 \end{pmatrix} (U,\{c\}) = \begin{pmatrix} c & h^1 & h^4 & h^5 \\ h^1 & 1 & 0.3 & 0.6 \\ h^4 & 0.1 & 1 & 0.6 \\ h^5 & 0.7 & 0.5 & 1 \end{pmatrix}$$

Then we obtain $(U \odot U, A)$ as

$$(U\odot U,\{b\}) = \begin{pmatrix} b & h^1 & h^4 & h^5 \\ h^1 & 1 & 0.2 & 0.4 \\ h^4 & 0 & 1 & 0 \\ h^5 & 0 & 0.2 & 1 \end{pmatrix} (U\odot U,\{c\}) = \begin{pmatrix} c & h^1 & h^4 & h^5 \\ h^1 & 1 & 0 & 0.2 \\ h^4 & 0 & 1 & 0.2 \\ h^5 & 0.4 & 0 & 1 \end{pmatrix}$$

Define $\mathcal{U} : S(X \times X, A) \to L$ as follows

$$\mathcal{U}((V,A)) = \begin{cases} 1, & \text{if } (V,A) = (1_{X\times X}, A), \\ 0.6, & \text{if } (U,A) \leq (V,A) \neq (1_{X\times X}, A), \\ 0.3, & \text{if } (U,A)\odot(U,A) \leq (V,A) \not\geq (U,A), \\ 0, & \text{otherwise.} \end{cases}$$

$$rN^{\mathcal{U}}_{h_1}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A)),\ \forall\ (F,A)\in S(X,A),\ x\in X, \tag{1}$$

$$\begin{aligned}
&rN^{\mathcal{U}}_{h^1}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= (F(b)(h^1)\wedge F(b)(h^4)\wedge F(b)(h^5)) \vee \Big(0.6\odot (F(b)(h^1)\wedge (0.9+F(b)(h^4))\wedge\\
&(0.6+F(b)(h^5)))\Big) \vee 0.3\odot \Big(F(b)(h^1))\Big) \vee (F(c)(h^1)\wedge F(c)(h^4)\wedge F(c)(h^5))\\
&\vee\Big(0.6\odot (0.7+F(c)(h^1))\wedge F(c)(h^4) \quad \wedge (0.5+F(c)(h^5)))\Big)\\
&\quad \vee 0.3\odot \Big(F(c)(h^1)\wedge (0.6+F(c)(h^5)))\Big)
\end{aligned}$$

$$\begin{aligned}
&rN^{\mathcal{U}}_{h^2}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= (F(b)(h^1)\wedge F(b)(h^4)\wedge F(b)(h^5)) \vee \Big(0.6\odot ((0.4+F(b)(h^1))\wedge F(b)(h^4)\wedge (0.4+F(b)(h^5)))\Big)\\
&\quad \vee (0.3\odot (0.8+F(b)(h^1))\wedge F(b)(h^4)\wedge (0.8+F(b)(h^5))) \vee (F(c)(h^1)\wedge F(c)(h^4)\wedge F(c)(h^5))\\
&\vee\Big(0.6\odot ((0.7+F(c)(h^1))\wedge F(c)(h^4)\wedge (0.5+F(c)(h^5)))\Big) \quad \vee 0.3\odot (F(c)(h^4))
\end{aligned}$$

$$\begin{aligned}
&rN^{\mathcal{U}}_{h^5}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[x],A),(F,A))\\
&= (F(b)(h^1)\wedge F(b)(h^4)\wedge F(b)(h^5)) \vee \Big(0.6\odot ((0.3+F(b)(h^1))\wedge (0.5+F(b)(h^4))\wedge F(b)(h^5))\Big)\\
&\quad \vee 0.3\odot \Big((0.6+F(b)(h^1))\wedge F(b)(h^5))\Big) \vee (F(c)(h^1)\wedge F(c)(h^4)\wedge F(c)(h^5))\\
&\vee\Big(0.6\odot ((0.4+F(c)(h^1))\wedge (0.4+F(c)(h^4))\wedge F(c)(h^5))\Big)\\
&\vee 0.3\odot \Big((0.8+F(c)(h^1))\wedge (0.8+F(c)(h^4))\wedge F(c)(h^5)\Big)
\end{aligned}$$

Put (F,A) be a fuzzy soft set as follow:

(F,A)	h^1	h^4	h^5
b	0.5	0.6	0.2
c	0.9	0.5	0.3

Since $rN^{\mathcal{U}}_{h^1}((F,A)) = 0.2, rN^{\mathcal{U}}_{h^4}((F,A)) = 0.2, rN^{\mathcal{U}}_{h^5}((F,A)) = 0.2,$

$$\begin{aligned}
&\mathcal{T}_N((F,A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x)\to rN_x((F,A)))\\
&= (F(b)(h^1)\to rN_{h^1}((F,A)))\wedge (F(b)(h^4)\to rN_{h^4}((F,A)))\wedge (F(b)(h^5)\to rN_{h^5}((F,A)))\\
&\wedge (F(c)(h^1)\to rN_{h^1}((F,A)))\wedge (F(c)(h^4)\to rN_{h^4}((F,A)))\wedge (F(c)(h^5)\to rN_{h^5}((F,A))) = 0.3.
\end{aligned}$$

(2)

$$lN^{\mathcal{U}}_{h^1}((F,A)) = \bigvee_{(U,A)\in S(X\times X,A)} \mathcal{U}((U,A)) \odot e_X((U[[x]],A),(F,A))$$
$$= (F(b)(h^1) \wedge F(b)(h^4) \wedge F(b)(h^5)) \vee \Big(0.6 \odot (F(b)(h^1) \wedge (0.4 + F(b)(h^4))$$
$$\wedge(0.3 + F(b)(h^5)))\Big) \vee 0.3 \odot \Big(F(b)(h^1) \wedge (0.8 + F(b)(h^4))$$
$$\wedge(0.6 + F(b)(h^5))\Big) \vee (F(c)(h^1) \wedge F(c)(h^4) \wedge F(c)(h^5))$$
$$\vee\Big(0.6 \odot (F(c)(h^1) \wedge (0.7 + F(c)(h^4)) \wedge (0.4 + F(c)(h^5)))\Big)$$
$$\vee 0.3 \odot \Big(F(c)(h^1) \wedge (0.8 + F(c)(h^5))\Big)$$

$$lN^{\mathcal{U}}_{h^2}((F,A)) = (F(b)(h^1) \wedge F(b)(h^4) \wedge F(b)(h^5))$$
$$\vee\Big(0.6 \odot (0.9 + F(b)(h^1)) \wedge F(b)(h^4) \wedge (0.5 + F(b)(h^5))\Big)$$
$$\vee(0.3 \odot F(b)(h^4)) \vee (F(c)(h^1) \wedge F(c)(h^4) \wedge F(c)(h^5))$$
$$\vee\Big(0.6 \odot (0.9 + F(c)(h^1)) \wedge F(c)(h^4) \wedge (0.4 + F(c))(h^5)\Big)$$
$$\vee 0.3 \odot (F(c)(h^4) \wedge (0.8 + F(c)(h^5)))$$

$$lN^{\mathcal{U}}_{h^5}((F,A)) = (F(b)(h^1) \wedge F(b)(h^4) \wedge F(b)(h^5))$$
$$\vee\Big(0.6 \odot (0.6 + F(b)(h^1)) \wedge (0.4 + F(b)(h^4)) \wedge F(b)(h^5)\Big)$$
$$\vee 0.3 \odot \Big((0.8 + F(b)(h^4)) \wedge F(b)(h^5)\Big) \vee (F(c)(h^1) \wedge F(c)(h^4) \wedge F(c)(h^5))$$
$$\vee\Big(0.6 \odot (0.3 + F(c)(h^1)) \wedge (0.5 + F(c)(h^4)) \wedge F(c)(h^5)\Big)$$
$$\vee 0.3 \odot \Big((0.6 + F(c)(h^1)) \wedge F(c)(h^5)\Big)$$

For a fuzzy soft set (F,A) in (1), since

$$lN^{\mathcal{U}}_{h^1}((F,A)) = 0.3, lN^{\mathcal{U}}_{h^4}((F,A)) = 0.2, lN^{\mathcal{U}}_{h^5}((F,A)) = 0.2,$$

we have

$$\mathcal{T}_N((F,A)) = (F(b)(h^1) \to lN_{h^1}((F,A)))$$
$$\wedge(F(b)(h^4) \to lN_{h^4}((F,A))) \wedge (F(b)(h^5) \to lN_{h^5}((F,A)))$$
$$\wedge(F(c)(h^1) \to lN_{h^1}((F,A))) \wedge (F(c)(h^4) \to lN_{h^4}((F,A)))$$
$$\wedge(F(c)(h^5) \to lN_{h^5}((F,A))) = 0.3.$$

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Existence of pseudo almost periodic mild solutions to impulsive partial stochastic differential equations in Hilbert spaces

Zuomao Yan* and Xiumei Jia

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Abstract: In this paper, we introduce the concept of p-mean piecewise pseudo almost periodic for a stochastic process. Also it establish a new composition theorem for such processes. With this new composition theorem and by virtue of the theory of operator semigroups, the stochastic analysis techniques and Leray-Schauder nonlinear alternative, we investigate the existence of p-mean piecewise pseudo almost periodic mild solutions for a class of impulsive partial stochastic differential equations. Finally, an example of impulsive stochastic heat equation is also provided to illustrate our results.

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1 Introduction

The study of almost periodic type functions constitutes one of the most attractive topics in qualitative theory of differential equations since their applications. Among them, pseudo almost periodic function was introduced by Zhang as a natural generalization of almost periodic function in [1]. Some contributions on pseudo almost periodic type solutions to abstract differential equations have recently been made [2-7] and the references therein. Recently, there has been an increasing interest in extending certain classical deterministic results to stochastic differential equations. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. The existence of almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions to some stochastic differential equations has been considered in many publications such as [8-17] and references therein. In particular, Bezandry and Diagana [18,19] introduced the concepts of p-mean pseudo pseudo almost periodicity, and studied the existence of p-mean pseudo almost periodic mild solutions to partial stochastic differential

equations. Diop et al. [20] obtained the existence, uniqueness and global attractiveness of an p-mean pseudo almost periodic solution for stochastic evolution equation driven by a fractional Brownian motion.

The theory of impulsive differential equations is an important branch of differential equations, which has an extensively physical background [21]. Therefore, it seems interesting to study the various types of impulsive differential equations. The asymptotic properties of solutions of impulsive differential equations have been considered by many authors. For example, Henríquez et al. [22], Liu and Zhang [23], Stamov et al. [24-26] discussed the piecewise almost periodic solutions of impulsive differential equations. Liu and Zhang [27], Chérif [28] established the existence and stability of piecewise pseudo almost periodic solutions to abstract impulsive differential equations. Bainov et al. [29] concerned with the asymptotic equivalence of impulsive differential equations. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. In recent years, several interesting results on impulsive partial stochastic systems have been reported in [30-32] and the references therein. Further, Zhang [33] obtained the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay by mean of the Banach contraction principle. In [34], the authors investigated the existence and stability of square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations by using Schauder's fixed point theorem.

In this paper, we study the existence of p-mean piecewise pseudo almost periodic mild solutions to the following impulsive partial stochastic differential equations:

$$dx(t) = [Ax(t) + g(t, x(t))]dt + f(t, x(t))dW(t), \quad t \in R, t \neq t_i, i \in Z, \tag{1}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in Z, \tag{2}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^p(P, H)$ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. g, f, I_i, t_i satisfy suitable conditions which will be established later. The notations $x(t_i^+), x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

To the best of our knowledge, the existence of p-mean piecewise pseudo almost periodic mild solutions for for nonlinear impulsive stochastic system (1)-(2) is an untreated original topic, which in fact is the main motivation of the present paper. In the paper, we will introduce the notion of p-mean piecewise pseudo almost periodic for stochastic processes, which, in turn generalizes all the above-mentioned concepts, in particular, the notion of piecewise almost periodic. Then we will establish a new composition theorem for p-mean pseudo almost periodic functions under non-Lipschitz conditions. As an application, we study and obtain the existence of p-mean piecewise pseudo almost periodic mild solutions to system (1)-(2) by using Leray-Schauder nonlinear alternative. Such a result generalizes most of known results on the existence of almost periodic

solutions of type system (1)-(2). It includes some results of almost periodic and pseudo almost periodic solutions to stochastic differential equations without impulse. Moreover, the results are also new for deterministic systems with impulse.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of p-mean piecewise pseudo almost periodic solutions for linear and nonlinear impulsive stochastic differential equations, respectively. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Throughout the paper, N, Z, R and R^+ stand for the set of natural numbers, integers, real numbers, positive real numbers, respectively. We assume that $(H, \| \cdot \|), (K, \| \cdot \|_K)$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, P)$ is supposed to be a filtered complete probability space. Define $L^p(P, H)$, for $p \geq 1$ to be the space of all H-valued random variables V such that $E \parallel V \parallel^p = \int_\Omega \parallel V \parallel^p \, dP < \infty$. Then $L^p(P, H)$ is a Banach space when it is equipped with its natural norm $\| \cdot \|_p$ defined by $\| V \|_p = (\int_\Omega E \parallel V \parallel^p \, dP)^{1/p} < \infty$ for each $V \in L^p(P, H)$. Let $C(R, L^p(P, H)), BC(R, L^p(P, H))$ stand for the collection of all continuous functions from R into $L^p(P, H)$, the Banach space of all bounded continuous functions from R into $L^p(P, H)$, equipped with the sup norm, respectively. We let $L(K, H)$ be the space of all linear bounded operators from K into H, equipped with the usual operator norm $\| \cdot \|_{L(K,H)}$; in particular, this is simply denoted by $L(H)$ when $K = H$. Furthermore, $L_2^0(K, H)$ denotes the space of all Q-Hilbert-Schmidt operators from K to H with the norm $\| \psi \|_{L_2^0}^2 = \mathrm{Tr}(\psi Q \psi^*) < \infty$ for any $\psi \in L(K, H)$.

Definition 2.1 ([18]). A stochastic process $x : R \to L^p(P, H)$ is said to be continuous provided that for any $s \in R$,

$$\lim_{t \to s} E \parallel x(t) - x(s) \parallel^p = 0.$$

Definition 2.2 ([18]). A stochastic process $x : R \to L^p(P, H)$ is said to be stochastically bounded provided that

$$\lim_{n \to \infty} \limsup_{t \in R} P\{\| x(t) \| > n\} = 0.$$

Let T be the set consisting of all real sequences $\{t_i\}_{i \in Z}$ such that $\alpha = \inf_{i \in Z}(t_{i+1} - t_i) > 0$, $\lim_{i \to \infty} t_i = \infty$, and $\lim_{i \to -\infty} t_i = -\infty$. For $\{t_i\}_{i \in Z} \in T$, let $PC(R, L^p(P, H))$ be the space consisting of all stochastically bounded piecewise continuous functions $f : R \to L^p(P, H)$ such that $f(\cdot)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i \in Z}$ and $f(t_i) = f(t_i^-)$ for all $i \in Z$; let $PC(R \times L^p(P, K), L^p(P, H))$ be the space formed by all stochastically piecewise continuous functions $f : R \times L^p(P, K) \to L^p(P, H)$ such that for any $x \in L^p(P, K)$, $f(\cdot, x) \in PC(R, L^p(P, H))$ and for any $t \in R$, $f(t, \cdot)$ is stochastically continuous at $x \in L^p(P, K)$.

Definition 2.3 ([18]). A function $f \in C(R, L^p(P,H))$ is said to be p-mean almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$ for all $t \in R$. Denote by $AP(R, L^p(P,H))$ the set of such functions.

Definition 2.4 (Compare with [21]). A sequence $\{x_n\}$ is called p-mean almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e., there exists a natural number $l = l(\varepsilon)$, such that for $k \in Z$, there is at least one number q in $[k, k+l]$, for which inequality $E \parallel x_{n+q} - x_n \parallel^p < \varepsilon$ holds for all $n \in N$. Denote by $AP(Z, L^p(P,H)))$ the set of such sequences.

Define $l^\infty(Z, L^p(P,H)) = \{x : Z \to L^p(P,H) : \parallel x \parallel = \sup_{n\in Z}(E \parallel x(n) \parallel^p)^{1/p} < \infty\}$, and

$$PAP_0(Z, L^p(P,H)) = \left\{ x \in l^\infty(Z, L^p(P,H)) : \lim_{n\to\infty} \frac{1}{2n} \sum_{j=-n}^{n} E \parallel x(n) \parallel^p dt = 0 \right\}.$$

Definition 2.5. A sequence $\{x_n\}_{n\in Z} \in l^\infty(Z,H)$ is called p-mean pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(Z, L^p(P,H)), x_n^2 \in PAP_0(Z, L^p(P,H))$. Denote by $PAP(Z, L^p(P,H))$ the set of such sequences.

Definition 2.6 (Compare with [21]). For $\{t_i\}_{i\in Z} \in T$, the function $f \in PC(R, L^p(P,H))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}, j \in Z$, is equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set Q_ε of R such that for each $\tau \in Q_\varepsilon$ there is an integer $q \in Z$ such that $|t_{i+q} - t_i - \tau| < \varepsilon$ for all $i \in Z$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of φ and $|t' - t''| < \tilde{\delta}$, then $E \parallel f(t') - f(t'') \parallel^p < \varepsilon$.

(iii) For every $\varepsilon > 0$, there exists a relatively dense set $\tilde{\Omega}(\varepsilon)$ in R such that if $\tau \in \tilde{\Omega}(\varepsilon)$, then
$$E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$$
for all $t \in R$ satisfying the condition $|t - t_i| > \varepsilon, i \in Z$. The number τ is called ε-translation number of f.

We denote by $AP_T(R, L^p(P,H))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(R, L^p(P,H))$ endowed with the sup norm defined by $\parallel f \parallel_\infty = \sup_{t\in R}(E \parallel f(t) \parallel^p)^{1/p}$ for any $f \in AP_T(R, L^p(P,H))$ is a Banach space. Let $UPC(R, L^p(P,H))$ be the space of all stochastic functions $f \in PC(R, L^p(P,H))$ such that f satisfies the condition (ii) in Definition 2.6.

Definition 2.7. The function $f \in PC(R \times L^p(P,K), L^p(P,H))$ is said to be p-mean piecewise almost periodic in $t \in R$ uniform in $x \in L^p(P,K)$ if for every

compact subset $\tilde{K} \subseteq L^p(P,K)$, $\{f(\cdot,x) : x \in \tilde{K}\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense subset Ω_ε such that

$$E \parallel f(t+\tau,x) - f(t,x) \parallel^p < \varepsilon$$

for all $x \in \tilde{K}, \tau \in \Omega_\varepsilon$, and $t \in R$ satisfying $|t - t_i| > \varepsilon$. Denote by $AP_T(R \times L^p(P,K), L^p(P,H))$ the set of all such functions.

Similarly as the proof of [21, Lemma 35], one has

Lemma 2.1. Assume that $f \in AP_T(R, L^p(P,H))$, the sequence $\{x_i\}_{i\in Z} \in AP(Z, L^p(P,H))$, and $\{t_i^j\}, j \in Z$ are equipotentially almost periodic. Then, for each $\varepsilon > 0$, there exist relatively dense sets Ω_ε of R and Ω_ε of Z such that

(i) $E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$ for all $t \in R, |t - t_i| > \varepsilon, \tau \in \Omega_\varepsilon$ and $i \in Z$.

(ii) $E \parallel x_{i+q} - x_i \parallel^p < \varepsilon$ for all $q \in \Omega_\varepsilon$ and $i \in Z$.

(iii) $E \parallel x_i^q - \tau \parallel^p < \varepsilon$ for all $q, \tau \in \Omega_\varepsilon$ and $i \in Z$.

Denote

$$PC_T^0(R, L^p(P,H)) = \left\{ f \in PC(R, L^p(P,H)) : \lim_{t\to\infty} E \parallel f(t) \parallel^p = 0 \right\},$$

$$PAP_T^0(R, L^p(P,H)) = \left\{ f \in PC(R, L^p(P,H)) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E \parallel f(t) \parallel^p dt = 0 \right\},$$

$$PAP_T^0(R \times L^p(P,K), L^p(P,H)) = \Bigg\{ f \in PC(R \times L^p(P,K), L^p(P,H)) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E \parallel f(t,x) \parallel^p dt = 0 \text{ uniformly with respect to } x \in \tilde{K}, \text{where } \tilde{K} \text{ is an arbitrary compact subset of } L^p(P,K) \Bigg\}.$$

Definition 2.8. A function $f \in PC(R, L^p(P,H))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(R, L^p(P,H))$ and $\varphi \in PAP_T^0(R, L^p(P,H))$.

Denoted by $PAP_T(R, L^p(P,H))$ the set of all such functions. $PAP_T(R, L^p(P,H))$ is a Banach space with the sup norm $\| \cdot \|_\infty$.

Similar to [1,27], one has

Remark 2.1. (i) $PAP_T^0(R, L^p(P,H))$ is a translation invariant set of $PC(R, L^p(P,H)))$. (ii) $PC_T^0(R, L^p(P,H)) \subset PAP_T^0(R, L^p(P,H))$.

Lemma 2.2. Let $\{f_n\}_{n\in N} \subset PAP_T^0(R, L^p(P,H))$ be a sequence of functions. If f_n converges uniformly to f, then $f \in PAP_T^0(R, L^p(P,H))$.

One can refer to Lemma 2.5 in [5] for the proof of Lemma 2.2.

Definition 2.9. A function $f \in PC(R \times L^p(P,K), L^p(P,H))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(R \times L^p(P,K), L^p(P,H))$ and $\varphi \in PAP_T^0(R \times L^p(P,K), L^p(P,H))$.

Denoted by $PAP_T(R \times L^p(P,K), L^p(P,H))$ the set of all such functions.

We need the following composition of p-mean pseudo almost periodic processes.

Lemma 2.3. Assume $f \in PAP_T(R \times L^p(P,K), L^p(P,H))$. Suppose that $f(t,x)$ satisfies

$$E \parallel f(t,x) - f(t,y) \parallel^p \leq \Lambda(E \parallel x - y \parallel^p) \tag{3}$$

for all $t \in R, x, y \in L^p(P,K)$, where Λ is a concave and continuous nondecreasing function from R^+ to R^+ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\Lambda(s)} = +\infty$. Here, the symbol $\int_{0^+}$ stands for $\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{+\infty}$. If $\phi(\cdot) \in PAP_T(R, L^p(P,K))$ then $f(\cdot, \phi(\cdot)) \in PAP_T(R, L^p(P,H))$.

Proof. Assume that $f = f_1 + f_2, \phi = \phi_1 + \phi_2$, where $f_1 \in AP_T(R \times L^p(P,K), L^p(P,H))$, $f_2 \in PAP_T^0(R \times L^p(P,K), L^p(P,H))$, $\phi_1 \in AP_T(R, L^p(P,H))$, and $\phi_2 \in PAP_T^0(R, L^p(P,H))$. Consider the decomposition

$$f(t, \phi(t)) = f_1(t, \phi_1(t)) + [f(t, \phi(t)) - f(t, \phi_1(t))] + f_2(t, \phi_1(t)).$$

Since $f_1(\cdot, \phi_1(\cdot)) \in AP_T(R, L^p(P,H))$, it remains to prove that both $[f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))]$ and $f_2(\cdot, \phi_1(\cdot))$ belong to $PAP_T^0(R, L^p(P,H))$. Indeed, using (3), it follows that

$$\begin{aligned} &\frac{1}{2r} \int_{-r}^{r} E \parallel f(t, \phi(t)) - f(t, \phi_1(t)) \parallel^p dt \\ &\quad \leq \frac{1}{2r} \int_{-r}^{r} \Lambda(E \parallel \phi(t) - \phi_1(t) \parallel^p) dt \\ &\quad = \frac{1}{2r} \int_{-r}^{r} \Lambda(E \parallel \phi_2(t) \parallel^p) dt, \end{aligned}$$

noting that Λ is concave, continuous and $\Lambda(0) = 0$, we deduce that

$$\begin{aligned} &\frac{1}{2r} \int_{-r}^{r} \Lambda(E \parallel \phi_2(t) \parallel^p) dt \\ &\quad \leq \Lambda\left(\frac{1}{2r} \int_{-r}^{r} E \parallel \phi_2(t) \parallel^p dt \right) \to 0 \quad \text{as } r \to \infty, \end{aligned}$$

which implies that $[f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))] \in PAP_T^0(R, L^p(P,H))$.

Since $\phi_1(R)$ is relatively compact in $L^p(P,K)$ and f_1 is uniformly continuous on sets of the form $R \times \tilde{K}$ where $\tilde{K} \subset L^p(P,K)$ is compact subset, for $\varepsilon > 0$ there exists $\xi \in (0, \varepsilon)$ such that

$$E \parallel f_1(t,z) - f_1(t,\tilde{z}) \parallel^p \leq \varepsilon, \quad z, \tilde{z} \in \phi_1(R)$$

with $|z-\tilde{z}|<\xi$. Now, fix $z_1,...,z_n\in\phi_1(R)$ such that $\phi_1(R)\subset\bigcup_{j=1}^n B_\xi(z_j,L^p(P,K))$. Obviously, the sets $D_j=\phi_1^{-1}(B_\xi(z_j))$ form an open covering of R, and therefore using the sets $B_1=D_1, B_2=D_2\backslash D_1$ and $B_j=D_j\backslash\bigcup_{k=1}^{j-1}D_k$ one obtains a covering of R by disjoint open sets. For $t\in B_j, \phi_1(t)\in B_\xi(z_j)$,

$$\begin{aligned}
&E\parallel f_2(t,\phi_1(t))\parallel^p\\
&\quad\le 3^{p-1}E\parallel f(t,\phi_1(t))-f(t,z_j)\parallel^p\\
&\qquad+3^{p-1}E\parallel -f_1(t,\phi_1(t))+f_1(t,z_j)\parallel^p+3^{p-1}E\parallel f_2(t,z_j)\parallel\\
&\quad\le 3^{p-1}\Lambda(E\parallel\phi_1(t)-z_j\parallel^p)+3^{p-1}\varepsilon+3^{p-1}E\parallel f_2(t,z_j)\parallel\\
&\quad\le 3^{p-1}\Lambda(\varepsilon)+3^{p-1}\varepsilon+3^{p-1}E\parallel f_2(t,z_j)\parallel.
\end{aligned}$$

Now using the previous inequality it follows that

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}E\parallel f_1(t,\phi_1(t))\parallel^p dt\\
&\quad=\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}E\parallel f_1(t,\phi_1(t))\parallel^p dt\\
&\quad\le 3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}E\parallel f(t,\phi_1(t))-f(t,z_j)\parallel^p dt\\
&\qquad+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}E\parallel f_1(t,\phi_1(t))-f_1(t,z_j)\parallel^p dt\\
&\qquad+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}E\parallel f_2(t,z_j)\parallel^p dt\\
&\quad\le 3^{p-1}\frac{1}{2r}\int_{-r}^{r}[\Lambda(\varepsilon)+\varepsilon]dt+3^{p-1}\sum_{j=1}^{n}\frac{1}{2r}\int_{-r}^{r}E\parallel f_2(t,z_j)\parallel^p dt.
\end{aligned}$$

In view of the above it is clear that $f_2(\cdot,\phi_1(\cdot))$ belongs to $PAP_T^0(R,L^p(P,H))$. This completes the proof.

Lemma 2.4. Assume the sequence of vector-valued functions $\{I_i\}_{i\in Z}$ is pseudo almost periodic, and there is a concave nondecreasing function from R^+ to R^+ such that $\Lambda_i(0)=0,\Lambda_i(s)>0$ for >0 and $\int_{0^+}\frac{ds}{\Lambda_i(s)}=+\infty$,

$$E\parallel I_i(x)-I_i(y)\parallel^p\le\Lambda_i(E\parallel x-y\parallel^p)$$

for all $x,y\in L^p(P,K)$, $i\in Z$. If $\phi\in PAP_T(R,L^p(P,H))\cap UPC(R,L^p(P,H))$ such that $R(\phi)\subset L^p(P,K)$, then $I_i(\phi(t_i))$ is pseudo almost periodic.

Proof. Assume that $\phi=\phi_1+\phi_2$, where $\phi_1\in AP_T(R,L^p(P,H))$, $\phi_2\in PAP_T^0(R,L^p(P,H))$. Fix $\phi\in PAP_T(R,L^p(P,H))\cap UPC(R,L^p(P,H))$, first we show $\phi(t_i)$ is pseudo almost periodic. One can refer to Lemma 37 in [21] that the sequence $\phi(t_i)$ is almost periodic. Next we need to show that $\phi(t_i)\in PAP_0(Z,L^p(P,H))$. By the hypothesis, $\phi,\phi_1\in UPC(R,L^p(P,H))$, so

$\phi_2 \in UPC(R, L^p(P,H))$. Let $0 < \varepsilon < 1$, there exists $0 < \xi < \min\{1, \gamma\}$ such that for $t \in (t_i - \xi, t_i), i \in Z$, we have

$$E \parallel \phi_2(t) \parallel^p \leq (1-\varepsilon) E \parallel \phi_2(t_i) \parallel^p, \quad i \in Z.$$

Since $t_i^j, \quad i \in Z, j = 0, 1, ...$ are equipotentially almost periodic, $\{t_i^1\}$ is an almost periodic sequence. Here we assume a bound of $\{t_i^1\}$ is M_t and $|t_i| \geq |t_{-i}|$; therefore,

$$\begin{aligned}
&\frac{1}{2t_i} \int_{-t_i}^{t_i} E \parallel \phi_2(t) \parallel^p dt \\
&\quad \geq \frac{1}{2t_i} \sum_{j=-i+1}^{i} \int_{t_j - \xi}^{t_j} E \parallel \phi_2(t) \parallel^p dt \\
&\quad \geq \frac{1}{2t_i} \sum_{j=-i+1}^{i} \xi(1-\varepsilon) E \parallel \phi_2(t_j) \parallel^p \\
&\quad \geq \frac{\xi(1-\varepsilon)}{M_t} \frac{1}{2t_i} \sum_{j=-i+1}^{i} E \parallel \phi_2(t_j) \parallel^p .
\end{aligned}$$

Since $\phi_2 \in PAP_T^0(R, L^p(P,H))$, it follows from the inequality above that $\phi_2(t_i) \in PAP_0(Z, L^p(P,H))$. Hence, $\phi(t_i)$ is pseudo almost periodic.

Now, we show $I_i(\phi(t_i))$ is pseudo almost periodic. Let

$$I(t,x) = (t-n) I_n(x), \quad n \leq t < n+1, n \in Z,$$

$$\vartheta(t) = (t-n) \phi_n(t_n), \quad n \leq t < n+1, n \in Z.$$

Since $I_n, \phi(t_n)$ are two pseudo almost periodic sequences, Refer to Lemma 1.7.12. in [36], we get that $I \in PAP(R \times L^p(P,K), L^p(P,H))$, $\vartheta \in PAP(R, L^p(P,K))$. For every $t \in R$, there exists a number $n \in Z$ such that $|t-n| \leq 1$, we have for $x_1, x_2 \in L^p(P,K)$,

$$\begin{aligned}
&E \parallel I(t,x_1) - I(t,x_2) \parallel^p \\
&\quad \leq E \parallel I_n(x_1) - I_n(x_2) \parallel^p \\
&\quad \leq \Lambda_n (E \parallel x_1 - x_2 \parallel^p).
\end{aligned}$$

Similar to the proof of Lemma 2.4, $I(\cdot, \vartheta(\cdot)) \in PAP(R, L^p(P,H))$. Again, similarly as the proof of Lemma 1.7.12 in [36], we have that $I(i, \vartheta(i))$ is a pseudo almost periodic sequence, that is, $I_i(\phi(t_i))$ is pseudo almost periodic. This completes the proof.

Next, we introduce a useful compactness criterion on $PC(R, L^p(P,H))$.

Let $h : R \to R^+$ be a continuous function such that $h(t) \geq 1$ for all $t \in R$ and $h(t) \to \infty$ as $|t| \to \infty$. Define

$$PC_h^0(R, L^p(P,H)) = \left\{ f \in PC(R, L^p(P,H)) : \lim_{|t| \to \infty} \frac{E \parallel f(t) \parallel^p}{h(t)} = 0 \right\}$$

endowed with the norm $\| f \|_h = \sup_{t\in R} \frac{E\|f(t)\|^p}{h(t)}$, it is a Banach space.

Lemma 2.5. A set $B \subseteq PC_h^0(R, L^p(P,H))$ is relatively compact if and only if it verifies the following conditions:

(i) $\lim\limits_{|t|\to\infty} \frac{E\|f(t)\|^p}{h(t)} = 0$ uniformly for $f \in B$.

(ii) $B(t) = \{f(t) : f \in B\}$ is relatively compact in $L^p(P,H)$ for every $t \in R$.

(iii) The set B is equicontinuous on each interval $(t_i, t_{i+1})(i \in Z)$.

One can refer to Lemma 4.1 in [27] for the proof of Lemma

Lemma 2.6 (Leray-Schauder nonlinear alternative [35]). Let X be a Banach space with $D \subset X$ closed and convex. Assume U is a relatively open subset of D with $0 \in U$ and $\Psi : \overline{U} \to D$ is a compact map, then either

(i) Ψ has a fixed point in $\overline{U}$, or

(ii) there is a point $x \in \partial U$ and $\lambda \in (0,1)$ with $x \in \lambda\Psi(x)$.

3 Main results

In this section, we investigate the existence of p-mean piecewise pseudo almost periodic mild solution for system (1)-(2). To do this, we first consider the existence of p-mean piecewise pseudo almost periodic mild solutions to the following linear stochastic differential equation

$$dx(t) = [Ax(t) + g(t)]dt + f(t)dW(t), \quad t \in R, t \neq t_k, i \in Z, \tag{4}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = \gamma_i, \quad i \in Z, \tag{5}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^p(P,H)$ such that for all $t \geq 0, \| T(t) \| \leq Me^{-\delta t}$ with $M, \delta > 0$. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. Furthermore, $g : R \to L^p(P,H)$, $f : R \to L^p(P, L_2^0)$ are two stochastic processes, γ_i is an p-mean pseudo almost periodic sequence.

Definition 3.1. An $\mathcal{F}_t$ -progressively measurable process $\{x(t)\}_{t\in R}$ is called a mild solution of system (4)-(5) if for any $t \in R, t > \sigma, \sigma \neq t_i, i \in Z$,

$$\begin{aligned} x(t) = T(t-\sigma)x(\sigma) &+ \int_\sigma^t T(t-s)g(s)ds \\ &+ \int_\sigma^t T(t-s)f(s)dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)\gamma_i. \end{aligned} \tag{6}$$

Theorem 3.1. *Assume $g \in PAP_T(R, L^p(P,H)), f \in PAP_T(R, L^p(P, L_2^0)), \{\gamma_i, i \in Z\}$ is an p-mean pseudo almost periodic sequence, then system (4)-(5) has a mild solution $x \in PAP_T(R, L^p(P,H))$.*

Proof. Consider for each $i \in Z$, the integrals

$$x(t) = \int_{-\infty}^{t} T(t-s)g(s)ds + \int_{-\infty}^{t} T(t-s)f(s)dW(s) + \sum_{t_i<t} T(t-t_i)\gamma_i \tag{7}$$

for each $t \in R$. Next we aim to prove (7) is an p-mean piecewise pseudo almost periodic mild solution of system (4)-(5).

Since $g \in PAP_T(R, L^p(P,H))$, $f \in PAP_T(R, L^p(P, L_2^0))$, $\gamma_i \in PAP(Z, L^p(P,H))$, there exist $g_1 \in AP_T(R, L^p(P,H))$, $f_1 \in AP_T(R, L^p(P, L_2^0))$, $\gamma_{1,i} \in AP(Z, L^p(P,H))$ and $g_2 \in PAP_T^0(R, L^p(P,H))$, $f_2 \in PAP_T^0(R, L^p(P, L_2^0))$, $\gamma_{2,i} \in PAP_0(Z, L^p(P,H))$, such that $g = g_1 + g_2, f = f_1 + f_2, \gamma_i = \gamma_{1,i} + \gamma_{2,i}$. Hence,

$$\begin{aligned}
&x(t)\\
&= \int_{-\infty}^{t} T(t-s)[g_1(s)+g_2(s)]ds + \int_{-\infty}^{t} T(t-s)[f_1(s)+f_2(s)]dW(s)\\
&\quad + \sum_{t_i<t} T(t-t_i)[\gamma_{1,i}+\gamma_{2,i}]\\
&= \Big[\int_{-\infty}^{t} T(t-s)g_1(s)ds + \int_{-\infty}^{t} T(t-s)f_1(s)dW(s)\\
&\quad + \sum_{t_i<t} T(t-t_i)\gamma_{1,i}\Big]\\
&\quad + \Big[\int_{-\infty}^{t} T(t-s)g_2(s)ds + \int_{-\infty}^{t} T(t-s)f_2(s)dW(s)\\
&\quad + \sum_{t_i<t} T(t-t_i)\gamma_{2,i}\Big]\\
&=: F(t) + \Phi(t).
\end{aligned}$$

In order to prove (7) is an p-mean pseudo almost periodic mild solution, we only need to verify $F(t) \in AP_T(R, L^p(P,H))$ and $\Phi(t) \in PAP_T^0(R, L^p(P,H))$. Thus, the following verification procedure is divided into three steps.

Step 1. $F \in UPC(R, L^p(P,H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup and $\| T(t) \| \leq Me^{-\delta t}, t \geq 0$, for any $\varepsilon > 0$, there exists $0 < \xi < \min\{(\frac{\varepsilon}{5\tilde{g}_1})^{1/p}, (\frac{\varepsilon}{5\tilde{f}_1})^{2(p-1)/p}\}$ such that $0 < t' - t'' < \xi$, we have for $p > 2$,

$$\begin{aligned}
&\| T(t'-t'') - I \|^p\\
&\qquad \leq \min\left\{\frac{\delta^p\varepsilon}{5\tilde{g}_1}, \frac{(\frac{p\delta}{p-2})^{(p-2)/2}\frac{p\delta}{2}\varepsilon}{5\tilde{f}_1}, \frac{(1-e^{-\delta\alpha})^p\varepsilon}{5\tilde{\gamma}_1}\right\},
\end{aligned}$$

where $\tilde{g}_1 = 6^{p-1}M^p \parallel g_1 \parallel_\infty^p$, $\tilde{f}_1 = 6^{p-1}M^pC_p \parallel f_1 \parallel_\infty^p$, $\tilde{\gamma}_1 = 3^{p-1}M^p \parallel \gamma_{1,i} \parallel_\infty^p$. Using Hölder's inequality and the Ito integral [37], we have

$$
\begin{aligned}
&E \parallel F(t') - F(t'') \parallel^p \\
&\quad \le 3^{p-1}E\left\| \int_{-\infty}^{t'} T(t'-s)g_1(s)ds - \int_{-\infty}^{t''} T(t''-s)g_1(s)ds \right\|^p \\
&\qquad +3^{p-1}E\left\| \int_{-\infty}^{t'} T(t'-s)f_1(s)dW(s) - \int_{-\infty}^{t''} T(t''-s)f_1(s)dW(s) \right\|^p \\
&\qquad +3^{p-1}E\left\| \sum_{t_i<t'} T(t'-t_i)\gamma_{1,i} - \sum_{t_i<t''} T(t''-t_i)\gamma_{1,i} \right\|^p \\
&\quad \le 6^{p-1}E\left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]g_1(s)ds \right\|^p \\
&\qquad +6^{p-1}E\left\| \int_{t''}^{t'} T(t'-s)g_1(s)ds \right\|^p \\
&\qquad +6^{p-1}E\left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]f_1(s)dW(s) \right\|^p \\
&\qquad +6^{p-1}E\left\| \int_{t''}^{t'} T(t'-s)f_1(s)dW(s) \right\|^p \\
&\qquad +3^{p-1}E\left\| \sum_{t_i<t''} T(t''-t_i)[T(t'-t'')-I]\gamma_{1,i} \right\|^p \\
&\quad \le 6^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds \right)^{p-1} \\
&\qquad \times\left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}E \parallel g_1(s) \parallel^p ds \right) \\
&\qquad +6^{p-1}M^p\left(\int_{t''}^{t'} e^{-\delta(t'-s)}ds \right)^{p-1}\left(\int_{t''}^{t'} e^{-\delta(t'-s)}E \parallel g_1(s) \parallel^p ds \right) \\
&\qquad +6^{p-1}M^pC_pE\left[\int_{-\infty}^{t''} e^{-2\delta(t''-s)} \parallel T(t'-t'')-I \parallel^2 \parallel f_1(s) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
&\qquad +6^{p-1}M^pC_pE\left[\int_{t''}^{t'} e^{-2\delta(t'-s)} \parallel f_1(s) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
&\qquad +3^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\sum_{t_i<t''} e^{-\delta(t''-t_i)} \right)^{p-1} \\
&\qquad \times\left(\sum_{t_i<t''} e^{-\delta(t''-t_i)}E \parallel \gamma_{1,i} \parallel^p \right) \\
&\quad \le 6^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds \right)^p \sup_{s\in R} E \parallel g_1(s) \parallel^p
\end{aligned}
$$

$$
\begin{aligned}
&+6^{p-1}M^p\left(\int_{t''}^{t'} e^{-\delta(t'-s)}ds\right)^p \sup_{s\in R} E \parallel g_1(s) \parallel^p \\
&+6^{p-1}M^pC_p \parallel T(t'-t'')-I \parallel^p \left(\int_{-\infty}^{t''} e^{-\frac{p}{p-2}\delta(t''-s)}ds\right)^{\frac{p-2}{p}} \\
&\times\left(\int_{-\infty}^{t''} e^{-\frac{p}{2}\delta(t''-s)}ds\right)\sup_{s\in R} \parallel f_1(s) \parallel^p_{L_2^0} \\
&+6^{p-1}M^pC_p\left(\int_{t''}^{t'} e^{-\frac{p}{p-2}\delta(t'-s)}ds\right)^{\frac{p-2}{p}} \\
&\times\left(\int_{t''}^{t'} e^{-\frac{p}{2}\delta(t'-s)}ds\right)\sup_{s\in R} \parallel f_1(s) \parallel^p_{L_2^0} \\
&+3^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\right)^p \sup_{i\in Z} E \parallel \gamma_{1,i} \parallel^p \\
\le\ & 6^{p-1}M^p \parallel g_1 \parallel^p_\infty \frac{\delta^p\varepsilon}{5\tilde{g}_1}\left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds\right)^p \\
&+6^{p-1}M^p \parallel g_1 \parallel^p_\infty \left[\left(\frac{\varepsilon}{5\tilde{g}_1}\right)^{1/p}\right]^p \\
&+6^{p-1}M^pC_p \parallel f_1 \parallel^p_\infty \frac{(\frac{p\delta}{p-2})^{(p-2)/p}\frac{p\delta}{2}\varepsilon}{5\tilde{f}_1}\left(\int_{-\infty}^{t''} e^{-\frac{p}{p-2}\delta(t''-s)}ds\right)^{\frac{p-2}{p}} \\
&\times\left(\int_{-\infty}^{t''} e^{-\frac{p}{2}\delta(t''-s)}ds\right) \\
&+6^{p-1}M^pC_p \parallel f_1 \parallel^p_\infty \left[\left(\frac{\varepsilon}{5\tilde{f}_1}\right)^{p/2(p-1)}\right]^{2(p-2)/p} \\
&+3^{p-1}M^p\frac{(1-e^{-\delta\alpha})^p\varepsilon}{5\tilde{\gamma}_1}\left(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\right)^p \parallel \gamma_{1,i} \parallel^p_\infty \\
<\ & \frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}=\varepsilon.
\end{aligned}
$$

Consequently, $F \in UPC(R, L^p(P,H))$.

Step 2. $F \in AP_T(R, L^p(P,H))$.

Let $t_i < t \le t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of R formed by ε-periods of F. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \alpha/2\}$, we have

$$
\begin{aligned}
&E \parallel F(t+\tau)-F(t) \parallel^p \\
&\quad\le 3^{p-1}E\left\| \int_{-\infty}^{t} T(t-s)[g_1(s+\tau)-g_1(s)]ds \right\|^p \\
&\qquad +3^{p-1}E\left\| \int_{-\infty}^{t} T(t-s)[f_1(s+\tau)-f_1(s)]dW(s) \right\|^p
\end{aligned}
$$

$$+3^{p-1}E\Big\| \sum_{t_i<t+\tau} T(t+\tau-t_i)\gamma_{1,i} - \sum_{t_i<t} T(t-t_i)\gamma_{1,i} \Big\|^p$$
$$= \sum_{k=1}^{3} J_k.$$

Using Höder's inequality, it follows that

$$\begin{aligned}
&J_1 \\
&\leq 3^{p-1}M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\quad\times\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\Big) \\
&\leq 3^{p-1}M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\quad\times\Big[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds \\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds \\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds \\
&\quad+\int_{t_i}^{t} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\Big].
\end{aligned}$$

Since $g_1 \in AP_T(R, L^p(P,H))$, one has

$$E\parallel g_1(t+\tau)-g_1(t)\parallel^p < \varepsilon$$

for all $t \in [t_j+\eta, t_{j+1}-\eta]$, $j \in Z, j \leq i$, and $t-s \geq t-t_i+t_i-(t_{j+1}-\eta) \geq t-t_i+\alpha(i-1-j)+\eta$. Then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p \\
&\quad\leq \varepsilon\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}ds \\
&\quad\leq \frac{\varepsilon}{\delta}\sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)} \\
&\quad\leq \frac{\varepsilon}{\delta}\sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j-1)}
\end{aligned}$$

$$\leq \frac{\varepsilon}{\delta(1-e^{-\delta\alpha})},$$

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s) \parallel^p ds \\
&\quad\leq 2^{p-1}\sup_{s\in R}E \parallel g_1(s) \parallel^p \sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\delta(t-s)}ds \\
&\quad\leq 2^{p-1} \parallel g_1 \parallel_\infty^p \varepsilon e^{\delta\eta}\sum_{j=-\infty}^{i-1} e^{-\delta(t-t_j)} \\
&\quad\leq 2^{p-1} \parallel g_1 \parallel_\infty^p \varepsilon e^{\delta\eta}e^{-\delta(t-t_i)}\sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j)} \\
&\quad\leq \frac{2^{p-1} \parallel g_1 \parallel_\infty^p e^{\delta\alpha/2}\varepsilon}{1-e^{-\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s) \parallel^p ds \leq \tilde{M}_1\varepsilon,$$

$$\int_{t_i}^{t} e^{-\delta(t-s)}E \parallel g_1(s+\tau)-g_1(s) \parallel^p ds \leq \tilde{M}_2\varepsilon,$$

where $\tilde{M}_1, \tilde{M}_2$ are some positive constants. Therefore, we get that $J_1 \leq \bar{N}_1\varepsilon$ for a positive constant $\bar{N}_1$. Using Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned}
&J_2 \\
&\quad\leq 3^{p-1}C_pE\bigg[\int_{-\infty}^{t} \parallel T(t-s) \parallel^2 \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^2 ds\bigg]^{p/2} \\
&\quad\leq 3^{p-1}C_pM^pE\bigg[\int_{-\infty}^{t} e^{-2\delta(t-s)} \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^2 ds\bigg]^{p/2} \\
&\quad\leq 3^{p-1}C_pM^p\bigg(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\bigg)^{\frac{p-2}{p}} \\
&\qquad\times\bigg[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \\
&\qquad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \\
&\qquad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds
\end{aligned}$$

$$+\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds\Big].$$

Since $f_1 \in AP_T(R, L^p(P, L_2^0))$, one has

$$E \parallel f_1(t+\tau)-f_1(t) \parallel_{L_2^0}^p < \varepsilon$$

for all $t \in [t_j+\eta, t_{j+1}-\eta]$ and $j \in Z, j \le i$. Then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \\
&\quad\le \varepsilon \sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)}ds \\
&\quad\le \frac{2}{\delta p}\sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta(t-t_{j+1}+\eta)} \\
&\quad\le \frac{2\varepsilon}{\delta p}\sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j-1)} \\
&\quad\le \frac{2\varepsilon}{\delta p(1-e^{-\delta\alpha})},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \\
&\quad\le 2^{p-1}\sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \sum_{j=-\infty}^{i-1}\int_{t_j}^{t_{j+1}+\eta} e^{-\frac{p}{2}\delta(t-s)}ds \\
&\quad\le 2^{p-1}\sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta}e^{-\frac{p}{2}\delta(t-t_i)}\sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\quad\le 2^{p-1}\sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta}e^{-\frac{p}{2}\delta(t-t_i)}\sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\quad\le \frac{2^{p-1} \parallel f_1 \parallel_\infty^p e^{\delta\alpha/4}\varepsilon}{1-e^{-\frac{p}{2}\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \le \tilde{M}_3\varepsilon,$$

$$\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel_{L_2^0}^p ds \le \tilde{M}_4\varepsilon,$$

where $\tilde{M}_3, \tilde{M}_4$ are some positive constants. Therefore, we get that $J_2 \leq \bar{N}_2\varepsilon$ for a positive constant $\bar{N}_2$. For $p = 2$, we have

$$\begin{aligned}
&J_2\\
&\quad\leq 3M^2E\int_{-\infty}^{t} e^{-2\delta(t-s)} \parallel f_1(s+\tau)-f_1(s) \parallel^2_{L_2^0} ds\\
&\quad\leq 3M^2\Big[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel^2_{L_2^0} ds\\
&\qquad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-2\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel^2_{L_2^0} ds\\
&\qquad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel^2_{L_2^0} ds\\
&\qquad+\int_{t_i}^{t} e^{-2\delta(t-s)}E \parallel f_1(s+\tau)-f_1(s) \parallel^2_{L_2^0} ds\Big].
\end{aligned}$$

Similarly, we get that $J_2 \leq \bar{N}_3\varepsilon$ for a positive constant $\bar{N}_3$. For any $\varepsilon > 0$, by Lemma 2.1, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, for every $\tau \in \Omega_\varepsilon$, there exists at least one number $q \in Q_\varepsilon$ such that $|t^q - \tau| < \varepsilon, i \in Z$ and $E \parallel \gamma_{1,i+q} - \gamma_{1,i} \parallel^p < \varepsilon, q \in Q_\varepsilon, i \in Z$. Then,

$$\begin{aligned}
&J_3\\
&\quad\leq 3^{p-1}E\Big[\sum_{t_i<t} \parallel T(t-t_i) \parallel\parallel \gamma_{1,i+q}-\gamma_{1,i} \parallel\Big]^p\\
&\quad\leq 3^{p-1}M^pE\Big[\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel \gamma_{1,i+q}-\gamma_{1,i} \parallel^p\Big)\Big]\\
&\quad\leq 3^{p-1}M^p\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^p E \parallel \gamma_{1,i+q}-\gamma_{1,i} \parallel^p\\
&\quad\leq \frac{3^{p-1}M^p\varepsilon}{(1-e^{-\delta\alpha})^p}.
\end{aligned}$$

Hence, $F \in AP_T(R, L^p(P, H))$.

Step 3. $\Phi \in PAP_T^0(R, L^p(P, H))$.

In fact, for $r > 0$, one has

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} E \parallel \Phi(t) \parallel^p dt\\
&\qquad\leq 3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\Big\| \int_{-\infty}^{t} T(t-s)g_2(s)ds\Big\|^p dt\\
&\qquad\quad+3^{p-1}\frac{1}{2r}\int_{-r}^{r} E\Big\| \int_{-\infty}^{t} T(t-s)f_2(s)dW(s)\Big\|^p dt
\end{aligned}$$

$$+3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Big\|\sum_{t_i<t}T(t-t_i)\gamma_{2,i}\Big\|^p dt.$$

Then, by Hölder's inequality, we obtains that

$$\begin{aligned}
&3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}T(t-s)g_2(s)ds\Big\|^p dt\\
&=3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Big\|\int_{0}^{\infty}T(s)g_2(t-s)ds\Big\|^p dt\\
&\le 3^{p-1}M^p\frac{1}{2r}\int_{-r}^{r}\Big(\int_0^\infty e^{-\delta s}ds\Big)^{p-1}\int_0^\infty e^{-\delta s}E\parallel g_2(t-s)\parallel^p dsdt\\
&=3^{p-1}M^p\Big(\int_0^\infty e^{-\delta s}ds\Big)^{p-1}\int_0^\infty e^{-\delta s}ds\frac{1}{2r}\int_{-r}^{r}E\parallel g_2(t-s)\parallel^p dt.
\end{aligned}$$

By Hölder's inequality and the Ito integral, we have for $p>2$,

$$\begin{aligned}
&3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}T(t-s)f_2(s)dW(s)\Big\|^p dt\\
&=3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Big\|\int_{0}^{\infty}T(s)f_2(t-s)dW(s)\Big\|^p dt\\
&\le 3^{p-1}C_p\frac{1}{2r}\int_{-r}^{r}E\Big[\int_0^\infty e^{-2s}\parallel f_2(t-s)\parallel^2_{L_2^0}ds\Big]^{p/2}dt\\
&\le 3^{p-1}M^pC_p\frac{1}{2r}\int_{-r}^{r}\Big(\int_0^\infty e^{-\frac{p}{p-2}\delta s}ds\Big)^{\frac{p-2}{p}}\\
&\quad\times\int_0^\infty e^{-\frac{p}{2}\delta s}E\parallel f_2(t-s)\parallel^p_{L_2^0}dsdt\\
&=3^{p-1}M^pC_p\Big(\int_0^\infty e^{-\frac{p-2}{p}\delta s}ds\Big)^{\frac{p-2}{p}}\int_0^\infty e^{-\frac{p}{2}\delta s}ds\\
&\quad\times\frac{1}{2r}\int_{-r}^{r}E\parallel f_2(t-s)\parallel^p_{L_2^0}dt.
\end{aligned}$$

For $p=2$, we have

$$\begin{aligned}
&\frac{3}{2r}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}T(t-s)f_2(s)dW(s)\Big\|^2 dt\\
&\le 3M^2\frac{1}{2r}\int_{-r}^{r}\int_0^\infty e^{-2s}E\parallel f_2(t-s)\parallel^2_{L_2^0}dsdt\\
&=3M^2\Big(\int_0^\infty e^{-2\delta s}ds\Big)\frac{1}{2r}\int_{-r}^{r}E\parallel f_2(t-s)\parallel^p_{L_2^0}dt.
\end{aligned}$$

Since $g_2\in PAP_T^0(R,L^p(P,H))$, $f_2\in PAP_T^0(R,L^p(P,L_2^0))$, it follows that $g_2(\cdot-s)\in PAP_T^0(R,\ L^p(P,H))$, $f_2(\cdot-s)\in PAP_T^0(R,L^p(P,L_2^0))$ for each $s\in R$ by

Remark 2.1, hence

$$3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\left\|\int_{-\infty}^{t}T(t-s)g_2(s)ds\right\|^p dt\to 0\quad \text{as } r\to\infty,$$

$$3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\left\|\int_{-\infty}^{t}T(t-s)f_2(s)dW(s)\right\|^p dt\to 0\quad \text{as } r\to\infty$$

for all $s\in R$.

For a given $i\in Z$, define the function $v(t)$ by $v(t)=T(t-t_i)\gamma_{2,i}, t_i<t\le t_{i+1}$, then

$$\begin{aligned}&\lim_{t\to\infty}E\parallel v(t)\parallel^p\\ &\quad=\lim_{t\to\infty}E\parallel T(t-t_i)\gamma_{2,i}\parallel^p\le\lim_{t\to\infty}M^pe^{-p\delta(t-t_i)}\sup_{i\in Z}E\parallel\gamma_{2,i}\parallel^p=0.\end{aligned}$$

Thus $v\in PC_T^0(R,L^p(P,H))\subset PAP_T^0(R,L^p(P,H))$. Define $v_j:R\to L^p(P,H)$ by

$$v_j(t)=T(t-t_{i-j})\gamma_{2,i-j},\quad t_i<t\le t_{i+1}, j\in N.$$

So $v_j\in PAP_T^0(R,L^p(P,H))$. Moreover,

$$\begin{aligned}&E\parallel v_j(t)\parallel^p\\ &\quad=E\parallel T(t-t_{i-j})\gamma_{2,i-j}\parallel^p\\ &\quad\le M^pe^{-p\delta(t-t_{i-j})}\sup_{i\in Z}E\parallel\gamma_{2,i}\parallel^p\\ &\quad\le M^pe^{-p\delta(t-t_i)}e^{-p\delta\alpha j}\sup_{i\in Z}E\parallel\gamma_{2,i}\parallel^p.\end{aligned}$$

Therefore, the series $\sum_{j=0}^{\infty}v_j$ is uniformly convergent on R. By Lemma 2.2, one has

$$\sum_{t_i<t}T(t-t_i)\gamma_{2,i}=\sum_{j=0}^{\infty}v_j(t)\in PAP_T^0(R,L^p(P,H)),$$

that is

$$3^{p-1}\frac{1}{2r}\int_{-r}^{r}E\left\|\sum_{t_i<t}T(t-t_i)\gamma_{2,i}\right\|^p dt\to 0\quad\text{as } r\to\infty.$$

Using Lebesgue's dominated convergence theorem, we have $\Phi\in PAP_T^0(R,L^p(P,H))$.

Finally, to prove that x satisfies (6) for all $t\ge s$, all $s\in R$. Fix $\sigma,\sigma\ne t_i,i\in Z$, we have

$$\begin{aligned}x(\sigma)=&\int_{-\infty}^{\sigma}T(\sigma-s)g(s)ds\\ &+\int_{-\infty}^{\sigma}T(\sigma-s)f(s)dW(s)+\sum_{t_i<\sigma}T(\sigma-t_i)\gamma_i.\end{aligned}$$

Since $\{T(t) : t \geq 0\}$ is a C_0-semigroup, we have for all $t \in R$

$$\begin{aligned}
&x(t)\\
&\quad= \int_{-\infty}^{t} T(t-s)g(s)ds + \int_{-\infty}^{t} T(t-s)f(s)dW(s) + \sum_{t_i<t} T(t-t_i)\gamma_i\\
&\quad= \int_{-\infty}^{\sigma} T(t-s)g(s)ds + \int_{-\infty}^{\sigma} T(t-s)f(s)dW(s) + \sum_{t_i<\sigma} T(t-t_i)\gamma_i\\
&\qquad+ \int_{\sigma}^{t} T(t-s)g(s)ds + \int_{\sigma}^{t} T(t-s)f(s)dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)\gamma_i\\
&\quad= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t} T(t-s)g(s)ds + \int_{\sigma}^{t} T(t-s)f(s)dW(s)\\
&\qquad+ \sum_{\sigma<t_i<t} T(t-t_i)\gamma_i.
\end{aligned}$$

Hence $x \in PAP_T(R, L^p(P,H))$ is an p-mean piecewise pseudo almost periodic mild solution to system (4)-(5). This completes the proof.

Now, we establish the existence theorem of p-mean piecewise pseudo almost periodic mild solutions to partial impulsive stochastic differential equation (1)-(2). For that, we make the following hypotheses:

(H1) A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t) : t \geq 0\}$. Moreover, $T(t)$ is compact for $t > 0$.

(H2) The functions $g \in PAP_T(R \times L^p(P,K), L^p(P,H))$, $f \in PAP_T(R \times L^p(P,K), L^p(P, L_2^0))$, and for each $t \in R$, $\psi_1, \psi_2 \in L^p(P,K)$,

$$E \parallel g(t,\psi_1) - g(t,\psi_2) \parallel^p + E \parallel f(t,\psi_1) - f(t,\psi_2) \parallel^p_{L_2^0} \leq \Lambda(E \parallel \psi_1 - \psi_2 \parallel^p),$$

where Λ is a concave and continuous nondecreasing function from R^+ to R^+ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\Lambda(s)} = +\infty$.

(H3) For any $\beta > 0$, there exist a constant $\mu > 0$ and nondecreasing continuous function $\Theta : R^+ \to R^+$ such that, for all $t \in R$, and $x \in L^p(P,K)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel g(t,x) \parallel^p + E \parallel f(t,x) \parallel^p_{L_2^0} \leq \beta\Theta(E \parallel x \parallel^p).$$

(H4) The functions $I_i \in PAP(Z, L^p(P,H))$, and for each $t \in R$, $\psi_1, \psi_2 \in L^p(P,H), i \in Z$,

$$E \parallel I_i(\psi_1) - I_i(\psi_2) \parallel^p \leq \tilde{\Lambda}_i(E \parallel \psi_1 - \psi_2 \parallel^p),$$

where $\tilde{\Lambda}_i$ are concave and continuous nondecreasing functions from R^+ to R^+ such that $\tilde{\Lambda}_i(0) = 0, \tilde{\Lambda}_i(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\tilde{\Lambda}_i(s)} = +\infty$.

(H5) For any $\beta > 0$, there exist a constant $\mu > 0$ and nondecreasing continuous function $\tilde{\Theta}_i : R^+ \to R^+, i \in Z$, such that, for all $t \in R$, and $x \in L^p(P, H)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel I_i(x) \parallel^p \leq \beta \tilde{\Theta}_i(E \parallel x \parallel^p).$$

Definition 3.2. An $\mathcal{F}_t$ -progressively measurable process $\{x(t)\}_{t\in R}$ is called a mild solution of system (1)-(2) if for any $t \in R, t > \sigma, \sigma \neq t_i, i \in Z$,

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)g(s, x(s))ds + \int_\sigma^t T(t-s)f(s, x(s))dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)I_i(x(t_i)). \quad (8)$$

Theorem 3.2. *Assume that assumptions* (H1)-(H5) *are satisfied. Then system* (1)-(2) *has a mild solution* $x \in PAP_T(R, L^p(P, H))$.

Proof. Consider the operator $\Psi : PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H)) \to PC(R, L^p(P, H))$ defined by

$$(\Psi x)(t) = \int_{-\infty}^t T(t-s)g(s, x(s))ds + \int_{-\infty}^t T(t-s)f(s, x(s))dW(s) + \sum_{t_i<t} T(t-t_i)I_i(x(t_i)), \quad t \in R.$$

We next show that Ψ has a fixed point in $PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$ and divide the proof into several steps.

Step 1. For every $x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$, $\Psi x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$.

Let $x(\cdot) \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$, by (H2), (H4) and Lemmas 2.3, 2.4, we deduce that $g(\cdot, x(\cdot)), f(\cdot, x(\cdot)) \in PAP_T(R, L^p(P, H))$ and $I_i(x(t_i)) \in PAP(Z, L^p(P, H))$ Similarly as the proof of Theorem 3.1, one has $\Psi x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H))$.

Step 2. Ψ maps bounded sets into bounded sets in $PAP_T(L^p(P, H)) \cap UPC(R, L^p(P, H))$.

Indeed, let $r^* > 0$ and $x \in B_{r^*} = \{x \in PAP_T(R, L^p(P, H)) \cap UPC(R, L^p(P, H)) : E \parallel x \parallel^p \leq r^*\}$. It is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $x \in B_{r^*}$ one has $E \parallel \Phi x \parallel^p \leq \mathcal{L}$. Let $\beta > 0$ be fixed. By (H3) and (H5) it follows that there exist a positive constant μ such that, for all $t \in R$ and $x \in L^p(P, H)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel g(t, x) \parallel^p + E \parallel f(t, x) \parallel^p_{L_2^0} \leq \beta\Theta(E \parallel x \parallel^p),$$

$$E \parallel I_i(x) \parallel^p \leq \beta\tilde{\Theta}_i(E \parallel x \parallel^p), i \in Z.$$

Let

$$\nu = \sup_{t\in R}\{E \parallel g(t, x) \parallel^p, E \parallel f(t, x) \parallel^p_{L_2^0} : E \parallel x \parallel^p \leq \mu\},$$

$$\nu_1 = \sup_{t\in R, i\in Z}\{E \parallel I_i(x) \parallel^p : E \parallel x \parallel^p \leq \mu\}.$$

Thus, for all $t \in R$ and $x \in L^p(P, H)$,

$$E \parallel g(t,x) \parallel^p + E \parallel f(t,x) \parallel^p_{L_2^0} \leq \beta\Theta(E \parallel x \parallel^p) + \nu, \tag{9}$$

$$E \parallel I_i(x) \parallel^p \leq \beta\tilde{\Theta}_i(E \parallel x \parallel^p) + \nu_1, i \in Z. \tag{10}$$

Note that, for β sufficiently small, we can choose $M^* > 0$ such that for $p > 2$,

$$\frac{M^*}{N_1[\beta\Theta(M^*) + \nu] + N_2[\beta \sup_{i\in Z}\tilde{\Theta}_i(M^*) + \nu_1]} > 1, \tag{11}$$

where $N_1 = 3^{p-1}M^p[\frac{1}{\delta^p} + C_p(\frac{p-2}{p\delta})^{\frac{p-2}{p}}\frac{2}{p\delta}]$, $N_2 = 3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}$. For the case of $p = 2$, take $N_1 = 3M^2[\frac{1}{\delta^2} + \frac{1}{2\delta}]$, $N_2 = 3M^2\frac{1}{(1-e^{-\delta\alpha})^2}$.

Let $x \in B_{r^*}$, and $t \in R$. By (H1), (9), (10), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned}
&E \parallel (\Psi x)(t) \parallel^p \\
&\leq 3^{p-1}E\left\| \int_{-\infty}^t T(t-s)g(s,x(s))ds \right\|^p \\
&\quad +3^{p-1}E\left\| \int_{-\infty}^t T(t-s)f(s,x(s))dW(s) \right\|^p \\
&\quad +3^{p-1}E\left\| \sum_{t_i<t} T(t-t_i)I_i(x(t_i)) \right\|^p \\
&\leq 3^{p-1}M^p\left(\int_{-\infty}^t e^{-\delta(t-s)}ds \right)^{p-1}\left(\int_{-\infty}^t e^{-\delta(t-s)}E \parallel g(s,x(s)) \parallel^p ds \right) \\
&\quad +3^{p-1}C_pM^pE\left(\int_{-\infty}^t e^{-2\delta(t-s)} \parallel f(s,x(s)) \parallel^2_{L_2^0} ds \right)^{p/2} \\
&\quad +3^{p-1}M^pE\left[\left(\sum_{t_i<t} e^{-\delta(t-t_i)} \right)^{p-1}\left(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p \right) \right] \\
&\leq 3^{p-1}M^p\frac{1}{\delta^{p-1}}\left(\int_{-\infty}^t e^{-\delta(t-s)}[\beta\Theta(E \parallel x(s) \parallel^p) + \nu]ds \right) \\
&\quad +3^{p-1}M^pC_p\left(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)}ds \right)^{\frac{p-2}{p}} \\
&\quad \times\left(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)}[\beta\Theta(E \parallel x(s) \parallel^p) + \nu]ds \right) \\
&\quad +3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}\left(\sum_{t_i<t} e^{-\delta(t-t_i)}[\beta\tilde{\Theta}_i(E \parallel x(t_i) \parallel^p) + \nu_1] \right) \\
&\leq 3^{p-1}M^p\frac{1}{\delta^p}[\beta\Theta(r^*) + \nu] + 3^{p-1}M^pC_p\left(\frac{p-2}{p\delta} \right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(r^*) + \nu]
\end{aligned}$$

$$+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]:=\mathcal{L}_1.$$

For $p=2$, we have

$$\begin{aligned}&E\parallel(\Psi x)(t)\parallel^2\\&\quad\le 3M^2\frac{1}{\delta^2}[\beta\Theta(r^*)+\nu]+3M^2\frac{1}{2\delta}[\beta\Theta(r^*)+\nu]\\&\qquad+3M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]:=\mathcal{L}_2.\end{aligned}$$

Take $\mathcal{L}=\max\{\mathcal{L}_1,\mathcal{L}_2\}$. Then for each $x\in B_{r^*}$, we have $E\parallel\Psi x\parallel^p\le\mathcal{L}$.

Step 3. $\Psi: PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))\to PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$ is continuous.

Let $\{x^{(n)}\}\subseteq B_{r^*}$ with $x^{(n)}\to x(n\to\infty)$ in $PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$, then there exists a bounded subset $\tilde{K}\subseteq L^p(P,K)$ such that $R(x)\subseteq\tilde{K}, R(x^n)\subseteq\tilde{K}, n\in N$. By the assumption (H2) and (H4), for any $\varepsilon>0$, there exists $\xi>0$ such that $x,y\in K$ and $E\parallel x-y\parallel^p<\xi$ implies that

$$E\parallel g(s,x(s))-g(s,y(s))\parallel^p<\varepsilon\quad\text{for all } t\in R,$$

$$E\parallel f(s,x(s))-f(s,y(s))\parallel^p_{L_2^0}<\varepsilon\quad\text{for all } t\in R,$$

and

$$E\parallel I_i(x)-I_i(y)\parallel^p<\varepsilon\quad\text{for all } i\in Z,$$

For the above ξ there exists n_0 such that $E\parallel x^{(n)}(t)-x(t)\parallel^p<\varepsilon$ for $n>n_0$ and $t\in R$, then for $n>n_0$, we have

$$E\parallel g(s,x^{(n)}(s))-g(s,x(s))\parallel^p<\varepsilon\quad\text{for all } t\in R,$$

$$E\parallel f(s,x^{(n)}(s))-f(s,x(s))\parallel^p_{L_2^0}<\varepsilon\quad\text{for all } t\in R,$$

and

$$E\parallel I_i(x^{(n)})-I_i(x)\parallel^p<\varepsilon\quad\text{for all } i\in Z.$$

Then, by (H2), (H4) and Hölder's inequality, we have that for $p>2$,

$$\begin{aligned}&E\parallel(\Psi x^{(n)})(t)-(\Psi x)(t)\parallel^p\\&\quad\le 3^{p-1}E\left\|\int_{-\infty}^t T(t-s)[g(s,x^{(n)}(s))-g(s,x(s))]ds\right\|^p\\&\qquad+3^{p-1}E\left\|\int_{-\infty}^t T(t-s)[f(s,x^{(n)}(s))-f(s,x(s))]dW(s)\right\|^p\\&\qquad+3^{p-1}E\left\|\sum_{t_i<t}T(t-t_i)[I_i(x^{(n)}(t_i))-I_i(x(t_i))]\right\|^p\\&\quad\le 3^{p-1}M^p\left(\int_{-\infty}^t e^{-\delta(t-s)}ds\right)^{p-1}\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}E\parallel g(s,x^{(n)}(s))-g(s,x(s))\parallel^p ds\Big)\\
&+3^{p-1}C_pM^p\Big(\int_{-\infty}^{t} e^{-2\delta(t-s)}E\parallel f(s,x^{(n)}(s))-f(s,x(s))\parallel^2_{L_2^0} ds\Big)^{p/2}\\
&+3^{p-1}M^pE\Big[\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\\
&\times\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\parallel I_i(x^{(n)}(t_i))-I_i(x(t_i))\parallel^p\Big)\Big]\\
&\le 3^{p-1}M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^p\varepsilon\\
&+3^{p-1}C_pM^p\Big(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\Big(\int_{-\infty}^{t} e^{-\frac{p}{2}\delta(t-s)}ds\Big)\varepsilon\\
&+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)\varepsilon\\
&\le 3^{p-1}M^p\Big[\frac{1}{\delta^p}+C_p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\frac{2}{p\delta}+\frac{1}{(1-e^{-\delta\alpha})^p}\Big]\varepsilon.
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi x^{(n)})(t)-(\Psi x)(t)\parallel^2\\
&\qquad\le 3M^2\Big[\frac{1}{\delta^2}+\frac{1}{2\delta}+\frac{1}{(1-e^{-\delta\alpha})^2}\Big]\varepsilon.
\end{aligned}
$$

Thus Ψ is continuous.

Step 4. Ψ maps bounded sets into equicontinuous sets of $PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$.

Let $\tau_1,\tau_2\in(t_i,t_{i+1}), i\in Z, \tau_1<\tau_2$, and $x\in B_{r^*}$. Then, by (H1), (9), (10), Hölder's inequality and the Ito integral, we have for $p>2$,

$$
\begin{aligned}
&E\parallel(\Psi x)(\tau_2)-(\Psi x)(\tau_1)\parallel^p\\
&\quad\le 3^{p-1}E\Big\|\int_{-\infty}^{\tau_2}T(\tau_2-s)g(s,x(s))ds-\int_{-\infty}^{\tau_1}T(\tau_1-s)g(s,x(s))ds\Big\|^p\\
&\qquad+3^{p-1}E\Big\|\int_{-\infty}^{\tau_2}T(\tau_2-s)f(s,x(s))dW(s)\\
&\qquad-\int_{-\infty}^{\tau_1}T(\tau_1-s)f(s,x(s))dW(s)\Big\|^p\\
&\qquad+3^{p-1}E\Big\|\sum_{t_i<\tau_2}T(\tau_2-t_i)I_i(x(t_i))-\sum_{t_i<\tau_1}T(\tau_2-t_i)I_i(x(t_i))\Big\|^p\\
&\quad\le 6^{p-1}E\Big\|\int_{-\infty}^{\tau_1}T(\tau_1-s)[T(\tau_2-\tau_1)-I]g(s,x(s))ds\Big\|^p
\end{aligned}
$$

$$
\begin{aligned}
&+6^{p-1}E\left\| \int_{\tau_1}^{\tau_2} T(\tau_2-s)g(s,x(s))ds \right\|^p \\
&+6^{p-1}E\left\| \int_{-\infty}^{\tau_1} T(\tau_1-s)[T(\tau_2-\tau_1)-I]f(s,x(s))dW(s) \right\|^p \\
&+6^{p-1}E\left\| \int_{\tau_1}^{\tau_2} T(\tau_2-s)f(s,x(s))dW(s) \right\|^p \\
&+3^{p-1}E\left\| \sum_{t_i<\tau_1} T(\tau_1-t_i)[T(\tau_2-\tau_1)-I]I_i(x(t_i)) \right\|^p \\
&\leq 6^{p-1}M^p \parallel T(\tau_2-\tau_1)-I \parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}ds \right)^{p-1} \\
&\times\left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}E \parallel g(s,x(s)) \parallel^p ds \right) \\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds \right)^{p-1} \\
&\times\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}E \parallel g(s,x(s)) \parallel^p ds \right) \\
&+6^{p-1}M^pC_pE\left[\int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1-s)} \parallel T(\tau_2-\tau_1)-I \parallel^2 \right. \\
&\left. \times \parallel f(s,x(s)) \parallel^2_{L_2^0} ds \right]^{p/2} \\
&+6^{p-1}M^pC_pE\left[\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)} \parallel f(s,x(s)) \parallel^2_{L_2^0} ds \right]^{p/2} \\
&+3^{p-1}M^p \parallel T(\tau_2-\tau_1)-I \parallel^p \left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)} \right)^{p-1} \\
&\times\left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}E \parallel I_i(x(t_i)) \parallel^p \right) \\
&\leq 6^{p-1}M^p \parallel T(\tau_2-\tau_1)-I \parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}ds \right)^{p-1} \\
&\times\left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}[\beta\Theta(E \parallel x(s) \parallel^p)+\nu]ds \right) \\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds \right)^{p-1} \\
&\times\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}[\beta\Theta(E \parallel x(s) \parallel^p)+\nu]ds \right) \\
&+6^{p-1}M^pC_p \parallel T(\tau_2-\tau_1)-I \parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\frac{p}{p-2}\delta(\tau_1-s)}ds \right)^{\frac{p-2}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_{-\infty}^{\tau_1} e^{-\frac{p}{2}\delta(\tau_1-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\right)\\
&+6^{p-1}M^pC_p\left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\right)^{\frac{p-2}{p}}\\
&\times\left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-s)}[\beta\Theta(E\parallel x(s)\parallel^p)+\nu]ds\right)\\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}\right)^{p-1}\\
&\times\left(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}[\beta\tilde{\Theta}_i(E\parallel x(t_i)\parallel^p)+\nu_1]\right)\\
\le\ & 6^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{\delta^p}[\beta\Theta(r^*)+\nu]\\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\right)^p[\beta\Theta(r^*)+\nu]\\
&+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(r^*)+\nu]\\
&+6^{p-1}M^pC_p\left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\right)^{\frac{p-2}{p}}\\
&\times\left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-s)}ds\right)[\beta\Theta(r^*)+\nu]\\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi x)(\tau_2)-(\Psi x)(\tau_1))\parallel^2\\
&\quad\le 6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{1}{\delta^2}[\beta\Theta(r^*)+\nu]\\
&\qquad+6M^2\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\right)^2[\beta\Theta(r^*)+\nu]\\
&\qquad+6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{2}{\delta}[\beta\Theta(r^*)+\nu]\\
&\qquad+6M^2\left(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)}ds\right)[\beta\Theta(r^*)+\nu]\\
&\qquad+3M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x\in B_{r^*}$ and tends to zero as $\tau_2\to\tau_1$, since the compactness of $T(t)$ for $t>0$ implies imply the continuity in the uniform operator topology. Thus, Ψ maps B_{r^*} into an equicontinuous family of functions.

Step 5. $V(t) = \{(\Psi x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(P, H)$ for each $t \in R$.

For each $t \in R$, and let ε be a real number satisfying $0 < \varepsilon < 1$. For $x \in B_{r^*}$, we define

$$\begin{aligned}
&(\Psi_\varepsilon x)(t)\\
&\quad = T(\varepsilon)\Big[\int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)g(s,x(s)ds\\
&\qquad + \int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)f(s,x(s)dW(s) + \sum_{t_i<t-\varepsilon} T(t-\varepsilon-t_i)I_i(x(t_i))\Big]\\
&\quad = T(\varepsilon)[(\Psi x)(t-\varepsilon)].
\end{aligned}$$

Since $T(t)(t > 0)$ is compact, then the set $V_\varepsilon(t) = \{(\Psi_\varepsilon x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(P, H)$ for each $t \in R$. Moreover, for every $x \in B_{r^*}$, we have for $p > 2$,

$$\begin{aligned}
&E \parallel (\Psi x)(t) - (\Psi_\varepsilon x)(t) \parallel^p\\
&\quad \le 3^{p-1}E\Big\| \int_{t-\varepsilon}^{t} T(t-s)g(s,x(s))ds \Big\|^p\\
&\qquad +3^{p-1}E\Big\| \int_{t-\varepsilon}^{t} T(t-s)f(s,x(s))dW(s) \Big\|^p\\
&\qquad +3^{p-1}E\Big\| \sum_{t-\varepsilon<t_i<t} T(t-t_i)I_i(x(t_i)) \Big\|^p\\
&\quad \le 3^{p-1}M^p\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds \Big)^{p-1}\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}E \parallel g(s,x(s)) \parallel^p ds \Big)\\
&\qquad +3^{p-1}C_pM^pE\Big(\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)} \parallel f(s,x(s)) \parallel^2_{L_2^0} ds \Big)^{p/2}\\
&\qquad +3^{p-1}M^pE\Big[\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)} \Big)^{p-1}\\
&\qquad \times\Big(\sum_{t-\epsilon<t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p \Big)\Big]\\
&\quad \le 3^{p-1}M^p\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds \Big)^{p-1}\\
&\qquad \times\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}[\beta\Theta(E \parallel x(s) \parallel^p) + \nu]ds \Big)\\
&\qquad +3^{p-1}C_pM^p\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds \Big)^{\frac{p-2}{p}}\\
&\qquad \times\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-s)}[\beta\Theta(E \parallel x(s) \parallel^p) + \nu]ds \Big)
\end{aligned}$$

$$
\begin{aligned}
&+3^{p-1}M^p\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\\
&\times\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}[\beta\tilde{\Theta}_i(E\parallel x_i(t_i)\parallel^p)+\nu_1]\Big)\\
&\le 3^{p-1}M^p\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds\Big)^{p}[\beta\Theta(r^*)+\nu]\\
&+3^{p-1}C_pM^p\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\\
&\times\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-s)}ds\Big)[\beta\Theta(r^*)+\nu]\\
&+3^{p-1}M^p\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{p}[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi x)(t)-(\Psi_\varepsilon x)(t)\parallel^2\\
&\quad\le 3M^2\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds\Big)^{2}[\beta\Theta(r^*)+\nu]\\
&\qquad+3M^2\Big(\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)}ds\Big)[\beta\Theta(r^*)+\nu]\\
&\qquad+3M^2\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{2}[\beta\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

Therefore, letting $\varepsilon\to 0$, it follows that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to $V(t)$ and hence $V(t)$ is also relatively compact in $L^p(P,H)$ for each $t\in R$. Since $\{\Psi x: x\in B_{r^*}\}\subset PC_h^0(R,L^p(P,H))$, then $\{\Psi x: x\in B_{r^*}\}$ is a relatively compact set by Lemma 2.5, then Ψ is a compact operator.

Step 6. We now show that there exists an open set $U\subseteq PAP_T(L^p(P,H))\cap UPC(R,L^p(P,H))$ with $x\notin\Psi x$ for $\lambda\in(0,1)$ and $x\in\partial U$.

Let $\lambda\in(0,1)$ and let $x\in L^p(P,H)$ be a possible solution of $x=\lambda\Psi(x)$ for some $0<\lambda<1$. Thus, for each $t\in R$,

$$
\begin{aligned}
x(t)&=\lambda(\Psi x)(t)\\
&=\lambda\int_{-\infty}^{t}T(t-s)g(s,x(s)ds+\lambda\int_{-\infty}^{t}T(t-s)f(s,x(s)dW(s)\\
&\quad+\lambda\sum_{t_i<t}T(t-t_i)I_i(x(t_i)).
\end{aligned}
$$

Then, by (H1), (9), (10), Hölder's inequality and the Ito integral, we have for $p>2$,

$$E\parallel x(t)\parallel^p$$

$$\leq 3^{p-1}M^p\frac{1}{\delta^p}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]$$
$$+3^{p-1}C_pM^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]$$
$$+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\tilde{\Theta}_i(E\parallel x(t_i)\parallel^p)+\nu_1].$$

For $p=2$, we have

$$E\parallel x(t)\parallel^2$$
$$\leq 3M^2\frac{1}{\delta^2}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]$$
$$+3M^2\frac{1}{2\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]$$
$$+3M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\tilde{\Theta}_i(E\parallel x(t_i)\parallel^2)+\nu_1].$$

Taking the supremum over t, we have for $p>2$,

$$\sup_{t\in R}E\parallel x(t)\parallel^p$$
$$\leq 3^{p-1}M^p\frac{1}{\delta^p}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]$$
$$+3^{p-1}C_pM^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu]$$
$$+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}[\beta\tilde{\Theta}_i(\sup_{s\in R}E\parallel x(s)\parallel^p)+\nu_1].$$

For $p=2$, we have

$$\sup_{t\in R}E\parallel x(t)\parallel^2$$
$$\leq 3M^2\frac{1}{\delta^2}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]$$
$$+3M^2\frac{1}{2\delta}[\beta\Theta(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu]$$
$$+3M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\beta\tilde{\Theta}_i(\sup_{s\in R}E\parallel x(s)\parallel^2)+\nu_1].$$

Therefore, we have for $p>2$,

$$\frac{\parallel x\parallel_\infty^p}{N_1[\beta\Theta(\parallel x\parallel_\infty^p)+\nu]+N_2[\beta\sup_{i\in Z}\tilde{\Theta}_i(\parallel x\parallel_\infty^p)+\nu_1]}\leq 1,$$

where $N_1=3^{p-1}M^p[\frac{1}{\delta^p}+C_p(\frac{p-2}{p\delta})^{\frac{p-2}{p}}\frac{2}{p\delta}]$, $N_2=3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}$. For the case of $p=2$, take $N_1=3M^2[\frac{1}{\delta^2}+\frac{1}{2\delta}]$, $N_2=3M^2\frac{1}{(1-e^{-\delta\alpha})^2}$. Then, by (11), there

exists M^* such that $\| x \|_\infty^p \neq M^*$. Set

$$U = \left\{ x \in PAP_T(R, L^p(P,H)) \cap UPC(R, L^p(P,H)) : \| x \|_\infty^p) < M^* \right\}.$$

As a consequence of Steps 1-6, it suffices to show that $\Psi : \overline{U} \to PAP_T(R, L^p(P, H)) \cap UPC(R,\ L^p(P,H))$ is a compact map. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda \Psi x$ for $\lambda \in (0,1)$. By Lemma 2.6, we deduce that Ψ has a fixed point $x \in \overline{U}$. The proof is complete.

4 An example

Consider following partial stochastic differential equations of the form

$$\begin{aligned} dz(t,x) = &\frac{\partial^2}{\partial x^2} z(t,x)dt + \varpi_1(t, z(t,x))dt \\ &+\varpi_2(t, z(t,x))dW(t), t \in R, t \neq t_i, \quad i \in Z, x \in [0,\pi], \end{aligned} \tag{12}$$

$$\Delta z(t_i, x) = \beta_i z(t_i, x), \quad i \in Z, x \in [0,\pi], \tag{13}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in R, \tag{14}$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. In this system, $\beta_i \in PAP(Z,R)$, $t_i = i + \frac{1}{4}|\sin i + \sin\sqrt{2i}|$, $\{t_i^j\}, i \in Z, j \in Z$ are equipotentially almost periodic and $\alpha = \inf_{i \in Z}(t_{i+1} - t_i) > 0$, one can see [21] for more details.

Let $H = L^2([0,\pi])$ with the norm $\| \cdot \|$ and define the operators $A : A(D) \subset H \to H$ by $Av = v''$ with the domain $D(A) := \{v \in H : v'' \in H, v(0) = v(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic compact semigroup $T(t)$ on H and $\| T(t) \| \leq e^{-t}$ for $t \geq 0$ with $M = \delta = 1$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in N$ and normalized eigenfunctions given by $v_n(\xi) := (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\xi)$. In addition, the following properties hold:

(a) The set of functions $\{v_n : n \in N\}$ is an orthonormal basis for H;

(b) For $v \in H$, $T(t)v = \sum_{n=1}^{\infty} \exp(-n^2 t)\langle v, v_n\rangle v_n$, and $Av = \sum_{n=1}^{\infty} n^2 \langle v, v_n\rangle$ $v_n,\ v \in D(A)$.

Taking

$$g(t,\psi)(\cdot) = \varpi_1(t, \psi(t,\cdot)),$$
$$f(t,\psi)(\cdot) = \varpi_2(t, \psi(t,\cdot)),$$

and

$$I_i(\psi)(\cdot) = \beta_i \psi(t_i, \cdot), \quad i \in Z.$$

Then, the above equation (12)-(14) can be written in the abstract form as the system (1)-(2).

From Theorem 3.2, it follows that the following proposition holds.

Proposition 4.1. Let ϖ_1, ϖ_2 satisfy (H2)-(H5), then system (12)-(14) has an p-mean piecewise pseudo almost periodic mild solution on R.

In the above example, we can take

$$\varpi_1(t, z(t,\cdot)) = \tilde{k}_1[\sin t + \sin\sqrt{2}t + l(t)]z(t,\cdot),$$

$$\varpi_1(t, z(t,\cdot)) = \tilde{k}_2[\sin t + \sin\sqrt{2}t + l(t)]z(t,\cdot),$$

and

$$\beta_i z(t_i,\cdot) = \tilde{c}_i[\sin i + \sin\sqrt{2}i + l(i)]z(t_i,\cdot), \quad i \in Z,$$

where $\tilde{k}_j > 0, j = 1, 2, 3$, and $\tilde{c}_i > 0, i \in Z$, $l \in UPC(R, R)$ defined by

$$l(t) = \begin{cases} 0, & \text{for } t \le 0, \\ e^{-t}, & \text{for } t \ge 0. \end{cases}$$

From [3], $\sin t + \sin\sqrt{2}t$ is almost periodic. On the other hand,

$$\frac{1}{2r}\int_{-r}^{r} |l(t)|^p dt = \frac{1}{2r}\int_{0}^{r} |l(t)|^p dt = \frac{1}{2r}\int_{0}^{r} e^{-pt} dt = \frac{1}{2r}\frac{1 - e^{-pr}}{p}.$$

Consequently

$$\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} |l(t)|^p dt = 0.$$

Then, all conditions in Theorem 3.2 are satisfied. Hence, the system (12)-(14) has an p-mean piecewise pseudo almost periodic mild solution.

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Zuomao Yan and Xiumei Jia
Department of Mathematics,
Hexi University,
Zhangye, Gansu
734000, P.R. China
* Corresponding author.
E-mail address: yanzuomao@163.com

Eventual periodicity of a max-type difference equation system

Taixiang Sun[1,2] Yaru Zhou[2,*] Guangwang Su[1] Bin Qin[1]

[1]College of Information and Statistics, Guangxi Univresity of Finance and Economics
Nanning, Guangxi 530003, P.R. China

[2]Guangxi Colleges and Universities Key Laboratory of Mathematics and Its Applications
Nanning, Guangxi 530004, P.R. China

Abstract In this paper, we investigate the eventual periodicity of the following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{\frac{1}{y_{n-m}}, \min\left\{1, \frac{\alpha}{x_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\frac{1}{x_{n-s}}, \min\left\{1, \frac{\beta}{y_{n-t}}\right\}\right\}, \end{cases} \quad n = 0, 1, 2, \cdots,$$

where $\alpha, \beta \in (0, +\infty)$, $m, s, r, t \in \{1, 2, \cdots\}$ with $r \neq m$ and $t \neq s$. We show that every solution of this system with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in (0, +\infty)$ is eventually periodic with period $m + s$, where $d = \max\{m, s, r, t\}$.

Keywords: System of difference equations; Solution; Eventual periodicity
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1. Introduction

Recently, there has been a great interest in studying difference equations and systems. A class of difference equations that has attracted recent attention is the class of, so called, max-type difference equations and and systems(see, e.g., [1-14,17,21-24]). On the other hand, some concrete classes of nonlinear systems of difference equations have also attracted some recent attention (see, e.g., [15,16,18-20, 25]).

In 2012, Stević [15] obtained the general solution of the following max-type system of difference equations

$$\begin{cases} x_{n+1} = \max\left\{\frac{\alpha}{x_n}, \frac{y_n}{x_n}\right\}, \\ y_{n+1} = \max\left\{\frac{\alpha}{y_n}, \frac{x_n}{y_n}\right\}, \end{cases} \quad n \in \mathbb{N}_0 \equiv \{0, 1, 2, \cdots\}, \tag{1.1}$$

where $\alpha \in \mathbb{R}_+ \equiv (0, +\infty)$ and the initial values $x_0, y_0 \in [\alpha, +\infty)$ and $y_0/x_0 \geq \max\{\alpha, 1/\alpha\}$.

In 2015, Yazlik et al. [25] studied the following max-type system of difference equations

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∗ Corresponding author: E-mail address: l2928w@163.com

$$\begin{cases} x_{n+1} = \max\left\{\frac{1}{x_n}, \min\left\{1, \frac{\alpha}{y_n}\right\}\right\}, \\ y_{n+1} = \max\left\{\frac{1}{y_n}, \min\left\{1, \frac{\alpha}{x_n}\right\}\right\}, \end{cases} \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $\alpha \in \mathbb{R}_+$ and the initial values $x_0, y_0 \in \mathbb{R}_+$, and obtained in an elegant way the general solution of (1.2).

Motivated by aforementioned papers, in this paper, we investigate the eventual periodicity of the following system of difference equations

$$\begin{cases} x_n = \max\left\{\frac{1}{y_{n-m}}, \min\left\{1, \frac{\alpha}{x_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\frac{1}{x_{n-s}}, \ \min\left\{1, \frac{\beta}{y_{n-t}}\right\}\right\}, \end{cases} \quad n \in \mathbb{N}_0 \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}_+$, $m, s, r, t \in \mathbb{N}$ with $r \neq m$ and $t \neq s$ and the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}$, $\cdots, x_{-1}, y_{-1} \in \mathbb{R}_+$ with $d = \max\{m, s, r, t\}$. The main result of this paper is the following theorem.

Theorem 1.1 Let $\alpha > 0$ and $\beta > 0$. Then every solution $\{(x_n, y_n)\}_{n \geq -d}$ with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in \mathbb{R}_+$ of system (1.3) is eventually periodic with period $m + s$.

2. Proof of Theorem 1.1

In this section, we will discuss the periodicity of solutions of system (1.3). Let $\{(x_n, y_n)\}_{n \geq -d}$ be a solution of (1.3) with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \cdots, x_{-1}, y_{-1} \in \mathbb{R}_+$. Write

$$A_n = \min\left\{1, \frac{\alpha}{x_{n-r}}\right\}, \qquad B_n = \min\left\{1, \frac{\beta}{y_{n-t}}\right\}.$$

It is easy to see that $A_n \leq 1$ and $B_n \leq 1$ for any $n \in \mathbb{N}_0$. The main result of this paper is established through the following lemmas.

Lemma 2.1 The following statements are true.

(1) $x_n y_{n-m} \geq 1$ (resp. $y_n x_{n-s} \geq 1$) for all $n \in \mathbb{N}_0$.

(2) $x_n \leq \max\{x_{n-m-s}, A_n\}$ (resp. $y_n \leq \max\{y_{n-m-s}, B_n\}$) for all $n \geq d$.

(3) If $x_n = 1/y_{n-m}$ (resp. $y_n = 1/x_{n-s}$) for some $n \geq d$, then $x_n \leq x_{n-m-s}$(resp. $y_n \leq y_{n-m-s}$). If $x_n = A_n > 1/y_{n-m}$ (resp. $y_n = B_n > 1/x_{n-s}$) for some $n \geq d$, then $x_n > x_{n-m-s}$(resp. $y_n > y_{n-m-s}$).

Proof (1) It follows from $x_n \geq 1/y_{n-m}$ (resp. $y_n \geq 1/x_{n-s}$) that $x_n y_{n-m} \geq 1$ (resp. $y_n x_{n-s} \geq 1$) for all $n \in \mathbb{N}_0$.

(2) Since $y_{n-m} x_{n-m-s} \geq 1$ ($n \geq m$), one know that for all $n \geq d$,

$$x_n = \max\left\{\frac{x_{n-m-s}}{y_{n-m} x_{n-m-s}}, A_n\right\} \leq \max\{x_{n-m-s}, A_n\}.$$

The other case is treated similarly, which detail is omitted.

(3) If $x_n = A_n > 1/y_{n-m}$ for some $n \geq d$, then combining (1.3) with $A_n \leq 1$ and $B_{n-m} \leq 1$

one know that

$$\begin{aligned} 1 < x_n y_{n-m} &= \max\left\{\frac{x_n}{x_{n-m-s}}, x_n B_{n-m}\right\} \\ &= \max\left\{\frac{x_n}{x_{n-m-s}}, A_n B_{n-m}\right\} \\ &= \frac{x_n}{x_{n-m-s}}, \end{aligned}$$

which implies $x_n > x_{n-m-s}$. If $x_n = 1/y_{n-m}$ for some $n \geq d$, then by $y_{n-m}x_{n-m-s} \geq 1$ $(n \geq m)$ one know that

$$x_n = \frac{x_{n-m-s}}{y_{n-m}x_{n-m-s}} \leq x_{n-m-s}.$$

The other case is treated similarly, which detail is omitted also. This completes the proof of Lemma 2.1.

Lemma 2.2 Let $N, k, l \in \mathbb{N}$ $(l \geq 2)$ such that the following statements hold.

(1) $x_{N+(m+s)\mu} = A_{N+(m+s)\mu} > 1/y_{N+(m+s)\mu-m}$ for every $\mu \in \{k, k+l\}$.

(2) $x_{N+(m+s)\mu} = 1/y_{N+(m+s)\mu-m}$ for every $\mu \in \{k+1, \cdots, k+l-1\}$.

Then $x_{N+(m+s)k} = x_{N+(m+s)(k+1)} = \cdots = x_{N+(m+s)(k+l-1)} < x_{N+(m+s)(k+l)}$.

Proof Combining the conditions (1) and (2) with Lemma 2.1 (3), one know that $x_{N+(m+s)(\mu+1)} \leq x_{N+(m+s)\mu}$ for every $\mu \in \{k, \cdots, k+l-2\}$ and $x_{N+(m+s)\mu} > x_{N+(m+s)(\mu-1)}$ for every $\mu \in \{k, k+l\}$.

Assume on the contrary that $x_{N+(m+s)(\mu+1)} < x_{N+(m+s)\mu}$ for some $k \leq \mu \leq k+l-2$. Then by $A_{N+(m+s)k}B_{N+(m+s)(\mu+1)-m} \leq 1$ one know that

$$\begin{aligned} x_{N+(m+s)k} = A_{N+(m+s)k} &\geq x_{N+(m+s)\mu} \\ &> x_{N+(m+s)(\mu+1)} \\ &= \frac{1}{y_{N+(m+s)(\mu+1)-m}} \\ &= \frac{1}{\max\left\{\frac{1}{x_{N+(m+s)\mu}}, B_{N+(m+s)(\mu+1)-m}\right\}} \\ &= \frac{1}{\frac{1}{x_{N+(m+s)\mu}}} \\ &= x_{N+(m+s)\mu}. \end{aligned}$$

A contradiction. This completes the proof of Lemma 2.2.

In a similar fashion as in the proof of Lemma 2.2, we can obtain the following Lemma 2.3.

Lemma 2.3 Let $N, k, l \in \mathbb{N}$ $(l \geq 2)$ such that the following statements hold.

(1) $y_{N+(m+s)\mu} = B_{N+(m+s)\mu} > 1/x_{N+(m+s)\mu-s}$ for every $\mu \in \{k, k+l\}$.

(2) $y_{N+(m+s)\mu} = 1/x_{N+(m+s)\mu-s}$ for every $\mu \in \{k+1, \cdots, k+l-1\}$.

Then $y_{N+(m+s)k} = y_{N+(m+s)(k+1)} = \cdots = y_{N+(m+s)(k+l-1)} < y_{N+(m+s)(k+l)}$.

By Lemma 2.2 and Lemma 2.3 and Lemma 2.1 (3) one obtain the following Lemma 2.4.

Lemma 2.4 Let $K, L \in \mathbb{N}$.

(1) If $x_{K+(m+s)\mu} = 1/y_{K+(m+s)\mu-m}$ (resp. $y_{K+(m+s)\mu} = 1/x_{N+(m+s)\mu-s}$) for all $\mu \geq L$, then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is nonincreasing eventually. If $(1 \geq)$ $x_{N+(m+s)\mu} = A_{N+(m+s)\mu} > 1/y_{N+(m+s)\mu-m}$ (resp. $(1 \geq)$ $y_{n+(m+s)\mu} = B_{N+(m+s)\mu} > 1/x_{N+(m+s)\mu-s}$) for all $\mu \geq L$, then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is increasing eventually .

(2) If $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is nonincreasing eventually and $x_{N+(m+s)\mu} < 1$ (resp. $y_{N+(m+s)\mu} < 1$) for all $\mu \geq L$, then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is a constant sequence eventually.

(3) If there are two sequences of positive integers $\{p_n\}$ and $\{q_n\}$ with $1 < p_1 < q_1 < p_2 < q_2 < \cdots < p_n < q_n < \cdots$ such that for every $p_n \leq k < q_n$,

$$x_{N+(m+s)k} = A_{N+(m+s)k} > \frac{1}{y_{N+(m+s)k-m}} \text{ (resp. } y_{N+(m+s)k} = B_{N+(m+s)k} > \frac{1}{x_{N+(m+s)k-s}})$$

and for every $p_n \leq k < q_{n+1}$,

$$x_{N+(m+s)k} = \frac{1}{y_{N+(m+s)k-m}} (\text{resp. } y_{N+(m+s)k} = \frac{1}{x_{N+(m+s)k-s}}) \ ,$$

then $x_{N+(m+s)\mu}$ (resp. $y_{N+(m+s)\mu}$) is nondecreasing eventually and $x_{N+(m+s)\mu} < 1$ (resp. $y_{N+(m+s)\mu} < 1$) eventually.

Proof (1) It follows from Lemma 2.1 (3).

(2) Noting that $1/x_{N+(m+s)\mu} > 1 \geq B_{N+(m+s)\mu+s}$ eventually, we have

$$y_{N+(m+s)\mu+s} = \max\{\frac{1}{x_{N+(m+s)\mu}}, B_{N+(m+s)\mu+s}\} = \frac{1}{x_{N+(m+s)\mu}}$$

eventually. Thus $y_{N+(m+s)\mu+s}$ is nondecreasing eventually. On the other hand, we know from (1) that $y_{N+(m+s)\mu+s}$ is nonincreasing eventually. Thus $y_{N+(m+s)\mu+s}$ and $x_{N+(m+s)\mu}$ are constant sequences eventually. The other case is treated similarly.

(3) Combining Lemma 2.1 (3) with Lemma 2.2 one know that $x_{N+(m+s)(k-1)} < x_{N+(m+s)k}$ for every $p_n \leq k < q_n$ and $x_{N+(m+s)(k-1)} = x_{N+(m+s)k}$ for every $q_n \leq k < p_{n+1}$. Thus $\{x_{N+(m+s)\mu}\}_{\mu \geq p_1}$ is nondecreasing and $x_{N+(m+s)\mu} < 1$ eventually. The other case is treated similarly, which detail is omitted also. This completes the proof of Lemma 2.4.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 It need only to show that $\{x_{(m+s)k+i}\}_{k\geq 1}$ and $\{y_{(m+s)k+i}\}_{k\geq 1}$ are constant sequences eventually for every $i \in \{0, 1, \cdots, m+s-1\}$.

Assume on the contrary that there is some $\theta \in \{0, 1, \cdots, m+s-1\}$ such that $\{x_{(m+s)k+\theta}\}_{k\geq 1}$ (resp. $\{y_{(m+s)k+\theta}\}_{k\geq 1}$) is not a constant sequence eventually. Then by Lemma 2.4 we know that $x_{(m+s)k+\theta}$ (resp. $y_{(m+s)k+\theta}$) is monotone eventually.

We claim that $x_{(m+s)k+\theta}$ is nondecreasing eventually. Indeed, if $x_{(m+s)k+\theta}$ is nonincreasing eventually, then $x_{(m+s)k+\theta} > 1$ eventually and

$$x_{(m+s)k+\theta} = \max\Big\{\frac{1}{y_{(m+s)k+\theta-m}}, A_{(m+s)k+\theta}\Big\} = \frac{1}{y_{(m+s)k+\theta-m}} \text{ eventually.}$$

Thus $y_{(m+s)k+\theta-m}$ is nondecreasing eventually and $y_{(m+s)k+\theta-m} < 1$ eventually. According to Lemma 2.4, there is a sequence of positive integers $k_{11} < k_{12} < \cdots < k_{1n} < \cdots$ such that

$$\begin{aligned} y_{(m+s)k_{1n}+\theta-m} &= \max\left\{\frac{1}{x_{(m+s)k_{1n}+\theta-m-s}}, B_{(m+s)k_{1n}+\theta-m}\right\} \\ &= B_{(m+s)k_{1n}+\theta-m} \\ &= \frac{\beta}{y_{(m+s)k_{1n}+\theta-m-t}} \\ &> \frac{1}{x_{(m+s)k_{1n}+\theta-m-s}}. \end{aligned}$$

From which we see that $y_{(m+s)k+\theta-m-t}$ is nonincreasing eventually and $y_{(m+s)k+\theta-m-t} > 1$ eventually and

$$y_{(m+s)k+\theta-m-t} = \max\left\{\frac{1}{x_{(m+s)k+\theta-m-t-s}}, B_{(m+s)k+\theta-m-t}\right\} = \frac{1}{x_{(m+s)k+\theta-m-t-s}} \text{ eventually.}$$

Thus $x_{(m+s)k+\theta-m-t-s}$ is nondecreasing eventually and $x_{(m+s)k+\theta-m-t-s} < 1$ eventually. Again according to Lemma 2.4, there is a sequence of positive integers $\lambda_{11} < \lambda_{12} < \cdots < \lambda_{1n} < \cdots$ such that

$$\begin{aligned} x_{(m+s)\lambda_{1n}+\theta-m-t-s} &= \max\left\{\frac{1}{y_{(m+s)\lambda_{1n}+\theta-m-t-s-m}}, A_{(m+s)\lambda_{1n}+\theta-m-t-s}\right\} \\ &= A_{(m+s)\lambda_{1n}+\theta-m-t-s} \\ &= \frac{\alpha}{x_{(m+s)\lambda_{1n}+\theta-m-t-s-r}} \\ &> \frac{1}{y_{(m+s)\lambda_{1n}+\theta-m-t-s-m}}. \end{aligned}$$

Therefore $x_{(m+s)\lambda_{1n}+\theta-m-t-s-r}$ is nonincreasing eventually and $x_{(m+s)\lambda_{1n}+\theta-m-t-s-r} > 1$ eventually. Write $\omega = m + s + r + t$. By induction, we may obtain that

(1) For every $0 \le \mu \le m + s - 1$, $x_{(m+s)k+\theta-\mu\omega} = 1/y_{(m+s)k+\theta-\mu\omega-m}$ is nonincreasing eventually and $x_{(m+s)k+\theta-\mu\omega} > 1$ eventually, and $y_{(m+s)k+\theta-\mu\omega-m-t} = 1/x_{(m+s)k+\theta-\mu\omega-m-t-s}$ is nonincreasing eventually and $y_{(m+s)k+\theta-\mu\omega-m-t} > 1$ eventually

(2) For every $1 \le \mu \le m + s$, $y_{(m+s)k+\theta-(\mu-1)\omega-m}$ is nondecreasing eventually and $y_{(m+s)k+\theta-(\mu-1)\omega-m} < 1$ eventually, and there is a sequence of positive integers $k_{\mu 1} < k_{\mu 2} < \cdots < k_{\mu n} < \cdots$ such that

$$\begin{aligned} & y_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m} \\ = & \max\left\{\frac{1}{x_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m-s}}, B_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m}\right\} \\ = & B_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m} \\ = & \frac{\beta}{y_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m-t}} \\ > & \frac{1}{x_{(m+s)k_{\mu n}+\theta-(\mu-1)\omega-m-s}}. \end{aligned}$$

(3) For every $1 \le \mu \le m + s$, $x_{(m+s)k+\theta-(\mu-1)\omega-m-t-s}$ is nondecreasing eventually and $x_{(m+s)k+\theta-(\mu-1)\omega-m-t-s} < 1$ eventually, and there is a sequence of positive integers $\lambda_{\mu 1} < \lambda_{\mu 2} <$

$\cdots < \lambda_{\mu n} < \cdots$ such that

$$\begin{aligned} & x_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s} \\ = & \max\Big\{\frac{1}{y_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s-m}}, A_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s}\Big\} \\ = & A_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s} \\ = & \frac{\alpha}{x_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-\omega}} \\ > & \frac{1}{y_{(m+s)\lambda_{\mu n}+\theta-(\mu-1)\omega-m-t-s-m}}. \end{aligned}$$

We may assume without loss of generality that $k_{\mu n} > \lambda_{\mu n} > k_{\mu+1,n} > \lambda_{\mu+1,n}$ for any $1 \le \mu \le m+s-1$. Furthermore, we can assume that

$$\lambda_{1n} = \max\{k \le k_{1n} : x_{(m+s)k+\theta-m-t-s} = A_{(m+s)k+\theta-m-t-s} > \frac{1}{y_{(m+s)k+\theta-m-t-s-m}}\}$$

and for $2 \le \mu \le m+s$,

$$\begin{aligned} k_{\mu n} = & \max\{k \le \lambda_{\mu-1,n} : y_{(m+s)k+\theta-(\mu-1)\omega-m} \\ & = B_{(m+s)k+\theta-(\mu-1)\omega-m} > \frac{1}{x_{(m+s)k+\theta-(\mu-1)\omega-m-s}}\} \end{aligned}$$

and

$$\begin{aligned} \lambda_{\mu n} = & \max\{k \le k_{\mu n} : x_{(m+s)k+\theta-(\mu-1)\omega-m-t-s} \\ & = A_{(m+s)k+\theta-(\mu-1)\omega-m-t-s} > \frac{1}{y_{(m+s)k+\theta-(\mu-1)\omega-m-t-s-m}}\}. \end{aligned}$$

Then it follows from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} & x_{(m+s)k_{1n}+\theta} \\ = & \frac{1}{y_{(m+s)k_{1n}+\theta-m}} = \frac{1}{B_{(m+s)k_{1n}+\theta-m}} = \frac{y_{(m+s)k_{1n}+\theta-m-t}}{\beta} \\ = & \frac{1}{\beta x_{(m+s)k_{1n}+\theta-m-t-s}} = \frac{1}{\beta x_{(m+s)\lambda_{1n}+\theta-m-t-s}} = \frac{x_{(m+s)\lambda_{1n}+\theta-\omega}}{\alpha\beta} \\ = & \frac{1}{\alpha\beta y_{(m+s)\lambda_{1n}+\theta-\omega-m}} = \frac{1}{\alpha\beta y_{(m+s)k_{2n}+\theta-\omega-m}} = \frac{y_{(m+s)k_{2n}+\theta-\omega-m-t}}{\beta^2\alpha} \\ = & \frac{1}{\beta^2\alpha x_{(m+s)k_{2n}+\theta-\omega-m-t-s}} = \frac{1}{\beta^2\alpha x_{(m+s)\lambda_{2n}+\theta-\omega-m-t-s}} = \frac{x_{(m+s)\lambda_{2n}+\theta-2\omega}}{\alpha^2\beta^2} \\ = & \cdots\cdots\cdots\cdots \\ = & \frac{x_{(m+s)\lambda_{m+s,n}+\theta-(m+s)\omega}}{(\alpha\beta)^{m+s}}. \end{aligned}$$

Let $\lim_{k\longrightarrow\infty} x_{(m+s)k+\theta} = \Phi$. Then $\Phi \ge 1$. By taking the limit in the following relationship.

$$x_{(m+s)k_{1n}+\theta} = \frac{x_{(m+s)\lambda_{m+s,n}+\theta-(m+s)\omega}}{(\alpha\beta)^{m+s}}$$

as $n \longrightarrow \infty$, we have $\Phi = \Phi/(\alpha\beta)^{m+s}$. Therefore $\alpha\beta = 1$ and

$$x_{(m+s)k_{1n}+\theta} = x_{(m+s)\lambda_{m+s,n}+\theta-(m+s)\omega} \ge x_{(m+s)(k_{1n}-1)+\theta} > x_{(m+s)k_{1n}+\theta}.$$

A contradiction.

By the above claim we see that $x_{(m+s)k+\theta}$ is nondecreasing eventually. Then $x_{(m+s)k+\theta} < 1$ eventually and there is a sequence of positive integers $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ such that

$$x_{(m+s)\lambda_n+\theta} = \max\left\{\frac{1}{y_{(m+s)\lambda_n+\theta-m}}, A_{(m+s)\lambda_n+\theta}\right\}$$

$$= \quad A_{(m+s)\lambda_n+\theta} = \frac{\alpha}{x_{(m+s)\lambda_n+\theta-r}} > \frac{1}{y_{(m+s)\lambda_n+\theta-m}}.$$

Thus $x_{(m+s)\lambda_n+\theta-r}$ is nonincreasing eventually and $x_{(m+s)\lambda_n+\theta-r} > 1$ eventually. This contradicts to the above claim.

Therefore, for every $i \in \{0, 1, \cdots, m+s-1\}$, $\{x_{(m+s)k+i}\}_{k\geq 1}$ is a constant sequence eventually. In a similar fashion, also we can show that for every $i \in \{0, 1, \cdots, m+s-1\}$, $\{y_{(m+s)k+i}\}_{k\geq 1}$ is a constant sequence eventually. This completes the proof of Theorem 2.5.

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On uni-soft implicative filters of BE-algebras

Jung Mi Ko[1] and Sun Shin Ahn[2,*]

[1] **Department of Mathematics, Gangneung-Wonju National University, Gangneung, 25457, Korea**
[2]**Department of Mathematics Education, Dongguk University, Seoul 04620, Korea**

Abstract. The notion of uni-soft implicative filters of a BE-algebra is introduced, and related properties are investigated. The problem of classifying uni-soft implicative by their γ-exclusive filter is solved. Also, as a generalization of uni-soft implicative filters, the foldness of uni-soft implicative filters are considered. Characterizations of uni-soft (n-fold) implicative filters are discussed.

1. Introduction

Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [11]. In response to this situation Zadeh [12] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [13]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [9]. Maji et al. [8] and Molodtsov [9] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [9] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set

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* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409
[0]**E-mail**: jmko@gwnu.ac.kr (J. M. Ko); sunshine@dongguk.edu (S. S. Ahn)
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theory are progressing rapidly. Maji et al. [8] described the application of soft set theory to a decision making problem. Maji et al. [7] also studied several operations on the theory of soft sets.

Ahn et al. [1] introduced the notion of an implicative vague filter in BE-algebras, and investigated some properties of it. Jun et al. [4] introduced the notion of int-soft implicative filter of a BE-algebra, and studied their properties.

In this paper, we introduce the notion of a uni-soft implicative filter of a BE-algebra, and investigate their properties. We solve the problem of classifying uni-soft implicative by their γ-exclusive filter. Also, as a generalization of uni-soft implicative filters, the foldness of uni-soft implicative filters are considered and discuss characterizations of uni-soft (n-fold) implicative filters.

2. **Preliminaries**

We recall some definitions and results discussed in [6].

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1 \in F$;
(F2) $x * y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in X$.

Definition 2.2. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called an *implicative filter* of X if

(F1) $1 \in F$;
(F2) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$ for all $x, y, z \in X$.

Proposition 2.3. *Let $(X; *, 1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.4. *Let $(X;*,1)$ be a self distributive BE-algebra. Then following hold: for any $x,y,z \in X$,*

(i) *if $x \leq y$, then $z*x \leq z*y$ and $y*z \leq x*z$.*
(ii) $y*z \leq (z*x)*(y*z)$.
(iii) $y*z \leq (x*y)*(x*z)$.

A BE-algebra $(X;*,1)$ is said to be *transitive* if it satisfies Proposition 2.4(iii).

A soft set theory is introduced by Molodtsov [9]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A,B,C,\cdots \subseteq X$.

Definition 2.5. A *soft set* (f,A) of X over U is defined to be the set of ordered pairs

$$(f,A) := \{(x,f(x)) : x \in X,\ f(x) \in \mathscr{P}(U)\},$$

where $f : X \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For a soft set (f,A) of X and a subset γ of U, the γ-*exclusive set* of (f,A), denoted by $e_A(f;\gamma)$, is defined to be the set $e_A(f;\gamma) := \{x \in A \mid f(x) \subseteq \gamma\}$.

For any soft sets (f,X) and (g,X) of X, we call (f,X) a *soft subset* of (g,X), denoted by $(f,X) \tilde{\subseteq} (g,X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of (f,X) and (g,X), denoted by $(f,X) \tilde{\cup} (g,X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of X over U in which $f \tilde{\cup} g$ is defined by

$$(f \tilde{\cup} g)(x) = f(x) \cup g(x) \text{ for all } x \in X.$$

The *soft intersection* of (f,X) and (g,X), denoted by $(f,X) \tilde{\cap} (g,X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of X over U in which $f \tilde{\cap} g$ is defined by

$$(f \tilde{\cap} g)(x) = f(x) \cap g(x) \text{ for all } x \in M.$$

3. **Uni-soft implicative filters**

In what follows, we take a BE-algebra X, as a set of parameters unless specified.

Definition 3.1.([5]) A soft set (f,X) of X over U is called a *union-soft filter* (briefly, *uni-soft filter*) of X if it satisfies:
(US1) $(\forall x \in X)(f(1) \subseteq f(x))$,
(US2) $(\forall x,y \in X)(f(y) \subseteq f(x*y) \cup f(x))$.

Proposition 3.2.([5]) Every uni-soft filter (f,X) of X over U satisfies the following properties:

$$(\forall x,y \in X)(x \leq y \Rightarrow f(y) \subseteq f(x)).$$

Definition 3.3. A soft set (f,X) of X over U is called a *union-soft implicative filter* (briefly, *uni-soft implicative filter*) of X if it satisfies (US1) and

(US3) $(\forall x, y, z \in X)\,(f(x * z) \subseteq f(x * (y * z)) \cup f(x * y))$.

Example 3.4. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{N} & \text{if } x \in \{1, a, b\} \\ \mathbb{Z} & \text{if } x \in \{c, d, 0\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft implicative filter of X.

Proposition 3.5. *Every uni-soft implicative filter of a BE-algebra X is a uni-soft filter of X.*

Proof. Let (f, X) be a uni-soft implicative filter of X. Using (US3) and (BE4), we have

$$\begin{aligned} f(x * z) \subseteq & f(x * (y * z)) \cup f(x * y) \\ = & f(y * (x * z)) \cup f(x * y) \end{aligned} \tag{3.1}$$

for any $x, y, z \in X$. Putting $x := 1$ in (3.1), we get $f(z) = f(1 * z) \subseteq f(y * (1 * z)) \cup f(1 * y) = f(y * z) \cup f(y)$. Hence (US2) holds. Therefore (f, X) is a uni-soft filter of X. □

The converse of Proposition 3.5 is not true in general as seen in the following example.

Example 3.6. Let $E = X$ be the set of parameters where $X := \{1, a, b, c\}$ is a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let (f, X) be a soft set of X over $U := \mathbb{Z}$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 3\mathbb{N} & \text{if } x \in \{1, c\} \\ 3\mathbb{Z} & \text{if } x \in \{a, b\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft filter of X. But it is not a uni-soft implicative filter of X, since $f(b * c) = f(a) = 3\mathbb{Z} \not\subseteq f(b * (a * c)) \cup f(b * a) = f(1) = 3\mathbb{N}$.

We provide conditions for a uni-soft filter to be a uni-soft implicative filter.

Proposition 3.7. *Let X be a self distributive BE-algebra. Let (f, X) be a soft filter over U satisfying*

$$(\forall x, y, z \in X)(f(y * z) \subseteq f(x * (y * (y * z))) \cup f(y * x)). \tag{3.2}$$

Then (f, X) is a uni-soft implicative filter over U.

Proof. Since $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z)) = x * (y * (x * z)) = y * (x * (x * z))$ for all $x, y \in X$, we have $f(x * (y * z)) \supseteq f(y * (x * (x * z)))$ by Proposition 3.2. Using (3.2), we have $f(x * z) \subseteq f(y * (x * (x * z)) \cup f(x * y) \subseteq f(x * (y * z)) \cup f(x * y)$. Thus (f, X) is a uni-soft implicative filter over U. □

Theorem 3.8. *Let X be a self distributive BE-algebra. Then the soft set (f, X) over U is a uni-soft implicative filter of X over U if and only if it is a uni-soft filter of X over U.*

Proof. By Proposition 3.5, every uni-soft implicative filter over U is a uni-soft filter over U.

Conversely, assume that (f, X) is a uni-soft filter of X. For any $x, y, z \in X$, using (US2) we have $f(x * z) \subseteq f((x * y) * (x * z)) \cup f(x * y) = f(x * (y * z)) \cup f(x * y)$. Hence (f, X) is a uni-soft implicative filter of X.

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = 1$.

Definition 3.9. A soft set (f, X) of a BE-algebra X over U is called a *uni-soft n-fold implicative filter* of X if it satisfies (US1) and

(US4) $(\forall x, y, z \in X)\,(f(x^n * z) \subseteq f(x^n * (y * z)) \cup f(x^n * y)))\,.$

Note that a uni-soft 1-fold implicative filter of a BE-algebra X is a uni-soft implicative filter of X.

Example 3.10. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra ([10]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in \{1, b, c\} \\ \gamma_2 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is a uni-soft n-fold implicative filter over U.

Theorem 3.11. *Every uni-soft n-fold implicative filter of a BE-algebra X is a uni-soft filter of X.*

Proof. Taking $x := 1$ in (US4) and using (BE3), we have $f(z) \subseteq f(y * z) \cup f(y)$. Hence (f, X) is a uni-soft filter over U. □

The converse of Theorem 3.10 is not true in general as seen the following example.

Example 3.12. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE-algebra as in Example 3.10. Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x = 1 \\ \gamma_2 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that (f, X) is a uni-soft filter of X. But it is not a uni-soft 1-fold implicative filter over U, since $f(d * c) = f(b) = \gamma_2 \not\subseteq \gamma_1 = f(1) = f(d * (b * c)) \cup f(d * b)$.

Theorem 3.13. *Let X be a transitive BE-algebra. For any uni-soft filter (f, X) of a BE-algebra X, the following are equivalent:*

(i) *(f, X) is a uni-soft n-fold implicative filter,*
(ii) $(\forall x, y \in X)\,(f(x^n * y) \subseteq f(x^{n+1} * y))$,
(iii) $(\forall x, y, z \in X)\,(f((x^n * y) * (x^n * z)) \subseteq f(x^n * (y * z)))$.

Proof. (i)⇒(ii) Assume that (f, X) is a uni-soft n-fold implicative filter of X. Putting $z := y, y := x$ in (US4), we have $f(x^n * y) \subseteq f(x^n * (x * y)) \cup f(x^n * x) = f(x^{n+1} * y) \cup f(1) = f(x^{n+1} * y)$. Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z))$, we have $f(x^n * ((x^n * y) * (x^n * z))) \subseteq f(x^n * (y * z))$. Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z))) = x^n * ((x^n * y) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} f(x^{n+1} * (x^{n-2} * ((x^n * y) * z)) =& f(x^n * (x^{n-1} * ((x^n * y) * z)) \\ \subseteq& f(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ =& f(x^n * ((x^n * y) * (x^n * z))) \\ \subseteq& f(x^n * (y * z)). \end{aligned} \tag{3.3}$$

It follows form (ii) and (3.3) that

$$\begin{aligned} f(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& f(x^n * (x^{n-2} * ((x^n * y) * z))) \\ \subseteq& f(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ \subseteq& f(x^n * (y * z)). \end{aligned}$$

Repeating this process, we conclude that

$$\begin{aligned} f((x^n * y) * (x^n * z)) =& f(x^n * ((x^n * y) * z)) \\ \subseteq& f(x^n * (y * z)). \end{aligned}$$

(iii)⇒(i) Let $x, y, z \in X$. Using (iii), we have

$$\begin{aligned} f(x^n * z) \subseteq& f((x^n * y) * (x^n * z)) \cap f(x^n * y) \\ \subseteq& f((x^n * (y * z)) \cap f(x^n * y). \end{aligned}$$

Hence (f, X) is a uni-soft n-fold implicative filter □

Definition 3.14. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 3.15. *In an n-fold implicative BE-algebra, the notion of uni-soft filters and uni-soft n-fold implicative filters coincide.*

Proof. Straightforward. □

Theorem 3.16. *Let X be a BE-algebra. A soft set (f, X) over U is a uni-soft n-fold implicative filter of a BE-algebra X if and only if it satisfies (US1) and*

(US5) $(\forall x, y, z \in X)\,(f(y^n * z) \subseteq f(x * (y^{n+1} * z)) \cup f(x))\,.$

Proof. Suppose that a soft set (f, X) over U is a uni-soft n-fold implicative filter. By Theorems 3.13, 3.11 and (US2), we have $f(y^n * z) \subseteq f(y^{n+1} * z) \subseteq f(x * (y^{n+1} * z)) \cup f(x)$ for any $x, y, z \in X$. Hence (US5) holds.

Conversely, assume that (f, X) satisfies (US1) and (US5). Using (BE3), we obtain $f(y) = f(1^n * y) \subseteq f(x * (1^{n+1} * y)) \cup f(x) = f(x * y) \cup f(x)$. Hence (US2) holds and so (f, X) is a uni-soft filter over U. By (US5), (US1) and (BE3), we get $f(x^n * y) \subseteq f(1 * (x^{n+1} * y)) \cup f(1) = f(x^{n+1} * y)$. By Theorem 3.13, (f, X) is a uni-soft n-fold implicative filter of X. □

Theorem 3.17. *Every implicative filter of a BE-algebra can be represented as a γ-exclusive set of a uni-soft implicative filter, i.e., given an implicative filter F a BE-algebra X, there exists a uni-soft implicative filter (f, X) of X over U such that F is the γ-exclusive set of (f, X) for a non-empty subset γ of U.*

Proof. Let F be an implicative filter of a BE-algebra X. For a subset γ of U, define a soft set (f, X) over U by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ U & \text{if } x \notin F. \end{cases}$$

Obviously, $F = e_X(f; \gamma)$. We now prove that (f, X) is a uni-soft implicative filter of X. Since $1 \in F = e_X(f; \gamma)$, we have $f(1) = \gamma \subseteq f(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x * (y * z), x * y \in F$, then $x * z \in F$ because F is an implicative filter of X. Hence $f(x * (y * z)) = f(x * y) = f(x * z) = \gamma$,

and so $f(x*(y*z))\cup f(x*y)\supseteq f(x*z)$. If $x*(y*z)\in F$ and $x*y\notin F$, then $f(x*(y*z))=\gamma$ and $f(x*y)=U$ which imply that $f(x*(y*z))\cup f(x*y)=\gamma\cup U=U\supseteq f(x*z)$. Similarly, if $x*(y*z)\notin F$ and $x*y\in F$, then $f(x*(y*z))\cup f(x*y)\supseteq f(x*z)$. Obviously, if $x*(y*z)\notin F$ and $x*y\notin F$, then $f(x*(y*z))\cup f(x*y)\supseteq f(x*z)$. Therefore (f,X) is a uni-soft implicative filter of X. □

For any elements a and b of X, consider a soft set $(f_{X(a,b)},X)$ over U where

$$f_{X(a,b)}:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\gamma_1 & \text{if } x\in X(a,b),\\ \gamma_2 & \text{otherwise,}\end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1\subsetneq\gamma_2$ and $X(a,b):=\{x\in X|a\leq b*x\}$. In the following example, we know that there exist $a,b\in X$ such that $(f_{X(a,b)},X)$ is not a uni-soft implicative filter of X.

Example 3.18. Consider the BE-algebra $X=\{1,a,b,c\}$ which is given in Example 3.6. Then $(f_{X(c,c)},X)$ is not a uni-soft implicative filter of X over U since $f_{X(c,c)}(b*(a*c)))\cup f_{X(c,c)}(b*a)=\gamma_1\not\supseteq f_{X(c,c)}(b*c)=\gamma_2$.

Now we provide a condition for the soft set $(f_{X(a,b)},X)$ to be a uni-soft implicative filter of X over U for all $a,b\in X$.

Theorem 3.19. *If X is a self distributive BE-algebra, then the soft set $(f_{X(a,b)},X)$ is a uni-soft implicative filter of X over U for all $a,b\in X$.*

Proof. Let $a,b\in X$. Obviously, $f_{X(a,b)}(1)\subseteq f_{X(a,b)}(x)$ for all $x\in X$. Let $x,y,z\in X$ be such that $a*(b*(x*(y*z)))\neq 1$ or $a*(b*(x*y))\neq 1$. Then $f_{X(a,b)}(x*(y*z))=\gamma_2$ or $f_{X(a,b)}(x*y)=\gamma_2$. Hence $f_{X(a,b)}(x*(y*z))\cup f_{X(a,b)}(x*y)=\gamma_2\supseteq f_{X(a,b)}(x*z)$. Assume that $a*(b*(x*(y*z)))=1$ and $a*(b*(x*y))=1$. Then

$$\begin{aligned}1&=a*(b*(x*(y*z)))\\&=a*(b*((x*y)*(x*z)))\\&=a*((b*(x*y))*(b*(x*z)))\\&=(a*(b*(x*y)))*(a*(b*(x*z)))\\&=1*(a*(b*(x*z)))\\&=a*(b*(x*z)),\end{aligned}$$

and so $f_{X(a,b)}(x*(y*z))\cup f_{X(a,b)}(x*y)=\gamma_1=f_{X(a,b)}(x*z)$. Therefore $(f_{X(a,b)},X)$ is a uni-soft implicative filter of X over U for all $a,b\in X$. □

Theorem 3.20. *If (f,X) and (g,X) are uni-soft implicative filters of a BE-algebra X, then the soft union $(f,X)\,\tilde{\cup}\,(g,X)$ of (f,X) and (g,X) is a uni-soft implicative filter of X.*

Proof. For any $x \in X$, we have $(f \tilde{\cup} g)(1) = f(1) \cup g(1) \subseteq f(x) \cup g(x) = (f \tilde{\cup} g)(x)$. Let $x, y, z \in X$. Then

$$\begin{aligned}(f \tilde{\cup} g)(x * z) &= f(x * z) \cup g(x * z) \\ &\subseteq (f(x * (y * z)) \cup f(x * y)) \cup (g(x * (y * z)) \cup g(x * y)) \\ &= (f(x * (y * z)) \cup g(x * (y * z))) \cup (f(x * y) \cup g(x * y)) \\ &= (f \tilde{\cup} g)(x * (y * z)) \cup (f \tilde{\cup} g)(x * y).\end{aligned}$$

Hence $(f, X) \tilde{\cup} (g, X)$ is an int-soft implicative filter of X. □

The following example shows that the soft intersection of uni-soft implicative filters of X may not be a uni-soft implicative filter of X.

Example 3.21. Let $E = X$ be the set of parameters and $U = X$ be the initial universe set, where $X = \{1, a, b, c, d\}$ is a BE-algebra with the following Cayley table ([6]):

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let (f, X) and (g, X) be soft sets of X over U defined, respectively, as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in \{1, b\} \\ \gamma_3 & \text{if } x \in \{a, c, d\} \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, a, c\} \\ \gamma_4 & \text{if } x \in \{b, d\} \end{cases}$$

where γ_1, γ_2, γ_3, and γ_4 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$. It is easy to check that (f, X) and (g, X) are uni-soft implicative filters of X over U. But $(f, X) \tilde{\cap} (g, X) = (f \tilde{\cap} g, X)$ is not a uni-soft implicative filter of X over U, since

$$\begin{aligned}(f \tilde{\cap} g)(1 * (c * d)) \cup (f \tilde{\cap} g)(1 * c) &= (f \tilde{\cap} g)(b) \cup (f \tilde{\cap} g)(c) \\ &= (f(b) \cap g(b)) \cup (f(c) \cap g(c)) \\ &= \gamma_1 \cup \gamma_2 = \gamma_2 \not\supseteq \gamma_3 = \gamma_3 \cap \gamma_4 \\ &= f(1 * d) \cap g(1 * d) \\ &= (f \tilde{\cap} g)(1 * d).\end{aligned}$$

Let (f, X) be a soft set of X. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$f[a^k; b] := \left\{x \in X \mid f\left(a^k * (b * x)\right) = f(1)\right\}$$

where $f(a^k * x) = f(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $a, b, 1 \in f[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

There exists a soft set (f, X) of X, $a, b \in X$ and $k \in \mathbb{N}$ such that $f[a^k; b]$ is not an implicative filter of X (see Example 3.22).

Example 3.22. Let $E = X$ be the set of parameters and $U = X$ be the initial universe set where $X = \{1, a, b, c\}$ is a BE-algebra as in Example 3.6. Let (f, X) be a soft set of X over U defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_1 & \text{if } x = 1 \\ \gamma_2 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then it is a soft set of X over U. But $f[1; c] = \{x \in X | f(1 * (c * x)) = f(1)\} = \{1, c\}$ is not an implicative filter, since $b * (a * c) = 1 \in f[1; c]$, $b * a = 1 \in f[1; c]$ and $b * c = a \notin f[1; c]$.

We provide conditions for a set $f[a^k; b]$ to be an implicative filter.

Theorem 3.23. *Let X be a self distributive BE-algebra. Let (f, X) be a soft set of X over U satisfying the condition (US1) and*

$$(\forall x, y \in X)\left(f(x * y) = f(x) \cup f(y)\right). \tag{3.4}$$

Then $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in f[a^k; b]$. Let $x, y, z \in X$ be such that $x * (y * z) \in f[a^k; b]$ and $x * y \in f[a^k; b]$. Then $f\left(a^k * (b * (x * y))\right) = f(1)$, which implies from (3.4) and (US1) that

$$\begin{aligned} f(1) &= f(a^k * (b * (x * (y * z)))) \\ &= f(a^{k-1} * (a * (b * (x * (y * z))))) \\ &= f(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\ &= \cdots \\ &= f((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\ &= f(a^k * (b * (x * y))) \cup f(a^k * (b * (x * z))) \\ &= f(1) \cup f(a^k * (b * (x * z))) \\ &= f(a^k * (b * (x * z))). \end{aligned}$$

Hence $x * z \in f[a^k; b]$ and therefore $f[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Francesco Altomare
Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Department of Mathematical Sciences
University of Memphis
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Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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New types of hesitant fuzzy ideals in $_{BCK-}$algebras

N. O. Alshehri[a] and H. A. Alshehri[b]
[a]Departement of Mathematics, Faculty of Science, King Abdulaziz University, Al-Faisaliah Campus, Jeddah, Saudi Arabia
[b]Departement of Natural and Engineeeing Sciences, Faculty Applied Studies and Community Service, King Saud University, Riyadh, Saudi Arabia

Abstract

The aim of this paper is to intrduoce the new types of ideals of $BCK-$algebras. We define and discuss the hesitant fuzzy (implicative, positive implicative and commutative) ideals in $BCK-$algebras and then we study some of thier proprties.

Key words: $BCK-$algebras, (implicative, positive implicative and commutative) ideals in $BCK-$algebras,fuzzy sets, hesitant fuzzy sets, heaitant fuzzy (subalgebras and ideals), hesitant fuzzy (implicative , positive implicative and commutative) ideals in $BCK-$algebras.

1 Introduction

A BCK-algebra is an important class of logical algebras introduced by Iséki (see [2,3]) and was extensively investigated by several researchers. The concept of fuzzy sets was introduced by Zadeh in [4]. Since then ideas have been applied to other algebraic structures such as semigroups, groups rings, modules, vector spaces and topologies. In 1991, Xi [6] applied the concept of fuzzy sets to BCK-algebra, and he got some results. Further, Jun and Roh [8] , Jun et al. [9] studied fuzzy commutative ideals and fuzzy positive implicative ideals, respectively. Meng [1] introduced the concept of implicative ideals in BCK-algebras and investigated the relationship of it with the concepts of positive implicative ideals and commutative ideals. Torra [7] introduced the notion of hesitant fuzzy sets which ara a very useful to express people hesitancy in daily life. The hesitant fuzzy set is a very uesful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. In 2011, Xi and Xu [5] introduced hesitant fuzzy information aggregation techniques and their applications for decision making. In 2016, Jun and Ahn [10] interoduced the notions of hesitsnt fuzzy subalgebras and hesitant fuzzy ideals of $BCK/BCI-$algebras. In this paper, we introduce the notions of hesitant fuzzy implicative ideals, hesitant fuzzy positive implicative ideals and hesitant fuzzy commutative ideals of BCK-algebras, and discuss some properties of them.

2 Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCK-algebra if it satisfies the axioms: for all $x,y,z \in X$

$(BCK-1)$ $((x*y)*(x*z))*(z*y)=0$,
$(BCK-2)$ $(x*(x*y))*y=0$,
$(BCK-3)$ $x*x=0$,
$(BCK-4)$ $0*x=0$,
$(BCK-5)$ $x*y=0$ and $y*x=0$ imply $x=y$.

Define a binary relation $\leq$ on X by letting $x \leq y$ if and only if $x*y=0$. Then $(X;\leq)$ is a partially ordered set with the least element 0. In any BCK-algebra X, the following hold :

(1) $(x*y)*z=(x*z)*y$,
(2) $x*y \leq x$,
(3) $x*0=x$,
(4) $(x*z)*(y*z) \leq x*y$,
(5) $x*(x*(x*y))=x*y$,
(6) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$, for all $x,y,z \in X$.

A non-empty subset I of X is called an ideal of X if (I_1) $0 \in I, (I_2)$ $x*y \in I$

and $y \in I$ imply $x \in I$. A non-empty subset I of is called an implicative ideal if it is satisfies (I_1) and (I_3) $x \in I$ whenever $(x*(y*x))*z \in I$ and $z \in I$. A commutative ideal if it satisfies (I_1) and (I_4) $x*(y*(y*x)) \in I$ whenever $(x*y)*z \in I$ and $z \in I$; a positive implicative ideal if it satisfies (I_1) and (I_5) $x*z \in I$ whenever $(x*y)*z \in I$ and $y*z \in I$.

A BCK-algebra X is said to be implicative if it satisfies: $\forall x,y \in X$
$x=x*(y*x)$.

A BCK-algebra X is said to be positive implicative if it satisfies:$\forall x,y,z \in X$
$(x*z)*(y*z)=(x*y)*z$.

A BCK-algebra X is said to be commutative if it satisfies:$\forall x,y \in X$
$x*(x*y)=y*(y*x)$

Definition 2.1: [4]
Let S be a set. A fuzzy set in S is a function $\mu : S \to [0,1]$.

Definition 2.2: [6]
A fuzzy set in X is said to be a fuzzy subalgebra of X if

$$\mu(x*y) \geq \min\{\mu(x),\mu(y)\}$$

for all $x,y \in X$.

Every fuzzy subalgebra μ of X satisfies the inequality $\mu(0) \geq \mu(x)$ for all $x \in X$.

Definition 2.3: [7]
Let E be a reference set. A hesitant fuzzy set on E is defined in terms of a funcation that when applied to E returns a subset of $[0,1]$ which can be viewed as the following mathematical representation :

$$H_E := \{(e, h_E(e)) \mid e \in E\}$$

where $h_E : E \to p([0,1])$

Definition 2.4 : [10]
Given a non-empty subset A of X , a hesitant fuzzy set $H_X := \{(x, h_X(x) \mid x \in X\}$ on X satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A$$

is called a hesitant fuzzy set related to A (brielfly , $A-$hesitant fuzzy set) on X,and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a maping from X to $p([0,1])$ with $h_A = \emptyset$ for all $x \notin A$.

Definition 2.5: [10]
Given a non-empty subset (subalgebra as much as possible) A of X , let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an $A-$hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a hesitant fuzzy subalgebra of X related to A (brielfly , $A-$hesitant fuzzy subalgebra of X) if it satisfies the following condition:

$$(\forall x, y \in A)\,(h_A(x * y) \supseteq h_A(x) \cap h_A(y)).$$

An $A-$hesitant fuzzy subalgebra of X with $A = X$ is called a hesitant fuzzy subalgebra of X.

Definition 2.6: [10]
Given a non-empty subset (subalgebra as much as possible) A of X , an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy ideal of X related to A (brielfly , $A-$hesitant fuzzy ideal of X) if it satisfies:

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(y) \subseteq h_A(x) \subseteq h_A(0))$$

An $A-$hesitant fuzzy ideal of X with $A = X$ is called a hesitant fuzzy ideal of X.

Proposition 2.7: [10]
Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an $A-$hesitant fuzzy ideal of X where A is a subalgebra of X . Then the following assertions ar valid.
(1) $(\forall x, y \in A)\ \ (x \leq y \ \Rightarrow\ h_A(x) \supseteq h_A(y))$.
(2) $(\forall x, y \in A)\ \ (x * y \leq z \Rightarrow h_A(x) \supseteq h_A(y) \cap h_A(z))$

Theorem 2.8: [10]
For a subalgebra A of a BCK-algebra X , every A-hesitant fuzzy ideal is an A-hesitant fuzzy subalgebra.

Proposition 2.9: [1]
In a BCK-algebra X the following hold: for all $x, y, z \in X$
(i) $((x * z) * z) * (y * z) \leq (x * y) * z$
(ii) $(x * z) * (x * (x * z)) = (x * z) * z$
(iii) $(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq x * y$.

3 Hesitant fuzzy implicative ideals

Definition 3.1:
Given a non-empty subset (subalgebra as much as possible) A of X, an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy implicative ideal of X related to A (brielfly, $A-$hesitant fuzzy implicative ideal of X) if it satisfies:
(H_1) $h_A(0) \supseteq h_A(x)$ for all $x \in A$
(H_2) $h_A(x) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$, for all $x, y, z \in A$
An $A-$hesitant fuzzy implicative ideal of X with $A = X$ is called a hesitant fuzzy implicative ideal of X.

Theorem 3.2:
Any $A-$hesitant fuzzy implicative ideal of X must be $A-$hesitant fuzzy ideal of X.
Proof:
Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy implicative ideal of X. subsitiuting x for y in (H_2) we obtain :

$$\begin{aligned} h_A(x) &\supseteq h_A((x * (x * x)) * z) \cap h_A(z) \\ &= h_A((x * 0) * z) \cap h_A(z) \\ &= h_A(x * z) \cap h_A(z) \end{aligned}$$

Hence $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy ideal of X . $\square$

Corollary 3.3:
Every $A-$hesitant fuzzy implicative ideal of X must be $A-$hesitant fuzzy subalgebra of X .

Remark 3.4:
The converse of Theorem (3.2)may not be true as shown in the following example.

Example 3.5:
Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	b	a	0

For a subalgebra $A = \{0, a, b\}$ of X, let $H_A = \{(x, h_A(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_A = \{(0, [\tfrac{1}{4}, \tfrac{3}{4}]), (a, (\tfrac{1}{4}, \tfrac{1}{2}), (b, [\tfrac{1}{2}, \tfrac{3}{4})), (c, \emptyset)\}$$

Then $H_A = \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy ideal of X, but it is not an $A-$hesitant fuzzy implicative ideal of X because $h_A(a) = (\frac{1}{4}, \frac{1}{2}) \not\supseteq h_A((a * (b * a)) * 0) \cap h_A(0) = \left[\frac{1}{4}, \frac{3}{4}\right]$

In the following theorem, we can see that the converse of theorem (3.2)also holds if X is an implicative BCK-algebra.

Theorem 3.6:

If X is an implicative BCK-algebra, then every $A-$hesitant fuzzy ideal of X is an $A-$hesitant fuzzy implicative ideal of X.

proof:

Let X is an implicative BCK-algebra, it follows that $x = x*(y*x), \forall x, y \in X$.

Let $H_A = \{(x, h_A(x)) \mid x \in X\}$ be an $A-$hesitant fuzzy ideal of X, then

$$\begin{aligned} h_A(x) &\supseteq h_A(x * z) \cap h_A(z) \\ &\supseteq h_A((x * (y * x)) * z) \cap h_A(z) \end{aligned}$$

for all $x, y, , z \in A$. Hence H_A is an $A-$hesitant fuzzy implicative ideal of X. The proof is complete . $\square$

Theorem 3.7:

Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be $A-$hesitant fuzzy set on X, where A is non-empty subset (subalgebra as much as possible) of X is a $A-$hesitant fuzzy implicative ideal of X if and only if, for any $\varepsilon \in p([0, 1])$,the set $H_A(\varepsilon) : \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is either empty or an implicative ideal of X.

The set $H_A(\varepsilon) : \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is called a $A-$hesitant fuzzy $\varepsilon-$level set of $H_A := \{(x, h_A(x)) \mid x \in X\}$.

proof:

suppose that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy implicative ideal of X and $H_A(\varepsilon) : \{x \in X \mid h_A(x) \supseteq \varepsilon\} \neq \emptyset$ for any $\varepsilon \in p([0, 1])$. It is clear that $0 \in H_A(\varepsilon)$ since $h_A(0) \supseteq \varepsilon$. Let $(x * (y * x)) * z \in H_A(\varepsilon)$ and $z \in H_A(\varepsilon)$, then $h_A((x * (y * x)) * z\) \supseteq \varepsilon$ and $h_A(z) \supseteq \varepsilon$. It follows from (H_2) that :

$h_A(x) \supseteq h_A((x * (y * x)) * z\) \cap h_A(z) \supseteq \varepsilon \Rightarrow x \in H_A(x)$. This show that $H_A(\varepsilon)$ is an implicative ideal of X.

conversely, suoppse that for each $\varepsilon \in p([0, 1]), H_A(\varepsilon)$ is either empty or an implicative ideal of X. For any $x \in A$, let $h_A(x) = \varepsilon$. Then $x \in H_A(\varepsilon)$.

Since $H_A(\varepsilon)(\neq \emptyset)$ is an implicative ideal of X, therefore $0 \in H_A(\varepsilon)$ and hence $h_A(0) \supseteq \varepsilon = h_A(x)$ for all $x \in A$.

Now we only need to show that H_A satisfies (H_2). If not , then there exist $x', y', z' \in A$ such that $h_A(x') \subseteq h_A((x' * (y' * x')) * z') \cap h_A(z')$. Taking

$\varepsilon_0 = \frac{1}{2}\{h_A(x') + (h_A(x' * (y' * x')) * z') \cap h_A(z'))\}$; then we have that $h_A(x') \subset \varepsilon_0 \subset \{h_A(x' * (y' * x')) * z') \cap h_A(z')\}$. Hence $x' \notin H_A(\varepsilon_0), ((x' * (y' * x')) * z') \in H_A(\varepsilon_0)$ and $z' \in H_A(\varepsilon_0)$. i.e, $H_A(\varepsilon_0)$ is not an implicative ideal of X, which is contradiction. Therefore, H_A is an $A-$hesitant fuzzy implicative ideal of X, completing the proof of the theorem. $\square$

Now we give characterizations of hesitant fuzzy implicative ideals.

Theorem 3.8 :

suppose that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy ideal of X. Then the following are equivalent:

(i) H_A is an $A-$hesitant fuzzy implicative ideal of X.

(ii) $h_A(x) \supseteq h_A(x * (y * x))$ for all $x, y \in A$.

(iii) $h_A(x) = h_A(x * (y * x))$ for all $x, y \in A$.

proof:

(i) $\Rightarrow$ (ii) Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be $A-$hesitant fuzzy implicative ideal of X. Then, by (H_2) we have

$$\begin{aligned} h_A(x) &\supseteq h_A((x * (y * x)) * 0) \cap h_A(0) \\ &= h_A((x * (y * x)) \cap h_A(0) \\ &= h_A((x * (y * x)) \end{aligned}$$

for all $x, y \in A$. Hence the condition (ii) holds.

(ii) $\Rightarrow$ (iii) Observe that in $BCK-$algebra X, $x * (y * x) \leq x$. Applying proposition 2.7(1) we have $h_A(x) \subseteq h_A(x * (y * x))$

It follows from (ii) that $h_A(x) = h_A(x * (y * x))$. Hence the condition (iii) holds.

(iii) $\Rightarrow$ (i) Suppose the condition (iii) holds. Since H_A is an $A-$hesitant fuzzy ideal , by definition 2.6 we have

$$h_A(x * (y * x)) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$$

combining (iii) we obtain: $h_A(x) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$. Thus H_A satisfies (H_2). Obviously, H_A satisfies (H_1).

Therefore, H_A is an $A-$hesitant fuzzy implicative ideal of X. Hence, the condition (i) holds. The proof is complete. $\square$

Now, we give an equivalent condition for which a hesitant fuzzy subalgebra of X is to be a hesitant fuzzy implicative ideal of X.

Theorem 3.9:

A hesitant fuzzy subalgebra H_A of X is a hesitant fuzzy implicative ideal if and only if it satisfies $(\mathrm{H}_3)(x * (y * x)) * z \leq u$ implies $h_A(x) \supseteq h_A(z) \cap h_A(u)$ for all $x, y, z, u \in X$.

proof:

Assume that H_A is a hesitant fuzzy implicative ideal of X and let $x, y, z, u \in X$ be such that $(x * (y * x)) * z \leq u$. Since H_A is also hesitant fuzzy ideal of X by theorem 3.2 it follows from proposition 2.7 that:

$$h_A(x * (y * x)) \supseteq h_A(z) \cap h_A(u)$$

Making use of the theorem 3.8 (iii) we obtain $h_A(x) \supseteq h_A(z) \cap h_A(u)$.

Conversely, suppose that H_A satisfies (H_3). Obviously, H_A satisfies (H_1) since $(x * (y * x)) * ((x * (y * x)) * z) \leq z$. It follows from (H_3) that:

$$h_A(x) \supseteq h_A((x * (y * x)) * z) \cap h_A(z)$$

which shows that H_A satisfies (H_2) and so H_A is a A-hesitant fuzzy implicative ideal of X. The proof is complete. □

4 Hesitant fuzzy positive implicative ideals

Definition 4.1:

Given a non-empty subset (subalgebra as much as possible) A of X, an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy positive implicative ideal of X related to A (brielfly, $A-$hesitant fuzzy positive implicative ideal of X) if it satisfies:

(H_1) $h_A(0) \supseteq h_A(x)$ for all $x \in A$

(H_4) $h_A(x * z) \supseteq h_A((x * y) * z) \cap h_A(y * z)$, for all $x, y, z \in A$

An $A-$hesitant fuzzy positive implicative ideal of X with $A = X$ is called a hesitant fuzzy positive implicative ideal of X.

Theorem 4.2:

Every $A-$hesitant fuzzy positive implicative ideal of X is $A-$hesitant fuzzy ideal of X.

Proof:

Assume that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an $A-$hesitant fuzzy positive implicative ideal of X . Putting $z = 0$ in (H_4) we obtain:

$$\begin{aligned} h_A(x * 0) &\supseteq h_A((x * y) * 0) \cap h_A(y * 0) \\ h_A(x) &\supseteq h_A(x * y) \cap h_A(y). \end{aligned}$$

So, H_A is an $A-$hesitant fuzzy ideal of X. □

Remark 4.3:

The converse of Theorem (4.2) may not be true as shown in the following example.

Example 4.4:

Let X be a BCK-algebra as in Example (3.5) It is easy to check that H_A is an $A-$hesitant fuzzy ideal of X , but it is not an $A-$hesitant fuzzy positive

implicative ideal of X because $h_A(a*0) = (\frac{1}{4}, \frac{1}{2}) \not\supseteq [\frac{1}{2}, \frac{3}{4}) = (h_A((a*b)*0) \cap h_A(b*0))$.

Proposition 4.5:
If X is positive implicative, then an $A-$hesitant fuzzy ideal of X is an $A-$hesitant fuzzy positive implicative ideal of X if and only if satisfies:

$$h_A(x*y) = h_A((x*y)*y) \text{ for all } x, y \in A.$$

Proof:
Suppose that the heaitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X is an $A-$hesitant fuzzy positive implicative ideal of X. So by (H_4)

$$h_A(x*z) \supseteq h_A((x*y)*z) \cap h_A(y*z)$$

subiting $z = y$ in (H_4) we have :

$$\begin{aligned} h_A(x*y) &\supseteq h_A((x*y)*y) \cap h_A(y*y) \\ &= h_A((x*y)*y) \cap h_A(0) \\ &= h_A((x*y)*y) \end{aligned}$$

On other hand, since $(x*y)*y \leq x*y$, it follows from Propsition 2.7(1) that $h_A((x*y)*y) \supseteq h_A(x*y)$. Thus we have $h_A(x*y) = h_A((x*y)*y)$ for all $x, y \in A$.

Conversely, assume that $h_A(x*y) = h_A((x*y)*y)$ for all $x, y \in A$. we want to prove that an $A-$hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X is an $A-$hesitant fuzzy positive implicative ideal of X. It is clear that, $h_A(0) \supseteq h_A(x)$ for all $x \in A$. Since H_A is A-hesitant fuzzy ideal, and X is positive implicative then

$$\begin{aligned} h_A(x*z) &= h_A((x*z)*z) \supseteq h_A((x*z)*(y*z)) \cap h_A(y*z) \\ h_A(x*z) &\supseteq h_A((x*y)*z) \cap h_A(y*z) \end{aligned}$$

Therefore, $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X is an $A-$hesitant fuzzy positive implicative ideal of X. The proof is complete. □

5 Hesitant fuzzy Commutative ideals

Definition 5.1:
Given a non-empty subset (subalgebra as much as possible) A of X, an $A-$hesitnat fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy commutative ideal of X related to A (brielfly, $A-$hesitant fuzzy commutative ideal of X) if it satisfies:
(H_1) $h_A(0) \supseteq h_A(x)$ for all $x \in A$
(H_C) $h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$, for all $x, y, z \in A$
An $A-$hesitant fuzzy commutative ideal of X with $A = X$ is called a hesitant fuzzy commutative ideal of X.

Example 5.2:
Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by $H_A = \{(0, [0, 1]), (a, [0.2, 0.7], (b, (0.2, 0.3]), (c, [0.6, 0.7))\}$. It is routine to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is hesitant fuzzy commutative ideal of X.

Theorem 5.3:
Any heaitant fuzzy commuatative ideal of BCK-algebra X is hesitant fuzzy ideal of X.

Proof:
Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy commutative ideal of a BCK-algebra X. For any $x, y, z, \in X$, we have

$$\begin{aligned} h(x * z) \cap h(z) &= h((x * 0) * z) \cap h(z) \\ &\subseteq h(x * (0 * (0 * x))) \\ &= h(x) \end{aligned}$$

i.e. H_X is hesitant fuzzy ideal of X. □

Remark 5.4:
A hesitant fuzzy ideal of a BCK- algebra X, may not be hesitant fuzzy commutative ideal. For instance, Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let $S_0, S_1, S_2 \in P([0, 1])$ be such that $S_0 = [0, 1) \supset S_1 = [0.2, 0.7] \supseteq S_2 = [0.4, 0.5]$. Define a mapping $h : X \to P([0, 1])$ by $h(0) = [0, 1), h(a) = [0.2, 0.7]$ and $h(b) = h(c) = h(d) = [0.4, 0.5)$. Routine calculations give that h is a hesitant fuzzy ideal of X. But it is not a hesitant fuzzy commutative ideal of X because $h(b * (c * (c * b))) \not\supseteq h((b * c) * 0) \cap h(0)$.

Theorem 5.5:
Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy ideal of a BCK-algebra X. Then H_X is hesitant fuzzy commutative ideal of X if and only if satisfies the condition

$$h(x * (y * (y * x))) \supseteq h(x * y) \text{ for all } x, y \in X \tag{1.1}$$

proof:
Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is hesitant fuzzy commutative ideal. Taking $z = 0$ in (H_C) and using (H_1). Also, we use $x * 0 = x$.

$$h(x * (y * (y * x))) \supseteq h((x * y) * 0) \cap h(0) = h(x * y).$$

Conversely, suppose that $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies the condition (1.1) as H_X is a hesitant fuzzy ideal. Hence

$$h(x * y) \supseteq h((x * y) * z) \cap h(z) \tag{1.2}$$

for all $x, y, z \in X$. combining (1.2) and(1.1) then we obtain (H_C). The proof is complete. □

Observing $x * y \leq x * (y * (y * x))$ and using propoition (2.7) we have $h(x * (y * (y * x))) \subseteq h(x * y)$. Hence, theorem (5.5) can be improved as follows:

Theorem 5.6:
A hesitant fuzzy ideal H_X of a BCK-algebra X is hesitant fuzzy commutative ideal of X if and only if satisfies the identity $h(x*y) = h(x*(y*(y*x)))$ for all $x, y \in X$.

Theorem 5.7:
In a commutative BCK-algebra X. Every hesitant fuzzy ideal is a hesitant fuzzy commutative ideal.
Proof:
Let H_X be a hesitant fuzzy ideal of $BCK-$algebra X. It is sufficient to show that H_X satisfies condition (H_C). Let $x, y, z \in X$.Then

$$\begin{aligned} & ((x * (y * (y * x))) * ((x * y) * z)) * z \\ = \; & ((x * (y * (y * x))) * z) * ((x * y) * z) \\ \leq \; & (x * (y * (y * x))) * (y * x) \\ = \; & (x * (y * y)) * (y * (y * x)) = 0 \end{aligned}$$

that is ,

$$(x * (y * (y * x))) * ((x * y) * z) \leq z$$

It follows from propostion (2.7)

$$h(x * (y * (y * x))) \supseteq h((x * y) * z) \cap h(z)$$

Thus (H_C) holds, and the proof is complete. □

Theorem 5.8:

Let X be a BCK-algebra and let H_A be A-hesitant fuzzy set in X. Then H_A is a A-hesitant fuzzy commutative ideal of X if and only if $H_A(\varepsilon)$ is commutative ideal of X for all $\varepsilon \in P([0,1]), H_A(0) \supseteq \varepsilon$.

Proof:

Suppose H_A is a A-hesitant fuzzy commutative ideal of X. For any fixed $\varepsilon \in P([0,1])$, if $h_A(0) \supseteq \varepsilon$ then $0 \in H_A(\varepsilon)$. Hence $H_A(\varepsilon)$ satisfie (H1). Let $(x*y)*z \in H_A(\varepsilon)$ and $z \in H_A(\varepsilon)$. It follows that $h_A((x*y)*z) \supseteq \varepsilon$ and $h_A(z) \supseteq \varepsilon$. By ($\mathrm{H}_C$) we have :

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z) \supseteq \varepsilon$$

namely $(x*(y*(y*x))) \in H_A(\varepsilon)$, so $H_A(\varepsilon)$ satisfies condition (C). Thus, $H_A(\varepsilon)$ is commutative ideal of X.

Conversely, assume that $H_A(\varepsilon)$ is commutative ideal of X for all $\varepsilon \in p([0,1])$, $h_A(0) \supseteq \varepsilon$. Let $h_A((x*y)*z) = \varepsilon_1$ and $h_A(z) = \varepsilon_2$ for $x, y, z \in A$. Then $(x*y)*z \in H_A(\varepsilon_1)$ and $z \in H_A(\varepsilon_2)$. Without loss of generality,we may assume that $\varepsilon_1 \subseteq \varepsilon_2$. Then $H_A(\ \varepsilon_2) \subseteq H_A(\ \varepsilon_1)$ and so $z \in H_A(\varepsilon_1)$. Since $H_A(\varepsilon_1)$ is commutative ideal of X by hypothesis we obtain that $x*(y*(y*x)) \in H_A(\varepsilon_1)$. Thus $h_A(x*(y*(y*x))) \supseteq \varepsilon_1 = h_A((x*y)*z) \cap h_A(z)$. It is clear that $h_A(0) \supseteq h_A(x)$ for all $x \in A$. Therefore H_A is a A-hesitant fuzzy commutative ideal of X. □

Definition 5.9:

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy commutative ideal of a BCK-algebra X. The commutative ideals $H_X(\ \varepsilon), \varepsilon \in p([0,1])$ and $H_X(\ 0) \supseteq \varepsilon$ are called hesitant $\varepsilon-$level commutative ideals of H_X.

Theorem 5.10:

Any commutative idael of BCK-algebra X can be realized as hesitant $\varepsilon-$level commutative ideal of some hesitant fuzzy commutative ideal of X.

Proof:

Suppose C is a commutative ideal of BCK-algebra X and let H_A be a hesitant fuzzy set in X defined by

$$H_A = \begin{cases} \varepsilon & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

where ε is a fixed interval in $p([0,1])$. Let $x, y, z \in A$. We will divide into the following cases to verify that H_A is $A-$hesitant fuzzy commutative ideal of X.

(i) If $((x*y)*z) \in C$ and $z \in C$, then $(x*(y*(y*x))) \in C$. Thus

$$h_A((x*y)*z) = h_A(z) = h_A(x*(y*(y*x))) = \varepsilon$$

and so

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$$

(ii) If $((x*y)*z) \notin C$ and $z \notin C$,then $h_A((x*y)*z) = h_A(z) = 0$. Hence

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$$

(iii) If exactly one of $(x*y)*z$ and z belongs to C then exactly one of $h_A((x*y)*z)$ and $h_A(z)$ is equal to 0. So

$$h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$$

The results above show $h_A(x*(y*(y*x))) \supseteq h_A((x*y)*z) \cap h_A(z)$, for all $x, y, z \in A$. It is clear that $h_A(0) \supseteq h_A(x)$ for all $x \in A$. Hence H_A is $A-$hesitant fuzzy commutative ideal of X and obviously $H_A(\varepsilon) = C$. The proof is completed. □

Theorem 5.11:

Suppose $H_A := \{(x, h_A(x)) \mid x \in X\}$ is a $A-$hesitant fuzzy commutative ideal of BCK-algebra X. Then two hesitant level commutative ideals $H_A(\varepsilon_1), H_A(\varepsilon_2)$ with $(\varepsilon_1 \subset \varepsilon_2)$ of H_A are equal if and only if there is no $x \in X$ such that $\varepsilon_1 \subseteq h_A(x) \subset \varepsilon_2$.

Proof:

Suppose $\varepsilon_1 \subset \varepsilon_2$ and $H_A(\varepsilon_1) = H_A(\varepsilon_2)$. If there exists $x \in X$ such that $\varepsilon_1 \subseteq h_A(x) \subset \varepsilon_2$, then $H_A(\varepsilon_2)$ is proper subset of $H_A(\varepsilon_1)$. This is impossible.

Conversely, suppose that there is no $x \in X$, such that $\varepsilon_1 \subseteq h_A(x) \subset \varepsilon_2$. Note that, $\varepsilon_1 \subset \varepsilon_2$ implies $H_A(\varepsilon_2) \subseteq H_A(\varepsilon_1)$. If $x \in H_A(\varepsilon_1)$, then $h_A(x) \supseteq \varepsilon_1$ and so $h_A(x) \supseteq \varepsilon_2$

because $h_A(x) \nsubseteq \varepsilon_2$. Hence $x \in H_A(\varepsilon_2)$ which says that $H_A(\varepsilon_1) \subseteq H_A(\varepsilon_2)$. Thus, $H_A(\varepsilon_1) = H_A(\varepsilon_2)$. This completes the proof. □

Let h_A be a A-hesitant fuzzy set in X and let $Im(h_A)$ denote the image of h_A.

Theorem 5.12:

Let X be a BCK-algebra and h_A a A-hesitant fuzzy commutative ideal of X. If $Im(h_A) = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ where $\varepsilon_1 \subset \varepsilon_2 \subset ... \subset \varepsilon_n$, then the family of commutative ideals $H_A(\varepsilon_i)(i = 1, 2, ..., n)$ consitiutes all the hesitant level commutative ideals of h_A.

Proof :

Let $\varepsilon \in p([0,1])$ and $\varepsilon \notin \mathrm{Im}(h_A)$. If $\varepsilon \subset \varepsilon_1$, then $h_A(\varepsilon_1) \subseteq h_A(\varepsilon)$. Since $h_A(\varepsilon_1) = X$, we have $h_A(\varepsilon) = X$ and $h_A(\varepsilon) = h_A(\varepsilon_1)$.

If $\varepsilon_i \subset \varepsilon \subset \varepsilon_{i+1}$ $(1 \leq i \leq n-1)$, then there is no $x \in X$ such that $\varepsilon \subseteq h_A(x) \subset \varepsilon_{i+1}$. From above theorem(5.10) it follows that $h_A(\varepsilon) = h_A(\varepsilon_{i+1})$. This shows that for any $\varepsilon \in p([0,1])$ with $h_A(0) \supseteq \varepsilon$, the hesitant level commutative ideal $h_A(\varepsilon)$ is in $\{h_A(\varepsilon_i) : 1 \leq i \leq n\}$.□

Lemma 5.13:

Let X be a finite $BCK-$algebra and H_A a $A-$hesitant fuzzy commutative ideal of X. If α and β belong to $\mathrm{Im}(h_A)$ such that $h_A(\alpha) = h_A(\beta)$, then $\alpha = \beta$.

Proof:

Suppose $\alpha \neq \beta$, say $\alpha < \beta$, then there is $x \in X$ such that $h_A(x) = \alpha < \beta$ and so $x \in h_A(\alpha)$ and $x \notin h_A(\beta)$. Thus $h_A(\alpha) \neq h_A(\beta)$ which is contradiction . The proof is complete. □

Theorem 5.14:

A hesitant fuzzy ideal H_A of X is hestiant fuzzy implicative ideal if and only if H_A is both hestiant fuzzy commutative ideal and hestiant fuzzy positive implicative ideal.

Proof:

Suppose H_A is hestiant fuzzy implicative ideal of X. For all $x, y, z \in A$ we have

$$\begin{aligned} h_A((x*y)*z) \cap h_A(y*z) &\subseteq h_A((x*z)*z), \quad [\text{by proposition}(2.9)(i)\text{and (2.7)}] \\ &= h_A((x*z)*(x*(x*z)), \quad [\text{by proposition}(2.9)(ii)] \\ &= h_A(x*z), \quad [\text{by theorem}(3.8)(iii)] \end{aligned}$$

So H_A is hesitant fuzzy positive implicative ideal of X. By proposition (2.7), $(2.9)(iii)$ and theorem $(3.8)(iii)$

$$h_A(x*y) \subseteq h_A(x*(y*(y*x)))*(y*(x*(y*(y*x))) = h_A(x*(y*(y*x))$$

It follows from proposition (5.5) that H_A is hesitant fuzzy commutative ideal of X.

Conversely, suppose that H_A is both hesitant fuzzy positive implicative and hesitant fuzzy commutative ideal of X. Since $(y*(y*x))*(y*x) \leq x*(y*x)$. It follows from proposition (2.7) that $h_A(x*(y*x)) \subseteq h_A(\ (y*(y*x))*(y*x))$. Using theorm (4.5)we have $h_A(\ (y*(y*x))*(y*x)) = h_A(y*(y*x))$ and so

$$h_A(x*(y*x)) \subseteq h_A(y*(y*x)) \qquad (*)$$

On the other hand, since $x*y \leq x*(y*x)$ implies $h_A(x*(y*x)) \subseteq h_A(\ x*y)$. Since H_A is hesitant fuzzy commutative by theorem (5.6) we have $h_A(x*y) = h_A(x*(y*(y*x)))$, hence $h_A(x*(y*x)) \subseteq h_A(x*(y*(y*x)))$. Combinging $(*)$ we obtain

$$h_A(x*(y*x)) \subseteq h_A(x*(y*(y*x))) \cap h_A(y*(y*x)) \subseteq h_A(x)$$

So H_A is a hesitant fuzzy implicative ideal of X by theorem (3.8) . The proof is complete. □

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L-fuzzy (K, E)-soft quasi-uniformities and L-fuzzy (K, E)-soft topologies

A.A.Ramadan, Yong Chan Kim[1]

Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

Department of Mathematics, Gangneung-Wonju University, Gangneung, Gangwondo 25457, Korea

Abstract The goal of this paper is to focus on the relationships between L-fuzzy (K, E)- soft quasi uniformities and L-fuzzy (K, E)- soft topologies in complete residuated lattices. As main results, we investigate the L-fuzzy (K, E)- soft topologies induced by L-fuzzy (K, E)- soft uniformities. Moreover, we study the L-fuzzy (K, E)- soft quasi uniformities induced by L-fuzzy (K, E)- soft topologies. We give their examples.

Key Words and Phrases: complete residuated lattice, L-fuzzy (K, E)- soft quasi uniformities , L-fuzzy (K, E)- soft topologies.

AMS Subject Classification: 03E72, 06A15, 06F07,54A05

1 Introduction

Molodtsov [18] introduced a completely new concept called soft set theory to model uncertainty, which associates a set with a set of parameters. Pei and Miao [19] showed that soft sets are a class of special information systems. Later, Maji et al. [15] introduced the concept of a fuzzy soft set which combines a fuzzy set and a soft set. Presently, the soft set theory is making progress rapidly [1,2,6,15-19,26,28,31,32]. The topological structures of soft sets have been developed by many researchers [3,5,8,23,27,29,30,33].

Hájek [9] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [4] investigated information systems and decision rules in complete residuated lattices. Höhle [10] introduced L-fuzzy topologies

[1]Corresponding author

with algebraic structure L(cqm, quantales, MV-algebra). Uniformities in fuzzy sets, have the following approach of Lowen [14] based on powersets of the form $L^{X\times X}$ as a viewpoint of the enourage approach, the uniform covering approach of Kotzé [13], the uniform operator approach of Rodabaugh [25] as a generalization of Hutton [11] based on powersets of the form $(L^X)^{(L^X)}$, the unification approach of Gutiérrez García [7]. Recently, Gutiérrez García introduced L-valued Hutton uniformity where a quadruple $(L, \leq, \otimes, \star)$ is defined by a GL-monoid $(L, \star)$ as an extension of a completely distrbutive lattice L. Kim [12] introduced the notion of L-fuzzy uniformities as an extension of Lowen in a strictly two-sided, commutative quantale. Moreover, he investigated the relations between L- fuzzy topological spaces and L- fuzzy uniform spaces. Ramadan et.al [23] introduced the notion of L-fuzzy (K, E)- soft topogenous orders and L-fuzzy (K, E)- soft quasi uniformities in complete residuated lattices.

The goal of this paper is to focus on the relationships between L-fuzzy (K, E)-soft quasi uniformities and L-fuzzy (K, E)- soft topologies in complete residuated lattices. As main results, we investigate the L-fuzzy (K, E)- soft topologies induced by L-fuzzy (K, E)- soft uniformities. Moreover, we study the L-fuzzy (K, E)- soft quasi uniformities induced by L-fuzzy (K, E)- soft topologies. We give their examples.

2 Preliminaries

Let $L = (L, \leq, \vee, \wedge, 0, 1)$ be a completely distributive lattice with the least element 0 and the greatest element 1 in L.

Definition 2.1. [4,9,11] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

Remark 2.2. Every completely distributive lattice $(L, \leq, \wedge, \vee, ^*)$ with order reversing involution * is a complete residuated lattice $(L, \leq, \odot, \oplus, ^*)$ with a strong nega-

tion * where $\odot = \wedge$, $\oplus = \vee$ and

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \leq, \odot, \to, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution $x^* = x \to 0$ which is defined by $x \oplus y = (x^* \odot y^*)^*$.

Lemma 2.3. [4,9,11] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \leq y$ iff $x \to y = 1$.

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(7) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(8) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(10) $x \odot y = (x \to y^*)^*$ and $x \oplus y = x^* \to y$,

(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(12) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

(14) $x \to y = y^* \to x^*$.

(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.

(16) $\bigvee_{i \in \Gamma} x_i \to \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \to y_i)$ and $\bigwedge_{i \in \Gamma} x_i \to \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \to y_i)$,

Throughout this paper, X refers to an initial universe, E and K are the sets of all parameters for X, and L^X is the set of all L-fuzzy sets on X.

Definition 2.4. [3,5,23] A map f is called an L- fuzzy soft set on X, where f is a mapping from E into L^X, i.e., $f_e := f(e)$ is an L- fuzzy set on X, for each $e \in E$. The family of all L- fuzzy soft sets on X is denoted by $(L^X)^E$. Let f and g be two L-fuzzy soft sets on X.

(1) f is an L-fuzzy soft subset of g and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L- fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L- fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L- fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An L- fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an L- fuzzy soft sets on X is denoted by f^*, where $f^* : E \to L^X$ is a mapping given by $f_e^* = (f_e)^*$, for each $e \in E$.

(7) f is called a null L- fuzzy soft set and is denoted by 0_X , if $f_e(x) = 0$, for each $e \in E$, $x \in X$.

(8) f is called an absolute L- fuzzy soft set and is denoted by 1_X , if $f_e(x) = 1$, for each $e \in E$, $x \in X$.

Definition 2.5. [3,5] A mapping $\mathcal{T} : K \to L^{(L^X)^E}$ (where $\mathcal{T}_k := \mathcal{T}(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an L-fuzzy (K, E)-soft topology on X if it satisfies the following conditions for each $k \in K$.

(SO1) $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$,

(SO2) $\mathcal{T}_k(f \odot g) \geq \mathcal{T}_k(f) \odot \mathcal{T}_k(g) \ \ \forall\ f, g \in (L^X)^E$,

(SO3) $\mathcal{T}_k(\bigsqcup_i f_i) \geq \bigwedge_{i \in I} \mathcal{T}_k(f_i)\ \forall\ f_i \in (L^X)^E,\ i \in I$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy (K, E)-soft topological space.

Definition 2.6. [3,5] A mapping $\mathcal{F} : K \to L^{(L^X)^E}$ is called an L-fuzzy (K, E)-soft cotopology on X if it satisfies the following conditions for each $e \in E$.

(SF1) $\mathcal{F}_k(0_X) = \mathcal{F}_k(1_X) = 1$,

(SF2) $\mathcal{F}_k(f \oplus g) \geq \mathcal{F}_k(f) \odot \mathcal{F}_k(g), \ \ \forall\ f, g \in (L^X)^E$,

(SF3) $\mathcal{F}_k(\sqcap_{i \in I} f_i) \geq \bigwedge_i \mathcal{F}_k(f_i),\ \forall\ f_i \in (L^X)^E,\ i \in I$.

The pair $(X, \mathcal{F})$ is called an L-fuzzy (K, E)-soft cotopological space.

Definition 2.7. [23] An L- fuzzy (K, E)-soft quasi uniformity is a mapping $\mathcal{U} : K \to L^{(L^{X \times X})^E}$ which satisfies the following conditions .

(SU1) There exists $u \in (L^{X \times X})^E$ such that $\mathcal{U}_k(u) = 1$.

(SU2) If $v \sqsubseteq u$, then $\mathcal{U}_k(v) \leq \mathcal{U}_k(u)$.

(SU3) For every $u, v \in (L^{X\times X})^E, \mathcal{U}_k(u \odot v) \geq \mathcal{U}_k(u) \odot \mathcal{U}_k(v)$.

(SU4) If $\mathcal{U}_k(u) \neq 0$ then $\top_\triangle \sqsubseteq u$ where, for each $e \in E$,

$$(\top_\triangle)_e(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

(SU5) $\mathcal{U}_k(u) \leq \bigvee\{\mathcal{U}_k(v) \mid v \circ v \sqsubseteq u\}$, where

$$u_e \circ v_e(x, z) = \bigvee_{y\in X} u_e(x, y) \odot v_e(y, z).$$

The pair $(X,\mathcal{U})$ is called an L-fuzzy (K, E)-soft quasi-uniform space.

An L-fuzzy (K, E)-soft quasi-uniform space $(X,\mathcal{U})$ is said to be an L-fuzzy (K, E)-soft uniform space if

(U) $\mathcal{U}_k(u) \leq \mathcal{U}_k(u^{-1})$, where $(u^{-1})_e(x, y) = u_e(y, x)$ for each $k \in K$ and $u \in (L^{X\times X})^E$.

An L-fuzzy (K, E)-soft quasi-uniformity $\mathcal{U}$ on X is said to be an $(L, \odot)$-fuzzy principle quasi-uniformity if

(P) $\mathcal{U}_k(\bigwedge_{i\in\Gamma} u_i) = \bigwedge_{i\in\Gamma} \mathcal{U}_k(u_i)$ for all $u_i \in (L^{X\times X})^E$.

Remark 2.8. Let $(X,\mathcal{U})$ be an L-fuzzy (K, E)-soft uniform space.

(1) By (SU1) and (SU2), we have $\mathcal{U}_k(1_{X\times X}) = 1$ because $u \sqsubseteq 1_{X\times X}$ for all $u \in (L^{X\times X})^E$.

(2) Since $\mathcal{U}_k(u) \leq \mathcal{U}_k(u^{-1}) \leq \mathcal{U}_k((u^{-1})^{-1}) = \mathcal{U}_k(u)$, then $\mathcal{U}_k(u) = \mathcal{U}_k(u^{-1})$.

3 L-fuzzy (K, E)-soft topologies induced by L-fuzzy (K, E)-soft uniformities

Lemma 3.1. For every $f \in (L^X)^E$, we define $(u_f), (u_f^{-1}) \in (L^{X\times X})^E$ by:

$$(u_f)_e(x, y) = f_e(x) \to f_e(y) \quad \forall e \in E,$$

$$(u_f^{-1})_e(x, y) = (u_f)_e(y, x) \ ,$$

then $\forall f, g \in (L^X)^E$ we have the following statements

(1) $1_{X\times X} = u_{0_X} = u_{1_X}$,

(2) For every $u_f \in (L^{X\times X})^E$, we have $u_f \circ u_f \sqsubseteq u_f$,

(3) $u_f \odot u_g \sqsubseteq u_{f\odot g}$,

(4) $u_f \odot u_g \sqsubseteq u_{f\oplus g}, u_f \oplus u_g \sqsubseteq u_{f\oplus g}$,

(5)$u_f^{-1} = u_{f^*}$,

(6)$u_{f\odot g}^{-1} = u_{f^*\oplus g^*}$,

(7)$u_{f\oplus g}^{-1} = u_{f^*\odot g^*}$,

Proof. (1) $(1_{X\times X})_e(x,y) = \top = (u_{0_X})_e(x,y) = (0_X)_e(x) \to (0_X)_e(y) = (1_X)_e(x) \to (1_X)_e(y) = (u_{1_X})_e(x,y)$.

(2)

$$\begin{aligned}&((u_f)_e \circ (u_f)_e)(x,z) = \bigvee_{y\in X}((u_f)_e(x,y) \odot (u_f)_e(y,z))\\ &= \bigvee_{y\in X}(f_e(x) \to f_e(y)) \odot (f_e(y) \to f_e(z)))\\ &\le (f_e(x) \to f_e(y)) \odot (f_e(y) \to f_e(z))\\ &\le f_e(x) \to f_e(z).\end{aligned}$$

Hence $(u_f \circ u_f) \le u_f$.

(3)

$$\begin{aligned}&(u_f \odot u_g)_e(x,y) = (u_f)_e(x,y) \odot (u_g)_e(x,y)\\ &\le (f_e(x) \to f_e(y)) \odot (g_e(x) \to g_e(y))\\ &\le f_e(x) \odot g_e(x) \to f_e(y) \odot g_e(y)\\ &= (u_{f\odot g})_e(x,y).\end{aligned}$$

(4)

$$\begin{aligned}&(u_f \odot u_g)(x,y) = u_f(x,y) \odot u_g(x,y)\\ &\le (f_e(x) \to f_e(y)) \odot (g_e(x) \to g_e(y))\\ &\le f_e(x) \oplus g_e(x) \to f_e(y) \oplus g_e(y)\\ &= u_{f\oplus g}(x,y).\end{aligned}$$

(5)

$$\begin{aligned}&(u_f^{-1})_e(x,y) = (u_f)_e(y,x) = f_e(y) \to f_e(x) = f_e^*(x) \to f_e^*(y)\\ &= (u_{f^*})_e(x,y).\end{aligned}$$

(6)

$$(u_{f\odot g}^{-1})_e = (u_{(f\odot g)^*})_e = (u_{f^*\oplus g^*})_e.$$

(7)

$$(u_{f\oplus g}^{-1})_e = (u_{(f\oplus g)^*})_e = (u_{f^*\odot g^*})_e.$$

Theorem 3.2. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft principle quasi-uniform space. Define the mapping $\mathcal{T}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by:

$$\mathcal{T}_k^{\mathcal{U}}(f) = \begin{cases} 1, & \text{if } f = 0_X \\ \mathcal{U}_k(u_f), & \text{if } f \neq 0_X. \end{cases}$$

Then, $\mathcal{T}^{\mathcal{U}}$ is an L-fuzzy (K, E)-soft topology on X.

Proof. (SO1) $\mathcal{T}_k^{\mathcal{U}}(0_X) = 1$ and $\mathcal{T}_k^{\mathcal{U}}(1_X) = \mathcal{U}_k(u_{1_X}) = \mathcal{U}_k(1_{X\times X}) = 1$.

(SO2) Since $u_f \odot u_g \sqsubseteq u_{f\odot g}$, for each $f, g \in (L^X)^E$, by (SU3), we have $\mathcal{U}_k(u_f \odot u_g) \leq \mathcal{U}_k(u_{f\odot g})$.

Hence, $\mathcal{T}_k^{\mathcal{U}}(f \odot g) = \mathcal{U}_k(u_{f\odot g}) \geq \mathcal{U}_k(u_f \odot u_g) \geq \mathcal{U}_k(u_f) \odot \mathcal{U}_k(u_g) = \mathcal{T}_k^{\mathcal{U}}(f) \odot \mathcal{T}_k^{\mathcal{U}}(g)$.

(SO3) Let $\{f_i\}_{i\in\Gamma}$ be a family of fuzzy soft sets in X. Then, by Lemma 2.3(8),we have

$$\begin{aligned} (u_{(\bigvee_{i\in\Gamma} f_i)})_e(x, y) &= (\bigvee_{i\in\Gamma} f_i)_e(x) \to (\bigvee_{i\in\Gamma} f_i)_e(y) \\ &\geq \bigwedge_i ((f_i)_e(x) \to (f_i)_e(y)) \\ &= \bigwedge_i (u_{f_i})_e(x, y). \end{aligned}$$

Then $\mathcal{T}_k^{\mathcal{U}}(\bigvee_{i\in\Gamma} f_i) = \mathcal{U}_k(u_{(\bigvee_{i\in\Gamma} f_i)}) \geq \mathcal{U}_k(\bigwedge_{i\in\Gamma} u_{f_i}) = \bigwedge_{i\in\Gamma} \mathcal{U}_k(u_{f_i}) = \bigwedge_{i\in\Gamma} \mathcal{T}_k^{\mathcal{U}}(f_i)$, for every $i \in \Gamma$.

Theorem 3.3. Let $(X, \mathcal{U})$ be a L-fuzzy (K, E)-soft principle quasi-uniform space. Define the mapping $\mathcal{F}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by:

$$\mathcal{F}_k^{\mathcal{U}}(f) = \begin{cases} 1, & \text{if } f = 1_X \\ \mathcal{U}_k(u_f^{-1}), & \text{if } f \neq 1_X. \end{cases}$$

Then, $\mathcal{F}^{\mathcal{U}}$ is an L- fuzzy (K, E)-soft cotopology on X.

Proof. (SF1) $\mathcal{F}_k^{\mathcal{U}}(1_X) = 1$ and $\mathcal{T}_k^{\mathcal{U}}(0_X) = \mathcal{U}_k(u_{0_X}^{-1}) = \mathcal{U}(u_{1_X}) = \mathcal{U}_k(1_{X\times X}) = 1$.

(SF2) Since $u_{f^*} \odot u_{g^*} \sqsubseteq u_{f^*\odot g^*} = u_{f\oplus g}^{-1}$, for each $f, g \in (L^X)^E$, by (U3), we have $\mathcal{U}_k(u_f^{-1} \odot u_g^{-1}) \leq \mathcal{U}_k(u_{f\odot g}^{-1})$. Hence, $\mathcal{F}_k^{\mathcal{U}}(f \oplus g) = \mathcal{U}_k(u_{f\oplus g}^{-1}) = \mathcal{U}_k(u_{f^*\odot g^*}) \geq \mathcal{U}_k(u_{f^*} \odot u_{g^*}) \geq \mathcal{U}_k(u_{f^*}) \odot \mathcal{U}(u_{g^*}) = \mathcal{F}_k^{\mathcal{U}}(f) \odot \mathcal{F}_k^{\mathcal{U}}(g)$.

(SF3) Let $\{f_i\}_{i\in\Gamma}$ be a family of fuzzy sets in X. Then, by Lemma 2.3(8),we have

$$\begin{aligned} (u_{(\bigvee_{i\in\Gamma} f_i^*)})_e(x, y) &= (\bigvee_{i\in\Gamma} f_i^*)_e(x) \to (\bigvee_{i\in\Gamma} f_i^*)_e(y) \\ &\geq \bigwedge_i ((f_i^*)_e(x) \to (f_i^*)_e(y)) \\ &= \bigwedge_i (u_{f_i^*})_e(x, y). \end{aligned}$$

Then $\mathcal{F}_k^{\mathcal{U}}(\bigwedge_{i\in\Gamma} f_i) = \mathcal{U}_k(u^{-1}_{(\bigwedge_{i\in\Gamma} f_i)}) = \mathcal{U}_k(u_{(\bigvee_{i\in\Gamma} f_i^*)}) \geq \mathcal{U}_k(\bigwedge_{i\in\Gamma} u_{f_i^*}) = \bigwedge_{i\in\Gamma} \mathcal{U}_k(u_{f_i^*}) = \bigwedge_{i\in\Gamma} \mathcal{F}_k^{\mathcal{U}}(f_i)$, for every $i \in \Gamma$.

Theorem 3.4 Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft quasi-uniform space. Define the mapping $\mathcal{T}^{\mathcal{U}} : K \to L^{(L^X)^E}$ by

$$\mathcal{T}_k^{\mathcal{U}}(f) = \bigwedge_{e\in E} \bigwedge_{x\in X} \{f_e^*(x) \vee \bigvee_{u[x]\sqsubseteq f} \mathcal{U}_k(u)\}$$

where $(u[x])_e(y) = u_e(y,x)$. Then, $\mathcal{T}^{\mathcal{U}}$ is an L-fuzzy (K,E)-soft topology on X.

Proof (SO1) It is easily checked.

(SO2) Suppose that

$$(\bigvee_{u[x]\sqsubseteq f} \mathcal{U}_k(u)) \odot (\bigvee_{v[x]\sqsubseteq g} \mathcal{U}_k(v)) \nleq \bigvee_{w[x]\sqsubseteq f\odot g} \mathcal{U}_k(w).$$

For each $i \in \{1,2\}$, there exists u_i with $u_i[x] \sqsubseteq f_i$ such that

$$\mathcal{U}_k(u_1) \odot \mathcal{U}_k(u_2) \nleq \bigvee_{w[x]\sqsubseteq f\odot g} \mathcal{U}_k(w).$$

It implies $(u_1 \odot u_2)[x] \sqsubseteq f \odot g$ such that

$$\bigvee_{w[x]\sqsubseteq f_1\odot f_2} \mathcal{U}_k(w) \geq \mathcal{U}_k(u_1 \odot u_2) \geq \mathcal{U}_k(u_1) \odot \mathcal{U}_k(u_2).$$

It is a contradiction.

$\mathcal{T}_k^{\mathcal{U}}(f) \odot \mathcal{T}_k^{\mathcal{U}}(g)$
$= \Big(\bigwedge_{e\in E}\bigwedge_{x\in X}\{f_e^*(x) \vee \bigvee_{u[x]\sqsubseteq f}\mathcal{U}_k(u)\}\Big) \odot \Big(\bigwedge_{e\in E}\bigwedge_{y\in X}\{g_e^*(y) \vee \bigvee_{v[y]\sqsubseteq g}\mathcal{U}_k(v)\}\Big)$
$\leq \bigwedge_{e\in E}\bigwedge_{x\in X}\Big(\{f_e^*(x) \vee \bigvee_{u[x]\sqsubseteq f}\mathcal{U}_k(u)\} \odot \{g_e^*(x) \vee \bigvee_{v[x]\sqsubseteq g}\mathcal{U}_k(v)\}\Big)$
(by Lemma 2.3)
$\leq \bigwedge_{e\in E}\bigwedge_{x\in X}\Big((f_e^* \oplus g_e^*)(x) \vee (\bigvee_{u[x]\sqsubseteq f}\mathcal{U}_k(u)) \odot (\bigvee_{v[x]\sqsubseteq g}\mathcal{U}_k(v))\Big)$
$\leq \bigwedge_{e\in E}\bigwedge_{x\in X}\Big((f_e^* \oplus g_e^*)(x) \vee (\bigvee_{u\odot v[x]\sqsubseteq f\odot g}\mathcal{U}_k(u \odot v)\Big)$
$\leq \mathcal{T}_k^{\mathcal{U}}(f \odot g).$

(SO3)

$$\begin{aligned}\mathcal{T}_k^{\mathcal{U}}(\bigvee_{j\in J} f_j) &= \bigwedge_{e\in E}\bigwedge_{x\in X}\{(\bigvee_{j\in J} f_j)_e^*(x)\vee\bigvee_{u[x]\sqsubseteq\vee_j f_j}\mathcal{U}(u)\}\\ &\quad\text{(since } L \text{ is a completely distributive lattice)}\\ &= \bigwedge_{e\in E}\bigwedge_{x\in X}\{\bigwedge_j\{(f_j)_e^*(x)\vee\bigvee_{u[x]\sqsubseteq\vee_j f_j}\mathcal{U}(u)\}\}\\ &= \bigwedge_j\{\bigwedge_{e\in E}\bigwedge_{x\in X}\{(f_j)_e^*(x)\vee\bigvee_{u[x]\sqsubseteq\vee_j f_j}\mathcal{U}(u)\}\}\\ &\geq \bigwedge_j\{\bigwedge_{e\in E}\bigwedge_{x\in X}\{(f_j)_e^*(x)\vee\bigvee_{u[x]\sqsubseteq f_j}\mathcal{U}(u)\}\}\\ &= \bigwedge_j \mathcal{T}_k^{\mathcal{U}}(f_j).\end{aligned}$$

Corollary 3.5. Let $(X,\mathcal{U})$ be an L-fuzzy (K,E)-soft quasi-uniform space. Define the mapping $\mathcal{F}^{\mathcal{U}}: K\to L^{(L^X)^E}$ by

$$\mathcal{F}_k^{\mathcal{U}}(f)=\bigwedge_{e\in E}\bigwedge_{x\in X}\{f_e(x)\vee\bigvee_{u[x]\sqsubseteq f^*}\mathcal{U}_k(u)\}.$$

Then, $\mathcal{F}^{\mathcal{U}}$ is an L-fuzzy (K,E)-soft cotopology on X.

4 L-fuzzy (K,E)-soft quasi-uniformities induced by L-fuzzy (K,E)-soft topologies

Theorem 4.1. Let $(X,\mathcal{T})$ be an L-fuzzy (K,E)-soft topological space. Define the mapping $\mathcal{U}^{\mathcal{T}}: K\to L^{(L^{X\times X})^E}$ by

$$\mathcal{U}_k^{\mathcal{T}}(u)=\bigvee\{\odot_{i=1}^n\mathcal{T}_k(f_i)\mid\odot_{i=1}^n u_{f_i}\sqsubseteq u, f_i\neq 0_X\},$$

where $\bigvee$ is taken over finite family $\{u_{f_i}\mid i=1,2,...,n\}$. Then $\mathcal{U}^{\mathcal{T}}$ is an L-fuzzy (K,E)-soft quasi -uniformity on X.

Proof.(1) Put $\mathcal{T}^0=\{f\in(L^X)^E\mid\mathcal{T}_k(f)\neq 0\}$.

(SU1) Since $u_{1_X}\sqsubseteq 1_{X\times X}$, we have

$$\mathcal{U}(1_{X\times X})\geq\mathcal{T}_k(1_X)=1.$$

(SU2) If $u_1\sqsubseteq u_2,\ u_1,u_2\in(L^{X\times X})^E$, then

$$\begin{aligned}\mathcal{U}_k(u_1) &= \bigvee\{\odot_{i=1}^n\mathcal{T}_k(f_i)\mid\odot_{i=1}^n u_{f_i}\sqsubseteq u_1, f_i\neq 0_X\}\\ &\leq \bigvee\{\odot_{i=1}^n\mathcal{T}_k(f_i)\mid\odot_{i=1}^n u_{f_i}\sqsubseteq u_2, f_i\neq 0_X\}\\ &= \mathcal{U}_k(u_2).\end{aligned}$$

(SU3) Let $f, g \in (L^X)^E$ and $u \in (L^{X\times X})^E$, then $u_f \odot u_g \sqsubseteq u_{f\odot g}$ and

$$\begin{aligned}
&\mathcal{U}_k^{\mathcal{T}}(u) \odot \mathcal{U}_k^{\mathcal{T}}(w)\\
&= \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \sqsubseteq u, f_i \neq 0_X\}\\
&\odot \bigvee\{\odot_{j=1}^m \mathcal{T}_k(g_j) \mid \odot_{j=1}^m u_{g_j} \sqsubseteq w, g_j \neq 0_X\}\\
&\leq \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \odot \odot_{j=1}^m \mathcal{T}_k(g_j) \mid\\
&\odot_{i=1}^n u_{f_i} \odot \odot_{j=1}^m u_{g_j} \sqsubseteq u \odot w, f_i \neq 0_X, g_j \neq 0_X\}\\
&\leq \mathcal{U}_k^{\mathcal{T}}(u \odot w).
\end{aligned}$$

(SU4) If $\mathcal{U}_k(u) \neq 0$, then there exists $f \in (L^X)^E$ with $\mathcal{T}_k(f) \neq 0$ such that $u_f \sqsubseteq u$. Hence, $1_\triangle \sqsubseteq u_f \sqsubseteq u$.

(SU5) Suppose there exists $u \in (L^{X\times X})^E$ such that

$$\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \not\geq \mathcal{U}_k^{\mathcal{T}}(u).$$

There exists a finite family $\{g_i \in \mathcal{T}^0 \mid \odot_{i=1}^m u_{g_i} \leq u\}$. such that

$$\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \not\geq \odot_{i=1}^m \mathcal{T}_k(g_i).$$

On the other hand, since $u_{g_i} \circ u_{g_i} \sqsubseteq u_{g_i}$, for each $i \in \{1, ..., m\}$, we have

$$\begin{aligned}
&(\odot_{i=1}^m u_{g_i}) \circ (\odot_{i=1}^m u_{g_i}) = \odot_{i=1}^m (u_{g_i} \circ u_{g_i})\\
&\sqsubseteq \odot_{i=1}^m u_{g_i}.
\end{aligned}$$

Put $v = \odot_{i=1}^m u_{g_i}$. Then $v \circ v \sqsubseteq u$ and

$$\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \geq \mathcal{U}_k^{\mathcal{T}}(v) \geq \odot_{i=1}^m \mathcal{T}_k(g_i).$$

It is a contradiction. Thus $\bigvee\{\mathcal{U}_k^{\mathcal{T}}(v) \mid v \circ v \sqsubseteq u\} \geq \mathcal{U}_k^{\mathcal{T}}(u)$.

Corollary 4.2. Let $(X, \mathcal{F})$ be an L-fuzzy (K, E)-soft cotopological space. Define the mapping $\mathcal{U}^{\mathcal{F}} : K \to L^{(L^{X\times X})^E}$ by

$$\mathcal{U}_k^{\mathcal{F}}(u) = \bigvee\{\odot_{i=1}^n \mathcal{F}_k(g_i^*) \mid \odot_{i=1}^n u_{g_i} \sqsubseteq u, g_i \neq 0_X\},$$

where the first $\bigvee$ is taken over every finite family $\{u_{G_i} \mid i = 1, ..., n\}$. Then $\mathcal{U}^{\mathcal{F}}$ is an L-fuzzy (K, E)-soft quasi-uniformity on X.

Example 4.3. Let $(L = [0, 1], \odot, \to)$ be a complete residuated lattice (ref. [4,9,11]) defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \to y = (1 - x + y) \wedge 1,$$

$$x \oplus y = (x + y) \wedge 1, \quad x^* = 1 - x.$$

Let $X = \{x, y\}$ be a set , $K = E = \{e_1, e_2\}$ and $f, g \in (L^X)^E$ such that

$$\begin{array}{c} f_{e_1}(x) = 0.6, f_{e_1}(y) = 0.5, \\ f_{e_2}(x) = 0.3, f_{e_2}(y) = 0.6. \\ g_{e_1}(x) = 0.2, g_{e_1}(y) = 0.5, \\ g_{e_2}(x) = 0.4, g_{e_2}(y) = 0.3. \end{array}$$

We obtain $f \odot f, f^*, g^*, f^* \oplus f^* \in (L^X)^E$ such that

$$\begin{array}{c} (f \odot f)_{e_1}(x) = 0.2, \quad (f \odot f)_{e_1}(y) = 0, \\ (f \odot f)_{e_2}(x) = 0, \quad (f \odot f)_{e_2}(y) = 0.2. \end{array}$$

$$\begin{array}{c} f^*_{e_1}(x) = 0.4, \quad f^*_{e_1}(y) = 0.5, \\ f^*_{e_2}(x) = 0.7, \quad f^*_{e_2}(y) = 0.4. \\ g^*_{e_1}(x) = 0.8, \quad g^*_{e_1}(y) = 0.5, \\ g^*_{e_2}(x) = 0.6, \quad g^*_{e_2}(y) = 0.7. \end{array}$$

$$\begin{array}{c} (f^* \oplus f^*)_{e_1}(x) = 0.8, \ (f^* \oplus f^*)_{e_1}(y) = 1, \\ (f^* \oplus f^*)_{e_2}(x) = 1, \ (f^* \oplus f^*)_{e_2}(y) = 0.8. \end{array}$$

(1) Define $\mathcal{T} : E \to L^{(L^X)^E}$ as follows

$$\mathcal{T}_{e_1}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.6, & \text{if } h = f, \\ 0.3, & \text{if } \ h = f \odot f, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_{e_2}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.4, & \text{if } h = g, \\ 0, & \text{otherwise.} \end{cases}$$

Since $0.3 = \mathcal{T}_{e_1}(f \odot f) \geq \mathcal{T}_{e_1}(f) \odot \mathcal{T}_{e_1}(f) = 0.2$, $\mathcal{T}$ is an L-fuzzy (E, E)-soft topology on X.

(2) From (1), we obtain an L-fuzzy (E, E)-soft cotopology $\mathcal{F} : E \to L^{(L^X)^E}$ with $\mathcal{F}_{e_i}(f) = \mathcal{T}_{e_i}(f^*)$ as follows

$$\mathcal{F}_{e_1}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.6, & \text{if } h = f^*, \\ 0.3, & \text{if } \ h = f^* \oplus f^*, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{F}_{e_2}(h) = \begin{cases} 1, & \text{if } h \in \{1_X, 0_X\} \\ 0.4, & \text{if } h = g^*, \\ 0, & \text{otherwise,} \end{cases}$$

(3) We obtain $u_f, u_g, u_{f\odot f}$ as

$$(u_f)_{e_1} = \begin{pmatrix} 1 & 0.9 \\ 1 & 1 \end{pmatrix} \quad (u_f)_{e_2} = \begin{pmatrix} 1 & 1 \\ 0.7 & 1 \end{pmatrix}$$

$$(u_{f\odot f})_{e_1} = \begin{pmatrix} 1 & 0.8 \\ 1 & 1 \end{pmatrix} \quad (u_{f\odot f})_{e_2} = \begin{pmatrix} 1 & 1 \\ 0.8 & 1 \end{pmatrix}$$

$$(u_g)_{e_1} = \begin{pmatrix} 1 & 1 \\ 0.7 & 1 \end{pmatrix} \quad (u_g)_{e_2} = \begin{pmatrix} 1 & 0.9 \\ 1 & 1 \end{pmatrix}$$

From Theorem 4.1, we obtain an L-fuzzy (E,E)-soft quasi-uniformity $\mathcal{U}^{\mathcal{T}} : E \to L^{(L^X \times L^X)^E}$ as follows

$$(\mathcal{U}^{\mathcal{T}})_{e_1}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X} \\ 0.6, & \text{if } u_f \sqsubseteq u \neq 1_{X\times X}, \\ 0.3, & \text{if } u_{f\odot f} \sqsubseteq u \not\supseteq u_f, \\ 0.2, & \text{if } u_f \odot u_f \sqsubseteq u \not\supseteq u_{f\odot f}, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mathcal{U}^{\mathcal{T}})_{e_2}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X} \\ 0.4, & \text{if } u_g \sqsubseteq u \neq 1_{X\times X}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.4. Let $\mathcal{T}$ be an L- fuzzy (K,E)-soft topology on X and $\mathcal{U}^{\mathcal{T}}$ the L-fuzzy (K,E)-soft principle quasi-uniformity. Then

(1) $\mathcal{T}^{\mathcal{U}^{\mathcal{T}}} \geq \mathcal{T}$,

(2) $\mathcal{U}^{\mathcal{T}^{\mathcal{U}}} \leq \mathcal{U}$.

Proof. (1) $\mathcal{T}_k^{\mathcal{U}^{\mathcal{T}}}(f) = \mathcal{U}_k^{\mathcal{T}}(u_f) = \bigvee\{\odot_{i=1}^n \mathcal{T}_k(f_i) \mid \odot_{i=1}^n u_{f_i} \leq u_f, f_i \neq 0_X\} \geq \mathcal{T}_k(f)$.

(2) Suppose that there exists $u \in (L^{X\times X})^E$ such that

$$\mathcal{U}_k^{\mathcal{T}^{\mathcal{U}}}(u) \not\leq \mathcal{U}_k(u).$$

There exists a family $\{u_{f_i} \mid \odot_{i=1}^m u_{f_i} \sqsubseteq u\}$ such that $\odot_{j=1}^m \mathcal{T}_k^{\mathcal{U}}(f_i) \not\leq \mathcal{U}_k(u)$. For each $i = 1, 2, ..., m$, by the definition of $\mathcal{T}^{\mathcal{U}}$, there exists $u_{f_i} \in (L^{X\times X})^E$ such that

$$\odot_{j=1}^m \mathcal{U}_k^{\mathcal{T}}(u_{f_i}) \not\leq \mathcal{U}_k(u).$$

On the other hand,

$$\mathcal{U}_k(u) \geq \mathcal{U}_k(\odot_{i=1}^m u_{f_i}) \geq \odot_{i=1}^m \mathcal{U}_k(u_{f_i}) = \odot_{i=1}^m \mathcal{T}_k(f_i).$$

It is a contradiction.

Example 4.5. Let $X = \{h_i \mid i = \{1, 2, 3\}\}$ with h_i=house and $E = \{e, b\}$ with e=expensive, b= beautiful. Define a binary operation $\odot$ on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\},\ x \to y = \min\{1 - x + y, 1\}$$

$$x \oplus y = \min\{1, x + y\},\ x^* = 1 - x$$

Then $([0, 1], \odot, \to, 0, 1)$ is a complete residuated lattice.

(1) Put $v, v \odot v, w \in ([0, 1]^{X\times X})^E$ as

$$v_e = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}$$

$$(v \odot v)_e = \begin{pmatrix} 1 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.2 & 1 \end{pmatrix} (v \odot v)_b = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}$$

$$w_e = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.3 & 1 & 0.5 \\ 0.2 & 0.3 & 1 \end{pmatrix}$$

We define $\mathcal{U} : E \to [0, 1]^{([0,1]^{X\times X})^E}$ as follows:

$$\mathcal{U}_e(u) = \begin{cases} 1, & \text{if } u = 1_{Y\times Y} \\ 0.6, & \text{if } v \sqsubseteq u \neq 1_{Y\times Y}, \\ 0.3, & \text{if } v \odot v \sqsubseteq u \not\supseteq v, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b(u) = \begin{cases} 1, & \text{if } u = 1_{Y\times Y} \\ 0.5, & \text{if } w \sqsubseteq u \neq 1_{Y\times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $v \circ v = v$, $w \circ w = w$ and $(v \odot v) \circ (v \odot v) = (v \odot v)$, $\mathcal{U}$ is a $[0, 1]$-fuzzy (E, E)-soft principle quasi-uniformity on X. From Theorem 3.2, we obtain a $[0, 1]$-fuzzy soft (E, E)-topology $\mathcal{T}^{\mathcal{U}} : E \to [0, 1]^{([0,1]^X)^E}$ such that $\mathcal{T}_e^{\mathcal{U}}(v[f]) = 0.6$ because, for $e \in \{e, b\}$, $f \neq 0_X$,

$$\begin{aligned} &(u_{v[f]})_e(y, z) = v[f]_e(y) \to v[f]_e(z) \\ &= \textstyle\bigvee_{x\in X}(v_e(x, y) \odot f_e(x)) \to \bigvee_{x\in X}(v_e(x, z) \odot f_e(x)) \\ &\geq \textstyle\bigwedge_{x\in X}((v_e(x, y) \odot f_e(x)) \to (v_e(x, z) \odot f_e(x))) \\ &\geq \textstyle\bigwedge_{x\in X}(v_e(x, y) \to v_e(x, z)) \\ &\geq v_e(y, z). \end{aligned}$$

Since $(u_{1_{X\times X}[\alpha_X]})_e(y, z) = 1_{X\times X}[\alpha_X]_e(y) \to 1_{X\times X}[\alpha_X]_e(z) = (1_{X\times X})_e(y, z)$, we have $\mathcal{T}_e^{\mathcal{U}}(1_{X\times X}[\alpha_X]) = \mathcal{T}_e^{\mathcal{U}}(\alpha_X) = 1$. Moreover, $\mathcal{T}_e^{\mathcal{U}}(v \odot v[f]) = 0.3$ and $\mathcal{T}_d^{\mathcal{U}}(w[f]) = 0.5$.

Hence

$$\mathcal{T}_e^{\mathcal{U}}(g) = \begin{cases} 1, & \text{if } g = \alpha_X, \\ 0.6, & \text{if } g = v[f], \\ 0.3, & \text{if } g = (v \odot v)[f] \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_b^{\mathcal{U}}(g) = \begin{cases} 1, & \text{if } g = \alpha_X, \\ 0.5, & \text{if } g = w[f], \\ 0, & \text{otherwise,} \end{cases}$$

From Theorem 4.1, we obtain $\mathcal{U}^{\mathcal{T}^{\mathcal{U}}} : E \to [0,1]^{([0,1]^{X\times X})^E}$ as follows:

$$\mathcal{U}_e^{\mathcal{T}^{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X} \\ 0.6, & \text{if } u_{v[f]} \sqsubseteq u \neq 1_{Y\times Y}, \\ 0.3, & \text{if } u_{(v\odot v)[f]} \sqsubseteq u \not\sqsupseteq u_{v[f]}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b^{\mathcal{T}^{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X} \\ 0.5, & \text{if } u_{v[f]} \sqsubseteq u \neq 1_{Y\times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\begin{aligned} &\textstyle\bigvee_{x\in X}(v_e(x,y) \odot f_e(x)) \to \bigvee_{x\in X}(v_e(x,z) \odot f_e(x)) \\ &\geq \textstyle\bigwedge_{x\in X}((v_e(x,y) \odot f_e(x)) \to (v_e(x,z) \odot f_e(x)) \\ &\geq \textstyle\bigwedge_{x\in X}(v_e(x,y) \to v_e(x,z)) \geq v_e(y,z), \end{aligned}$$

$u_{v[f]} \sqsubseteq v$ and $u_{(v\odot v)[f]} \sqsubseteq v$. Hence $\mathcal{U}^{\mathcal{T}^{\mathcal{U}}} \leq \mathcal{U}$.

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E-mail: aramadan58@hotmail.com

E-mail: yck@gwnu.ac.kr

A FIXED POINT APPROACH TO STABILITY OF ADDITIVE MAPPINGS IN MODULAR SPACES WITHOUT Δ_2-CONDITIONS

CHOONKIL PARK, ABASALT BODAGHI, AND SANG OG KIM*

Abstract. In this article, we prove the generalized Hyers-Ulam-Rassias stability of the following equivalent functional equations

$$f(x+y) = f(x) + f(y),$$
$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x)$$

via fixed point method. We obtain the result in the framework of modular spaces without Δ_2-conditions.

1. Introduction and preliminaries

We recall some basic facts concerning modular spaces.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies for all $\alpha, \beta \in \mathbb{K},\ x, y \in X$

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.
If we replace (iii) by
(iii′) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$

then we say that ρ is a *convex modular*.

A modular ρ defines a corresponding *modular space*, denoted by X_ρ, given by

$$X_\rho = \{x \in X \mid \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

X_ρ is a vector subspace of X.

Definition 1.2. Let X_ρ be a modular space and $\{x_n\}$ be a sequence in X_ρ.

(1) $\{x_n\}$ *ρ-converges* to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$. The point x is called the *ρ-limit* of the sequence $\{x_n\}$, which is denoted by $x_n \xrightarrow{\rho} x$.
(2) $\{x_n\}$ is a *ρ-Cauchy sequence* if $\rho(x_n - x_m) \to 0$ as $m, n \to \infty$.
(3) A subset $S \subseteq X$ is called *ρ-complete* if every ρ-Cauchy sequence in S is ρ-convergent to an element of S.

Remark 1.3. Note that for a fixed $x \in X_\rho$, the valuation $\lambda \in \mathbb{K} \mapsto \rho(\lambda x)$ is increasing. In case the modular ρ is convex, one has $\rho(x) \leq \delta\rho\left(\frac{1}{\delta}x\right)$ for all $x \in X_\rho$, provided $0 < \delta \leq 1$. In

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* Corresponding author.

particular, $\rho(x) \leq \frac{1}{2^n}\rho(2^n x)$ for all $x \in X_\rho$ and $n \in \mathbb{N}$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences in X_ρ that ρ-converge to a, b, c, d, respectively. It is easy to show that

$$\rho\left(\frac{1}{4}(a_n - a) + \frac{1}{4}(b_n - b) + \frac{1}{4}(c_n - c) + \frac{1}{4}(d_n - d)\right) \leq \frac{1}{4}\big(\rho(a_n - a) + \rho(b_n - b) + \rho(c_n - c) + \rho(d_n - d)\big).$$

Unlike a norm, a modular need not be continuous. The convergence of a sequence $\{x_n\}$ does not imply that of $\{cx_n\}$ for a scalar c. In order to avoid such difficulties, some additional conditions are imposed on the modular so that the multiple of $\{x_n\}$ converges naturally. One of such conditions is the so-called Δ_2-condition. A modular ρ is said to satisfy the Δ_2-*condition* if there exists $\kappa \geq 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in X_\rho$.

Example 1.4. A convex function $\phi(t)$ defined on $[0, \infty)$, nondecreasing and continuous for $t \geq 0$ and such that $\phi(0) = 0, \phi(t) > 0$ for $t > 0$, and $\phi(t) \to \infty$ as $t \to \infty$, is called an Orlicz function. Let (Ω, Σ, μ) be a measure space. Let $L^0(\mu)$ be the set of all measurable real-valued (or complex-valued) functions on Ω. Define for $f \in L^0(\mu)$,

$$\rho_\phi(f) = \int_\Omega \phi(|f|)d\mu.$$

Then ρ_ϕ is a modular and the associated modular function space is called an Orlicz space and denoted by

$$L^\phi = \{f \in L^0(\mu) \mid \rho_\phi(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

It is known that L^ϕ is ρ_ϕ-complete. Moreover, $(L^\phi, \|\cdot\|_{\rho_\phi})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\phi}$is defined as follows;

$$\|f\|_{\rho_\phi} = \inf\left\{\lambda > 0 \;\Big|\; \int_\Omega \phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\right\}.$$

Note that if μ is the Lebesgue measure on $\mathbb{R}$ and $\phi(t) = e^t - 1$, then ρ_ϕ does not satisfy the Δ_2-condition.

The notion of modulars on linear spaces and the corresponding theory of modular spaces, as a generalization of metric spaces, were initiated by Nakano [15] in connection with the theory of ordered spaces. Further and the most complete development of the theories are due to Luxemburg, Musielak, Orlicz, Mazur [11, 12, 13, 14] and their collaborators. In the present time, the theory of modular and modular spaces are extensively applied, in particular, in the theory of various Orlicz spaces and interpolation theory, which have broad applications. For a review of Musielak-Orlicz space and modular spaces, the reader is referred to the book of Musielak and Orlicz [13].

The stability problem of functional equations originated from a question of Ulam [23] in 1940, concerning the stability of group homomorphisms. In 1941, Hyers [7] gave the first affirmative answer to the problem of Ulam for Banach spaces. Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [16] for linear mappings by considering an unbounded Cauchy difference. Generalizations of the Rassias' theorem were obtained by Forti [4] and Găvruta [5] who permitted the Cauchy difference to become arbitrary unbounded. Since then a wide spectrum of stability problems has been investigated for a variety of functional equations and spaces. A large list of references concerning the stability of various functional equations can be found e.g., in [2, 3, 8, 9, 18, 19, 20, 21, 22]. Recently, Sadeghi [17] showed a fixed point method

to prove the stabilities of the Cauchy and Jensen functional equations on modular spaces with the Δ_2-conditions. Wongkum et al. [24] obtained a stability result of the quadratic functional equation without the Δ_2-conditions. Also, Eshaghi Gordji et al. [6] proved a generalized Ulam-Hyers-Rassias stability of Cauchy mappings in modular spaces endowed with a partial order without the Δ_2-condition.

Motivated by the ideas and results of [6], [17] and [24], we prove the generalized Ulam-Hyers-Rassias stability of additive functional equations in the framework of modular spaces via fixed point theory. It is very important to note that we dropped the Δ_2-condition. The results are interesting and many results on the stability of additive functional equations in normed spaces can be reformulated. Note that the functional equations we are considering in Theorems 2.2 and 3.1 are equivalent, but in the stability of those functional equations the control functions are different.

2. Stability of the functional equation $\mathbf{f(x+y)=f(x)+f(y)}$

Let X_ρ be a modular space, $C \subset X_\rho$, and $T : C \to C$ be a mapping. The *orbit* of T at $x \in C$ is the set

$$\mathcal{O}(x) = \{x, Tx, T^2x, \ldots\}.$$

The quantity $\delta_\rho(x) = \sup\{\rho(T^n(x) - T^m(x)) \mid n, m \in \mathbb{N}\}$ is called the *ρ-diameter* of T at x.

We start with a known fixed point theorem in modular spaces.

Theorem 2.1. ([10]) *Let X_ρ be a modular space whose induced modular is lower semicontinuous and let $C \subset X_\rho$ be a ρ-complete subset. If $T : C \to C$ is a ρ-contraction, that is, there is a constant $k \in [0, 1)$ such that*

$$\rho(Tx - Ty) \le k\rho(x - y), \quad x, y \in C,$$

and $\delta_\rho(x_0) < \infty$ for $x_0 \in C$, then the sequence $\{T^n(x_0)\}$ is ρ-convergent to a point $w \in C$. If $\rho\big(w - T(w)\big) < \infty$ and $\rho\big(x_0 - T(w)\big) < \infty$, then w is a fixed point of T.

Theorem 2.2. *Let V be a linear space, X_ρ be a ρ-complete modular space where ρ is lower semicontinuous and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. Let $0 \le L < 1$ be a constant and suppose that*

$$\rho\big(2f(x + y) - 2f(x) - 2f(y)\big) \le \varphi(x, y), \quad x, y \in V, \tag{2.1}$$

where $\varphi : V \times V \to [0, \infty)$ is a nonnegative real-valued function with the following properties;

$$\begin{aligned} &\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \\ &\varphi(2x, 2x) \le 2L\varphi(x, x), \quad x, y \in V. \end{aligned} \tag{2.2}$$

Then there exists a unique additive mapping $w : V \to X_\rho$ such that

$$\rho\big(w(x) - f(x)\big) \le \frac{1}{4(1 - L)}\varphi(x, x) \tag{2.3}$$

for all $x \in V$.

Proof. Let

$$M = \{g \mid g : V \to X_\rho,\ g(0) = 0\}.$$

Then by a standard argument as in [24, Lemma 10] or [17, Theorem 2.1], M is a linear space, and the generalized function $\widetilde{\rho} : M \to [0, \infty]$ defined by

$$\widetilde{\rho}(g) = \inf\{c > 0 \mid \rho\big(g(x)\big) \le c\varphi(x, x),\ x \in V\}$$

is a convex modular on M. Moreover the corresponding modular space $M_{\widetilde{\rho}}$ is the whole space M and is $\widetilde{\rho}$-complete. Also $\widetilde{\rho}$ is lower semicontinuous.

Define $T : M_{\widetilde{\rho}} \to M_{\widetilde{\rho}}$ by

$$T(g)(x) = \frac{1}{2}g(2x), \quad g \in M_{\widetilde{\rho}},\ x \in V.$$

We first show that T is a $\widetilde{\rho}$-contraction. Let $x \in V,\ g, h \in M_{\widetilde{\rho}}$ and c be an arbitrary constant with $\widetilde{\rho}(g-h) \le c$. Then we have

$$\rho\big(g(2x) - h(2x)\big) \le c\varphi(2x, 2x),$$

so that by (2.2)

$$\rho\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right) \le \frac{1}{2}\big(g(2x) - h(2x)\big) \le \frac{c}{2}\varphi(2x, 2x) \le Lc\varphi(x, x).$$

Hence we have $\widetilde{\rho}(Tg - Th) \le L\widetilde{\rho}(g-h)$, from which T is a $\widetilde{\rho}$-contraction.

Next, we show that T has a bounded orbit at f. For that, we first show by induction on $n \in \mathbb{N}$

$$\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \le \sum_{i=1}^{n} \frac{1}{2^i}\varphi\left(2^{i-1}x, 2^{i-1}x\right), \quad x \in V. \tag{2.4}$$

In fact, for $n = 1$, letting $x = y$ in (2.1), we have

$$\rho\big(f(2x) - 2f(x)\big) \le \frac{1}{2}\rho\big(2f(2x) - 4f(x)\big) \le \frac{1}{2}\varphi(x, x).$$

Assume that (2.4) holds for $n-1$. Then, for all $x \in V$,

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \\
&\le \frac{1}{2}\rho\left(\frac{f(2^n x)}{2^{n-2}} - 2f(2x)\right) + \frac{1}{2}\rho\big(2f(2x) - 4f(x)\big) \\
&\le \frac{1}{2}\sum_{i=1}^{n-1} \frac{1}{2^i}\varphi\left(2^i x, 2^i x\right) + \frac{1}{2}\varphi(x, x) \\
&= \sum_{i=1}^{n} \frac{1}{2^i}\varphi\left(2^{i-1}x, 2^{i-1}x\right),
\end{aligned}$$

from which it follows that (2.4) holds for every $n \in \mathbb{N}$. Hence we deduce that

$$\begin{aligned}
\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) &\le \sum_{i=1}^{n} \frac{1}{2^i}\varphi(2^{i-1}x, 2^{i-1}x) \\
&\le \frac{1}{2}\varphi(x, x)\sum_{i=1}^{n} L^{i-1} \\
&\le \frac{1}{2}\varphi(x, x)\sum_{i=1}^{\infty} L^{i-1} \\
&\le \frac{1}{2(1-L)}\varphi(x, x)
\end{aligned} \tag{2.5}$$

for all $x \in V$.

Now for $m, n \in \mathbb{N}$, we have by (2.5)

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right) \\
&\leq \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) + \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^m} - 2f(x)\right) \\
&\leq \frac{2}{4(1-L)}\varphi(x,x) \\
&= \frac{1}{2(1-L)}\varphi(x,x)
\end{aligned} \tag{2.6}$$

for all $x \in V$. We also have

$$\rho\left(\frac{f(2^n x)}{2^n} - f(x)\right) \leq \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) \leq \frac{1}{4(1-L)}\varphi(x,x) \tag{2.7}$$

for all $x \in V$. Hence it follows by (2.6) that for $m, n \in \mathbb{N}$,

$$\widetilde{\rho}(T^n f - T^m f) \leq \frac{1}{2(1-L)} < \infty,$$

and hence, the $\widetilde{\rho}$-diameter of T at f is finite, i.e., $\delta_{\widetilde{\rho}}(f) < \infty$.

By Theorem 2.1, there exists an element $w \in M_{\widetilde{\rho}}$ such that $T^n f \xrightarrow{\widetilde{\rho}} w$. By the $\widetilde{\rho}$-contractivity of T, one has

$$\widetilde{\rho}(Tw - T^{n+1} f) \leq L\widetilde{\rho}(w - T^n f). \tag{2.8}$$

Letting $n \to \infty$ in (2.8) and applying the lower semicontinuity of $\widetilde{\rho}$, we have

$$\begin{aligned}
\widetilde{\rho}(Tw - w) &\leq \liminf_{n\to\infty} \widetilde{\rho}(Tw - T^{n+1} f) \\
&\leq \liminf_{n\to\infty} L\widetilde{\rho}(w - T^n f) \\
&= 0,
\end{aligned}$$

so that w is a fixed point of T, i.e., $w(2x) = 2w(x)$ for all $x \in V$.

Replacing (x, y) by $(2^n x, 2^n y)$ in (2.1), we have

$$\rho\Big(2f\big(2^n(x+y)\big) - 2f(2^n x) - 2f(2^n y)\Big) \leq \varphi(2^n x, 2^n y).$$

Then by using Remark 1.3 and (2.2), we have

$$\begin{aligned}
&\rho\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right) \\
&\leq \rho\left(\frac{2f(2^n(x+y))}{2^n} - \frac{2f(2^n x)}{2^n} - \frac{2f(2^n y)}{2^n}\right) \\
&\leq \frac{1}{2^n}\rho\Big(2f\big(2^n(x+y)\big) - 2f(2^n x) - 2f(2^n y)\Big) \\
&\leq \frac{1}{2^n}\varphi(2^n x, 2^n y) \to 0
\end{aligned} \tag{2.9}$$

as $n \to \infty$.

Applying the lower semicontinuity of ρ, Remark 1.3 and (2.9), we deduce that

$$w(x+y) = w(x) + w(y)$$

for all $x, y \in V$, that is, w is additive. Moreover, by (2.7), it follows that

$$\rho\big(w(x) - f(x)\big) \leq \frac{1}{4(1-L)}\varphi(x,x)$$

for all $x \in V$. Hence the inequality (2.3) is proved.

Finally, we show the uniqueness of the additive mapping w. Assume that w_1, w_2 are additive mappings that satisfy (2.3). Then we deduce that

$$\begin{aligned}
&\rho\left(\frac{w_1(x)}{2} - \frac{w_2(x)}{2}\right)\\
&= \rho\left(\frac{w_1(2^n x)}{2^{n+1}} - \frac{f(2^n x)}{2^{n+1}} + \frac{f(2^n x)}{2^{n+1}} - \frac{w_2(2^n x)}{2^{n+1}}\right)\\
&\leq \frac{1}{2^n}\rho\left(\frac{w_1(2^n x)}{2} - \frac{f(2^n x)}{2} + \frac{f(2^n x)}{2} - \frac{w_2(2^n x)}{2}\right)\\
&\leq \frac{1}{2^n}\left(\frac{1}{2}\rho\big(w_1(2^n x) - f(2^n x)\big) + \frac{1}{2}\rho\big(f(2^n x) - w_2(2^n x)\big)\right)\\
&\leq \frac{1}{2^{n+1}} \cdot \frac{2}{4(1-L)}\varphi(2^n x, 2^n x)\\
&\leq \frac{1}{2^{n+2}} \cdot \frac{1}{1-L} \cdot (2L)^n \varphi(x,x)\\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $x \in V$, from which it follows that $w_1 = w_2$. This completes the proof. □

If the control function φ is replaced by a constant, we obtain the following result.

Corollary 2.3. *Let V be a linear space, X_ρ be a ρ-complete modular space where ρ is lower semicontinuous and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. If there exists a constant $\delta > 0$ such that*

$$\rho\big(2f(x+y) - 2f(x) - 2f(y)\big) \leq \delta, \quad x, y \in V,$$

then there exists a unique additive mapping $w : V \to X_\rho$ such that

$$\rho\big(w(x) - f(x)\big) \leq \frac{\delta}{2}$$

for all $x \in V$.

Proof. It is easy to see that we can take $L = \frac{1}{2}$ in Theorem 2.2 if $\varphi(x,y) = \delta$ for all $x, y \in V$. □

It is known that every normed space is a modular space with $\rho(x) = \|x\|$. Applying Theorem 2.2, we have the following result.

Corollary 2.4. *Let V be a linear space, $(X, \|\cdot\|)$ be a Banach space and $f : V \to X$ be a mapping with $f(0) = 0$. Let $0 \leq L < 1$ be a constant and suppose that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y), \quad x, y \in V,$$

where $\varphi : V \times V \to [0, \infty)$ is a nonnegative real-valued function with the following properties;

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0,\\
&\varphi(2x, 2x) \leq 2L\varphi(x,x), \quad x, y \in V.
\end{aligned}$$

Then there exists a unique additive mapping $w : V \to X$ such that

$$\|w(x) - f(x)\| \leq \frac{1}{2(1-L)}\varphi(x,x)$$

for all $x \in V$.

Example 2.5. Let ϕ be an Orlicz function and L^ϕ be the Orlicz space. Let $f : V \to L^\phi$ be a mapping with $f(0) = 0$. Let $0 \leq L < 1$ be a constant and suppose that

$$\int_\Omega \phi\left(|2f(x+y) - 2f(x) - 2f(y)|\right) d\mu \leq \varphi(x,y), \quad x, y \in V,$$

where $\varphi : V \times V \to [0,\infty)$ is a nonnegative real-valued function with the following properties;

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0,$$
$$\varphi(2x, 2x) \leq 2L\varphi(x,x), \quad x, y \in V.$$

Then there exists a unique additive mapping $w : V \to L^\phi$ such that

$$\int_\Omega \phi\left(|w(x) - f(x)|\right) d\mu \leq \frac{1}{4(1-L)}\varphi(x,x)$$

for all $x \in V$.

3. Stability of the functional equation $\mathbf{f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x)}$

Theorem 3.1. *Let V be a linear space, X_ρ be a ρ-complete modular space where ρ is lower semicontinuous and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. Let $0 \leq L < 1$ be a constant and suppose that*

$$\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - 2f(x)\right) \leq \varphi(x,y), \quad x, y \in V, \tag{3.1}$$

where $\varphi : V \times V \to [0,\infty)$ is a nonnegative real-valued function with the following properties;

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0,$$
$$\varphi(2x, 0) \leq 2L\varphi(x,0), \quad x, y \in V. \tag{3.2}$$

Then there exists a unique additive mapping $w : V \to X_\rho$ such that

$$\rho\big(w(x) - f(x)\big) \leq \frac{L}{2(1-L)}\varphi(x,0) \tag{3.3}$$

for all $x \in V$.

Proof. Let

$$M = \{g \mid g : V \to X_\rho,\ g(0) = 0\}.$$

Then as in the proof of Theorem 2.2, M is a linear space, and the generalized function $\widetilde{\rho} : M \to [0,\infty]$ defined by

$$\widetilde{\rho}(g) = \inf\{c > 0 \mid \rho\big(g(x)\big) \leq c\varphi(x,0),\ x \in V\}$$

is a convex modular on M. Moreover the corresponding modular space $M_{\widetilde{\rho}}$ is the whole space M and is $\widetilde{\rho}$-complete. Also $\widetilde{\rho}$ is lower semicontinuous. Define $T : M_{\widetilde{\rho}} \to M_{\widetilde{\rho}}$ by

$$T(g)(x) = \frac{1}{2}g(2x), \quad g \in M_{\widetilde{\rho}},\ x \in V.$$

We first show that T is a $\widetilde{\rho}$-contraction. Let $x \in V,\ g, h \in M_{\widetilde{\rho}}$ and c be an arbitrary constant with $\widetilde{\rho}(g-h) \leq c$. Then we have

$$\rho\big(g(2x) - h(2x)\big) \leq c\varphi(2x, 0),$$

so that by (3.2)

$$\rho\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right) \leq \frac{1}{2}\rho\big(g(2x) - h(2x)\big) \leq \frac{c}{2}\varphi(2x, 0) \leq Lc\varphi(x, 0).$$

Hence we have $\widetilde{\rho}(Tg - Th) \leq L\widetilde{\rho}(g-h)$, from which it follows that T is a $\widetilde{\rho}$-contraction.

Next, we show that T has a bounded orbit at f. For that, we first show by induction on $n \in \mathbb{N}$

$$\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \leq (L^n + \ldots + L^2 + L)\varphi(x, 0), \quad x \in V. \tag{3.4}$$

In fact, for $n = 1$, letting $y = 0$ in (3.1), we have

$$\rho\left(4f\left(\frac{x}{2}\right) - 2f(x)\right) \leq \varphi(x, 0), \quad x \in V.$$

Then we have

$$\rho\big(4f(x) - 2f(2x)\big) \leq \varphi(2x, 0) \leq 2L\varphi(x, 0), \quad x \in V,$$

and hence

$$\rho\big(f(2x) - 2f(x)\big) \leq \frac{1}{2}\rho\big(4f(x) - 2f(2x)\big) \leq L\varphi(x, 0), \quad x \in V.$$

Assume that (3.4) holds for $n-1$. Then, for all $x \in V$,

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^{n-1}} - 2f(x)\right) \\
&\leq \frac{1}{2}\rho\left(\frac{f(2^n x)}{2^{n-2}} - 2f(2x)\right) + \frac{1}{2}\rho\big(2f(2x) - 4f(x)\big) \\
&\leq \frac{1}{2}(L^{n-1} + \ldots + L^2 + L)\varphi(2x, 0) + \frac{1}{2} \cdot 2L\varphi(x, 0) \\
&\leq (L^n + \ldots + L^2 + L)\varphi(x, 0),
\end{aligned} \tag{3.5}$$

from which it follows that (3.4) holds for every $n \in \mathbb{N}$.

Now for $m, n \in \mathbb{N}$, we get by (3.5)

$$\begin{aligned}
&\rho\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right) \\
&\leq \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) + \frac{1}{2}\rho\left(2\frac{f(2^m x)}{2^m} - 2f(x)\right) \\
&\leq \frac{1}{2}\left(\frac{L(1-L^n)}{1-L} + \frac{L(1-L^m)}{1-L}\right)\varphi(x, 0) \\
&\leq \frac{L}{(1-L)}\varphi(x, 0)
\end{aligned} \tag{3.6}$$

for all $x \in V$. We also have by (3.5)

$$\rho\left(\frac{f(2^n x)}{2^n} - f(x)\right) \leq \frac{1}{2}\rho\left(2\frac{f(2^n x)}{2^n} - 2f(x)\right) \leq \frac{L}{2(1-L)}\varphi(x, 0) \tag{3.7}$$

for all $x \in V$. Hence it follows by (3.6) that for $m, n \in \mathbb{N}$,

$$\widetilde{\rho}(T^n f - T^m f) \leq \frac{L}{1-L} < \infty,$$

which implies that the $\widetilde{\rho}$-diameter of T at f is finite, i.e., $\delta_{\widetilde{\rho}}(f) < \infty$. By Theorem 2.1, there exists an element $w \in M_{\widetilde{\rho}}$ such that $T^n f \xrightarrow{\widetilde{\rho}} w$. By the $\widetilde{\rho}$-contractivity of T, one has

$$\widetilde{\rho}(Tw - T^{n+1} f) \leq L\widetilde{\rho}(w - T^n f). \tag{3.8}$$

Letting $n \to \infty$ in (3.8) and applying the lower semicontinuity of $\widetilde{\rho}$, we have

$$\begin{aligned} \widetilde{\rho}(Tw - w) &\leq \liminf_{n\to\infty} \ \widetilde{\rho}(Tw - T^{n+1} f) \\ &\leq \liminf_{n\to\infty} \ L\widetilde{\rho}(w - T^n f) \\ &= 0, \end{aligned}$$

so that w is a fixed point of T, i.e., $w(2x) = 2w(x)$ for all $x \in V$.

Replacing (x, y) by $(2^{n+1}x, 2^{n+1}y)$ in (3.1), we have

$$\rho\big(2f(2^n(x+y)) + 2f(2^n(x-y)) - 2f(2^{n+1}x)\big) \leq \varphi(2^{n+1}x, 2^{n+1}y).$$

Then by using Remark 1.3 and (3.2), we get

$$\begin{aligned} &\rho\left(\frac{2f\big(2^n(x+y)\big)}{2^{n+1}} + \frac{2f\big(2^n(x-y)\big)}{2^{n+1}} - \frac{2f(2^{n+1}x)}{2^{n+1}}\right) \\ &\leq \frac{1}{2^{n+1}}\rho\Big(2f\big(2^n(x+y)\big) + 2f\big(2^n(x-y)\big) - 2f(2^{n+1}x)\Big) \\ &\leq \frac{1}{2^{n+1}}\varphi\left(2^{n+1}x, 2^{n+1}y\right) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

Since $w(2x) = 2w(x)$, applying the lower semicontinuity of ρ, Remark 1.3 and (3.9), we deduce that

$$w(x+y) + w(x-y) = 2w(x) \tag{3.10}$$

for all $x, y \in V$. Since $w(0) = 0$, letting $x = 0$ in (3.10), it follows that

$$w(-y) = -w(y)$$

for all $y \in V$. Replacing (x, y) by (y, x) in (3.10), we have

$$w(x+y) + w(y-x) = 2w(y) \tag{3.11}$$

for all $x, y \in V$. From (3.10) and (3.11), we obtain that w is additive, that is,

$$w(x+y) = w(x) + w(y)$$

for all $x, y \in V$. Moreover, by (3.7), it follows that

$$\rho\big(w(x) - f(x)\big) \leq \frac{L}{2(1-L)}\varphi(x, 0)$$

for all $x \in V$. Hence the inequality (3.3) is proved. Finally, we show the uniqueness of the additive mapping w. Assume that w_1, w_2 are additive mappings that satisfy (3.3). Then we

deduce that

$$\begin{aligned}
&\rho\left(\frac{w_1(x)}{2}-\frac{w_2(x)}{2}\right)\\
&=\rho\left(\frac{w_1(2^nx)}{2^{n+1}}-\frac{f(2^nx)}{2^{n+1}}+\frac{f(2^nx)}{2^{n+1}}-\frac{w_2(2^nx)}{2^{n+1}}\right)\\
&\le\frac{1}{2^n}\rho\left(\frac{w_1(2^nx)}{2}-\frac{f(2^nx)}{2}+\frac{f(2^nx)}{2}-\frac{w_2(2^nx)}{2}\right)\\
&\le\frac{1}{2^n}\left(\frac{1}{2}\rho\big(w_1(2^nx)-f(2^nx)\big)+\frac{1}{2}\rho\big(f(2^nx)-w_2(2^nx)\big)\right)\\
&\le\frac{1}{2^{n+1}}\cdot\frac{2L}{2(1-L)}\varphi(2^nx,0)\\
&\le\frac{1}{2^{n+1}}\cdot\frac{L}{(1-L)}\cdot(2L)^n\varphi(x,0)\\
&\to 0 \text{ as } n\to\infty
\end{aligned}$$

for all $x\in V$, from which it follows that $w_1=w_2$. This completes the proof. □

Remark 3.2. It is curious that the multiple of 2 in the inequalities (2.1) and (3.1) appears. It is an interesting question whether the constant 2 in (2.1) and (3.1) can be dropped.

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Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

E-mail address: baak@hanyang.ac.kr

Young Researchers and Elite Club, Islamshahr Branch, Islamic Azad University, Islamshahr, Iran

E-mail address: abasalt.bodaghi@gmail.com

Department of Mathematics, Hallym University, Chuncheon 24252, Korea

E-mail address: sokim@hallym.ac.kr

FUZZY STABILITY OF ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following additive-quadratic ρ-functional inequalities

$$N\Big(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y),-\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big],t\Big)\geq\frac{t}{t+\phi(x,y)},\ \Big(\rho\neq 1,\frac{1}{2}\Big)$$

and

$$N\Big(4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)-\rho[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\Big],t\Big)\geq\frac{t}{t+\phi(x,y)},\ (\rho\neq 0,1,2)$$

in fuzzy Banach spaces.

1. Introduction and preliminaries

The concept of a fuzzy norm on a linear space was introduced by Katsaras [12] in 1984. Later, Cheng and Mordeson [3] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for all $x, y \in X$ and all $c, s, t \in \mathbb{R}$,

(N1) $N(x,\ t) = 0$ for all $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,\ t) = 1$ for all $t > 0$;
(N3) $N(cx,\ t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y,\ s+t) \geq min\{N(x,s),\ N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,\ t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space*.

Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$, $t > 0$, there is an $m \in N$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$. It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space

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* Corresponding author.

is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

In 1940, Ulam proposed the following stability problem (cf. [24]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam·s problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings, and by Rassias [23] for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([4], [5], [6], [17]).

In 2008, for the first time, Mirmostafaee and Moslehian [15], [16] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of the stability for the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.2}$$

Glányi [10] and Fechner [8] proved the Hyers-Ulam stability of the following functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|.$$

Park, Cho, and Han [22] proved the generalized Hyers-Ulam stability of the following functional inequalities associated with the following Cauchy additive functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|.$$

and the following Cauchy-Jesen additive functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|2f(\frac{x + y}{2} + z)\|.$$

Park [19, 20, 21] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities.

Now, we consider the following fixed point theorem on generalized metric spaces.

Theorem 1.2. [7] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and*
(4) *$d(y, y^*) \leq \frac{1}{1 - L}\, d(y, Jy)$ for all $y \in Y$.*

In this paper, we investigate the solution of following additive-quadratic ρ-functional inequalities

$$\begin{aligned} &N\Big(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\\ &\quad-\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big],t\Big)\\ &\geq \frac{t}{t+\phi(x,y)}\ \Big(\rho\neq 1\frac{1}{2}\ \Big), \end{aligned} \tag{1.3}$$

$$\begin{aligned} &N\Big(4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\\ &\quad-\rho[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\Big],t\Big)\\ &\geq \frac{t}{t+\phi(x,y)}\ (\rho\neq 0,1,2), \end{aligned} \tag{1.4}$$

and prove the generalized Hyers-Ulam stability for them in fuzzy Banach spaces.

Throughout this paper, we assume that $(X,\|\cdot\|)$ is a linear space and (Y,N) is a fuzzy Banach space.

2. Solutions and stability of (1.3) and (1.4)

In this section, we investigate the solution and prove the generalized Hyers-Ulam stability of the ρ-functional inequalities (1.3) and (1.4) in fuzzy Banach spaces. For any mapping $f:X\longrightarrow Y$, let

$$f_o(x)=\frac{f(x)-f(-x)}{2},\ \ f_e(x)=\frac{f(x)+f(-x)}{2},$$

$$\begin{aligned} D_1f(x,y)&=f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\\ &\quad-\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big] \end{aligned}$$

and

$$\begin{aligned} D_2f(x,y)&=4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\\ &\quad-\rho[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\Big]. \end{aligned}$$

Lemma 2.1. *Let $\rho\neq 1,\frac{1}{2}(\rho\neq 0,1,2$, resp.) A mapping $f:X\longrightarrow Y$ saisfies $f(0)=0$ and $D_1f(x,y)=0$ ($D_2f(x,y)=0$, resp.) if and only if f is an additive-quadratic mapping.*

Proof. Suppose that f satisfies $f(0)=0$ and $D_1f(x,y)=0$. Setting $y=x$ in $D_1f(x,y)=0$, we have

$$f(2x)=3f(x)-f(-x)$$

for all $x\in X$ and so we get

$$f\Big(\frac{x}{2}\Big)=\frac{3}{8}f(x)-\frac{1}{8}f(-x)$$

for all $x \in X$. Hence we have

$$\begin{aligned}
&f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\\
&=\rho\Big[4f\Big(\frac{x+y}{2}\Big)+4f\Big(\frac{x-y}{2}\Big)-3f(x)+f(-x)-f(y)-f(-y)\Big]\\
&=\rho\Big[\frac{3}{2}f(x+y)+\frac{3}{2}f(x-y)-\frac{1}{2}f(-x-y)-\frac{1}{2}f(y-x)-3f(x)+f(-x)-f(y)-f(-y)\Big]
\end{aligned}$$

for all $x \in X$ and thus

$$(1-2\rho)[f_o(x+y)+f_o(x-y)-2f_o(x)]=0,$$

and

$$(1-\rho)[f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)]=0.$$

Since $\rho \neq 1, \frac{1}{2}$, f_o is additive and f_e is quadratic and thus $f = f_o + f_e$ is additive-quadratic. The converse is trivial.

The proof for $D_2 f(x,y) = 0$ is similar to that for $D_1 f(x,y) = 0$. □

Now, we will prove the generalized Hyers-Ulam stability for (1.3) in fuzzy normed spaces.

Theorem 2.2. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x,y) \leq \frac{L}{4}\phi(2x,2y) \tag{2.1}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_1 f(x,y),t) \geq \frac{t}{t+\phi(x,y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 1, \frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x)-F(x),t) \geq \min\Big\{\frac{Lt}{Lt+4(1-L)\phi(x,x)}, \frac{Lt}{Lt+4(1-L)\phi(-x,-x)}\Big\} \tag{2.3}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big), \quad F_e(x) = N - \lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big) \tag{2.4}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned}
d(g,h) = &\inf\{c \in [0,\infty) \mid N(g(x)-h(x),ct)\\
&\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\},\ \forall x \in X,\ \forall t > 0\}.
\end{aligned}$$

Then (S,d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big)$$

for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (2.1), we have

$$\begin{aligned} N(Jg(x)-Jh(x),cLt) &= N\Big(3g\Big(\frac{x}{2}\Big)+g\Big(-\frac{x}{2}\Big)-3h\Big(\frac{x}{2}\Big)-h\Big(-\frac{x}{2}\Big),cLt\Big) \\ &\ge \min\Big\{N\Big(g\Big(\frac{x}{2}\Big)-h\Big(\frac{x}{2}\Big),\frac{1}{4}cLt\Big),N\Big(g(-\frac{x}{2}\Big)-h\Big(-\frac{x}{2}\Big),\frac{1}{4}cLt\Big)\Big\} \\ &\ge \min\Big\{\frac{\frac{Lt}{4}}{\frac{Lt}{4}+\phi\Big(\frac{x}{2},\frac{x}{2}\Big)},\frac{\frac{Lt}{4}}{\frac{Lt}{4}+\phi\Big(-\frac{x}{2},-\frac{x}{2}\Big)}\Big\} \\ &\ge \min\Big\{\frac{t}{t+\phi(x,x)},\frac{t}{t+\phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg,Jh) \le Ld(g,h)$ for any $g,h \in S$ and so J is a strictly contractive mapping.

Putting $y = x$ in (2.2), we get

$$N(f(2x)-3f(x)-f(-x),t) \ge \frac{t}{t+\phi(x,x)} \tag{2.5}$$

for all $x \in X$, $t > 0$ and hence

$$N\Big(f(x)-3f\Big(\frac{x}{2}\Big)-f\Big(-\frac{x}{2}\Big),t\Big) \ge \frac{t}{t+\phi\Big(\frac{x}{2},\frac{x}{2}\Big)} \ge \frac{\frac{4}{L}t}{\frac{4}{L}t+\phi(x,x)}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$N\Big(f(x)-Jf(x),\frac{L}{4}t\Big) \ge \frac{t}{t+\phi(x,x)} \ge \min\Big\{\frac{t}{t+\phi(x,x)},\frac{t}{t+\phi(-x,-x)}\Big\}$$

for all $x \in X$, $t > 0$ and so we have $d(f,Jf) \le \frac{L}{4} < \infty$. By Theorem 1.2, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. By induction, we have

$$J^n f(x) = \frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Hence we have

$$N - \lim_{n\to\infty}\Big[\frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)\Big] = F(x) \tag{2.6}$$

for all $x \in X$ and so we get (2.4). Replacing x, y, and t by $\frac{x}{2^n}$, $\frac{y}{2^n}$, and $\frac{t}{2^{2n}}$ in (2.2), respectively, by (2.2), (N3), and (N4), we have

$$\begin{aligned} &N\Big(2^{2n}D_1 f_e\Big(\frac{x}{2^n},\frac{y}{2^n}\Big),2^{2n}t\Big) \\ &\ge \min\Big\{N\Big(2^{2n}D_1 f\Big(\frac{x}{2^n},\frac{y}{2^n},\Big),2^{2n}t\Big),N\Big(2^{2n}D_1 f\Big(-\frac{x}{2^n},-\frac{y}{2^n}\Big),2^{2n}t\Big)\Big\} \\ &\ge \min\Big\{\frac{t}{t+\phi\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)},\frac{t}{t+\phi\Big(-\frac{x}{2^n},-\frac{y}{2^n}\Big)}\Big\} \end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. Hence

$$N\Big(2^{2n}D_1 f_e\Big(\frac{x}{2^n},\frac{y}{2^n}\Big),t\Big) \ge \min\Big\{\frac{t}{t+L^n\phi(x,y)},\frac{t}{t+L^n\phi(-x,-y)}\Big\} \tag{2.7}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.7), by (2.4),

$$D_1F_e(x,y)=0$$

for all $x,y\in X$. Similarly, we have

$$D_1F_o(x,y)=0$$

for all $x,y\in X$. Hence

$$D_1F(x,y)=0$$

for all $x,y\in X$ and by Lemma 2.1, F is an additive-quadratic mapping. Since $d(f,Jf)\leq\frac{L}{4}$, by Theorem 1.2, we have (2.3).

Now, we show the uniqueness of F. Let G be an additive-quadratic mapping with (2.3). Then clearly, G is a fixed point of J and

$$d(Jf,G)=d(Jf,JG)\leq Ld(f,G)\leq\frac{L}{4(1-L)}<\infty \tag{2.8}$$

and hence by (3) in Theorem 1.2, $F=G$. □

As examples of $\phi(x,y)$ in Theorem 2.2, we can take $\phi(x,y)=\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.3. *Let $\epsilon\geq 0$ and p be a real number with $1<p$. Let $f:X\longrightarrow Y$ be a mapping such that $f(0)=0$ and*

$$N(D_1f(x,y),t)\geq\frac{t}{t+\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}$$

for all $x,y\in X$ and all $t>0$. Suppose that $\rho\neq 1,\frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F:X\longrightarrow Y$ such that

$$N(f(x)-F(x),t)\geq\frac{t}{t+3\epsilon(2^{2p}-4)\|x\|^{2p}}$$

for all $x\in X$ and all $t>0$. Moreover

$$F_o(x)=N-\lim_{n\to\infty}2^nf_o\Big(\frac{x}{2^n}\Big),\ \ F_e(x)=N-\lim_{n\to\infty}2^{2n}f_e\Big(\frac{x}{2^n}\Big)$$

for all $x\in X$.

Related with Theorem 2.2, we can also have the following theorem. And the proof is similar to that of Theorem 2.2.

Theorem 2.4. *Assume that $\phi:X^3\longrightarrow Z$ is a function such that*

$$\phi(x,y)\leq 2L\phi\Big(\frac{x}{2},\frac{y}{2}\Big) \tag{2.9}$$

for all x,y and some L with $0<L<1$. Let $f:X\longrightarrow Y$ be a mapping such that $f(0)=0$ and

$$N(D_1f(x,y),t)\geq\frac{t}{t+\phi(x,y)} \tag{2.10}$$

for all $x,y\in X$ and all $t>0$. Suppose that $\rho\neq 1,\frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F:X\longrightarrow Y$ such that

$$N(f(x)-F(x),t)\geq\min\Big\{\frac{t}{t+2(1-L)\phi(x,x)},\frac{t}{t+2(1-L)\phi(-x,-x)}\Big\} \tag{2.11}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \quad F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = &\inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \\ &\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\}, \ \forall x \in X, \ \forall t > 0\}. \end{aligned}$$

Then (S,d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (2.9), we have

$$\begin{aligned} N(Jg(x) - Jh(x), cLt) &= N\Big(\frac{3}{8}g(2x) - \frac{1}{8}g(-2x) - \frac{3}{8}h(2x) + \frac{1}{8}h(-2x), cLt\Big) \\ &\geq \min\Big\{N\Big(\frac{3}{8}\Big[g(2x) + h(2x)\Big], \frac{3}{4}cLt\Big), N\Big(\frac{1}{8}\Big[g(-2x) + h(-2x)\Big], \frac{1}{4}cLt\Big)\Big\} \\ &\geq \min\Big\{\frac{2Lt}{2Lt+\phi(2x,2x)}, \frac{2Lt}{2Lt+\phi(-2x,-2x)}\Big\} \\ &\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (2.5), we have

$$\begin{aligned} &N\Big(f(x) - Jf(x), \frac{1}{2}t\Big) \\ &= N\Big(\frac{3}{8}\Big[f(2x) - 3f(x) - f(-x)\Big] - \frac{1}{8}\Big[f(-2x) - 3f(-x) - f(x)\Big], \frac{1}{2}t\Big) \\ &\geq \min\{N(f(2x) - 3f(x) - f(-x), t), N(f(-2x) - 3f(-x) - f(x), t)\} \\ &\geq \min\Big\{\frac{t}{t+\phi(x,x)}, \frac{t}{t+\phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{1}{2} < \infty$. By Theorem 1.2, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. The rest of the proof is similar to that of Theorem 2.2. □

As examples of $\phi(x,y)$ in Theorem 2.4, we can take $\phi(x,y) = \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.5. *Let $\epsilon \geq 0$ and p be a real number with $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_1 f(x,y), t) \geq \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 1, \frac{1}{2}$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \frac{t}{t + 3\epsilon(2^{2p} - 2)\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \;\; F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Now, we will prove the stability of the functional inequality (1.4) in fuzzy Banach spaces.

Theorem 2.6. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x, y) \leq \frac{L}{4}\phi(2x, 2y) \tag{2.12}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_2 f(x, y), t) \geq \frac{t}{t + \phi(x, y)} \tag{2.13}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

(2.14)

$$N\Big(f(x) - F(x), t\Big) \geq \min\Big\{\frac{Lt}{Lt + 4\rho(1 - L)\phi(x, x)}, \frac{Lt}{Lt + 4\rho(1 - L)\phi(-x, -x)}\Big\}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big), \;\; F_e(x) = N - \lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g, h) = &\inf\{c \in [0, \infty) \mid N(g(x) - h(x), ct) \\ &\geq \min\Big\{\frac{t}{t + \phi(x, x)}, \frac{t}{t + \phi(-x, -x)}\Big\}, \; \forall x \in X, \; \forall t > 0\}. \end{aligned}$$

Then (S, d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big)$$

for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (2.12), we have

$$\begin{aligned} N(Jg(x) - Jh(x), cLt) &= N\Big(3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big) - 3h\Big(\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), cLt\Big) \\ &\geq \min\Big\{N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), \frac{1}{4}cLt\Big), N\Big(g(-\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), \frac{1}{4}cLt\Big)\Big\} \\ &\geq \min\Big\{\frac{\frac{Lt}{4}}{\frac{Lt}{4} + \phi\Big(\frac{x}{2}, \frac{x}{2}\Big)}, \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \phi\Big(-\frac{x}{2}, -\frac{x}{2}\Big)}\Big\} \\ &\geq \min\Big\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = x$ in (2.13), we get

$$N(\rho[f(2x) - 3f(x) - f(-x)], t) \geq \frac{t}{t + \phi(x,x)} \tag{2.15}$$

for all $x \in X$, $t > 0$ and hence

$$N(f(x) - Jf(x), \frac{t}{\rho}) \geq \frac{t}{t + \phi\Big(\frac{x}{2}, \frac{x}{2}\Big)} \geq \frac{t}{t + \frac{L}{4}\phi(x,x)}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$N\Big(f(x) - Jf(x), \frac{L}{4\rho}t\Big) \geq \frac{t}{t + \phi(x,x)} \geq \min\Big\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\Big\}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{L}{4\rho} < \infty$. The rest of the proof is similar to that of Theorem 2.2. □

As examples of $\phi(x,y)$ in Theorem 2.6, we can take $\phi(x,y) = \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.7. *Let $\epsilon \geq 0$ and p be a real number with $1 < p$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_2 f(x,y), t) \geq \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \frac{t}{t + 3\epsilon\rho(2^{2p} - 4)\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} 2^n f_o\Big(\frac{x}{2^n}\Big), \;\; F_e(x) = N - \lim_{n\to\infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$.

Theorem 2.8. *Assume that $\phi : X^3 \longrightarrow Z$ is a function such that*

$$\phi(x,y) \leq 2L\phi\Big(\frac{x}{2}, \frac{y}{2}\Big) \tag{2.16}$$

for all x, y and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_2 f(x,y), t) \geq \frac{t}{t + \phi(x,y)} \tag{2.17}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \min\Big\{\frac{t}{t + 2\rho(1-L)\phi(x,x)}, \frac{t}{t + 2\rho(1-L)\phi(-x,-x)}\Big\} \tag{2.18}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \;\; F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = &\inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \\ &\geq \min\Big\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\Big\}, \; \forall x \in X, \; \forall t > 0\}. \end{aligned}$$

Then (S, d) is a complete metric space(see [18]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{3}{8} g(2x) - \frac{1}{8} g(-2x)$$

for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (2.16), we have

$$\begin{aligned} N(Jg(x) - Jh(x), cLt) &= N\Big(\frac{3}{8} g(2x) - \frac{1}{8} g(-2x) - \frac{3}{8} h(2x) + \frac{1}{8} h(-2x), cLt\Big) \\ &\geq \min\Big\{N\Big(\frac{3}{8}\Big[g(2x) + h(2x)\Big], \frac{3}{4} cLt\Big), N\Big(\frac{1}{8}\Big[g(-2x) + h(-2x)\Big], \frac{1}{4} cLt\Big)\Big\} \\ &\geq \min\Big\{\frac{2Lt}{2Lt + \phi(2x,2x)}, \frac{2Lt}{2Lt + \phi(-2x,-2x)}\Big\} \\ &\geq \min\Big\{\frac{t}{t + \phi(x,x)}, \frac{t}{t + \phi(-x,-x)}\Big\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (2.15), we have

$$\begin{aligned} &N(\rho[f(x) - Jf(x)], \frac{t}{2}) \\ &= N\Big(\frac{3}{8}\rho\Big[f(2x) - 3f(x) - f(-x)\Big] - \frac{1}{8}\rho\Big[f(-2x) - 3f(-x) - f(x)\Big], \frac{t}{2}\Big) \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{1}{2\rho} < \infty$. By Theorem 1.2, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. The rest of the proof is similar to that of Theorem 2.2. □

As examples of $\phi(x,y)$ in Theorem 2.6, we can take $\phi(x,y) = \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$. Then we can formulate the following corollary

Corollary 2.9. *Let $\epsilon \geq 0$ and p be a real number with $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(D_2 f(x,y),t) \geq \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Suppose that $\rho \neq 0, 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x),t) \geq \frac{t}{t + 3\epsilon\rho(2 - 2^{2p})\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$. Moreover

$$F_o(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f_o(2^n x), \quad F_e(x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x)$$

for all $x \in X$.

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Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea
E-mail address: kci206@hanmail.net

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea
E-mail address: gilhan@dankook.ac.kr

Symmetric identities for Dirichlet-type multiple twisted q-l-function

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we give some interesting symmetric identities for Dirichlet-type multiple twisted q-l-function in complex field.

Key words : Symmetric properties, power sums, Euler numbers and polynomials, twisted q-Euler numbers and polynomials, generalized twisted q-Euler polynomials of higher order, Dirichlet-type multiple twisted q-l-function.

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1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the q- extension of Euler numbers and polynomials(see [1, 2, 3, 5, 6, 7, 8, 9, 11, 13]). Recently, D. Kim *et al.*[4] derived some identities of symmetry for (h, q)-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[2] derived some identities of symmetry for higher-order generalized q-Euler polynomials. Y. He studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [3]). In this paper, we establish some interesting symmetric identities for generalized twisted q-Euler polynomials of higher order in complex field.

The purpose of this paper is to present a systemic study of the generalized twisted q-Euler numbers and polynomials of higher-order by using the multiple q-Euler zeta function. Throughout this paper, the notations $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 5])} .$$

Note that $\lim_{q\to 1}[x] = x$. Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and ε be the p^N-th root of unity(see [10, 12, 13]).

In [5], T. Kim introduced the multiple q-Euler zeta function which interpolates higher-order q-Euler polynomials at negative integers as follows:

$$\zeta_{q,r}(s,x) = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1+\cdots+m_r+x]_q^s}, \tag{1}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

Recently, D. V. Dolgy *et al.*[2] considered some symmetric identities for higher-order generalized q-Euler polynomials. The generalized Euler polynomials of order $r \in \mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2\sum_{l=0}^{d-1} \frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt}+1}\right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x)\frac{t^m}{m!}. \tag{2}$$

When $x=0, E_{n,\chi}^{(r)}=E_{n,\chi}^{(r)}(0)$ are called the generalized Euler numbers $E_{n,\chi}^{(r)}$ attached to χ(see [2, 4]).

For $h\in\mathbb{Z}, \alpha, k\in\mathbb{N}$, and $n\in\mathbb{Z}_+$, we introduced the higher order twisted q-Euler polynomials with weight α as follows(see [7]):

$$\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k|x)=\frac{[2]_q^k}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{\alpha lx}}{(1+\varepsilon q^{\alpha l+h})\cdots(1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x=0$, $\widetilde{E}_{n,q,w}^{(\alpha)}(h,k|0)=\widetilde{E}_{n,q,w}^{(\alpha)}(h,k)$ are called the higher-order twisted q-Euler numbers with weight α.

We consider the higher order generalized q-Euler polynomials of order r attached to χ twisted by ramified roots of unity as follows(see [10]):

$$\sum_{n=0}^{\infty}E_{n,\chi,\zeta,q}^{(r)}(x)\frac{t^n}{n!}=2^r\sum_{m_1,\ldots,m_r=0}^{\infty}(-\zeta)^{\sum_{j=0}^{r}m_j}\left(\prod_{i=1}^{r}\chi(m_i)\right)e^{[x+\sum_{j=1}^{r}m_j]_q t}.$$

In the special case $x=0$, the sequence $E_{n,\chi,\zeta,q}^{(r)}(0)=E_{n,\chi,\zeta,q}^{(r)}$ are called the n-th generalized q-Euler numbers of order r attached to χ twisted by ramified roots of unity.

As is well known, the higher-order generalized twisted q-Euler polynomials $E_{n,\chi,q,\varepsilon}^{(k)}(x)$ attached to χ are defined by the following generating function to be

$$\begin{aligned}\widetilde{F}_{\chi,q,\varepsilon}^{(k)}(t,x)&=[2]_q^k\sum_{m_1,\cdots,m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\varepsilon^{m_1+\cdots+m_k}\left(\prod_{j=1}^{k}\chi(m_j)\right)e^{[m_1+\cdots+m_k+x]_q t}\\&=\sum_{n=0}^{\infty}E_{n,\chi,q,\varepsilon}^{(k)}(x)\frac{t^n}{n!},\end{aligned}\tag{3}$$

where $k\in\mathbb{N}$. When $x=0, E_{n,\chi,q,\varepsilon}^{(h,k)}=E_{n,\chi,q,\varepsilon}^{(k)}(0)$ are called the higher-order generalized twisted q-Euler numbers $E_{n,\chi,q,\varepsilon}^{(k)}$ attached to χ. Observe that if $q\to 1, \varepsilon\to 1$, then $E_{n,\chi,q,\varepsilon}^{(k)}\to E_{n,\chi}^{(k)}$ and $E_{n,\chi,q,\varepsilon}^{(k)}(x)\to E_{n,\chi}^{(k)}(x)$.

By using (3) and Cauchy product, we have

$$\begin{aligned}E_{n,\chi,q,\varepsilon}^{(k)}(x)&=\sum_{l=0}^{n}\binom{n}{l}q^{lx}E_{l,\chi,q,\varepsilon}^{(k)}[x]_q^{n-l}\\&=(q^xE_{\chi,q,\varepsilon}^{(k)}+[x]_q)^n,\end{aligned}\tag{4}$$

with the usual convention about replacing $(E_{\chi,q,\varepsilon}^{(k)})^n$ by $E_{n,\chi,q,\varepsilon}^{(k)}$.

By using complex integral and (3), we can also obtain the Dirichlet-type multiple twisted q-l-function as follows:

$$\begin{aligned}l_{\chi,q,\varepsilon}^{(k)}(s,x)&=\frac{1}{\Gamma(s)}\int_0^\infty\widetilde{F}_{\chi,q,\varepsilon}^{(k)}(-t,x)t^{s-1}dt\\&=[2]_q^k\sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^{k}m_j}\left(\prod_{j=1}^{k}\chi(m_j)\right)\varepsilon^{\sum_{j=1}^{k}m_j}}{[m_1+\cdots+m_k+x]_q^s},\end{aligned}\tag{5}$$

where $s\in\mathbb{C}$ and $x\in\mathbb{R}$, with $x\neq 0,-1,-2,\ldots$

By using Cauchy residue theorem, the value of Dirichlet-type multiple twisted q-l-function at negative integers is given explicitly by the following theorem:

Theorem 1. Let $k\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We obtain

$$l_{\chi,q,\varepsilon}^{(k)}(-n,x)=E_{n,\chi,q,\varepsilon}^{(k)}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order generalized twisted q-Euler polynomials $E_{n,\chi,q,\varepsilon}^{(k)}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted q-l-function. In this paper, if we take $\chi^0 = 1, \varepsilon = 1$, then [4] is the special case of this paper. If we take $\varepsilon = 1$ in all equations of this article, then [2] are the special case of our results.

2. Symmetry identities for multiple twisted q-l-function

In this section, we investigate some symmetric identities for higher-order generalized twisted q-Euler polynomials $E_{n,\chi,q,\varepsilon}^{(k)}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted q-l-function. We assume that χ is a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and ε be the p^N-th root of unity. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for Dirichlet-type multiple twisted q-l-function.

Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (5), we derive next result by substitute $w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)$ for x in and replace q and ε by q^{w_1} and ε^{w_1}, respectively.

$$\begin{aligned}
&\frac{1}{[2]_{q^{w_1}}^k} l_{\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)) \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \left(\prod_{j=1}^k \chi(m_j)\right) \varepsilon^{w_1(m_1+\cdots+m_k)}}{[m_1 + \cdots + m_k + w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)]_{q^{w_1}}^s} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \left(\prod_{j=1}^k \chi(m_j)\right) \varepsilon^{w_1(m_1+\cdots+m_k)}}{\left[\frac{w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)}{w_1}\right]_{q^{w_1}}^s} \\
&= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{dw_2-1} \frac{(-1)^{\sum_{j=1}^k m_j} \left(\prod_{j=1}^k \chi(m_j)\right) \varepsilon^{w_1(m_1+\cdots+m_k)}}{[w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)]_q^s} \\
&= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{dw_2-1} (-1)^{\sum_{j=1}^k m_j} (-1)^{\sum_{j=1}^k i_j} \left(\prod_{j=1}^k \chi(i_j)\right) \\
&\quad \times \varepsilon^{dw_1 w_2 \sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k i_j} \\
&\quad \times \left([w_1 w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]_q^s\right)^{-1}
\end{aligned} \tag{6}$$

Thus, from (6), we can derive the following equation.

$$\begin{aligned}
&\frac{[w_2]_q^s}{[2]_{q^{w_1}}^k} \sum_{j_1,\cdots,j_k=0}^{dw_1-1} (-1)^{\sum_{l=1}^k j_l} \left(\prod_{l=1}^k \chi(j_l)\right) \varepsilon^{w_2(j_1+\cdots+j_k)} \\
&\qquad \times l_{\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)) \\
&= [w_1]_q^s [w_2]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{dw_2-1} \sum_{j_1,\cdots,j_k=0}^{dw_1-1} (-1)^{\sum_{l=1}^k (j_l+i_l+m_l)} \left(\prod_{l=1}^k \chi(j_l)\right) \left(\prod_{l=1}^k \chi(i_l)\right) \\
&\quad \times \varepsilon^{dw_1 w_2 \sum_{l=1}^k m_l} \varepsilon^{w_1 \sum_{l=1}^k i_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} \\
&\quad \times \left([w_1 w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]_q^s\right)^{-1}
\end{aligned} \tag{7}$$

By using the same method as (7), we have

$$\begin{aligned}
&\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\
&\quad\times l_{\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k))\\
&=[w_1]_q^s[w_2]_q^s\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}\sum_{i_1,\cdots,i_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k(j_l+i_l+m_l)}\left(\prod_{l=1}^k\chi(j_l)\right)\left(\prod_{l=1}^k\chi(i_l)\right)\\
&\quad\times\varepsilon^{dw_1w_2\sum_{l=1}^k m_l}\varepsilon^{w_2\sum_{l=1}^k i_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(j_1+\cdots+j_k)+w_2(i_1+\cdots+i_k)]_q^s\right)^{-1}
\end{aligned}\tag{8}$$

Therefore, by (7) and (8), we have the following theorem.

Theorem 2. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1\pmod 2$, $w_2\equiv 1\pmod 2$. For $k\in\mathbb{N}$, we have

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\
&\qquad\times l_{\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(s,w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_1(j_1+\cdots+j_k)}\\
&\qquad\times l_{\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}\left(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)
\end{aligned}\tag{9}$$

By (9) and Theorem 1, we obtain the following theorem.

Theorem 3. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1\pmod 2$, $w_2\equiv 1\pmod 2$. For $k\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\
&\qquad\times E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)\varepsilon^{w_1(j_1+\cdots+j_k)}\\
&\qquad\times E_{n,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right).
\end{aligned}\tag{10}$$

From (4), we note that

$$\begin{aligned}
E_{n,\chi,q,\varepsilon}^{(k)}(x+y)&=(q^{x+y}E_{n,\chi,q,\varepsilon}^{(k)}+[x+y]_q)^n\\
&=\sum_{i=0}^n\binom{n}{i}q^{xi}E_{i,\chi,q,\varepsilon}^{(k)}(y)[x]_q^{n-i}.
\end{aligned}\tag{11}$$

with the usual convention about replacing $(E_{\chi,q,\varepsilon}^{(k)})^n$ by $E_{n,\chi,q,\varepsilon}^{(k)}$.

By (11), we have

$$\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)$$
$$=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\sum_{i=0}^{n}\binom{n}{i}E_{i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{n-i}$$
$$=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\sum_{i=0}^{n}\binom{n}{i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{i}$$

Hence we have the following theorem.

Theorem 4. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)=$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}[j_1\cdots+j_k]_{q^{w_2}}^i.$$

For each integer $n \geq 0$, let

$$\mathcal{S}_{n,i,\chi,q,\varepsilon}^{(k)}(w)=\sum_{j_1,\cdots,j_k=0}^{w-1}(-1)^{(j_1+\cdots+j_k)}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{(j_1+\cdots+j_k)}[j_1\cdots+j_k]_q^i.$$

The above sum $\mathcal{S}_{n,i,\chi,q,\varepsilon}^{(k)}(w)$ is called the alternating generalized q-power sums.

By Theorem 4, we have

$$\begin{aligned}&[2]_{q^{w_2}}^k[w_1]_q^n\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_2(j_1+\cdots+j_k)}\\&\quad\times E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\&=[2]_{q^{w_2}}^k\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}(dw_1)\end{aligned}\tag{12}$$

By using the same method as in (12), we have

$$\begin{aligned}&[2]_{q^{w_1}}^k[w_2]_q^n\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^{k}j_l}\left(\prod_{l=1}^{k}\chi(j_l)\right)\varepsilon^{w_1(j_1+\cdots+j_k)}\\&\quad\times E_{n,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)\\&=[2]_{q^{w_1}}^k\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1x)\mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(dw_2)\end{aligned}\tag{13}$$

Therefore, by (12), (13), and Theorem 3, we have the following theorem.

Theorem 5. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_2}}^k\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}(dw_1)\\&=[2]_{q^{w_1}}^k\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1x)\mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(dw_2).\end{aligned}$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order generalized twisted q-Euler numbers $E_{n,\chi,q,\varepsilon}^{(k)}$ in complex field.

Corollary 6. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$[2]_{q^{w_2}}^k \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}(dw_1) E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}$$
$$= [2]_{q^{w_1}}^k \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(k)}(dw_2) E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(k)}.$$

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ON A q-ANALOGUE OF DEGENERATE λ-CHANGHEE POLYNOMIALS

SANG JO YUN[1], SANG-KI LEE[2], AND JIN-WOO PARK[3,*]

Abstract. In this paper, we derive the q-analog of degenerate λ-Changhee polynomials, and found some new and interesting identities and properties of those numbers and polynomials.

1. Introduction

Let d be a fixed positive integer and let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. the p-adic norm is defined $|p|_p = \frac{1}{p}$

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic fermionic integral on* $\mathbb{Z}_p$ is defined by Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \text{ (see [8, 9, 10])}. \quad (1.1)$$

If we put $f_1(x) = f(x+1)$, then, by (1.1), we can get the following well-known integral identity

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (1.2)$$

where $f'(0) = \frac{df(x)}{dx}|_{x=0}$.

It is well known that the *q-Euler polynomials of order k* are defined by the generating function to be

$$\left(\frac{[2]_q}{1+qe^t}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \text{ (see [1, 5, 8, 9, 15, 18])}.$$

In the special case $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the *n-th q-Euler numbers.*

Recently, D. S. Kim et. al introduced the *Changhee polynomials* as follows :

$$Ch_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y), (n \geq 0), \text{ (see [6, 13, 17])}.$$

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* corresponding author.

When $x = 0$, $Ch_n = Ch_n(0)$ are called the *Changhee numbers*. In [12], authors defined the *q-Changhee polynomials* as follows.

$$\frac{1+q}{q(1+t)+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}.$$

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \geq 0), \tag{1.3}$$

and the *Stirling numbers of the second kind* is defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \tag{1.4}$$

(see [3, 16]). Note that

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!}, \ (n \geq 0), \tag{1.5}$$

(see [3, 4, 6, 11-14]).

In [2], L. Carlitz consider the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{x}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.6}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers*. Note that $\lim_{\lambda\to 0}\beta_n(\lambda) = B_n$.

It is well known that

$$e^t = \lim_{u\to 0}(1+ut)^{\frac{1}{u}}, \ (\text{see } [2, 5, 7]).$$

The function $(1+ut)^{\frac{1}{u}}$ is called the *degenerate function of* e^t. Thus, for $t = \log e^t$, we have $\log(1+ut)^{\frac{1}{u}}$ as the degenerate function.

Recently, Changhee numbers and polynomials are introduced by Kim et. al., and found interesting identities by many researchers(see [6, 12, 13, 14, 17]).

In this paper, we derive the q-analog of degenerate λ-Changhee polynomials and found some new and interesting identities and properties of those numbers and polynomials.

2. λ-q-Changhee polynomials of the first kind

In this section, we assume that $u, t \in \mathbb{C}_p$ with $|ut|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

Now, as a generalization of Changhee polynomials, we consider the *degenerate λ-q-Changhee polynomials* :

$$\frac{1+q}{1+q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}}\left(1+\frac{1}{u}\log(1+ut)\right)^x = \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $Ch_{n,\lambda,q}(u) = Ch_{n,\lambda,q}(0|u)$ are called the *degenerate λ-q-Changhee numbers*. It is easy to check that $\lim_{u\to 0} Ch_{n,\lambda,q}(x|u) = Ch_{n,\lambda,q}(x)$ and $Ch_{n,q} = Ch_{n,1,q}(0)$.

Let us take $f(x) = \left(1 + \frac{1}{u}\log(1+ut)\right)^{\lambda x}$. From (1.2), we have

$$\int_{\mathbb{Z}_p} \left(1 + \frac{1}{u}\log(1+ut)\right)^{\lambda x} d\mu_{-q}(x) = \frac{1+q}{1+q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{t^n}{n!} =& \frac{1+q}{1+q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}}\left(1+\frac{1}{u}\log(1+ut)\right)^{x} \\ =& \int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y+x} d\mu_{-q}(y),\end{aligned} \tag{2.3}$$

and, by (1.5),

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y+x} d\mu_{-q}(y) \\ =& \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n} u^{-n}\left(\log(1+ut)\right)^n d\mu_{-q}(y) \\ =& \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n} u^{-n} n! \sum_{k=n}^{\infty} S_1(k,n)\frac{u^k t^k}{k!} d\mu_{-q}(y) \\ =& \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} u^{n-m} S_1(n,m)\int_{\mathbb{Z}_p}(\lambda y+x)_m d\mu_q(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Thus, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$Ch_{n,\lambda,q}(x|u) = \sum_{m=0}^{n} u^{n-m} S_1(n,m)\int_{\mathbb{Z}_p}(\lambda y+x)_m d\mu_{-q}(y).$$

By replacing t by $\frac{1}{u}\left(e^{ut}-1\right)$ in (2.1), we get

$$\begin{aligned}\frac{1+q}{q(1+t)^{\lambda}+1}(1+t)^x =& \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{ut}-1\right)\right)^n \\ =& \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{1}{n!} n! \sum_{l=n}^{\infty} S_2(l,n) u^{-n}\frac{(ut)^l}{l!} \\ =& \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_{m,\lambda,q}(x|u) u^{n-m} S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

On the other hands, by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.1), we have

$$\begin{aligned}\frac{1+q}{1+qe^{\lambda t}}e^{xt} &= \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n \\ &= \sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x|u)\frac{1}{n!}u^{-n}n!\sum_{l=n}^{\infty} S_2(l,n)\frac{u^l}{l!}\left(e^t-1\right)^l \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} Ch_{m,\lambda,q}(x|u)S_2(n,m)\frac{u^{n-m}}{n!}\left(e^t-1\right)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

and

$$\frac{1+q}{1+qe^{\lambda t}}e^{xt} = \frac{1+q}{1+qe^{\lambda t}}e^{\lambda t\left(\frac{x}{\lambda}\right)} = \sum_{n=0}^{\infty}\lambda^n E_{n,q}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}. \tag{2.7}$$

By (1.3) and Theorem 2.1,

$$\begin{aligned}Ch_{n,\lambda,q}(x|u) &= \sum_{k=0}^{n} u^{n-k}S_1(n,k)\int_{\mathbb{Z}_p}(\lambda y+x)_k d\mu_{-q}(y) \\ &= \sum_{k=0}^{n} u^{n-k}S_1(n,k)\sum_{l=0}^{k} S_1(k,l)\lambda^l\int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)d\mu_{-q}(y) \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k} u^{n-k}\lambda^l S_1(n,k)S_1(k,l)E_{l,q}\left(\frac{x}{\lambda}\right).\end{aligned} \tag{2.8}$$

Therefore, by (2.4), (2.5), (2.6) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$Ch_{n,\lambda,q}(x) = \sum_{m=0}^{n} Ch_{m,\lambda,q}(x|u)u^{n-m}S_2(n,m),$$

and

$$Ch_{n,\lambda,q}(x|u) = \sum_{k=0}^{n}\sum_{l=0}^{k} u^{n-k}\lambda^l S_1(n,k)S_1(k,l)E_{l,q}\left(\frac{x}{\lambda}\right).$$

In addition,

$$\lambda^n E_{n,q}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}(x|u)u^{m-l}S_2(m,l)S_2(n,m).$$

Let us consider the *degenerate λ-q-Changhee polynomials of the first kind* with order $k(\in\mathbb{N})$ as follows:

$$Ch_{n,\lambda,q}^{(k)}(x|u) = \sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k). \tag{2.9}$$

From (2.9), we can derive the generating function of $Ch_{n,\lambda,q}^{(k)}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,\lambda,q}^{(k)}(x|u)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
&=\sum_{l=0}^{\infty} u^{-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\frac{1}{l!}l!\sum_{m=l}^{\infty}S_1(m,l)\frac{(ut)^m}{m!}\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\binom{\lambda y_1+\cdots+\lambda y_k+x}{l}u^{-l}\left(\log(1+ut)\right)^l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y_1+\cdots+\lambda y_k+x} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k).
\end{aligned}\tag{2.10}$$

Note that by (1.3),

$$\begin{aligned}
&Ch_{n,\lambda,q}^{(k)}(x|u)\\
&=\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
&=\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{s=0}^{l}S_1(l,s)(\lambda y_1+\cdots+\lambda y_k+x)^s d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
&=\sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(y_1+\cdots+y_k+\frac{x}{\lambda}\right)^s d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_k)\\
&=\sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s E_{s,q}^{(k)}\left(\frac{x}{\lambda}\right).
\end{aligned}\tag{2.11}$$

From (2.6) and (2.10), we have

$$\begin{aligned}
\left(\frac{1+q}{1+qe^{\lambda t}}\right)^k e^{xt} &= \sum_{n=0}^{\infty} Ch_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\right)\frac{t^n}{n!}
\end{aligned}\tag{2.12}$$

and

$$\left(\frac{1+q}{1+qe^{\lambda t}}\right)^k e^{\lambda t\left(\frac{x}{\lambda}\right)} = \sum_{n=0}^{\infty}\lambda^n E_{n,q}^{(k)}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}.\tag{2.13}$$

Thus, by (2.11), (2.12) and (2.13), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, $k\in\mathbb{N}$, we have*

$$\lambda^n E_{n,q}^{(k)}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{l=0}^{m} Ch_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m),$$

and

$$\begin{aligned} Ch_{n,\lambda,q}^{(k)}(x|u) =& \sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l} S_1(n,l) S_1(l,s) \lambda^s E_{s,q}^{(k)}\left(\frac{x}{\lambda}\right) \\ =& \sum_{l=0}^{n}\sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m} u^{n+m-l-r}\lambda^{-s} Ch_{r,\lambda,q}^{(k)}(x|u) S_1(n,l) S_1(l,s) S_2(s,m) S_2(m,r). \end{aligned}$$

3. λ-q-Changhee polynomials of the second kind

For $n \geq 0$, the *rising factorial sequence* is defined by

$$\begin{aligned} x^{(n)} =& x(x+1)\cdots(x+n-1) = (-1)^n(-x)_n \\ =& \sum_{l=0}^{n}(-1)^{n-l} S_1(n,l) x^l. \end{aligned} \tag{3.1}$$

Let us define the *degenerate λ-q-Changhee polynomials of the second kind* as follows:

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda,q}(x|u)\frac{t^n}{n!} = \frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}\left(1+\frac{1}{u}\log(1+ut)\right)^{x}. \tag{3.2}$$

When $\lambda = 1$, $\widehat{Ch}_{n,1,q}(x|u) = \widehat{Ch}_{n,q}(x|u)$ are called the *degenerate q-Changhee polynomials of the second kind*. In particular, if $x = 0$, then $\widehat{Ch}_{n,\lambda,q}(0|u) = \widehat{Ch}_{n,\lambda,q}(u)$ are called the *degenerate λ-q-Changhee numbers of the second kind*, and $\widehat{Ch}_{n,1,q}(0|u) = \widehat{Ch}_{n,q}(u)$ are called the *degenerate q-Changhee numbers of the second kind*.

Let us take $f(x) = (1+t)^{-\lambda x}$. Then, by (1.2), we have

$$\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda x} d\mu_{-q}(x) = \frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}, \tag{3.3}$$

and so

$$\begin{aligned} &\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda y+x} d\mu_q(y) \\ =& \frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}\left(1+\frac{1}{u}\log(1+ut)\right)^{x} \\ =& \sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

By (3.4), we get

$$\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}(x|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} u^{n-m} S_1(n,m)\int_{\mathbb{Z}_p}(-\lambda y+x)_m\, d\mu_{-q}(y)\frac{t^n}{n!}. \tag{3.5}$$

By (3.3) and (3.5), we get

$$\begin{aligned}\widehat{Ch}_{n,\lambda,q}(x|u) &= \sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} (-\lambda y + x)_m \, d\mu_{-q}(y) \\ &= \sum_{m=0}^{n} \sum_{l=0}^{m} u^{n-m} (-\lambda)^l S_1(n,m) S_1(m,l) \int_{\mathbb{Z}_p} \left(y - \frac{x}{\lambda}\right)^l d\mu_{-q}(y) \\ &= \sum_{m=0}^{n} \sum_{l=0}^{m} u^{n-m} (-\lambda)^l S_1(n,m) S_1(m,l) E_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{3.6}$$

By (3.4), we get

$$\begin{aligned}\frac{q+1}{qe^{-\lambda t}+1} e^{xt} &= \sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda,q}(x|u) \frac{1}{n!} \left(\frac{1}{u} e^{u(e^t-1)} - 1\right)^n \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \widehat{Ch}_{n,\lambda,q}(x|u) u^{m-n} S_2(m,n) \right) \frac{t^m}{m!},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\frac{q+1}{qe^{-\lambda t}+1} e^{xt} &= \frac{q+1}{qe^{-\lambda t}+1} e^{\left(-\frac{x}{\lambda}\right)(-\lambda t)} \\ &= \sum_{m=0}^{\infty} (-\lambda)^m E_{m,q}\left(-\frac{x}{\lambda}\right) \frac{t^m}{m!}.\end{aligned} \tag{3.8}$$

Therefore, by (3.6), (3.7) and (3.8), we obtain the following theorem.

Theorem 3.1. *For $m \geq 0$, we have*

$$\widehat{Ch}_{m,\lambda,q}(x|u) = \sum_{n=0}^{m} \sum_{l=0}^{n} u^{m-n} (-\lambda)^l S_1(m,n) S_1(n,l) E_{l,q}\left(-\frac{x}{\lambda}\right),$$

and

$$\begin{aligned}(-\lambda)^m E_{m,q}\left(-\frac{x}{\lambda}\right) &= \sum_{n=0}^{m} \widehat{Ch}_{n,\lambda,q}(x|u) u^{m-n} S_2(m,n) \\ &= \sum_{n=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{k} u^{m-k} (-\lambda)^l S_1(n,k) S_1(k,l) S_2(m,n) E_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned}$$

By the Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda,q}(u) = \sum_{n=0}^{m} \sum_{l=0}^{n} u^{m-n} (-\lambda)^l S_1(m,n) S_1(n,l) E_{l,q},$$

and

$$\begin{aligned}E_{m,q} &= (-\lambda)^{-m} \sum_{n=0}^{m} \widehat{Ch}_{n,\lambda,q}(u) u^{m-n} S_2(m,n) \\ &= \sum_{n=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{k} u^{m-k} (-\lambda)^l S_1(n,k) S_1(k,l) S_2(m,n) E_{l,q}.\end{aligned}$$

As the special case of the Corollary 3.2, $\lambda = 1$ and $u = 1$, we have

$$\widehat{Ch}_{n,q} = \sum_{n=0}^{m}\sum_{l=0}^{n}\frac{(-1)^l}{n!}S_1(m,n)S_1(n,l)E_{l,q}$$

and

$$\begin{aligned}E_{m,q} =&(-1)^{-m}\sum_{n=0}^{m}\widehat{Ch}_{n,q}S_2(m,n)\\ =&\sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^l S_1(n,k)S_1(k,l)S_2(m,n)E_{l,q}.\end{aligned}$$

For $k \in \mathbb{N}$, we define the *degenerate λ-q-Changhee polynomials of the second kind with order k*:

$$\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u) = \sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1-\cdots-\lambda x_k+x)_m d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{3.9}$$

From (3.9), we can derive the generating function of

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1-\cdots-\lambda x_k+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\frac{t^n}{n!}\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda x_1-\cdots-\lambda x_k+x}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ =&\left(\frac{1+q}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}+1}\right)^k\left(1+\frac{1}{u}\log(1+ut)\right)^x.\end{aligned} \tag{3.10}$$

Replacing t by $\frac{1}{u}(e^{ut}-1)$ in (3.10), we get

$$\begin{aligned}&\left(\frac{q+1}{q(1+t)^{-\lambda}+1}\right)^k(1+t)^x\\ =&\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{\left(\frac{1}{u}(e^{ut}-1)\right)^n}{n!}\\ =&\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}u^{-n}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{(ut)^l}{l!}\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda,q}^{(k)}(x|u)u^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.11}$$

On the other hands, by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (3.10), we have

$$\begin{aligned}
&\left(\frac{q+1}{qe^{-\lambda t}+1}\right)^k e^{tx} \\
&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n \\
&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)\frac{1}{n!}u^{-n}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{u^l}{l!}\left(e^t-1\right)^l \\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda,q}^{(k)}(x|u)u^{n-m}S_2(n,m)\frac{(e^t-1)^n}{n!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\widehat{Ch}_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\left(\frac{q+1}{qe^{-\lambda t}+1}\right)^k e^{tx} &= \left(\frac{q+1}{qe^{-\lambda t}+1}\right)^k e^{-\lambda t\left(-\frac{\lambda}{x}\right)} \\
&=\sum_{n=0}^{\infty}(-\lambda)^n E_{n,q}^{(k)}\left(-\frac{x}{\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.13}$$

By (1.1) and (3.9), we get

$$\begin{aligned}
&\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u) \\
&=\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1-\cdots-\lambda x_k+x)_m d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}u^{n-m}S_1(n,m)S_1(m,l)(-\lambda)^l E_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned} \tag{3.14}$$

Hence, by (3.11), (3.12), (3.13) and (3.14), we obtain the following theorem.

Theorem 3.3. *For $n\geq 0$, we have*

$$\widehat{Ch}_{n,\lambda,q}^{(k)}(x|u)=\sum_{m=0}^{n}\sum_{l=0}^{m}u^{n-m}S_1(n,m)S_1(m,l)(-\lambda)^l E_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right),$$

and

$$Ch_{n,-\lambda,q}^{(k)}(x)=\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda,q}^{(k)}(x|u)u^{n-m}S_2(n,m).$$

In addition,

$$\begin{aligned}
(-\lambda)^n E_{n,q}^{(k)}\left(-\frac{x}{\lambda}\right) &= \sum_{m=0}^{n}\sum_{l=0}^{m}\widehat{Ch}_{l,\lambda,q}^{(k)}(x|u)u^{m-l}S_2(m,l)S_2(n,m) \\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\sum_{s=0}^{r}u^{m-r}S_1(l,r)S_1(r,s)S_2(m,l)S_2(n,m)(-\lambda)^s E_{s,q}^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned}$$

As the special case of Theorem 3.3, if we put $x = 0$, $\lambda = 1$ and $u = 1$, then

$$\widehat{Ch}_{n,q}^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) S_1(m,l) (-\lambda)^l E_{l,q}^{(k)},$$

and

$$\begin{aligned} E_{n,q}^{(k)} =& (-1)^n \sum_{m=0}^{n} \sum_{l=0}^{m} \widehat{Ch}_{l,\lambda,q}^{(k)} S_2(m,l) S_2(n,m) \\ =& \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{r=0}^{l} \sum_{s=0}^{r} S_1(l,r) S_1(r,s) S_2(m,l) S_2(n,m) (-1)^{n+s} E_{s,q}^{(k)}. \end{aligned}$$

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[1] Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea.
E-mail address: pitt0202@hanmail.net

[2] Department of Mathematics education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: sanklee@daegu.ac.kr

[3] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: a0417001@knu.ac.kr

Existence of Generalized Solutions for Fuzzy Impulsive Retarded Differential Equations*

Qiang Ma[1], Yabin Shao [2], † Zizhong Chen[3]

1. Network Information Management Center

Northwest University for Nationalities, Lanzhou 730030, Gansu, P.R. China

2. School of Science

Chongqing University of Posts and Telecommunications

Nan'an 400065, Chongqing, P. R. China

3. School of Computer Science

Chongqing University of Posts and Telecommunications

Nan'an 400065, Chongqing, P. R. China

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Abstract

In this paper, we study the existence of solution for fuzzy initial valued problems of fuzzy impulsive retarded differential equations in the setting of a generalized Hukuhara derivative and by using the Strong Lusin Condition of fuzzy Henstock integrable functions.

Keywords: Fuzzy number; Strong Lusin Condition; Discontinuous impulsive retarded differential equations; Fuzzy Henstock integrals.

1 INTRODUCTION

It is known that the theory of retarded functional differential equations has been well known when the right side function is continuous, hence Riemann integral. Hale [11] prove that the results still hold true when continuity of right function is weakened to satisfaction of a Carathéodory condition. The further step of generalisation was done in [3] and [22] which applies the Henstock integrals to the study of retarded functional differential equations with finite delays and unbounded delays. By using generalized differential equation theory [16], M. Federson and P. Táboas [6] proved that a local flow can be constructed for a

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†Corresponding author. E-mail: yb-shao@163.com(Y. Shao)

general class of non-autonomous retarded functional differential equations. On the other hand, it is meaningful to study the fuzzy retarded function differential. In [19], Lupulescu applied a successive approximation method to discuss the fuzzy differential equations with distributed delays. Guo et al. [10] discussed the oscillation properties of a class of fuzzy delay differential equation of second order and provided an oscillation criterion. In [15], Kloeden and Lorenz removed the assumption of fuzzy convexity of fuzzy differential equations and discussed the fuzzy delay differential equations in this perspective. In [14], Khastan et al. provided sufficient conditions for the global existence of a unique (ii)-solution to an initial value problem for fuzzy functional differential equations using generalized derivative and were of broader applicability than those using Hukuhara derivative.

However, there are discontinuous systems in which the right-hand side functions $\tilde{f} : [a, b] \times E^n \to E^n$ are not integrable in the sense of Kaleva [13] on certain intervals and their solutions are not absolute continuous functions. Recently, Wu and Gong [25, 26] have combined the fuzzy set theory [28] and nonabsolute integration theory [12], and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva[13] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, Gong and Shao [7, 8] have defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem. So, in [20, 23, 24], the authors used the strong fuzzy Henstock integrals [8], and deal with the Cauchy problem of discontinuous fuzzy systems. In this paper, we use the fuzzy Henstock integral to establish the existence of generalized solutions as well as continuous dependence on the interval conditions of fuzzy impulsive retarded differential equations

$$\begin{cases} x'(t) = \tilde{f}(t, x_t), & t \neq t_k, \\ \triangle\, x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots\cdots, m, \\ x_{t_0} = \phi, \quad x(t_0) = x_0 \end{cases} \tag{1}$$

where $\tilde{f} : I \times FH([-r, 0], \mathbb{R}_{\mathcal{F}}) \to \mathbb{R}_{\mathcal{F}}$, and $I_k : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ are give fuzzy mapping, $\phi \in FH([-r, 0], \mathbb{R}_{\mathcal{F}})$ and $x_0 \in \mathbb{R}_{\mathcal{F}}$.

To make our analysis possible, in section 2, we will first recall some basic results of fuzzy numbers and given the definition of Strong Lusin Condition of fuzzy-number-valued functions. Under this notion, we give another look at the fundamental theorem of calculus of fuzzy Henstock integrals. In section 3, we deal with the Cauchy problem of discontinuous fuzzy impulsive retarded differential equations. And in section 4, we present some concluding remarks.

2 PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and

B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric [5]:

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel b-a \parallel\}.$$

Denote $E^n = \{u : R^n \to [0,1] | u \text{ satisfies (1)-(4) below}\}$ is a fuzzy number space. where

(1)u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2)u is fuzzy convex, i.e. $u(\lambda x+(1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3)u is upper semi-continuous,

(4)$[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [5]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy see that D is a metric in E^n. Using the results [4], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(3)$D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

A fuzzy-number-valued function $f : [a,b] \to E^n$ is said to satisfy the condition (H) on $[a,b]$, if for any $x_1 < x_2 \in [a,b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call u is the H-difference of $f(x_2)$ and $f(x_1)$, denoted $f(x_2) -_H f(x_1)$ ([13]).

For brevity, we always assume that it satisfies the condition (H) when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

In this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [1] and [2].

Definition 1 ([1]) *Let $\tilde{f} : (a,b) \to E^n$ and $x_0 \in (a,b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that*

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0+h) -_H \tilde{f}(x_0), \tilde{f}(x_0) -_H \tilde{f}(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

Definition 2 ([18]) *Let $\delta(\xi)$ be a positive real function on a closed set $[a, b]$. A division $P = \{(\xi_i, [x_{i-1}, x_i])\}$ is said to be δ-fine, if the following conditions are satisfied:*

(1) $a = x_1 < x_2 < \cdots < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

Definition 3 ([25, 26]) *A fuzzy-number-valued function $\tilde{f}$ is said to be Henstock integrable on $[a, b]$ if there exists a fuzzy number $\tilde{A}$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u, v], \xi\}$ of $[a, b]$, we have*

$$D\bigg(\sum \tilde{f}(\xi)(v - u), \tilde{A}\bigg) < \varepsilon.$$

We write $(FH)\int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{A}$ and $\tilde{f} \in FH[a, b]$.

Remark 1 *If the fuzzy-number-valued function $\tilde{f}$ and fuzzy number $\tilde{A}$ are replaced by a real valued function and real number, respectively, then the real valued function f is said to be Henstock integrable on $[a, b]$ and we write $f \in H[a, b]$.*

Remark 2 *When the function $\delta : [a, b] \to R^+$ is constant, then we obtain the Riemann integrability for fuzzy-number-valued functions. In this case, $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ is called the Riemann integral of $\tilde{f}$ on $[a, b]$, beingdenoted by $(FR)\int_a^b \tilde{f}(x)\mathrm{d}x$. Consequently, the fuzzy Riemann integrability is a particular case of the fuzzy Henstock integrability.*

Definition 4 *A fuzzy-number-valued function $\tilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to satisfy Strong Lusin Condition, that is, $\tilde{f} \in SL([a, b], \mathbb{R}_{\mathcal{F}})$, if for every $\varepsilon > 0$ and $E \subset [a, b]$ with $|E| = 0$, there exists a $\delta-$fine division of $[a, b]$ with $I_i = [t_{i-1}, t_i]$ and $\xi_i \in E$ such that*

$$\sum_i D(\tilde{f}(t_i), \tilde{f}(t_{i-1})) < \varepsilon.$$

Remark 3 *If $AC([a, b], \mathbb{R}_{\mathcal{F}})$ denotes the space of all absolutely continuous functions, the inclusions $AC([a, b], \mathbb{R}_{\mathcal{F}}) \subset SL([a, b], \mathbb{R}_{\mathcal{F}}) \subset C([a, b], \mathbb{R}_{\mathcal{F}})$ hold.*

Theorem 1 ([9]) *Let $\tilde{f} \in AC([a, b], \mathbb{R}_{\mathcal{F}})$ be a fuzzy-number-valued function. Then there exists a absolutely Henstock integrable (i.e. Kaleva integrable) fuzzy-number-valued function $\tilde{f}$ such that*

$$\tilde{F}(x) = (K)\int_a^x \tilde{f}(t)\mathrm{d}t + \tilde{F}(a)$$

for any $x \in [a, b]$.

Now, by Theorem 1 and Definition 4 (Strong Lusin Condition), we give the fundamental theorem of calculus of fuzzy Henstock integrals.

Theorem 2 *For any $\tilde{f} \in SL([a,b],\mathbb{R}_{\mathcal{F}})$ $(GH)-$differentiable a.e., and $\tilde{F}' = \tilde{f}$ a.e. Then $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$ and*

$$\tilde{F}(x) = (FH)\int_a^x \tilde{f}(t)\mathrm{d}t + \tilde{F}(a)$$

or

$$\tilde{F}(x) = (FH)\int_a^x \tilde{f}(t)\mathrm{d}t \ominus_H (-1)\cdot\tilde{F}(a)$$

for any $x \in [a,b]$.

Conversely, if $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$, then $\tilde{f} \in SL([a,b],\mathbb{R}_{\mathcal{F}})$ and there exists $\tilde{F}' = \tilde{f}$ for almost every $t \in [a,b]$.

Theorem 3 *If $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$, then there exists a sequence of closed sets $X_i \subset X_{i+1} \subset [a,b]$, for every $i \in \mathbb{N}$ and $\cup X_i = [a,b]$ such that $\tilde{f} \in K([a,b],\mathbb{R}_{\mathcal{F}})$, and*

$$\lim_{i\to\infty}(K)\int_{X_i\cap[a,x]} \tilde{f}(t)\mathrm{d}t = (FH)\int_a^x \tilde{f}(t)\mathrm{d}t$$

uniformly for $x \in [a,b]$.

By the definition of improper fuzzy Riemann integral in [27], we can also give the proposition of the improper fuzzy Henstock integral.

Proposition 1 *If $\tilde{f} \in FH([a,c),\mathbb{R}_{\mathcal{F}})$ for every $c \in [a,b)$, then $\tilde{f} \in FH([a,b],\mathbb{R}_{\mathcal{F}})$ and*

$$\int_a^b \tilde{f}(t)\mathrm{d}t = \lim_{c\to b}\int_a^c \tilde{f}(t)\mathrm{d}t.$$

3 MAIN RESULTS

We start this section by defining some basic concept. Given a fuzzy-number-valued function $x : [t_0, t_0+a] \to \mathbb{R}_{\mathcal{F}}$, let $\triangle\, x(t_k)$ denote the jump of $x(t)$ at $t = t_k$ for $k = 1, 2, \cdots\cdots, m$, where $t_0 < t_1 < \cdots\cdots < t_k \cdots\cdots < t_m \le t_0 + a$, that is $\triangle\, x(t_k) = x(t_k+) \ominus_H x(t_k-) = I_k(x(t_k))$ is the $(H)-$difference, and $I_k(x(t_k)) : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$.

Definition 5 *Denote C_{t_0} is the set of all functions $x : [t_0, t_0+a] \to \mathbb{R}_{\mathcal{F}}$ such that*

(C1) x is continuous at $t \neq t_k, k = 1, 2, \cdots\cdots, m$;

(C2) x is left continuous at $t = t_k, k = 1, 2, \cdots\cdots, m$;

(C3) there exists the right limit $x(t_k+)$ for $k = 1, 2, \cdots\cdots, m$.

The same as above, let SL_{t_0} (respectively AC_{t_0}) be the set of all functions $x : [t_0 - r, t_0 + a] \to \mathbb{R}_{\mathcal{F}}$ such that $x \in SL([I, \mathbb{R}_{\mathcal{F}}])$ (resp. $x \in \mathbb{R}_{\mathcal{F}}$) for every close interval $I \subset (t_k, t_{k+1})$ and such (C1) and (C3) holds.

Let $r > 0, x_0 \in \mathbb{R}_{\mathcal{F}}$ and $\phi \in FH([-r, 0], \mathbb{R}_{\mathcal{F}})$. We consider the set:

$$C_{\phi,x_0} = \{x : [t_0 - r, t_0 + a] \to \mathbb{R}_{\mathcal{F}} | x(t_0) = x_0; x_{t_0} = \phi; x|_{[t_0,t_0+a]} \in C_{t_0}\}$$
$$SL_{\phi,x_0} = \{x \in C_{\phi,x_0} : x|_{[t_0,t_0+a]} \in SL_{t_0}\}$$
$$AC_{\phi,x_0} = \{x \in C_{\phi,x_0} : x|_{[t_0,t_0+a]} \in AC_{t_0}\}$$

Definition 6 *Consider a function $x(t) = x(t, t_0, x_0, \phi)$ and the following conditions:*

(i) $x \in C_{\phi,x_0}$*;*

(ii) $(t, x_t) \in [t_0, t_0 + a] \times FH([-r, 0], \mathbb{R}_{\mathcal{F}})$*, a.e.;*

(iii) $x(t) = x(s) + \int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$ *for every* $t_{k-1} < s \le t \le t_k$ *and* $t_m < s \le t \le t_0 + a, k = 1, 2 \cdots, m$*;*

(iv) $x(t) = x(s) \ominus_H (-\int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma)$ *for every* $t_{k-1} < s \le t \le t_k$ *and* $t_m < s \le t \le t_0 + a, k = 1, 2 \cdots, m$*;*

(v) $I_(k)(x(t_k)) = x(t_k+) \ominus_H x(t_k), k = 1, 2 \cdots, m.$

If x satisfies conditions (i)-(iii) and (v), we say that it is called to be a generalized (i)-solution of problem (5) through (t_0, x_0, ϕ). If x satisfies conditions (i),(ii) (iv) and (v), we say that it is called to be a generalized (ii)-solution of problem (5) through (t_0, x_0, ϕ).

Lemma 1 *If x is a generalized (i)-solution of problem (5), then x satisfies the following integral equation*

$$\begin{cases} x(t) = x_0 + \int_{t_0}^t \tilde{f}(s, x_s)\mathrm{d}s + \sum\limits_{t_0 < t_k < t} [x(t_k+) \ominus_H x(t_k)], \\ x_{t_0} = \phi, \quad x(t_0) = x_0, \end{cases} \tag{2}$$

where $t \in [t_0, t_0 + a]$.

Proof. Let $k \in \{0, 1, 2, \cdots, m\}$ and $t_k < s \le u \le t_{k+1}$. By Theorem 53, $x(u) = x(s) + \int_s^u \tilde{f}(t, x_t)\mathrm{d}t$. By Definition 57, we have $x'(t) = \tilde{f}(t, x_t)$ for $t_k < t \le t_{k+1}$. Hence $x(u) = x(s) + \int_s^u x'(t)\mathrm{d}t$. By Proposition 55, for $t_k < t \le t_{k+1}$, there exists $\int_{t_k}^t x'(\sigma)\mathrm{d}\sigma = \lim\limits_{s \to t_k} \int_s^t x'(\sigma)\mathrm{d}\sigma = x(t) \ominus_H x(t_k+)$. Thus, for $t \in (t_0, t_0 + a]$,

there exists $k \in \{0, 1, 2, \cdots, m\}$ such that $t_k < t \leq t_{k+1}$ and we have

$$\begin{aligned}\int_{t_0}^{t} x'(\sigma)\mathrm{d}\sigma &= \int_{t_0}^{t_1} x'(\sigma)\mathrm{d}\sigma + \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} x'(\sigma)\mathrm{d}\sigma \\ &\quad + \int_{t_k}^{t} x'(\sigma)\mathrm{d}\sigma \\ &= x(t_1) \ominus_H [x_0 + \sum_{i=1}^{k-1}(x(t_{i+1}) \ominus_H x(t_i+))] + [x(t) \ominus_H x(t_k+)] \\ &= x(t) \ominus_H [x_0 + \sum_{i=1}^{k-1}(x(t_i+) \ominus_H x(t_i))].\end{aligned}$$

Therefore, we have

$$\begin{aligned}x(t) &= x_0 + \int_{t_0}^{t} x'(\sigma)\mathrm{d}\sigma + \sum_{t_0<t_k<t}(x(t_k+) \ominus_H x(t_k)) \\ &= x_0 + \int_{t_0}^{t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma + \sum_{t_0<t_k<t}(x(t_k+) \ominus_H x(t_k)).\end{aligned}$$

Since $x'(t) = \tilde{f}(t, x_t)$, for $t_k < t \leq t_{k+1}$ and $k \in \{0, 1, 2, \cdots, m\}$, and due to Theorem 2.8 in [26]. The proof is complete.

Lemma 2 *If x is a generalized (ii)-solution of problem (5), then x satisfies the following integral equation*

$$\begin{cases} x(t) = x_0 \ominus_H (-\int_{t_0}^{t} \tilde{f}(s, x_s)\mathrm{d}s) + \sum\limits_{t_0<t_k<t}[x(t_k+) \ominus_H x(t_k)], \\ x_{t_0} = \phi, \quad x(t_0) = x_0, \end{cases} \tag{3}$$

where $t \in [t_0, t_0 + a]$.

Proof. The proof of this result is analogous to that of Lemma 1.

Lemma 3 *If x is a generalized (i)-solution of the problem (6), the x is a generalized (i)-solution of the problem (5) and $x \in SL_{\phi,x_0}$.*

Proof. Given $\varepsilon > 0$ then

$$x(t_k + \varepsilon) = x(t_k) + \int_{t_k}^{t_k+\varepsilon} \tilde{f}(s, x_s)\mathrm{d}s + I_k(x(t_k))$$

which tend to $I_k(x(t_k))$ as $\varepsilon \to 0$. By hypothesis, given $x \in C_{\phi,x_0}$, and the fuzzy Henstock integral $\int_{t_0,t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$ exists, for every $t \in [t_0, t_0 + a]$. By Theorem 53, we have $\int_{t_0}^{t_0+a} \tilde{f}(t, x_t)\mathrm{d}t \in SL([[t_0, a], \mathbb{R}_{\mathcal{F}})$ and $x'(t) = \mathrm{d}/\mathrm{d}t \int_{t_0}^{t} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma = \tilde{f}(t, x_t)$ a.e. on $[t_0, a]$. That is, $x(t) = x(s) + \int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$.

Now, we shall prove that $x(t_{k+1}) = x(s) + \int_s^{t_{k+1}} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma$ for $s \in (t_k, t_{k+1}]$. By Proposition 55, we have

$$\begin{aligned}\int_s^{t_{k+1}} \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma &= \lim_{t\to t_{k+1}} \int_s^t \tilde{f}(\sigma, x_\sigma)\mathrm{d}\sigma = \lim_{t\to t_{k+1}} D(x(s), x(t)) \\ &= x(t_{k+1}-) \ominus_H x(s) = x(t_{k+1}) \ominus_H x(s)\end{aligned}$$

This completed the proof.

Lemma 4 *If x is a generalized (ii)-solution of the problem (7), the x is a generalized (ii)-solution of the problem (5) and $x \in SL_{\phi,x_0}$.*

Theorem 4 *Let T be the operator given by*

$$Tx(t) = x_0 + \int_{t_0}^t \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_k<t} [x(t_k+) \ominus_H x(t_k)]$$

for $t \in [t_0.t_0 + a]$. Then T maps C_{ϕ,x_0} into C_{ϕ,x_0}.

Proof. Let $x : [t_0 - r, t_0 + a] \to \mathbb{R}_{\mathcal{F}}$ be such that $x \in C_{\phi,x_0}$. Then $Tx(t_0) = x_0$. Given $\varepsilon > 0$, we have $Tx(t)$ is continuous for $t \in [t_0-r, t_0+a]\backslash\{t_1, t_2, \cdots\cdots, t_m\}$. On the other hand, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0} Tx(t_k - \varepsilon) \\ =\ &\lim_{\varepsilon\to 0}\left[x_0 + \int_{t_0}^{t_k-\varepsilon} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i<t_k-\varepsilon} [x(t_i+) \ominus_H x(t_i)]\right] \\ =\ &x_0 + \int_{t_0}^{t_k} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i<t_k} [x(t_i+) \ominus_H x(t_i)] \\ =\ &Tx(t_k).\end{aligned}$$

Hence, $Tx(t_k)$ is left continuous for $k = 1, 2, \cdots, m$. Finally, we consider

$$\begin{aligned}&\lim_{\varepsilon\to 0} Tx(t_k + \varepsilon) \\ =\ &\lim_{\varepsilon\to 0}\left[x_0 + \int_{t_0}^{t_k+\varepsilon} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i\le t_k+\varepsilon} [x(t_i+) \ominus_H x(t_i)]\right] \\ =\ &x_0 + \int_{t_0}^{t_k} \tilde{f}(s, x_s)\mathrm{d}s + \sum_{t_0<t_i\le t_k} [x(t_i+) \ominus_H x(t_i)] \\ =\ &Tx(t_k) + [x(t_k+) \ominus_H x(t_k)].\end{aligned}$$

and hence there exists $Tx(t_k+)$ for $k = 1, 2, \cdots, m$.

Theorem 5 *Suppose that $\tilde{f}(t, x_t)$ is fuzzy Henstock integrable for any $x \in C_{\phi,x_0}$. If there exist positive number M and $a_1, a_2, \cdots, a_m$ such that*

(i) $D(\tilde{f}(t,\varphi_1),\tilde{f}(t,\varphi_2)) \leq M \cdot D(\varphi_1,\varphi_2)$ *for every* $t \in [t_0, t_0+a]$ *and every* $\varphi_1,\varphi_2 \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$*;*

(ii) $D(I_k(u),I_k(v)) \leq a_k \cdot D(u,v)$ *for every* $u,v \in \mathbb{R}_{\mathcal{F}}$ *and* $k \in \{1,2,\cdots,m\}$*;*

(iii) $a^2 \cdot M + \sum_{k=1}^{m} a_k < 1$*;*

then there exist two generalized solutions $x(t) = x(t,x_0,\phi)$ *of problem (1),*$x(\cdot,x_0,\phi) \in SL_{\phi,x_0}$ *depending continously on each variable.*

Proof. Let $r,a > 0$ and $F = \{y \in C_0 : y(0) = \tilde{0}\}$. For y^1, $y^2 \in F$, we define the distence

$$H(y^1_{\phi,x_0}, y^2_{\phi,x_0}) = \sup_{t\in[0,a]} D(y^1_{\phi,x_0}, y^2_{\phi,x_0}).$$

For any $y \in F, x_0 \in \mathbb{R}_{\mathcal{F}}$ and $\phi \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$ we define an auxiliary fuzzy-number-valued function $y_{\phi,x_0} : [-r,a] \to \mathbb{R}_{\mathcal{F}}$ by

$$y_{\phi,x_0}(t) = \begin{cases} y(t), & t \in [0,a] \\ \phi(t) \ominus_H x_0, & x \in [-r,0]. \end{cases}$$

Then, $(y_{\phi,x_0})_t \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$ for all $t \in [0,a]$ and $(y_{\phi,x_0})_t(\theta) = y_{\phi,x_0}(t+\theta), \theta \in [-r,0]$.

We consider the family

$$\mathcal{U}_F = \{U_{\phi,x_0} : \phi \in FH([-r,0],\mathbb{R}_{\mathcal{F}}), x_0 \in \mathbb{R}_{\mathcal{F}}\}$$

of operators from F to C_0 given by

$$U_{\phi,x_0}y(t) = \int_{t_0}^{t+t_0} \tilde{f}(s,(y_{\phi,x_0})_{s-t_0} + x_0)\mathrm{d}s + \sum_{t_0<t_k<t+t_0} I_k(y(t_k - t_0))$$

for all $t \in [0,a]$.

Firstly, we prove that $\mathcal{U}_F$ is a contraction operators. In fact, by using hypothesis (i) and (ii)

$$\begin{aligned} & H(U_{\phi,x_0}y^1(t), U_{\phi,x_0}y^2(t)) = \sup_{t\in[0,a]} D(U_{\phi,x_0}y^1(t), U_{\phi,x_0}y^2(t)) \\ = & \sup_{t\in[0,a]} D(\int_{t_0}^{t+t_0} \tilde{f}(s,(y^1_{\phi,x_0})_{s-t_0} + x_0)\mathrm{d}s, \int_{t_0}^{t+t_0} \tilde{f}(s,(y^2_{\phi,x_0})_{s-t_0} + x_0)\mathrm{d}s) \\ + & \sum_{t_0<t_k<t+t_0} I_k(y^1(t_k - t_0) \ominus_H I_k(y^2(t_k - t_0)) \\ \leq & \sup_{t\in[0,a]} D(\int_{t_0}^{t+t_0} \tilde{f}(s,(y^1_{\phi,x_0})_{s-t_0})\mathrm{d}s, \int_{t_0}^{t+t_0} \tilde{f}(s,(y^2_{\phi,x_0})_{s-t_0})\mathrm{d}s) \\ \leq & M \cdot ((L)\int_{t_0}^{a+t_0} D((y^1_{\phi,x_0})_{s-t_0},(y^2_{\phi,x_0})_{s-t_0})\mathrm{d}s) + \sum_{t_0<t_k<t+t_0} a_k D(y^1,y^2). \end{aligned}$$

In addition, since $(y^1_{\phi,x_0})_{s-t_0}, (y^2_{\phi,x_0})_{s-t_0} \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$, we have

$$\begin{aligned} & D(y^1_{\phi,x_0})_{s-t_0}, (y^2_{\phi,x_0})_{s-t_0}) \\ = & \sup_{-v\in[-r,0]} D(\int_{-r}^{-v}(y^1_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta, \int_{-r}^{-v}(y^2_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta). \end{aligned}$$

Thus, we have

$$\begin{aligned} & D(\int_{-r}^{-v}(y^1_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta, \int_{-r}^{-v}(y^2_{\phi,x_0})_{s-t_0}(\theta)\mathrm{d}\theta) \\ = & D(\int_{t_0-s}^{-v} y^1(s-t_0+\theta)\mathrm{d}\theta, \int_{t_0-s}^{-v} y^2(s-t_0+\theta)\mathrm{d}\theta) \\ \le & (L)\int_0^{-v+s-t_0} D(y^1(u),y^2(u))\mathrm{d}u \\ \le & (-v+s-t_0)D(y^1,y^2) \le aD(y^1,y^2). \end{aligned}$$

This implies that $\mathcal{U}_F$ is a contraction operators.

Next, we shall prove $\mathcal{U}_F$ is a continuous at x_0 and ϕ. Given $\varepsilon > 0$, let $\phi_1, \phi_2 \in FH([-r,0],\mathbb{R}_{\mathcal{F}})$ and $0 < \delta \le \varepsilon/2$ be such that $D(\phi_1,\phi_2) < \delta$. Then, by hypothesis (i), we have

$$\begin{aligned} & H(T_{\phi_1,x_0}y(t), T_{\phi_2,x_0}y(t)) = \sup_{t\in[0,a]} D(T_{\phi_1,x_0}y(t), T_{\phi_2,x_0}y(t)) \\ = & D(\int_{t_0}^{t} +t_0\tilde{f}(s,(y_{\phi_1,x_0})_{s-t_0}+x_0)\mathrm{d}s, \int_{t_0}^{t} +t_0\tilde{f}(s,(y_{\phi_2,x_0})_{s-t_0}+x_0)\mathrm{d}s) \\ \le & M\cdot(L)\int_{t_0}^{a} +t_0 D((y_{\phi_1,x_0})_{s-t_0},(y_{\phi_2,x_0})_{s-t_0})\mathrm{d}s. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & D(\int_{-r}^{-v}(y_{\phi_1,x_0})_{s-t_0}(\theta)\mathrm{d}\theta, \int_{-r}^{-v}(y_{\phi_2,x_0})_{s-t_0}(\theta)\mathrm{d}\theta) \\ = & D(\int_{-r}^{t_0-s}\phi_1(s-t_0+\theta)\mathrm{d}\theta, \int_{-r}^{t_0-s}\phi_2(s-t_0+\theta)\mathrm{d}\theta) \\ = & D(\int_{-r-t_0+s}^{0}\phi_1(u)\mathrm{d}u, \int_{-r-t_0+s}^{0}\phi_2(u)\mathrm{d}u) \\ \le & D(\int_{-r}^{0}\phi_1(u)\mathrm{d}u, \int_{-r}^{0}\phi_2(u)\mathrm{d}u) + D(\int_{-r}^{-r-t_0+s}\phi_1(u)\mathrm{d}u, \int_{-r}^{-r-t_0+s}\phi_2(u)\mathrm{d}u) \\ \le & 2\cdot\sup_{-v\in[-r,0]} D(\int_{-r}^{-v}\phi_1(u)\mathrm{d}u, \int_{-r}^{-r}\phi_2(u)\mathrm{d}u) < \varepsilon. \end{aligned}$$

At last, since $FH([-r,0],\mathbb{R}_{\mathcal{F}})\times\mathbb{R}_{\mathcal{F}}$ is a complete space and by using Lemma 3, we conclude that $x(\cdot,x_0,\phi) \in SL_{\phi,x_0}$ and it is a generalized solutions $x(t) = x(t,x_0,\phi)$ of problem (1).

4 CONCLUSIONS

In this paper, we study the Cauchy problem of discontinuous fuzzy impulsive retarded differential equations involving the fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrablility. Our result improves the result given in Ref. [14] and [17] (where uniform continuity was required), as well as those referred therein.

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PRODUCT-TYPE SYSTEM OF DIFFERENCE EQUATIONS WITH MULTIPLICATIVE COEFFICIENTS SOLVABLE IN CLOSED FORM

STEVO STEVIĆ*, BRATISLAV IRIČANIN, AND ZDENĚK ŠMARDA

Abstract. Solvability of the following system of difference equations

$$z_{n+1} = \alpha z_n^a w_{n-1}^b, \quad w_{n+1} = \beta w_n^c z_{n-1}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$, $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, is studied.

1. Introduction

Numerous concrete nonlinear difference equations and systems have attracted some recent attention (see, e.g., [1]-[6], [8], [9], [13]-[49]). There has been also a renewed interest in the problem of their solvability (see, e.g., [1]-[4], [6], [17], [20], [21], [25]-[41], [43]-[49]), especially after the publication of [20] where S. Stević used a nice change of variables for explaining and extending the formula for solutions to the second-order difference equation in [6]. This motivated several authors to develop the idea and apply it for the case of some extensions of the equation, as well as some other equations and systems (see, e.g., [1], [3], [4], [17], [21], [25]-[30], [32]-[36], [39], [41], [43]-[47], [49]). Some classical methods for solving difference equations and systems can be found, e.g., in [7], [10]-[12]. On the other hand, there has been some recent interest in close to symmetric systems of difference equations (see, e.g., [4], [13]-[15], [18], [19], [25], [27]-[29], [31]-[44], [46], [48], [49]). Many of the above mentioned classes of difference equations and systems are obtained from the product-type ones by the translation or max-type operators (see, e.g., [16], [22]-[24], [42]). Some max-type systems are even solvable ([31]).

Product-type equations are solvable if the initial values and coefficients are positive, which is not the case in general. This suggests investigation of product-type equations and systems for the case when

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*Corresponding author.

the coefficients and initial values are not positive. The problem was studied for the first time in [40] (an extension of the system in [40] was later studied in [38]), but with methods not so close to those in [20] (or in [26], [30] etc.). They are, in fact, more related to the ones in [22], [23] and [42]. In [48] was investigated the problem of solvability of the system

$$z_{n+1}=\frac{z_n^{\hat{a}}}{w_{n-1}^{\hat{b}}},\quad w_{n+1}=\frac{w_n^{\hat{c}}}{z_{n-1}^{\hat{d}}},\quad n\in\mathbb{N}_0, \tag{1}$$

where $\hat{a},\hat{b},\hat{c},\hat{d}\in\mathbb{Z}$ (the condition is posed to avoid multi-valued solutions) and $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}\setminus\{0\}$. Soon after that, in [37], was shown that for the case of another product-type system some coefficients can be included to get again a solvable system.

Here we show that the same holds for the next natural extension of system (1)

$$z_{n+1}=\alpha z_n^a w_{n-1}^b,\quad w_{n+1}=\beta w_n^c z_{n-1}^d,\quad n\in\mathbb{N}_0, \tag{2}$$

where $a,b,c,d\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{C}$ and $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}$. Since the cases $\alpha=0$ and $\beta=0$ are trivial, we will assume that $\alpha,\beta\in\mathbb{C}\setminus\{0\}$.

Note that the domain of undefinable solutions ([32]) to system (2) is a subset of

$$\mathcal{U}=\{(z_{-1},z_0,w_{-1},w_0)\in\mathbb{C}^4: z_{-1}=0 \text{ or } z_0=0 \text{ or } w_{-1}=0 \text{ or } w_0=0\}.$$

This is why we will also assume that $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}\setminus\{0\}$.

2. Main result

The problem of solvability of system (2) is studied in this section.

Theorem 1. *Assume that $a,b,c,d\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ and $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}\setminus\{0\}$. Then system (2) is solvable in closed form.*

Proof. Case $b=0$. Since $b=0$ system (2) is

$$z_{n+1}=\alpha z_n^a,\quad w_{n+1}=\beta w_n^c z_{n-1}^d,\quad n\in\mathbb{N}_0. \tag{3}$$

The first equation in (3) yields

$$z_n=\alpha^{\sum_{j=0}^{n-1}a^j} z_0^{a^n},\quad n\in\mathbb{N}. \tag{4}$$

Hence, if $a\neq 1$ we have

$$z_n=\alpha^{\frac{1-a^n}{1-a}} z_0^{a^n},\quad n\in\mathbb{N}, \tag{5}$$

while if $a=1$ we have

$$z_n=\alpha^n z_0,\quad n\in\mathbb{N}. \tag{6}$$

Using (4) in the second equation in (3) we have

$$w_n = \beta\alpha^{d\sum_{j=0}^{n-3}a^j} z_0^{da^{n-2}} w_{n-1}^c, \tag{7}$$

for $n \geq 3$.

From (7) we get

$$\begin{aligned} w_n &= \beta\alpha^{d\sum_{j=0}^{n-3}a^j} z_0^{da^{n-2}} (\beta\alpha^{d\sum_{j=0}^{n-4}a^j} z_0^{da^{n-3}} w_{n-2}^c)^c \\ &= \beta^{1+c}\alpha^{d\sum_{j=0}^{n-3}a^j + dc\sum_{j=0}^{n-4}a^j} z_0^{da^{n-2}+dca^{n-3}} w_{n-2}^{c^2}, \end{aligned} \tag{8}$$

for $n \geq 4$.

Equalities (7) and (8) suggest that for a $k \in \mathbb{N}$

$$w_n = \beta^{\sum_{i=0}^{k-1}c^i}\alpha^{d\sum_{i=0}^{k-1}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\sum_{i=0}^{k-1}c^i a^{n-i-2}} w_{n-k}^{c^k}, \tag{9}$$

for $n \geq k+2$. Assume that (9) holds for some $k \in \mathbb{N}$ and every $n \geq k+2$.

Applying (7) with $n \to n-k$ in (9), it follows that

$$\begin{aligned} w_n &= \beta^{\sum_{i=0}^{k-1}c^i}\alpha^{d\sum_{i=0}^{k-1}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\sum_{i=0}^{k-1}c^i a^{n-i-2}} (\beta\alpha^{d\sum_{j=0}^{n-k-3}a^j} z_0^{da^{n-k-2}} w_{n-k-1}^c)^{c^k} \\ &= \beta^{\sum_{i=0}^{k}c^i}\alpha^{d\sum_{i=0}^{k}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\sum_{i=0}^{k}c^i a^{n-i-2}} w_{n-k-1}^{c^{k+1}}, \end{aligned} \tag{10}$$

for $n \geq k+3$.

Equalities (7), (10) along with the induction show that (9) holds for all natural numbers k and n such that $1 \leq k \leq n-2$.

For $k = n-2$ equality (9) becomes

$$w_n = \beta^{\sum_{i=0}^{n-3}c^i}\alpha^{d\sum_{i=0}^{n-3}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\sum_{i=0}^{n-3}c^i a^{n-i-2}} w_2^{c^{n-2}}, \tag{11}$$

for $n \geq 3$.

From (11) and since

$$w_2 = \beta w_1^c z_0^d = \beta(\beta w_0^c z_{-1}^d)^c z_0^d = \beta^{1+c} z_{-1}^{cd} z_0^d w_0^{c^2},$$

we have

$$\begin{aligned} w_n &= \beta^{\sum_{i=0}^{n-3}c^i}\alpha^{d\sum_{i=0}^{n-3}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\sum_{i=0}^{n-3}c^i a^{n-i-2}} (\beta^{1+c} z_{-1}^{cd} z_0^d w_0^{c^2})^{c^{n-2}} \\ &= \beta^{\sum_{i=0}^{n-1}c^i}\alpha^{d\sum_{i=0}^{n-3}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\sum_{i=0}^{n-2}c^i a^{n-i-2}} z_{-1}^{dc^{n-1}} w_0^{c^n}, \end{aligned} \tag{12}$$

for $n \geq 3$. A direct verification shows that (12) also holds for $n = 2$.

Case $a \neq c$. In this case from (12) we get

$$w_n = \beta^{\sum_{i=0}^{n-1}c^i}\alpha^{d\sum_{i=0}^{n-3}c^i\sum_{j=0}^{n-i-3}a^j} z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}} z_{-1}^{dc^{n-1}} w_0^{c^n}, \quad n \geq 2. \tag{13}$$

Subcase $a \neq 1 \neq c$. In this case we have

$$\begin{aligned} w_n =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\sum_{i=0}^{n-3}c^i\frac{1-a^{n-i-2}}{1-a}}z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\ =&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d}{1-a}\left(\frac{1-c^{n-2}}{1-c}-a\frac{a^{n-2}-c^{n-2}}{a-c}\right)}z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\ =&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d(a-c+(1-a)c^{n-1}+(c-1)a^{n-1})}{(1-a)(1-c)(a-c)}}z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}, \quad n\geq 2. \end{aligned} \tag{14}$$

Subcase $a\neq 1=c$. In this case we have

$$\begin{aligned} w_n =&\beta^n\alpha^{d\sum_{i=0}^{n-3}\frac{1-a^{n-i-2}}{1-a}}z_0^{d\frac{a^{n-1}-1}{a-1}}z_{-1}^{d}w_0\\ =&\beta^n\alpha^{\frac{d}{1-a}\left(n-2-a\frac{a^{n-2}-1}{a-1}\right)}z_0^{d\frac{a^{n-1}-1}{a-1}}z_{-1}^{d}w_0\\ =&\beta^n\alpha^{\frac{d(a^{n-1}-(n-1)a+n-2)}{(a-1)^2}}z_0^{d\frac{a^{n-1}-1}{a-1}}z_{-1}^{d}w_0, \quad n\geq 2. \end{aligned} \tag{15}$$

Subcase $a=1\neq c$. In this case we have

$$\begin{aligned} w_n =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\sum_{i=0}^{n-3}c^i(n-i-2)}z_0^{d\frac{1-c^{n-1}}{1-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\ =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\left((n-2)\frac{1-c^{n-2}}{1-c}-c\frac{1-(n-2)c^{n-3}+(n-3)c^{n-2}}{(1-c)^2}\right)}z_0^{d\frac{1-c^{n-1}}{1-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\ =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\frac{n-2-(n-1)c+c^{n-1}}{(1-c)^2}}z_0^{d\frac{1-c^{n-1}}{1-c}}z_{-1}^{dc^{n-1}}w_0^{c^n}, \quad n\geq 2. \end{aligned} \tag{16}$$

Case $a=c$. In this case from (12) we get

$$w_n=\beta^{\sum_{i=0}^{n-1}c^i}\alpha^{d\sum_{i=0}^{n-3}c^i\sum_{j=0}^{n-i-3}c^j}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}, \tag{17}$$

for $n\geq 2$.

Subcase $a=c\neq 1$. In this case we have

$$\begin{aligned} w_n =&\beta^{\frac{1-c^n}{1-c}}\alpha^{d\sum_{i=0}^{n-3}c^i\frac{1-c^{n-i-2}}{1-c}}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\ =&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d}{1-c}\left(\frac{1-c^{n-2}}{1-c}-(n-2)c^{n-2}\right)}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}\\ =&\beta^{\frac{1-c^n}{1-c}}\alpha^{\frac{d(1-(n-1)c^{n-2}+(n-2)c^{n-1})}{(1-c)^2}}z_0^{(n-1)dc^{n-2}}z_{-1}^{dc^{n-1}}w_0^{c^n}, \quad n\geq 2. \end{aligned} \tag{18}$$

Subcase $a=c=1$. In this case we have

$$\begin{aligned} w_n =&\beta^n\alpha^{d\sum_{i=0}^{n-3}(n-i-2)}z_0^{(n-1)d}z_{-1}^{d}w_0\\ =&\beta^n\alpha^{d\frac{(n-2)(n-1)}{2}}z_0^{(n-1)d}z_{-1}^{d}w_0, \quad n\geq 2. \end{aligned} \tag{19}$$

Case $d=0$. Since $d=0$ system (2) is

$$z_{n+1}=\alpha z_n^a w_{n-1}^b, \quad w_{n+1}=\beta w_n^c, \quad n\in\mathbb{N}_0. \tag{20}$$

Second equation in (20) yields

$$w_n=\beta^{\sum_{j=0}^{n-1}c^j}w_0^{c^n}, \quad n\in\mathbb{N}, \tag{21}$$

from which it follows that

$$w_n = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n}, \quad n \in \mathbb{N}, \tag{22}$$

if $c \neq 1$, while if $c = 1$, then

$$w_n = \beta^n w_0, \quad n \in \mathbb{N}. \tag{23}$$

Using (21) in the first equation in (20) we have

$$z_n = \alpha\beta^{b\sum_{j=0}^{n-3} c^j} w_0^{bc^{n-2}} z_{n-1}^a, \tag{24}$$

for $n \geq 3$.

From (24) we get

$$\begin{aligned} z_n &= \alpha\beta^{b\sum_{j=0}^{n-3} c^j} w_0^{bc^{n-2}} (\alpha\beta^{b\sum_{j=0}^{n-4} c^j} w_0^{bc^{n-3}} z_{n-2}^a)^a \\ &= \alpha^{1+a}\beta^{b\sum_{j=0}^{n-3} c^j + ba\sum_{j=0}^{n-4} c^j} w_0^{bc^{n-2}+bac^{n-3}} z_{n-2}^{a^2}, \end{aligned}$$

for $n \geq 4$.

Assume that for a $k \in \mathbb{N}$ we have proved

$$z_n = \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b\sum_{i=0}^{k-1} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{k-1} a^i c^{n-i-2}} z_{n-k}^{a^k}, \tag{25}$$

for $n \geq k+2$.

Applying (24) with $n \to n-k$ in (25), it follows that

$$\begin{aligned} z_n &= \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b\sum_{i=0}^{k-1} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{k-1} a^i c^{n-i-2}} (\alpha\beta^{b\sum_{j=0}^{n-k-3} c^j} w_0^{bc^{n-k-2}} z_{n-k-1}^a)^{a^k} \\ &= \alpha^{\sum_{i=0}^{k} a^i} \beta^{b\sum_{i=0}^{k} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{k} a^i c^{n-i-2}} z_{n-k-1}^{a^{k+1}}, \end{aligned} \tag{26}$$

for $n \geq k+3$.

Equalities (24), (26) along with the induction show that (25) holds for all natural numbers k and n such that $1 \leq k \leq n-2$.

For $k = n-2$ equality (25) becomes

$$z_n = \alpha^{\sum_{i=0}^{n-3} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{n-3} a^i c^{n-i-2}} z_2^{a^{n-2}}, \tag{27}$$

for $n \geq 3$.

From (27) and since

$$z_2 = \alpha z_1^a w_0^b = \alpha(\alpha z_0^a w_{-1}^b)^a w_0^b = \alpha^{1+a} w_{-1}^{ab} w_0^b z_0^{a^2},$$

we have

$$\begin{aligned} z_n &= \alpha^{\sum_{i=0}^{n-3} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{n-3} a^i c^{n-i-2}} (\alpha^{1+a} w_{-1}^{ab} w_0^b z_0^{a^2})^{a^{n-2}} \\ &= \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\sum_{i=0}^{n-2} a^i c^{n-i-2}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \end{aligned} \tag{28}$$

for $n \geq 3$. A direct verification shows that (28) also holds for $n = 2$.

Case $a \neq c$. In this case from (28) we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} c^j} w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \quad n \geq 2. \tag{29}$$

Subcase $a \neq 1 \neq c$. In this case we have

$$\begin{aligned} z_n =&\alpha^{\frac{1-a^n}{1-a}} \beta^{b\sum_{i=0}^{n-3} a^i \frac{1-c^{n-i-2}}{1-c}} w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =&\alpha^{\frac{1-a^n}{1-a}} \beta^{\frac{b}{1-c}\left(\frac{1-a^{n-2}}{1-a} - c\frac{a^{n-2}-c^{n-2}}{a-c}\right)} w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =&\alpha^{\frac{1-a^n}{1-a}} \beta^{\frac{b(a-c+(1-a)c^{n-1}+(c-1)a^{n-1})}{(1-b)(1-c)(a-c)}} w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \quad n \geq 2. \end{aligned} \tag{30}$$

Subcase $a \neq 1 = c$. In this case we have

$$\begin{aligned} z_n =&\alpha^{\frac{1-a^n}{1-a}} \beta^{b\sum_{i=0}^{n-3} a^i(n-i-2)} w_0^{b\frac{a^{n-1}-1}{a-1}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =&\alpha^{\frac{1-a^n}{1-a}} \beta^{b\left((n-2)\frac{1-a^{n-2}}{1-a} - a\frac{1-(n-2)a^{n-3}+(n-3)a^{n-2}}{(1-a)^2}\right)} w_0^{b\frac{a^{n-1}-1}{a-1}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =&\alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{n-2-(n-1)a+a^{n-1}}{(1-a)^2}} w_0^{b\frac{a^{n-1}-1}{a-1}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \quad n \geq 2. \end{aligned} \tag{31}$$

Subcase $a = 1 \neq c$. In this case we have

$$\begin{aligned} z_n =&\alpha^n \beta^{b\sum_{i=0}^{n-3} \frac{1-c^{n-i-2}}{1-c}} w_0^{b\frac{1-c^{n-1}}{1-c}} w_{-1}^{b} z_0 \\ =&\alpha^n \beta^{\frac{b}{1-c}\left((n-2) - c\frac{1-c^{n-2}}{1-c}\right)} w_0^{b\frac{1-c^{n-1}}{1-c}} w_{-1}^{b} z_0 \\ =&\alpha^n \beta^{b\frac{n-2-(n-1)c+c^{n-1}}{(1-c)^2}} w_0^{b\frac{1-c^{n-1}}{1-c}} w_{-1}^{b} z_0, \quad n \geq 2. \end{aligned} \tag{32}$$

Case $a = c$. In this case from (28) we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b\sum_{i=0}^{n-3} a^i \sum_{j=0}^{n-i-3} a^j} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \tag{33}$$

for $n \geq 2$.

Subcase $a = c \neq 1$. In this case we have

$$\begin{aligned} z_n =&\alpha^{\frac{1-a^n}{1-a}} \beta^{b\sum_{i=0}^{n-3} a^i \frac{1-a^{n-i-2}}{1-a}} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =&\alpha^{\frac{1-a^n}{1-a}} \beta^{\frac{b}{1-a}\left(\frac{1-a^{n-2}}{1-a} - (n-2)a^{n-2}\right)} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n} \\ =&\alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{(1-(n-1)a^{n-2}+(n-2)a^{n-1})}{(1-a)^2}} w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}} z_0^{a^n}, \end{aligned} \tag{34}$$

for $n \geq 2$.

Subcase $a = c = 1$. In this case we have

$$\begin{aligned} z_n =&\alpha^n \beta^{b\sum_{i=0}^{n-3}(n-i-2)} w_0^{b(n-1)} w_{-1}^{b} z_0 \\ =&\alpha^n \beta^{b\frac{(n-2)(n-1)}{2}} w_0^{b(n-1)} w_{-1}^{b} z_0, \end{aligned} \tag{35}$$

for $n \geq 2$.

Case $bd \neq 0$. Since $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, it easily follows that $z_n \neq 0 \neq w_n$ for every $n \geq -1$. Hence, from the first equation in (2), we have

$$w_{n-1}^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0. \tag{36}$$

Taking both sides of the second equation in (2) to the b-th power we get

$$w_{n+1}^b = \beta^b w_n^{bc} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{37}$$

From (36) and (37) it follows that

$$\frac{z_{n+3}}{\alpha z_{n+2}^a} = \beta^b \frac{z_{n+2}^c}{\alpha^c z_{n+1}^{ac}} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0,$$

that is,

$$z_{n+3} = \alpha^{1-c} \beta^b z_{n+2}^{a+c} z_{n+1}^{-ac} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{38}$$

which is a fourth order product-type difference equation.

We also have

$$z_1 = \alpha z_0^a w_{-1}^b \quad \text{and} \quad z_2 = \alpha^{1+a} z_0^{a^2} w_{-1}^{ab} w_0^b. \tag{39}$$

Let $\gamma := \alpha^{1-c}\beta^b$,

$$a_1 = a + c, \quad b_1 = -ac, \quad c_1 = 0, \quad d_1 = bd, \quad x_1 = 1. \tag{40}$$

Using (40), equation (38) can be written as

$$z_{n+3} = \gamma^{x_1} z_{n+2}^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \quad n \in \mathbb{N}_0. \tag{41}$$

Using (41) with $n \to n-1$ into (41) it follows that

$$\begin{aligned} z_{n+3} &= \gamma^{x_1} (\gamma z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1})^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \\ &= \gamma^{x_1+a_1} z_{n+1}^{a_1 a_1 + b_1} z_n^{b_1 a_1 + c_1} z_{n-1}^{c_1 a_1 + d_1} z_{n-2}^{d_1 a_1} \\ &= \gamma^{x_2} z_{n+1}^{a_2} z_n^{b_2} z_{n-1}^{c_2} z_{n-2}^{d_2}, \end{aligned} \tag{42}$$

for $n \in \mathbb{N}$, where

$$a_2 := a_1 a_1 + b_1, \quad b_2 := b_1 a_1 + c_1, \quad c_2 := c_1 a_1 + d_1, \quad d_2 := d_1 a_1, \tag{43}$$

$$x_2 := x_1 + a_1. \tag{44}$$

Assume that for some $2 \leq k \leq n$, we have proved that

$$z_{n+3} = \gamma^{x_k} z_{n+3-k}^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k}, \tag{45}$$

for $n \geq k-1$, and that

$$a_k = a_1a_{k-1} + b_{k-1}, \qquad b_k = b_1a_{k-1} + c_{k-1},$$
$$c_k = c_1a_{k-1} + d_{k-1}, \qquad d_k = d_1a_{k-1}, \tag{46}$$

$$x_k := x_{k-1} + a_{k-1}. \tag{47}$$

Then by using (41) with $n \to n-k$ into (45) we have

$$\begin{aligned} z_{n+3} &= \gamma^{x_k}(\gamma z_{n+2-k}^{a_1} z_{n+1-k}^{b_1} z_{n-k}^{c_1} z_{n-1-k}^{d_1})^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k} \\ &= \gamma^{x_k+a_k} z_{n+2-k}^{a_1a_k+b_k} z_{n+1-k}^{b_1a_k+c_k} z_{n-k}^{c_1a_k+d_k} z_{n-1-k}^{d_1a_k} \\ &= \gamma^{x_{k+1}} z_{n+2-k}^{a_{k+1}} z_{n+1-k}^{b_{k+1}} z_{n-k}^{c_{k+1}} z_{n-1-k}^{d_{k+1}}, \end{aligned} \tag{48}$$

for $n \geq k$, where

$$a_{k+1} : = a_1a_k + b_k, \qquad b_{k+1} := b_1a_k + c_k,$$
$$c_{k+1} : = c_1a_k + d_k, \qquad d_{k+1} := d_1a_k, \tag{49}$$

$$x_{k+1} := x_k + a_k. \tag{50}$$

Equalities (48)-(50) along with (42)-(44) and the method of induction show that (45)-(47) hold for $2 \leq k \leq n+1$ ((45) also holds for $k = 1$).

Hence, for $k = n+1$, (45) becomes

$$\begin{aligned} z_{n+3} =& \gamma^{x_{n+1}} z_2^{a_{n+1}} z_1^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}} \\ =& (\alpha^{1-c}\beta^b)^{x_{n+1}} (\alpha^{1+a} z_0^{a^2} w_{-1}^{ab} w_0^b)^{a_{n+1}} (\alpha z_0^a w_{-1}^b)^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}} \\ =& \alpha^{(1-c)x_{n+1}+(1+a)a_{n+1}+b_{n+1}} \beta^{bx_{n+1}} z_0^{a^2a_{n+1}+ab_{n+1}+c_{n+1}} \\ & \times z_{-1}^{d_{n+1}} w_0^{ba_{n+1}} w_{-1}^{aba_{n+1}+bb_{n+1}}, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{51}$$

From (46) we easily see that $(a_k)_{k\geq 4}$ is a solution to the linear difference equation

$$a_k = a_1a_{k-1} + b_1a_{k-2} + c_1a_{k-3} + d_1a_{k-4}. \tag{52}$$

This and the relations $b_{k-1} = a_k - a_1a_{k-1}$, $c_{k-1} = b_k - b_1a_{k-1}$ and $d_k = d_1a_{k-1}$ show that $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are also solutions to equation (52).

On the other hand, from (47) and since $x_1 = 1$, we have that

$$x_k = 1 + \sum_{j=1}^{k-1} a_j. \tag{53}$$

From (46) and (47) with $k = 1$ we get

$$a_1 = a_1a_0 + b_0, \quad b_1 = b_1a_0 + c_0, \quad c_1 = c_1a_0 + d_0,$$
$$d_1 = d_1a_0, \quad x_1 = x_0 + a_0. \tag{54}$$

Since $bd = d_1 \neq 0$, from the fourth equation in (54) we get $a_0 = 1$. Using this fact in the other equalities in (54) we get $b_0 = c_0 = d_0 = 0$ and $x_0 = 0$ (since $x_1 = 1$).

From this and by (46) and (47) with $k = 0$ we get

$$1 = a_0 = a_1 a_{-1} + b_{-1}, \quad 0 = b_0 = b_1 a_{-1} + c_{-1},$$
$$0 = c_0 = c_1 a_{-1} + d_{-1}, \quad 0 = d_0 = d_1 a_{-1}, \quad 0 = x_0 = x_{-1} + a_{-1}. \quad (55)$$

Since $d_1 \neq 0$, from the fourth equation in (55) we get $a_{-1} = 0$. Using this fact in the other equalities in (55) we get $b_{-1} = 1$, $c_{-1} = d_{-1} = 0$ and $x_{-1} = 0$.

From this and by (46) and (47) with $k = -1$ we get

$$0 = a_{-1} = a_1 a_{-2} + b_{-2}, \quad 1 = b_{-1} = b_1 a_{-2} + c_{-2},$$
$$0 = c_{-1} = c_1 a_{-2} + d_{-2}, \ 0 = d_{-1} = d_1 a_{-2}, \ 0 = x_{-1} = x_{-2} + a_{-2}. \quad (56)$$

Since $d_1 \neq 0$, from the fourth equation in (56) we get $a_{-2} = 0$. Using this fact in the other equalities in (56) we get $b_{-2} = 0$, $c_{-2} = 1$ and $d_{-2} = 0$ and $x_{-2} = 0$.

From this and by (46) and (47) with $k = -2$ we get

$$0 = a_{-2} = a_1 a_{-3} + b_{-3}, \ 0 = b_{-2} = b_1 a_{-3} + c_{-3},$$
$$1 = c_{-2} = c_1 a_{-3} + d_{-3}, \ 0 = d_{-2} = d_1 a_{-3}, \ 0 = x_{-2} = x_{-3} + a_{-3}. \quad (57)$$

Since $d_1 \neq 0$, from the fourth equation in (57) we get $a_{-3} = 0$. Using this fact in the other equalities in (57) we get $b_{-3} = 0$, $c_{-3} = 0$ and $d_{-3} = 1$ and $x_{-3} = 0$.

Hence, sequences $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ and $(d_k)_{k\geq -3}$ are solutions to linear difference equation (52) satisfying the next (shifted) initial conditions

$$\begin{aligned} &a_{-3} = 0, \quad a_{-2} = 0, \quad a_{-1} = 0, \quad a_0 = 1; \\ &b_{-3} = 0, \quad b_{-2} = 0, \quad b_{-1} = 1, \quad b_0 = 0; \\ &c_{-3} = 0, \quad c_{-2} = 1, \quad c_{-1} = 0, \quad c_0 = 0; \\ &d_{-3} = 1, \quad d_{-2} = 0, \quad d_{-1} = 0, \quad d_0 = 0, \end{aligned} \quad (58)$$

respectively, while $(x_k)_{k\geq -3}$ is given by (53) and satisfies the conditions

$$x_{-3} = x_{-2} = x_{-1} = x_0 = 0 \quad \text{and} \quad x_1 = 1. \quad (59)$$

Since difference equation (52) is solvable, it follows that closed form formulas for $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ and $(d_k)_{k\geq -3}$, can be found. From this, (53), (59) and by using some known sums, $(x_k)_{k\geq -3}$ can also be calculated. From this fact and (51) we see that equation (38) is solvable too.

The second equation in (2) yields

$$z_{n-1}^d = \frac{w_{n+1}}{\beta w_n^c}, \quad n \in \mathbb{N}_0. \tag{60}$$

Since $d \neq 0$, from the first equation in (2) we have

$$z_{n+1}^d = \alpha^d z_n^{ad} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{61}$$

From (60) and (61) we obtain

$$\frac{w_{n+3}}{\beta w_{n+2}^c} = \alpha^d \frac{w_{n+2}^a}{\beta^a w_{n+1}^{ac}} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0,$$

which can be written as

$$w_{n+3} = \alpha^d \beta^{1-a} w_{n+2}^{a+c} w_{n+1}^{-ac} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{62}$$

which differs from equation (38) only by the coefficient $\alpha^d \beta^{1-a}$. Beside this, sequence $(w_n)_{n\geq -1}$ satisfies the following initial conditions

$$w_1 = \beta w_0^c z_{-1}^d \quad \text{and} \quad w_2 = \beta^{1+c} w_0^{c^2} z_{-1}^{cd} z_0^d. \tag{63}$$

Let $\delta := \alpha^d \beta^{1-a}$. Then the above procedure can be repeated and obtained that for $1 \leq k \leq n+1$

$$w_{n+3} = \delta^{x_k} w_{n+3-k}^{a_k} w_{n+2-k}^{b_k} w_{n+1-k}^{c_k} w_{n-k}^{d_k}, \tag{64}$$

where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ satisfy recurrent relations (46) with initial conditions (40), while $(x_k)_{k\in\mathbb{N}}$ is given by (53) and (59).

From (64) with $k = n+1$ and by using (63) we get

$$\begin{aligned} w_{n+3} =& \delta^{x_{n+1}} w_2^{a_{n+1}} w_1^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ =& (\alpha^d \beta^{1-a})^{x_{n+1}} (\beta^{1+c} w_0^{c^2} z_{-1}^{cd} z_0^d)^{a_{n+1}} (\beta w_0^c z_{-1}^d)^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ =& \alpha^{dx_{n+1}} \beta^{(1-a)x_{n+1}+(1+c)a_{n+1}+b_{n+1}} w_0^{c^2 a_{n+1}+cb_{n+1}+c_{n+1}} \\ & \times w_{-1}^{d_{n+1}} z_0^{da_{n+1}} z_{-1}^{cda_{n+1}+db_{n+1}}, \end{aligned} \tag{65}$$

for $n \in \mathbb{N}_0$.

As above the solvability of equation (52) shows that closed form formulas for $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ and $(d_k)_{k\geq -3}$, can be found. From this, (53), (59) and by using some known sums, $(x_k)_{k\geq -3}$ can also be calculated. This facts along with (65) imply that equation (62) is solvable too. Direct calculation shows that formulas (51) and (65) are really solutions to system (2), finishing the proof. □

Remark 1. Since linear difference equation (52) is of the fourth order it is not only theoretically solvable, but also practically.

The following corollary is a consequence of Theorem 1.

Corollary 1. *Consider system* (2) *with* $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$. *Assume that* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) If $b = 0$ *and* $a \neq 1 \neq c$, *then the general solution to system* (2) *is given by* (5) *and* (14).

(b) If $b = 0$ *and* $a \neq 1 = c$, *then the general solution to system* (2) *is given by* (5) *and* (15).

(c) If $b = 0$ *and* $a = 1 \neq c$, *then the general solution to system* (2) *is given by* (6) *and* (16).

(d) If $b = 0$ *and* $a = c \neq 1$, *then the general solution to system* (2) *is given by* (5) *and* (18).

(e) If $b = 0$ *and* $a = c = 1$, *then the general solution to system* (2) *is given by* (6) *and* (19).

(f) If $d = 0$ *and* $a \neq 1 \neq c$, *then the general solution to system* (2) *is given by* (22) *and* (30).

(g) If $d = 0$ *and* $a \neq 1 = c$, *then the general solution to system* (2) *is given by* (23) *and* (31).

(h) If $d = 0$ *and* $a = 1 \neq c$, *then the general solution to system* (2) *is given by* (22) *and* (32).

(i) If $d = 0$ *and* $a = c \neq 1$, *then the general solution to system* (2) *is given by* (22) *and* (34).

(j) If $d = 0$ *and* $a = c = 1$, *then the general solution to system* (2) *is given by* (23) *and* (35).

(k) If $bd \neq 0$, *then the general solution to system* (2) *is given by* (51) *and* (65), *where* $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$ *and* $(d_k)_{k\geq -3}$ *are solutions to equation* (52) *satisfying the conditions* (58), *while* $(x_k)_{k\geq -3}$ *is given by* (53) *and* (59).

Remark 2. The formulas appearing in the proof of Theorem 1 can be used in studying of the behavior of solutions to system (2). We leave it to the reader as some exercises.

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Stevo Stević, Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia

Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: sstevic@ptt.rs

Bratislav Iričanin, Faculty of Electrical Engineering, Belgrade University, Bulevar Kralja Aleksandra 73, 11000 Beograd, Serbia

Brno University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics, Technicka 3058/10, CZ - 616 00 Brno, Czech Republic
E-mail address: iricanin@etf.rs

Zdeněk Šmarda, Brno University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics, Technicka 3058/10, CZ - 616 00 Brno, Czech Republic
E-mail address: smarda@feec.vutbr.cz

ON SOME RESULTS IN METRIC SPACES USING AUXILIARY SIMULATION FUNCTIONS VIA NEW FUNCTIONS

XIAO-LAN LIU*, ARSLAN HOJAT ANSARI, SUMIT CHANDOK, STOJAN RADENOVIĆ

ABSTRACT. In this paper, we investigate the existence and uniqueness of coincidence points for two nonlinear operators by generalizing the notion of simulation function introduced by F. Khojasteh et al. [F. Khojasteh, S. Shukla and S. Radenović, Filomat 29 (6), 1189-1194 (2015)]. Our results improve, extend, complement and generalize several existing results in the literature. Also, some examples are provided to illustrate the usability of our results.

1. Introduction and Preliminaries

There are many disciplines in which various authors are working, but due to the possible applications, two of them, fixed point theory and equilibrium theory have received major attention from the last few decades. Both of them have many ways to face, from a theoretical point of view, to various problems which arise from real-world contexts. Fixed point theory has been applied in various topics, convex minimization and split feasibility, construction and structure of fixed points, as well as for finding zeros of contractive operators (see [1, 2, 3] and [4]). It also has lots of applications; in particular, in image recovery and signal processing, and in transition operators for initial valued problems of differential inclusions. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems and Nash equilibrium problems as special cases. For other important questions in the fixed point theory with various approach see [7]-[25].

Recently, Khojasteh et al. [5] introduced the notion of simulation function in order to express different contractivity conditions in a unified way, and they obtained some fixed point results. In this paper, we generalized the simulation function using a class of C-function and investigate the existence and uniqueness of coincidence points for two operators in the setting of

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* Corresponding author.

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metric spaces. Some examples are also provided to illustrate the importance of the results obtained.

To begin with, we give some basic notions and definitions.

In the sequel, let X be a nonempty set and $T, g : X \to X$ be two mappings. For simplicity, we denote $T(x)$ by Tx. A mapping g is injective on $A \subseteq X$ if we can deduce that $a = b$ for all $a, b \in A$ such that $ga = gb$. We denote the nth iterate of T by T^n , that is, $T^1 = T$ and $T^{n+1} = T \circ T^n$ for all $n \in \mathbb{N}$.

Definition 1. *A point $x \in X$ is called a*
(1) fixed point of the operator T if $Tx = x$.
(2) coincidence point of T and g if $Tx = gx$.
(3) common fixed point of T and g if $Tx = gx = x$.

Definition 2. *Let $T, g : X \to X$ be mappings on a metric space(X, d). We say that T and g are compatible if $\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$ for all sequence $\{x_n\} \subseteq X$ such that the sequences $\{gx_n\}$ and $\{Tx_n\}$ are convergent and have the same limit.*

Remark 3. *If T and g are commuting (that is, $Tgx = gTx$ for all $x \in X$), then T and g are compatible.*

Definition 4. *Given two self-mappings $T, g : X \to X$ and a sequence $\{x_n\}_{n\geq 0} \subseteq X$, we say that $\{x_n\}$ is a Picard-Jungck sequence of the pair (T, g) (based on x_0) if $gx_{n+1} = Tx_n$ for all $n \geq 0$.*

We say that X verifies the CLR (T, g)-property at a point $x_0 \in X$ if there exists on X a Picard-Jungck sequence of (T, g) based on x_0.

Remark 5. *1. It is well known that if T and g are two self-mappings such that $T(X) \subseteq g(X)$, then there exists a Picard-Jungck sequence of (T, g) based on any point $x_0 \in X$. In other words, if $T(X) \subseteq g(X)$, then X verifies the CLR (T, g)-property at each point $x \in X$.*

2. If $g = I_X$ is the identity mapping on X, then there exists a unique Picard sequence of (T, I_X) based at each $x_0 \in X$, which is given by $x_{n+1} = Tx_n$ for all $n \geq 0$. Therefore, X satisfies the CLR (T, I_X) -property at every point, whatever the mapping T .

Definition 6. *(see [6]) A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following conditions:*
(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

Note that for some F we have that $F(0, 0) = 0$. We denote C-class functions as $\mathcal{C}$.

Also special case of C-class functions can be found in [7]. For examples of C-class function see [6], [8].

Definition 7. [9] *An altering distance function is a continuous, non-decreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{0\}$.*

2. C-class Simulation Function

In this section, we generalized the simulation function introduced by Khojasteh et al. [5] using the function of C-class as follows:

Definition 8. *A mapping $F : [0,\infty)^2 \to \mathbb{R}$ has property C_F, if there exists an $C_F \geq 0$ such that*
(1) $F(s,t) > C_F \Rightarrow s > t$;
(2) $F(t,t) \leq C_F$, for all $t \in [0,\infty)$.

Example 9. *The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$ that have property C_F, for all $s,t \in [0,\infty)$:*
(1) $F(s,t) = s - t, C_F = r,\ r \in [0,\infty)$
(2) $F(s,t) = \frac{s}{(1+t)^r}, r \in (0,\infty); C_F = 1$
(3) $F(s,t) = \frac{s}{1+kt}; k \geq 1, C_F = \frac{r}{1+k},\ r \in [2,\infty)$
(4) $F(s,t) = (s+l)^{\frac{1}{1+T}} - l, l > 1, C_F = 0, 1$
(5) $F(s,t) = s - (\frac{2+t}{1+t})t; C_F = 0,$
(6) $F(s,t) = \frac{ks}{1+t}; 0 < k < 1, C_F = k, 1$
(7) $F(s,t) = \frac{ks}{1+kt}; 0 < k, C_F = \frac{k+1}{k}, 1$
(8) $F(s,t) = \frac{s}{1+t}; 0 < k, C_F = 1, 2$

Definition 10. *A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following axioms:*
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < F(s,t)$ for all $t,s > 0$, here function $F : [0,\infty)^2 \to \mathbb{R}$ is element of $\mathcal{C}$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The third condition is symmetric in both arguments of ζ but, in proofs, this property is not necessary. In fact, in practice, the arguments of ζ have different meanings and they play different roles. Then, we slightly modify the previous definition in order to highlight this difference and to enlarge the family of all simulation functions.

Definition 11. *A C_F-simulation function is a mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:*
(ζ_a) $\zeta(0,0) = 0$;
(ζ_b) $\zeta(t,s) < F(s,t)$ for all $t,s > 0$; here function $F : [0,\infty)^2 \to \mathbb{R}$ is element of $\mathcal{C}$ which has property C_F
(ζ_c) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_F$.

Let Z_F be the family of all C_F-simulation functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$. Every simulation function as in Definition 10 is also a C_F-simulation function as in Definition 11, but the converse is not true, for this see Example 3.3 in [10] using C-class function $F(s,t) = s - t$.

Example 12. *Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kF(s,t) - t$, here $C_F = 0$. Note that $\zeta(t,s) = kF(s,t) - t \leq kF(s,t) < F(s,t) \leq s$ and $\zeta(t,t) = kF(t,t) - t \leq kt - t < 0$.*

It is easy to verify that ζ verifies (ζ_a), and (ζ_b).

If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty} (kF(s_n, t_n) - t_n) \leq (k-1)\delta < 0$.

Also, if $F(s,t) = ms$, $m < 1$, then of $\zeta(t,s) = kF(s,t) - t$, we have $\zeta(t,s) = kms - t$, or $\zeta(t,s) = \lambda s - t, \lambda \in (0,1)$.

Therefore ζ is an C_F-simulation function as in Definition 11.

Example 13. *Let $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kF(s,t)$, here take $C_F = 1$. Here, $\zeta(t,s) = kF(s,t) \leq ks < s$ and $\zeta(t,t) = kF(t,t) < 1$.*

it is easy to verify, ζ verifies (ζ_a), and (ζ_b).

If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty} (kF(s_n, t_n)) \leq kF(\delta,\delta) < 1$.

Therefore ζ is an C_F-simulation function as in Definition 11.

Also, if take $F(s,t) = \frac{s}{1+t}$, ζ is an C_F-simulation function.

Example 14. *Let $F : [0,\infty)^2 \to \mathbb{R}$ be C-class, $F(\psi(s), \varphi(t)) - t < F(s,t), \psi(t) < t$,and let $\zeta : [0,\infty) \times [0,\infty) \to R$ be the function defined by $\zeta(t,s) = F(\psi(s), \varphi(t)) - t$, here $C_F = 0$, note that $\zeta(0,0) = F(\psi(0), \varphi(0)) - 0 = 0$, and $\zeta(t,t) = F(\psi(t), \varphi(t)) - t < F(t,t) - t \leq t - t = 0$*

Clearly, ζ verifies (ζ_1), and (ζ_2) follows from the following:

If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) = \limsup_{n\to\infty} (F(\psi(s_n), \varphi(t_n)) - t_n) \leq \psi(\delta) - \delta < 0$.

3. A coincidence point theorem for (Z,g)-contractions

In this section we establish some results on the existence and uniqueness of coincidence point by using simulation functions on metric spaces.

Definition 15. *Let (X,d) be a metric space and $T, g : X \to X$ be self-mappings. A mapping T is called a (Z_F, g)-contraction if there exists $\zeta \in Z_F$ such that*

$$\zeta(d(Tx,Ty), d(gx,gy)) \geq C_F \tag{1}$$

for all $x, y \in X$ such that $gx \neq gy$.

For clarity, we use the term $(Z_{F,d}, g)$-contraction when we want to highlight that T is a (Z_F, g)-contraction on a metric space involving the metric d. In such a case, we say that T is a $(Z_{F,d}, g)$-contraction with respect to ζ.

If g is the identity mapping on X, T is a $Z_{F,d}$-contraction with respect to ζ .

Remark 16. *1. By axiom (ζ_b), it is clear that a simulation function must verify $\zeta(r,r) < C_F$ for all $r > 0$.*

2. Furthermore, if T is a $(Z_{F,d}, g)-$contraction with respect to $\zeta \in Z_F$, then

$$d(Tx,Ty) < d(gx,gy), \tag{2}$$

for all $x, y \in X$ such that $gx \neq gy$.

To prove it, assume that $gx \neq gy$. Then $d(gx,gy) > 0$. If $Tx = Ty$, then $d(Tx,Ty) = 0 < d(gx,gy)$. On the contrary case, if $Tx \neq Ty$, then $d(Tx,Ty) > 0$, and applying (ζ_b) and (1), we have that

$C_F \leq \zeta(d(Tx,Ty), d(gx,gy)) < F(d(gx,gy), d(Tx,Ty))$, so (2) holds.

Next, we observe some useful properties of $(Z_{F,d}, g)$-contractions in the context of metric spaces.

Lemma 17. *If T is a $(Z_{F,d}, g)$-contraction in a metric space (X,d) and $x, y \in X$ are coincidence points of Tand g, then $Tx = gx = gy = Ty$. In particular, the following properties hold.*

1. If g (or T) is injective on the set of all coincidence points of T and g (or, simply, injective), then Tand g have, at most, a unique coincidence point.

2. If T and g have a common fixed point, then it is unique.

Proof. We will show that $gx = gy$ reasoning by contradiction. Suppose, on the contrary, that $gx \neq gy$. Then $d(gx,gy) > 0$. By (1), we have $C_F \leq \zeta(d(Tx,Ty), d(gx,gy)) = \zeta(d(gx,gy), d(gx,gy))$, which is a contradiction due to item 1 of Remark 16. Therefore, if x and y are coincidence points of T and g, then $Tx = gx = gy = Ty$. Hence the result. □

Theorem 18. *Let T be a $(Z_{F,d},g)$-contraction in a metric space (X,d) and suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\geq 0}$ of (T,g). Also assume that, at least, one of the following conditions hold.*

(a) $(g(X),d)$ (or $(T(X),d)$) is complete.

(b) (X,d) is complete and T and g are continuous and compatible.

(c) (X,d) is complete and T and g are continuous and commuting.

Then T and g have, at least, a coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of T and g or, at least, one of the following properties holds.

In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$such that $gv = u$ is a coincidence point of T and g.

In cases (b) and (c), the sequence $\{gx_n\}$ converges to a coincidence point of T and g.

In addition to this, if $x, y \in X$ are coincidence points of T and g, then $Tx = gx = gy = Ty$. And if g (or T) is injective on the set of all coincidence points of T and g (or, simply, it is injective), then T and g have a unique coincidence point.

Proof. If $\{x_n\}$ contains a coincidence point of T and g, the proof is finished. Assume that $\{x_n\}$does not contain any coincidence point of T and g, that is,

$$gx_n \neq Tx_n = gx_{n+1}, \tag{3}$$

for all $n \geq 0$. In such a case,

$$d(gx_n, gx_{n+1}) > 0, \tag{4}$$

for all $n \geq 0$. We divide the proof in three steps.

Step 1. We claim that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{5}$$

Using (ζ_b) and (1), for all $n \geq 0$,

$$\begin{aligned} C_F &\leq \zeta(d(Tx_n, Tx_{n+1}), d(gx_n, gx_{n+1})) \\ &= \zeta(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) \\ &< F(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})), \end{aligned} \tag{6}$$

which means that $0 < d(gx_{n+1}, gx_{n+2}) < d(gx_n, gx_{n+1})$ for all $n \geq 0$. Then $\{d(gx_n, gx_{n+1})\}$ is a non-increasing sequence of nonnegative real numbers, so it is convergent.

Let $r = \lim_{n\to\infty} d(gx_n, gx_{n+1})$. We prove that $r = 0$ reasoning by contradiction. Assume that $r > 0$. Applying the axiom (ζ_c) to the sequences $\{t_n = d(gx_{n+1}, gx_{n+2})\}$ and $\{s_n = d(gx_n, gx_{n+1})\}$ (which have the same limit $r > 0$ and verify $t_n < s_n$ for all n), it follows that

$$\limsup_{n\to\infty} \zeta(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) = \lim_{n\to\infty} \sup \zeta(t_n, s_n) < C_F,$$

which contradicts with (6) because $\zeta(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) \geq C_F$ for all $n \geq 0$. This contradiction prove that $lim_{n\to\infty} d(gx_n, gx_{n+1}) = r = 0$, that is, (5) holds.

Step 2. Now, we claim that the sequence $\{gx_n\}$ is a Cauchy in (X, d). On the contrary, we assume that $\{gx_n\}$ is not a Cauchy sequence in (X, d). In this case, there exists $\varepsilon_0 > 0$ such that, for all $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ verifying $m > n > N$ and $d(gx_m, gx_n) > \varepsilon_0$. Using (5), there exists $n_0 \in N$ such that

$$d(gx_{n+1}, gx_n) < \varepsilon_0, \tag{7}$$

for all $n \geq n_0$.

Taking successive values for n, we can find two partial subsequences $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that

$$n_0 \leq n(k) < m(k) < n(k+1) \text{ and } d(gx_{m(k)}, gx_{n(k)}) > \varepsilon_0 \tag{8}$$

for all $k \in N$.

If we choose $m(k)$ as the least natural number $m \in \{n(k), n(k)+1, n(k)+2, ...\}$ such that (8) holds, then we also have that

$$d(gx_{m(k)-1}, gx_{n(k)}) \leq \varepsilon_0, \tag{9}$$

for all $k \in N$.

Notice that $m(k) \geq n(k)+1$ for all $k \in N$. Indeed, joining (7) and (8), we deduce that the case $m(k) = n(k)+1$ is impossible. Therefore, $m(k) \geq n(k)+2$ for all $k \in N$. It follows that

$n(k)+1 < m(k) < m(k)+1$ for all $k \in N$.

Taking into account (8) and (9), we deduce that

$\varepsilon_0 < d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)-1}) + \varepsilon_0$

for all k. Using (5), we have that

$$\lim_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}) = \varepsilon_0. \tag{10}$$

Moreover, by

$d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{n(k)})$

and

$d(gx_{m(k)+1}, gx_{n(k)+1}) \leq d(gx_{m(k)+1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{n(k)}) + d(gx_{n(k)}, gx_{n(k)+1})$

for all k, and also using (5), it follows that

$$\lim_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) = \varepsilon_0. \tag{11}$$

In particular, there exists $n_1 \in N$ such that

$d(gx_{m(k)}, gx_{n(k)}) > \frac{\varepsilon_0}{2} > 0$

and

$d(gx_{m(k)+1}, gx_{n(k)+1}) > \frac{\varepsilon_0}{2} > 0$ for all $k \geq n_1$.

Using the fact that T is a $(Z_{F,d}, g)$-contraction with respect to ζ and the axiom (ζ_b), we deduce that

$C_F \leq \zeta(d(Tx_{m(k)}, Tx_{n(k)}), d(gx_{m(k)}, gx_{n(k)})) = \zeta(d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})$

$< F(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)+1}, gx_{n(k)+1}))$, for all $k \geq n_1$. In particular,

$0 < d(gx_{m(k)+1}, gx_{n(k)+1}) < d(gx_{m(k)}, gx_{n(k)})$ for all $k \geq n_1$

Employing the sequences $\{t_k = d(gx_{m(k)+1}, gx_{n(k)+1})\}$ and $\{s_k = d(gx_{m(k)}, gx_{n(k)})\}$ (which have the same positive limit by (10) and (11) and verify $t_k < s_k$ for all k) in axiom (ζ_c), we conclude that which is a contradiction. Therefore, we must admit that the sequence $\{gx_n\}$ is Cauchy in (X, d). Hence, Step 2 holds.

Step 3. Now, we prove that T and g have a coincidence point by taking the assumptions (a), or (b), or (c).

Case (a): Assume that $(g(X), d)$ (or $(T(X), d)$) is complete. In this case, notice that $gx_{n+1} = Tx_n \in T(X) \subseteq g(X)$ for all $n \geq 0$,which means that the sequence $\{gx_{n+1}\}$ is included in $T(X) \subseteq g(X)$. Taking into account that $g(X)$ (or $T(X)$) is d-complete,then there exists $u \in g(X)$ such that $\{gx_n\} \to u$, that is,

$$\lim_{n\to\infty} d(gx_n, u) = 0. \tag{12}$$

Since $Tx_n = gx_{n+1}$ for all n, we also have that

$$\lim_{n\to\infty} d(Tx_n, u) = 0. \tag{13}$$

Let $v \in X$ be any point such that $gv = u$. We will show that v is a coincidence point of T and g. On the contrary, we assume that v is not a coincidence point of T and g, that is, $u = gv \neq Tv$. In such a case, $\delta = d(Tv, gv) > 0$. Using(13), let $n_0 \in N$ be such that $d(gx_n, gv) < \delta$ for all $n \geq n_0$. This means that

$d(gx_n, gv) < \delta = d(Tv, gv)$ for all $n \geq n_0$.

In particular, $gx_n \neq Tv$ for all $n \geq n_0$, that is,

$$d(Tx_n, Tv) = d(gx_{n+1}, Tv) > 0, \tag{14}$$

for all $n \geq n_0$.

On the other hand, by (4), it is impossible the condition

$\exists n_1 \in N$ such that $gx_n = gv$ for all $n \geq n_1$.

Therefore, there exists a partial subsequence $\{gx_{\delta(n)}\}$ of $\{gx_n\}$ such that

$$gx_{\delta(n)} \neq gv \ \forall n \tag{15}$$

Now, let $n_2 \in N$ be such that $\delta(n_2) \geq n_0$. Therefore, by (14) and (15),

$d(gx_{\delta(n)}, gv) > 0$ and $d(Tx_{\delta(n)}, Tv) > 0$ for all $n \geq n2$.

Using (ζ_b),

$C_F \leq \zeta(d(Tx_{\delta(n)}, Tv), d(gx_{\delta(n)}, gv)) < F(d(gx_{\delta(n)}, gv), d(Tx_{\delta(n)}, Tv))$

for all $n \geq n_2$, which means that

$0 \leq d(Tx_{\delta(n)}, Tv) < d(gx_{\delta(n)}, gv) = d(gx_{\delta(n)}, u)$,

for all $n \geq n_2$. In particular, by (12), $\{Tx_{\delta(n)}\} \to Tv$. However, $\{Tx_{\delta(n)}\} = \{gx_{\delta(n)+1}\}$ is a partial subsequence of $\{gx_n\}$, which converges to gv. By the unicity of the limit, we conclude that $gv = Tv$, which is a contradiction with the fact that we have supposed that $gv \neq Tv$. This contradiction yields v is a coincidence point of T and g.

Case (b): Assume that (X, d) is complete and T and g are continuous and compatible. In this case, the sequence $\{gx_n\}$ is a Cauchy sequence in the complete metric space (X, d) , so there exists $u \in X$such that $\{gx_n\} \to u$. Since T and g are continuous,it follows that $\{ggx_n\} \to gu$ and $\{Tgx_n\} \to Tu$. Moreover, as T and g are compatible and the sequences$\{Tx_n = gx_{n+1}\}$and $\{gx_n\}$ have the same limit, we deduce that

$lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$. It follows that

$d(Tu, gu) = lim_{n\to\infty} d(Tgx_n, ggx_{n+1}) = lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$.

Therefore, $Tu = gu$ and we conclude that u is a coincidence point of T and g.

Case (c): Assume that (X, d) is complete and T and g are continuous and commuting. It follows from case (b) taking into account Remark 16. Last statement follows from Lemma 17. □

Now if let $F(s, t) = s - t$, we have the following result of [10].

Corollary 19. *Let T be a $(Z_{d,}g)$-contraction in a metric space (X,d) and suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\geq 0}$ of (T,g). Also assume that, at least, one of the following conditions holds.*

(a) $(g(X),d)$ (or $(T(X),d)$) is complete.

(b) (X,d) is complete and T and g are continuous and compatible.

(c) (X,d) is complete and T and g are continuous and commuting.

Then T and g have, at least, a coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of T and g or, at least, one of the following properties holds.

In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

In cases (b) and (c), the sequence $\{gx_n\}$ converges to a coincidence point of T and g.

In addition to this, if $x, y \in X$ are coincidence points of T and g, then $Tx = gx = gy = Ty$. And if g (or T) is injective on the set of all coincidence points of T and g (or, simply, it is injective), then T and g have a unique coincidence point.

In the following example, we describe how to use our main result in order to guarantee existence and uniqueness of a solution for nonlinear equations.

Example 20. *Let $X = [0,\infty)$ provided with the Euclidean metric $d(x,y) = |x-y|$ for all $x, y \in X$, and consider the operators $T, g : X \to X$ given, for all $x \in X$, by*

$$Tx = x + 2,\ gx = 4x + e^{2x}$$

It is clear that T is not a contraction in the classical Banach sense (in fact, it is an isometry). In order to solve the nonlinear equation

$$x + 2 = 4x + e^{2x}, \tag{16}$$

Theorem 18 can be applied using the simulation function $\zeta(t,s) = \frac{9}{10}[\psi(s) - (\frac{2+\varphi(t)}{1+\varphi(t)})\varphi(t)]$,
where ψ and φ are the altering distance functions given by $\psi(t) = t$ and $\varphi(t) = t$ for all $t \geq 0$ and $C_F = 0, F(s,t) = s - (\frac{2+t}{1+t})t$, now we have that

$$\begin{aligned}
& \zeta(d(Tx,Ty), d(gx,gy)) \\
= \ & \frac{9}{10}[\psi(d(gx,gy)) - (\frac{2+\varphi(d(Tx,Ty))}{1+\varphi(d(Tx,Ty))})\varphi(d(Tx,Ty))] \\
= \ & \frac{9}{10}[d(gx,gy) - (\frac{2+d(Tx,Ty)}{1+d(Tx,Ty)})d(Tx,Ty)] \\
= \ & \frac{9}{10}[\left|4(x-y) + (e^{2x} - e^{2y})\right| - (\frac{2+|x-y|}{1+|x-y|})\,|x-y|] \\
\geq \ & 0.
\end{aligned}$$

Therefore T is a (Z_d, g)-contraction with respect to ζ_1. As all conditions of Theorem 4.8 are satisfied (for instance, $T(X) = [2, \infty) \subset [1, \infty) = g(X)$ and $g(X)$ is complete), it is ensured that T and g have a unique coincidence point, which is the only solution of equation (16). □

4. Competing interests

The authors declare that they have no competing interests.

5. Authors contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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X.Liu School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China, &, Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, 643000, China
E-mail address: stellalwp@163.com

A.H. Ansari Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
E-mail address: mathanalsisamir4@gmail.com

S. Chandok School of Mathematics, Thapar University, Patiala 147-004, Punjab, India.
E-mail address: chansok.s@gmail.com, sumit.chandok@thapar.edu

S. Radenović Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia, &, State University of Novi Pazar, Novi Pazar, Serbia

E-mail address: stojan.radenovic@tdt.edu.vn

The uniqueness of meromorphic functions and weight-shared *

Jin Tu[a], Hong Yan Xu[b]†, Hong Zhang[c]

[a]Institute of Mathematics and informatics, Jiangxi Normal University,
Nanchang, Jiangxi, 330022, China
<e-mail: tujin2008@sina.com>

[a]Department of Informatics and Engineering, Jingdezhen Ceramic Institute,
Jingdezhen, Jiangxi, 333403, China
<e-mail: xhyhhh@126.com>

[c] School of the Tourism and Urban Management,Jiangxi University of Finance and Economics
Nanchang, Jiangxi, 330032, China
<e-mail: zhanghongjxcd@163.com>

Abstract

The purpose of this paper is to study some uniqueness problem of meromorphic functions sharing two sets and three sets with finite weight. Our results are improvement and complement of some results given by Zhang-Xu, Fang-Xu, Lahiri-Banerjee.
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1 Introduction and main results

This purpose of this paper is to study some uniqueness problem of meromorphic functions sharing two sets and three sets with finite weight. The fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used (see Hayman [6] and Yi and Yang [12]). Let f be a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$ and S be a subset of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Define

$$E(S,f) = \bigcup_{a\in S} \{z : f(z) - a = 0, \ \textbf{counting} \quad \textbf{multiplicity}\},$$

$$\overline{E}(S,f) = \bigcup_{a\in S} \{z : f(z) - a = 0, \ \textbf{ignoring} \quad \textbf{multiplicity}\}.$$

If $E(S,f) = E(S,g)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}(S,f) = \overline{E}(S,g)$, we say that f and g share the set S IM. Especially, let $S = \{a\}$, we say f and g share the value a CM. If $E(S,f) = E(S,g)$, and we say that f and g share the value a IM if $\overline{E}(S,f) = \overline{E}(S,g)$ (see [5]).

Let m be a nonnegative integer, we denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a;f)$ the set of distinct a-points of f with multiplicities not greater than m. If for

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†Corresponding author.

some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a;f) = E_{\infty)}(a;g)$ we say that f, g share the value a CM. For any positive integer m, we define

$$E_{m)}(S,f) = \bigcup_{a\in S} E_{m)}(a;f), \quad \text{and} \quad \overline{E}_{m)}(S,f) = \bigcup_{a\in S} \overline{E}_{m)}(a;f).$$

In 1977, Gross [4] proved that there exist three finite sets $S_j (j = 1, 2, 3)$, such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ must be identical, and posed the following question:

Question A *Does there exist a finite set S such that, for any pair of nonconstant entire functions f and g, $E(S, f) = E(S, g)$ implies $f \equiv g$?*

If the answer of this question is affirmative, a natural question is the following:

Question B *What is the smallest cardinal of S?*

In 1995, Yi [11] first proved that such that a set exist. In fact, Yi proved the following theorem

Theorem A [11]. *There exists a set S with 7 elements such that $E(S, f) = E(S, g)$ implies $f \equiv g$, for any pair of nonconstant entire functions f and g.*

For meromorphic functions, the present best answer to Question B was obtained by Frank and Reinders [3].

Theorem B [3]. *There exists a set S with 11 elements such that $E(S, f) = E(S, g)$ implies $f \equiv g$ for any pair of nonconstant meromorphic functions f and g.*

A natural problem arises: *What can we say if nonconstant meromorphic functions f and g have "few" poles?*

Lahiri and Banerjee [8] investigated the situation for $\Theta(\infty; f) \leq \frac{1}{2}$ and $\Theta(\infty; g) \leq \frac{1}{2}$ in Theorem C and proved the following result.

Theorem C [8]. *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}, S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and $n(\geq 4)$ is an integer. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty; f) + \Theta(\infty; g) > 0$. If $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$, then $f \equiv g$.*

Recently, Zhang and Xu [13] proved the following result:

Theorem D [13]. *Let $S = \{z : z^7 - z^6 = 1\}$. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty; f) + \Theta(\infty; g) > 1$. If $E(S, f) = E(S, g)$ and $E(\infty, f) = E(\infty, g)$, then $f \equiv g$.*

Now considering all the above theorems it is natural to ask the following question:

Question 1.1 *Is it possible in any way to further relax the nature of sharing the set S_1, S_2, S_3 in Theorem C?*

In the present paper we shall investigate this problem and obtain two results which will improve all the previous theorems mentioned earlier. Also we shall provide an answer to the question of Gross in a more compact and convenient way than the previous authors have given.

Now we state the following theorems which are the main results of this paper.

Theorem 1.1 *Let S_1, S_3 be as in Theorem C. Suppose that f, g are nonconstant meromorphic functions satisfying $E_{1)}(S_1, f) = E_{1)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 3$, $E(S_3, f) = E(S_3, g)$ and satisfy one of the following conditions*

(i) $n = 8$ and $\Theta(\infty; g) + \Theta(\infty; g) > \frac{3}{2}$;

(ii) $n = 9$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{4}$;

(iii) $n \geq 10$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, n *is a positive integer.*
Then $f \equiv g$.

Theorem 1.2 *Let* S_1, S_3 *be as in Theorem C. Suppose that* f, g *are nonconstant meromorphic functions satisfying* $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 4$, $E(S_3, f) = E(S_3, g)$ *and satisfy one of two conditions:*
(i) $n = 7$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 1$;
(ii) $n \geq 8$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, n *is a positive integer.*
Then $f \equiv g$.

Remark 1.1 *When* $n = 7$, *it obviously that Theorem 1.2 is an improvement of Theorem D.*

Theorem 1.3 *Let* S_1, S_3 *be as in Theorem C. Suppose that* f, g *are nonconstant meromorphic functions satisfying* $E_{1)}(S_1, f) = E_{1)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 3$, $E(S_2, f) = E(S_2, g), E(S_3, f) = E(S_3, g)$ *and satisfy one of the conditions:*
(i) $n = 6$ *and* $\Theta(\infty; g) + \Theta(\infty; g) > \frac{3}{2}$;
(ii) $n = 7$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{4}$;
(iii) $n \geq 8$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, n *is a positive integer.*
Then $f \equiv g$.

Theorem 1.4 *Let* S_1, S_2, S_3 *be as in Theorem C. Suppose that* f, g *are nonconstant meromorphic functions satisfying* $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, $\overline{E}_{m)}(S_1, f) = \overline{E}_{m)}(S_1, g)$, $m \geq 4$, $E(S_2, f) = E(S_2, g), E(S_3, f) = E(S_3, g)$ *and satisfy one of the conditions:*
(i) $n = 5$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 1$;
(ii) $n \geq 6$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 0$.
Then $f \equiv g$.

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.1 *(see [1]). Let* k *and* r *be two positive integers such that* $1 \leq r < k - 1$ *and for* $a \in \mathbf{C}$, $\overline{E}_{k)}(a, f) = \overline{E}_{k)}(a, g), E_{r)}(a, f) = E_{r)}(a, g)$. *Let* z_0 *be a zero of* $f - a$ *of multiplicity* p *and a zero of* $g - a$ *of multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ *the reduced counting function of those a-points of* f *and* g *for which* $p > q \geq r + 1 (q > p \geq r + 1)$, *by* $\overline{N}_E^{(r+1}(r, a; f)$ *the reduced counting function of those a-points of* f *and* g *for which* $p = q \geq r + 1$, *by* $\overline{N}_{f \geq k+1}(r, a; f|g \neq a)(\overline{N}_{g \geq k+1}(r, a; g|f \neq a))$ *the reduced counting functions of those a-points of* f *and* g *for which* $p \geq k + 1$ *and* $q = 0 (q \geq k + 1$ *and* $p = 0)$.

Definition 1.2 *(see [1]). If* $r = 0$ *in definition 1.1 then we use the same notations as in definition 1.1 except by* $\overline{N}_E^{1)}(r, a; f)$ *we mean the common simple a-points of* f *and* g *and by* $\overline{N}_E^{(2}(r, a; f)$ *we mean the reduced counting functions of those a-points of* f *and* g *for which* $p = q \geq 2$.

Definition 1.3 *(see [9]). Let* $a, b \in \mathbb{C} \cup \{\infty\}$, *We denote by* $N(r, a; f|g = b)$ *the counting function of those* $a-$*points of* f, *counted according to multiplicity, which are b-points of g; by* $N(r, a; f|g \neq b)$ *the counting function of those a-points of* f, *counted according to multiplicity, which are not the b-points of g.*

2 Some Lemmas

In this section we shall denote by H and V the following two functions

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}),$$

where F, G are two nonconstant meromorphic functions.

Lemma 2.1 *(see [12]). Let f be a nonconstant meromorphic function and let*

$$R(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = max\{n, m\}$.

Lemma 2.2 *(see [14]). Let F, G be two nonconstant meromorphic functions such that $E_{1)}(1; F) = E_{1)}(1; F)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 2.3 *(see [8, Lemma 9]). Let f, g be two nonconstant meromorphic functions. Let*

$$F = \frac{f^{n-1}(f+a)}{-b}, \quad and \quad G = \frac{g^{n-1}(g+a)}{-b}. \tag{1}$$

If $F \equiv G$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, where $n \geq 4$, then $f \equiv g$.

Lemma 2.4 *(see [8, Lemma 3]). Let f, g be two nonconstant meromorphic functions sharing $(0, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 0$. Then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where $n(\geq 2)$ is an integer and a is a nonzero finite constant.*

Lemma 2.5 *(see [7, Lemma 5]). If two nonconstant meromorphic functions f, g share $(\infty, 0)$ then for $n \geq 2$,*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 2.6 *(see [1, Lemma 2.2]). Let $\overline{E}_{m)}(1, F) = \overline{E}_{m)}(1, G), E_{1)}(1, F) = E_{1)}(1, G)$ and $H \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} & N(r, \infty; H) \\ \leq\ & \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_L(r, \infty; F) + \overline{N}_L(r, \infty; G) \\ & + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_{F \geq m+1}(r, 1; F|G \neq 1) \\ & + \overline{N}_{G \geq m+1}(r, 1; G|F \neq 1) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'), \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

From Lemma 2.6, we can get the following lemmas easily.

Lemma 2.7 *Let $\overline{E}_{m)}(1, F) = \overline{E}_{m)}(1, G), E_{1)}(1, F) = E_{1)}(1, G)$, $E(\infty, F) = E(\infty, G)$ and $H \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} N(r, \infty; H) \leq & \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\ & + \overline{N}_{F \geq m+1}(r, 1; F|G \neq 1) + \overline{N}_{G \geq m+1}(r, 1; G|F \neq 1) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'), \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ and $\overline{N}_0(r, 0; G')$ is stated as in Lemma 2.6.

Lemma 2.8 *Let $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$, $E(\infty;F)=E(\infty;G)$, $E(0,F)=E(0,G)$ and $H\not\equiv 0$, where $m\geq 3$. Then*

$$\begin{aligned}N(r,\infty;H)\leq&\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)\\&+\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ and $\overline{N}_0(r,0;G')$ is stated as in Lemma 2.6.

Lemma 2.9 *(see [1, Lemma 2.6]). Let $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$ and $H\not\equiv 0$, where $m\geq 3$. Then*

$$\begin{aligned}&2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}^{(2}_E(r,1;F)\\&+m\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)-\overline{N}_{F>2}(r,1;G)\\\leq&N(r,1;G)-\overline{N}(r,1;G).\end{aligned}$$

Lemma 2.10 *(see [10]). If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geq k)+S(r,f).$$

Lemma 2.11 *(see [1, Lemma 2.9]). Let $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$, where $m\geq 3$. Then*

$$\begin{aligned}&2\overline{N}_{F>2}(r,1;G)+2\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)\\\leq&\frac{2}{3}\overline{N}(r,0;F)+\frac{2}{3}\overline{N}(r,\infty;F)-\frac{2}{3}N_0(r,0;F')+S(r,F).\end{aligned}$$

Lemma 2.12 *Let $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$ and $E(\infty,f)=E(\infty,g)$, where $m\geq 3$ and $H\not\equiv 0$, then*

$$T(r,F)\leq N_2(r,0;F)+\frac{8}{3}\overline{N}(r,\infty;F)+N_2(r,0;G)+\frac{2}{3}\overline{N}(r,0;F)+S(r,F)+S(r,G).\qquad(2)$$

Proof: From the condition of Lemma 2.12, and by Lemma 2.7, we have

$$\begin{aligned}&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\\leq\;&N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}^{(2}_E(r,1;F)+\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)+\overline{N}(r,1;G)\\\leq\;&\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\&+\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\&+\overline{N}^{(2}_E(r,1;F)+\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)+T(r,G)-m(r,1;G)\\&+O(1)-2\overline{N}_L(r,1;F)-2\overline{N}_L(r,1;G)-\overline{N}^{(2}_E(r,1;F)\\&-m\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)+\overline{N}_{F>2}(r,1;G)+\overline{N}_0(r,0;F')\\&+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\\leq\;&\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+T(r,G)-m(r,1;G)+2\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)\\&+\overline{N}_{F>2}(r,1;G)-(m-1)\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)+\overline{N}_0(r,0;F')\\&+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}$$

By Lemma 2.11, it follows that

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\leq&\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+T(r,G)-m(r,1;G)\qquad(3)\\&+\frac{2}{3}\overline{N}(r,0;F)+\frac{2}{3}\overline{N}(r,\infty;F)-(m-1)\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)\\&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}$$

By the second fundamental theorem, we have

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) - \overline{N}_0(r,0;F') + S(r,F), \tag{4}$$

$$T(r,G) \leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) - \overline{N}_0(r,0;G') + S(r,G). \tag{5}$$

Adding (4) and (5) and from (3), it follows

$$\begin{aligned} T(r,F)+T(r,G) \leq & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;F) \\ & + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq & N_2(r,0;F) + \frac{8}{3}\overline{N}(r,\infty;F) + N_2(r,0;G) + T(r,G) - m(r,1;G) \\ & + \frac{2}{3}\overline{N}(r,0;F) - (m-1)\overline{N}_{G\geq m+1}(r,1;G|F\neq 1) + S(r,F) + S(r,G). \end{aligned} \tag{6}$$

Thus, we can get (2) from (6) easily. □

Similar to the above argument and by Lemma 2.8, we can get the following lemma.

Lemma 2.13 *Let* $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{1)}(1,F) = E_{1)}(1,G)$, $E(\infty,f) = E(\infty,g)$ *and* $E(0,F) = E(0,G)$, *where* $m \geq 3$ *and* $H \not\equiv 0$, *then*

$$T(r,F) \leq \frac{8}{3}\overline{N}(r,0;F) + \frac{8}{3}\overline{N}(r,\infty;F) + S(r,F) + S(r,G). \tag{7}$$

Lemma 2.14 *If* $\overline{E}_{m)}(1,F) = \overline{E}_{m)}(1,G), E_{2)}(1,F) = E_{2)}(1,G)$ *and* $E(\infty,F) = E(\infty,G)$, *where* $m \geq 4$ *and* $H \not\equiv 0$, *then*

$$T(r,F)+T(r,G) \leq 2N_2(r,0;F) + 2N_2(r,0;G) + 4\overline{N}(r,\infty;F) + S(r,F) + S(r,G). \tag{8}$$

Proof: From (4), (5) and by Lemma 2.7, we have

$$\begin{aligned} T(r,F)+T(r,G) \leq & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;F) \\ & + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + N(r,1;F|=1) \\ & + \overline{N}(r,1;F|\geq 2) + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') \\ & + S(r,F) + S(r,G) \\ \leq & \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,1;G) \\ & + \overline{N}(r,1;F|\geq 2) + \overline{N}_L(r,1;G) + \overline{N}_L(r,1;F) + \overline{N}_{F\geq m+1}(r,1;F|G\neq 1) \\ & + \overline{N}_{G\geq m+1}(r,1;G|F\neq 1) + S(r,F) + S(r,G). \end{aligned} \tag{9}$$

Since

$$\overline{N}(r,1;F|=m;G|=m-1) + \cdots + \overline{N}(r,1;F|=m;G|=3) \leq \overline{N}(r,1;F|=m);$$

and

$$\overline{N}(r,1;G|=m;F|=m-1) + \cdots + \overline{N}(r,1;G|=m;F|=3) \leq \overline{N}(r,1;G|=m),$$

we see that

$$\begin{aligned}
& \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq m+1}(r,1;F|G\neq 1)\\
& +\overline{N}_{G\geq m+1}(r,1;G|F\neq 1)+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;G)\\
\leq\ & \overline{N}(r,1;F|=m;G|=m-1)+\cdots+\overline{N}(r,1;F|=m;G|=3)\\
& +\overline{N}(r,1;F|\geq m+2)+\overline{N}(r,1;G|=m;F|=m-1)+\cdots\\
& +\overline{N}(r,1;G|=m;F|=3)+\overline{N}(r,1;G|\geq m+2)\\
& +\overline{N}(r,1;G|\geq m+2)+\overline{N}(r,1;F|\geq m+1)\\
& +\overline{N}(r,1;G|\geq m+1)+\overline{N}(r,1;F|=2)+\cdots\\
& +\overline{N}(r,1;F|=m)+\overline{N}(r,1;F|\geq m+1)+\overline{N}(r,1;G|=1)\\
& +\cdots+\overline{N}(r,1;G|=m)+\overline{N}(r,1;G|\geq m+1)\\
\leq\ & \tfrac{1}{2}N(r,1;F|=1)+\overline{N}(r,1;F|=2)+\cdots+2\overline{N}(r,1;F|=m)\\
& +2\overline{N}(r,1;F|\geq m+1)+\overline{N}(r,1;F|\geq m+2)+\tfrac{1}{2}N(r,1;G|=1)\\
& +\overline{N}(r,1;G|=2)+\cdots+2\overline{N}(r,1;G|=m)+2\overline{N}(r,1;G|\geq m+1)\\
& +\overline{N}(r,1;G|\geq m+2)\\
\leq\ & \tfrac{1}{2}[N(r,1;F)+N(r,1;G)]\\
\leq\ & \tfrac{1}{2}[T(r,F)+T(r,G)].
\end{aligned}\tag{10}$$

From (9) and (10), we can get (8) easily.

Thus, this completes the proof of Lemma 2.14. □

Similar to the argument as in Lemma 2.14, and by Lemma 2.8, we can get the following lemma

Lemma 2.15 *If* $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{2)}(1,F)=E_{2)}(1,G)$, $E(0,F)=E(0,G)$ *and* $E(\infty,F)=E(\infty,G)$, *where* $m\geq 4$ *and* $H\not\equiv 0$, *then*

$$T(r,F)+T(r,G)\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+2\overline{N}(r,0;F)+2\overline{N}(r,0;G)+S(r,F)+S(r,G).$$

3 Proofs of the Theorems

3.1 Proof of Theorem 1.1

Proof: Let F,G be stated as in (1). Suppose that $H\not\equiv 0$. From the condition of Theorem 1.1, we have $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G), E_{1)}(1,F)=E_{1)}(1,G)$ and $E(\infty,f)=E(\infty,g)$, where $m\geq 3$.

From the definitions of F,G, it follows that

$$\begin{aligned}
& N_2(r,0;F)\leq 2\overline{N}(r,0;f)+N(r,0;f+a)+S(r,f);\\
& N_2(r,0;G)\leq 2\overline{N}(r,0;g)+N(r,0;g+a)+S(r,g);\\
& \overline{N}(r,0;F)=\overline{N}(r,0;f)+\overline{N}(r,0;f+a);\\
& \overline{N}(r,0;G)=\overline{N}(r,0;g)+\overline{N}(r,0;g+a);\\
& \overline{N}(r,\infty;F)=\overline{N}(r,\infty;f),\quad \overline{N}(r,\infty;G)=\overline{N}(r,\infty;g);\\
& T(r,F)=nT(r,f)+S(r,f),\quad T(r,G)=nT(r,g)+S(r,g).
\end{aligned}\tag{11}$$

It follows from (11) and Lemma 2.12 that

$$(n-\frac{13}{3})T(r,f)\leq 3T(r,g)+\frac{8}{3}\overline{N}(r,\infty;f)+S(r,f)+S(r,g).\tag{12}$$

Similarly, we have

$$(n-\frac{13}{3})T(r,g)\leq 3T(r,f)+\frac{8}{3}\overline{N}(r,\infty;g)+S(r,f)+S(r,g).\tag{13}$$

Combining (12) and (13), we have

$$\left\{n-10+\frac{4}{3}[\Theta(\infty,f)+\Theta(\infty,g)-2\varepsilon]\right\}T(r)\leq S(r), \tag{14}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$ as $r\to\infty$.

If $n=8$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{3}{2}$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g)-\frac{3}{2})$. Thus, we can get a contradiction from (14).

If $n=9$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{3}{4}$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g)-\frac{3}{4})$. Thus, we can get a contradiction from (14).

If $n\geq 10$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n-1}$, we can take any real number $\varepsilon(>0)$, we also get a contradiction from (14).

Thus, it follows $H\equiv 0$, that is

$$\frac{F''}{F'}-\frac{2F'}{F-1}\equiv\frac{G''}{G'}-\frac{2G'}{G-1}.$$

By a simple calculate, we have either $F\cdot G\equiv 1$ or $F\equiv G$.

If $F\cdot G\equiv 1$, that is $f^{n-1}(f+a)g^{n-1}(g+a)\not\equiv b^2$. By Lemma 2.6, we can get a contradiction.

If $F\equiv G$, that is $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$. From the condition of Theorem 1.1 and Lemma 2.3, we can get $f\equiv g$.

Thus, we complete the proof of Theorem 1.1. □

3.2 The proof of Theorem 1.2

Proof: From the condition of Theorem 1.2, we have $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G)$, $E_{2)}(1,F)=E_{2)}(1,G)$ and $E(\infty,F)=E(\infty,G)$,where $m\geq 4$. Let $H\equiv 0$. Then it follows from (11) and Lemma 2.14 that

$$(n-6)[T(r,f)+T(r,g)]\leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+S(r,f)+S(r,g),$$

that is,

$$\{n-8+\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\}\,T(r)\leq S(r), \tag{15}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$ as $r\to\infty$.

If $n=7$ and $\Theta(\infty,f)+\Theta(\infty,g)>1$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g)-1)$. Thus, we can get a contradiction from (15).

If $n\geq 8$ and $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n-1}>0$, we can take any real number $\varepsilon(0<2\varepsilon<\Theta(\infty,f)+\Theta(\infty,g))$. Thus, we can get a contradiction from (15).

Suppose that $H\equiv 0$, by using the same argument as in Theorem 1.1, we can get the conclusion of Theorem 1.2 easily.

Thus, we complete the proof of Theorem 1.2. □

3.3 The proof of Theorem 1.3

Proof: From the condition of Theorem 1.3, we have $\overline{E}_{m)}(1,F)=\overline{E}_{m)}(1,G)$, $E_{1)}(1,F)=E_{1)}(1,G)$, $E(\infty,F)=E(\infty,G)$ and $E(0,F)=E(0,G)$, where $m\geq 3$.

Suppose that $H\not\equiv 0$. From (11) and Lemma 2.13, it follows

$$(n-\frac{16}{3})T(r,f)\leq\frac{8}{3}\overline{N}(r,\infty;f)+S(r,f)+S(r,g),$$

that is,

$$\left(n-8+\frac{4}{3}(\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon)\right)T(r)\leq S(r). \tag{16}$$

If $n = 6$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{3}{2}$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g) - \frac{3}{2})$. Thus, we can get a contradiction from (16).

If $n = 7$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{3}{4}$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g) - \frac{3}{4})$. Thus, we can get a contradiction from (16).

If $n \geq 8$ and $\Theta(\infty, f) + \Theta(\infty, g) > 0$, we can take any real number $\varepsilon(> 0)$, we also get a contradiction from (16).

Let $H \equiv 0$,that is

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{G''}{G'} - \frac{2G'}{G-1}.$$

By a simple calculate, we have either $F \cdot G \equiv 1$ or $F \equiv G$.

If $F \cdot G \equiv 1$, that is $f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2$. By Lemma 2.5, we can get a contradiction.

If $F \equiv G$, that is $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$. Since $E(0; f) = E(0, g)$, from the condition of Theorem 1.3 and Lemma 2.4, we can get $f \equiv g$.

Thus, we complete the proof of Theorem 1.3. □

3.4 The proof of Theorem 1.4

Proof: From the condition of Theorem 1.4, we have $\overline{E}_{m)}(1, F) = \overline{E}_{m)}(1, G), E_{2)}(1, F) = E_{2)}(1, G)$, $E(\infty, F) = E(\infty, G)$ and $E(0, F) = E(0, G)$, where $m \geq 4$.

Suppose that $H \not\equiv 0$. From (11) and Lemma 2.15, it follows

$$(n-4)[T(r, f) + T(r, g)] \leq 2\overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + S(r, f) + S(r, g),$$

that is,

$$(n - 6 + \Theta(\infty; f) + \Theta(\infty; g) - 2\varepsilon)\, T(r) \leq S(r). \tag{17}$$

If $n = 5$ and $\Theta(\infty, f) + \Theta(\infty, g) > 1$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g) - 1)$. Thus, we can get a contradiction from (17).

If $n \geq 6$ and $\Theta(\infty, f) + \Theta(\infty, g) > 0$, we can take any real number $\varepsilon(0 < 2\varepsilon < \Theta(\infty, f) + \Theta(\infty, g))$. Thus, we can get a contradiction from (17).

Suppose that $H \equiv 0$, by using the same argument as in Theorem 1.3, we can get the conclusion of Theorem 1.4 easily.

Thus, we complete the proof of Theorem 1.4. □

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Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + \frac{x_n}{x_{n-1}\, x_{n-2}}$

Keying Liu[1,2], Peng Li[2], Fei Han[3], Weizhou Zhong[1,4]

1. School of Economics and Finance, Xi'an Jiaotong University, Xi'an 710061, China
2. School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450045, China
3. The Hilltiop Institute, University of Maryland Baltimore County, Baltimore, Maryland 21250, USA
4. College of Business Administration, Huaqiao University, Quanzhou 362021, China

Correspondence should be addressed to Zhong Weizhou, weizhou@mail.xjtu.edu.cn.

Abstract

In this paper, global dynamics of a kind of nonlinear difference equation was investigated, which had only one positive equilibrium. Every positive solution of the equation either converges to its positive equilibrium if it is locally asymptotically stable or nonhyperbolic or oscillates about its positive equilibrium if it is a saddle point. For the latter case, the length of positive or negative semicycles is no more than four.

Key words: Difference equations; Stable; Semicycles

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1 Introduction

In this paper, we focus on the nonlinear difference equation of third order

$$x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}, \qquad n = 0, 1, \cdots, \tag{1.1}$$

with the parameter A and initial conditions $x_{-2},\ x_{-1},\ x_0$ being positive. We will investigate its global dynamics and present the following results:

If $A \geq 1/\sqrt{2},$ then every positive solution of (1.1) converges to $\bar{x} = (A + \sqrt{A^2+4})/2$.

If $0 < A < 1/\sqrt{2},$ then every positive solution of (1.1) oscillates about $\bar{x}$.

Equation (1.1) can be regarded as a variation of the following equation

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^q\, x_{n-2}^r}, \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

In 2009, Iričanin and Stević [1] studied the boundedness character of positive solutions of (1.2) when all parameters A, p, q and r are positive.

It can be also regarded as a variation of the following systems of difference equations.

In 2015, Zhang et al.[2] investigated the following system

$$x_{n+1} = A + \frac{x_n}{y_{n-1}\, y_{n-2}}, \qquad y_{n+1} = A + \frac{y_n}{x_{n-1}\, x_{n-2}}, \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where A is a positive constant and initial conditions $x_{-i},\ y_{-i} \in (0, \infty)$ for $i = 0, -1, -2$.

The equilibria of system (1.3) satisfy the following system of equations:

$$\bar{x} = A + \frac{\bar{x}}{\bar{y}^2}, \qquad \bar{y} = A + \frac{\bar{y}}{\bar{x}^2}, \tag{1.4}$$

By eliminating y from (1.4), we obtain

$$(A^2 - 1)\bar{x}^4 - A^3\bar{x}^3 + 2\bar{x}^2 - 1 = 0. \tag{1.5}$$

Denote $c = (A + \sqrt{A^2 + 4})/2$, $a = 2/(A - \sqrt{4 - 3A^2})$ and $b = 2/(A + \sqrt{4 - 3A^2})$.

If $A > 0$, then system (1.3) always has the positive equilibrium $(c,\, c)$.

If $1 < A < 2/\sqrt{3}$, then system (1.3) has two additional positive equilibrium $(a,\, b),\ (b,\, a)$.

The main results[2] are listed in the following:

(A.1) If $A > 1$, every positive solution of system (1.3) is bounded.

(A.2) If $A > 2/\sqrt{3},\ (c,\, c)$ is locally asymptotically stable.

(A.3) If $A > \sqrt{3}$, every positive solution of system (1.3) approaches $(c,\, c)$.

(A.4) If $1 < A < 2/\sqrt{3},\ (a,\, b)$ and $(b,\, a)$ are locally asymptotically stable.

In 2005, Yang[3] studied the global behavior of the following system

$$x_n = A + \frac{y_{n-1}}{x_{n-p}\, y_{n-q}}, \qquad y_n = A + \frac{x_{n-1}}{x_{n-r}\, y_{n-s}}, \qquad n = 1, 2, \ldots, \tag{1.6}$$

where $p, q, r, s \geq 2$, A is a positive constant and initial conditions are positive real numbers. System (1.6) has the unique positive equilibrium $(c,\, c)$. He obtained the following results:

(B.1) If $A > 1,$ every positive solution of system (1.6) is bounded.

(B.2) If $A > 2/\sqrt{3},\ (c,\, c)$ is locally asymptotically stable.

(B.3) If $A > \sqrt{2},$ every positive solution of system (1.6) approaches $(c,\, c)$.

For these two systems, they gave the same condition $A > 2/\sqrt{3}$ to obtain the local asymptotical stability of $(c,\, c)$ by the same method. Other related articles see[4]-[8]. Other methods include Routh-Hurwitz criterion[9] and Rouche's Theorem[10]. Other types of equation and boundedness nature and so on see [11]-[18].

2 Local Stability

First of all, we obtain the local stability of $\bar{x}$ of (1.1).

As for the definition of stability and the method of linearized stability, see [17],[18] and related papers[2–10].

As is known, $\bar{x}$ of (1.1) satisfies $\bar{x} = A + 1/\bar{x}$ and $\bar{x} = c$ is positive, which is unique.

The linearized equation of (1.1) about $\bar{x} = c$ is

$$x_{n+1} = c^{-2}x_n - c^{-2}x_{n-1} - c^{-2}x_{n-2} \tag{2.2}$$

and its characteristic polynomial is

$$f(\lambda) = \lambda^3 - c^{-2}\lambda^2 + c^{-2}\lambda + c^{-2}, \tag{2.3}$$

from which we have

$$f(0) = c^{-2} > 0, \tag{2.4}$$
$$f(1) = c^{-2} + 1 > 0, \tag{2.5}$$
$$f(-0.5) = \frac{1}{8c^2}(2 - c^2), \tag{2.6}$$
$$f(-1) = -c^{-2} - 1 < 0, \tag{2.7}$$

which lead to the existence of a solution λ_0 of $f(\lambda) = 0$ in the interval $(-1,\, 0)$.

On the other hand,

$$f'(\lambda) = 3\lambda^2 - 2c^{-2}\lambda + c^{-2} > 0 \tag{2.8}$$

because $\Delta = 4c^{-4} - 12c^{-2} = 4c^{-4}(1 - 3c^2) < 0$ for $A > 0$ and $c > 1$. Hence, we claim that $f(\lambda) = 0$ has a pair of conjugate complex roots $\lambda_{1,\,2}$.

Let $f(\lambda)$ be factorize as the following

$$\begin{aligned} f(\lambda) &= \lambda^3 - c^{-2}\lambda^2 + c^{-2}\lambda + c^{-2} \\ &= (\lambda - \lambda_0)(\lambda^2 + a\lambda + b) \\ &= \lambda^3 + (a - \lambda_0)\lambda^2 + (b - a\lambda_0)\lambda - b\lambda_0, \end{aligned} \tag{2.9}$$

from which we have

$$a - \lambda_0 = -c^{-2}, \qquad b - a\lambda_0 = c^{-2}, \qquad -b\lambda_0 = c^{-2}.$$

Thus, we have the modulus of the complex roots $\lambda_{1,\,2}$

$$|\lambda_{1,\,2}|^2 = b = \lambda_0^2 - c^{-2}\lambda_0 + c^{-2}. \tag{2.10}$$

We claim that $|\lambda_{1,\,2}| < 1$ if $A > 1/\sqrt{2}$.

In fact, we have that

$$|\lambda_{1,\,2}| < 1 \Longleftrightarrow b < 1 \Longleftrightarrow h(\lambda_0) < 0 \tag{2.11}$$

with $h(x) = x^2 - c^{-2}x + c^{-2} - 1$ for $\lambda_0 \in (-1,\, 0)$. As is known, $h(x) = 0$ has two distinct roots: $x_1 = 1$ and $x_2 = 2c^{-2} - 2$. Further, $h(-0.5) = 3(2 - c^2)/(4c^2)$.

If $A > 1/\sqrt{2}$, then we have that $c^2 > 2$ and $\lambda_0 \in (-0.5,\, 0)$, from which $|\lambda_{1,\,2}| < 1$.

If $A = 1/\sqrt{2}$, then $c = \sqrt{2}$ and $\lambda_0 = -0.5$, thus $|\lambda_{1,\,2}| = 1$.

If $A < 1/\sqrt{2}$, then $c < \sqrt{2}$ and $\lambda_0 \in (-1,\, -0.5)$, thus $|\lambda_{1,\,2}| > 1$.

A nonhyperbolic equilibrium point of difference equations or system of difference equations is called non-hyperbolic point of stable type (resp. of unstable type) if the other characteristic value of the Jacobian matrix about it is in interval (-1, 1) (resp. outside of interval [-1, 1]).

We generalize the above into the following theorem.

Theorem 2.1. *Assume that* $\bar{x} = c$ *is the positive equilibrium of* (1.1).

1. *If* $A > 1/\sqrt{2}$, *then* $\bar{x} = c$ *is locally asymptotically stable.*
2. *If* $A = 1/\sqrt{2}$, *then* $\bar{x} = c$ *is nonhyperbolic of stable type.*
3. *If* $0 < A < 1/\sqrt{2}$, *then* $\bar{x} = c$ *is a saddle point .*

3 Global Dynamics

In this section, we present the main results by investigating the global dynamics of (1.1), which has only one positive equilibrium – being locally asymptotically stable or nonhyperbolic of stable type or a saddle point depending on the parameter A .

First, we show the boundedness of positive solutions of (1.1).

By Theorem 3[1], if $p^2 < 3q$, then every positive solution of (1.2) is bounded. Thus, every positive solution of (1.1) is bounded and $x_{n+1} \leq A + 1/A + 1/A^3$ for $n > 4$.

Second, we show the semicycles of solutions of (1.1).

From (1.1), we obtain

$$x_{n+1} - \bar{x} = \frac{x_n}{x_{n-1}\, x_{n-2}} - \frac{1}{\bar{x}} = \frac{\bar{x}}{x_{n-1}\, x_{n-2}} \left(\frac{x_n}{\bar{x}} - \frac{x_{n-1}}{\bar{x}} \frac{x_{n-2}}{\bar{x}} \right) \tag{3.1}$$

and

$$x_{n+2} \;=\; A + \frac{x_{n+1}}{x_n\, x_{n-1}} = A + \frac{A}{x_n\, x_{n-1}} + \frac{1}{x_{n-1}^2\, x_{n-2}} \tag{3.2}$$

Then the following statements are true:

1. If for some $N \geq 0$, x_{N-2}, $x_{N-1} \leq \bar{x}$ and $x_N \geq \bar{x}$, then $x_{N+1} \geq \bar{x}$.
2. If for some $N \geq 0$, x_{N-2}, $x_{N-1} \geq \bar{x}$ and $x_N \leq \bar{x}$, then $x_{N+1} \leq \bar{x}$.
3. If for some $N \geq 0$, x_{N-2}, x_{N-1}, $x_N \geq \bar{x}$, then $x_{N+2} \leq \bar{x}$.
4. If for some $N \geq 0$, x_{N-2}, x_{N-1}, $x_N \leq \bar{x}$, then $x_{N+2} \geq \bar{x}$.

From the above, we obtain that every positive solution of (1.1) is bounded and converges to $\bar{x}$ or oscillates about $\bar{x}$.

Now, we try to analyze the global dynamics of (1.1) for three cases.

3.1 Case $A > 1/\sqrt{2}$

In this case, the positive equilibrium $\bar{x} = c$ of (1.1) is locally asymptotically stable. We will show that it is an attractor and is globally asymptotically stable.

Theorem 3.1. *Assume that* $\bar{x} = c$ *is the positive equilibrium of* (1.1).
If $A > 1/\sqrt{2}$, *then* $\bar{x} = c$ *is globally asymptotically stable.*

Proof.

Let $(m,\, M) \in [A,\, A+1/A+1/A^3\,]^2$ be a solution of the system

$$m = A + \frac{m}{M^2}, \qquad M = A + \frac{M}{m^2}, \tag{3.3}$$

From (3.3), we could derive

$$\frac{m^3}{m^2-1} = \frac{M^3}{M^2-1}. \tag{3.4}$$

As is known, the function $k(x) = x^3/(x^2-1)$ is strictly increasing for $x > \sqrt{3}$ and is strictly decreasing for $0 < x < 1$ and $1 < x < \sqrt{3}$. Therefore, (3.4) holds only for $A \le m = M$.

By Theorem A.0.5 [17], every positive solution of (1.1) converges to $\bar{x} = c$ for $A > 1/\sqrt{2}$, that is, $\bar{x}$ is a global attractor. Hence, we obtain the result. □

It is worth pointing out that under the condition $A > 1/\sqrt{2}$, $\bar{x} = c$ of (1.1) is locally asymptotically stable which implies that it is globally asymptotically stable, that is, locally asymptotically stable $\Longrightarrow$ globally asymptotically stable.

3.2 Case $A = 1/\sqrt{2}$

In this case, the positive equilibrium $\bar{x} = \sqrt{2}$ of (1.1) is nonhyperbolic of stable type.

Theorem 3.2. *If $A = 1/\sqrt{2}$, then every positive solution of* (1.1) *converges to $\bar{x}$.*

Proof.

Let $\{x_n\}$ be a positive solution of (1.1). We assume that $m = \liminf_{n\to\infty} x_n$ and $M = \limsup_{n\to\infty} x_n$, then $A \le m \le M < \infty$. From (1.1), we obtain

$$m \ge A + \frac{m}{M^2}, \qquad M \le A + \frac{M}{m^2}, \tag{3.5}$$

from which we get

$$m \ge \frac{AM^2}{M^2-1}, \qquad m^2 \le \frac{M}{M-A}. \tag{3.6}$$

It follows that

$$(A^2-1)M^4 - A^3M^3 + 2M^2 - 1 \le 0. \tag{3.7}$$

Similarly, we obtain

$$(A^2-1)m^4 - A^3m^3 + 2m^2 - 1 \ge 0, \tag{3.8}$$

As is known, for $A = 1/\sqrt{2}$, the equation $(A^2-1)x^4 - A^3x^3 + 2x^2 - 1 = 0$, which is (1.5), has exactly one root $c = \sqrt{2}$ greater than A, which implies that $\lim\limits_{n\to\infty} x_n = c = \sqrt{2}$

□

3.3 Case $0 < A < 1/\sqrt{2}$

In this case, the positive equilibrium $\bar{x} = c$ of (1.1) is a saddle point.

Theorem 3.3. *Suppose that $0 < A < 1/\sqrt{2}$ and let $\{x_n\}$ be a nontrivial solution of* (1.1). *Then the solution oscillates about $\bar{x}$ with semicycles of length no more than four.*

4 Numerical Results

Some numerical simulations are given to support our theoretical analysis with Matlab7.0.

Exam1 For $A = 0.8 > 1/\sqrt{2}$ and $(x_{-2},\ x_{-1},\ x_0) = (2.2,\ 2.3,\ 2)$, the solution of (1.1) converges to $\bar{x} = 1.4770$ by Theorem 3.1. See Figure 1.

Exam2 For $A = 1/\sqrt{2}$ and $(x_{-2},\ x_{-1},\ x_0) = (0.2,\ 1.8,\ 0.5)$, the solution of (1.1) converges to $\bar{x} = 1.4142$ by Theorem 3.2. See Figure 2.

Exam3 For $A = 0.5 < 1/\sqrt{2}$ and $(x_{-2},\ x_{-1},\ x_0) = (1.5,\ 1.6,\ 2.5)$, the solution of (1.1) oscillates about $\bar{x} = 1.2808$ by Theorem 3.3. See Figure 3. The length of positive or negative semicycles is no more than four.

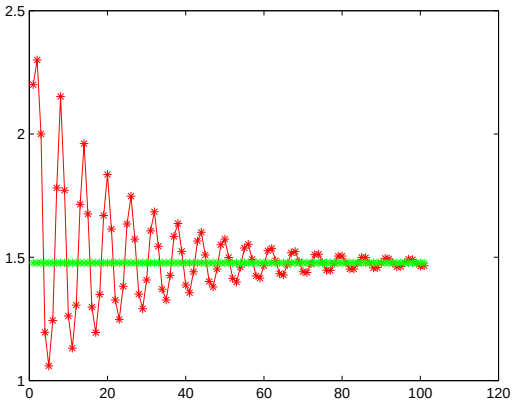


Figure 1: The solution of (1.1) converges to $\bar{x} = 1.4770$.

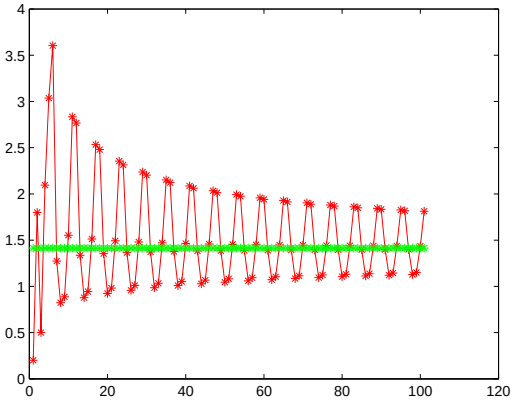


Figure 2: The solution of (1.1) converges to $\bar{x} = 1.4142$.

5 Conclusion

Difference equation appeals more and more attention in recent years. It is of great interest to

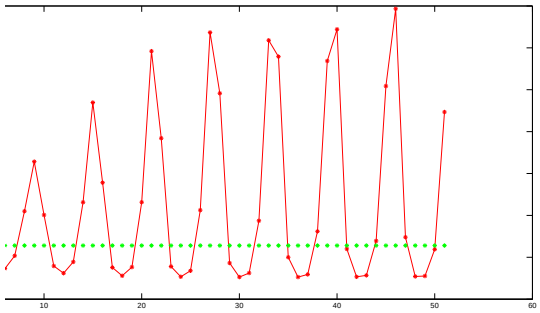


Figure 3: The solution of (1.1) oscillates about $\bar{x} = 1.2808$.

investigate of the monotonicity, periodicity, boundedness nature and global dynamics of kinds of difference equations and system of difference equations.

It is known that the techniques in the investigation of the behavior of difference equations can be used in investigating equations arising in mathematical models describing real life situations in biology, economics, physics, sociology, control theory and vice versa.

Here, we consider the local stability of the positive equilibrium of such a system by investigating the distribution of roots of the corresponding characteristic polynomial with order three. For the global dynamics, we analyze three cases according to the positive equilibrium, especially we consider the the distribution of roots of an auxiliary equation of order four which arises naturally in the proof. In one word, we consider the distribution of roots of two particular polynomial of higher order. The popular methods including the Schur-Cohn criterion, Rouché Theorem, Routh-Hurwitz criterion and Jury criterion and so on.

The complex dynamics of such a equation we considered should play an important role on the dynamics of systems such as (1.3) and (1.6) and so on. We believe that they should be paid more attention on the global dynamics.

Conflict of Interests

The authors declare that they have no competing interests.

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Regular hesitant fuzzy filters and MV-hesitant fuzzy filters of residuated lattices

G. Muhiuddin[a,*] and Abdullah M. Al-roqi[b]

[a] *Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia*

[b] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

Abstract. The notion of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters are introduced, and related properties are investigated. Characterizations of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters are considered, relations between regular hesitant fuzzy filters and MV-hesitant fuzzy filters are discussed. Extension property for Boolean hesitant fuzzy filters (resp. MV-hesitant fuzzy filters) is established.

1. Introduction

Since the original definition of fuzzy sets by Zadeh in 1965, several extensions have been proposed for fuzzy sets, for example, type 2 fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and fuzzy multisets etc. Another extension of fuzzy sets, so called hesitant fuzzy sets, has been proposed in [7]. The motivation for introducing hesitant fuzzy sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values. As a non-classical logic system, residuated lattices are a formal and useful tool for computer science to deal with uncertain and fuzzy information. Using the notion of hesitant fuzzy sets, Jun and Song [4] have studied filter theory in MTL-algebras. Also, Muhiuddin [5] have discussed filter theory in residuated lattices.

In this paper, we introduce the notion of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters, and investigate related properties. We consider

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*Corresponding author.

E-mail: chishtygm@gmail.com (G. Muhiuddin), aalroqi@kau.edu.sa (Abdullah M. Al-roqi)

characterizations of regular hesitant fuzzy filters, Boolean hesitant fuzzy filters and MV-hesitant fuzzy filters. We discuss relations between regular hesitant fuzzy filters and MV-hesitant fuzzy filters. We establish extension property for Boolean hesitant fuzzy filters (resp. MV-hesitant fuzzy filters).

2. Preliminaries

Definition 2.1 ([1, 2, 3])**.** A residuated lattice is an algebra $\mathcal{L} := (L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(2) $(L, \odot, 1)$ is a commutative monoid.
(3) $\odot$ and $\to$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \to z \;\Leftrightarrow\; x \odot y \leq z)\,.$$

A regular residuated lattice is a residuated lattice $\mathcal{L}$ satisfying the following regularity equation:

$$(\forall x \in L)\,((x \to 0) \to 0 = x)\,. \tag{2.1}$$

A residuated lattice L is called an MTL-algebra if it satisfies:

$$(\forall x, y \in L)\,((x \to y) \vee (y \to x) = 1)\,. \tag{2.2}$$

A residuated lattice L is called a BL-algebra if it satisfies the condition (2.2) and

$$(\forall x, y \in L)\,(x \wedge y = x \odot (x \to y))\,. \tag{2.3}$$

In a residuated lattice L, the ordering $\leq$ is defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \to y = 1)$$

and x' will be reserved for $x \to 0$, and $x'' = (x')'$, etc. for all $x \in L$.

Proposition 2.2 ([1, 2, 3, 8, 9]). *In a residuated lattice $\mathcal{L}$, the following properties are valid.*

$$1 \to x = x,\ x \to 1 = 1,\ x \to x = 1,\ 0 \to x = 1,\ x \to (y \to x) = 1. \tag{2.4}$$

$$x \to (y \to z) = (x \odot y) \to z = y \to (x \to z). \tag{2.5}$$

$$x \leq y \ \Rightarrow\ z \to x \leq z \to y,\ y \to z \leq x \to z. \tag{2.6}$$

$$z \to y \leq (x \to z) \to (x \to y),\ z \to y \leq (y \to x) \to (z \to x). \tag{2.7}$$

$$x' = x''',\ x \leq x'',\ 1' = 0,\ 0' = 1. \tag{2.8}$$

$$x' \wedge y' = (x \vee y)'. \tag{2.9}$$

$$x \vee x' = 1 \ \Rightarrow\ x \wedge x' = 0. \tag{2.10}$$

Definition 2.3 ([6]). A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a filter of $\mathcal{L}$ if it satisfies the conditions:

$$(\forall x, y \in L)\,(x, y \in F \ \Rightarrow\ x \odot y \in F)\,. \tag{2.11}$$

$$(\forall x, y \in L)\,(x \in F,\ x \leq y \ \Rightarrow\ y \in F)\,. \tag{2.12}$$

Proposition 2.4 ([6]). *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

$$1 \in F. \tag{2.13}$$

$$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \ \Rightarrow\ y \in F)\,. \tag{2.14}$$

Definition 2.5 ([6]). A nonempty subset F of $\mathcal{L}$ is called a *Boolean filter* of a residuated lattice $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x \in L)\,(x \vee x' \in F)\,. \tag{2.15}$$

Zhu and Xu [10] introduced the notion of a regular filter in a residuated lattice.

Definition 2.6 ([10]). A filter F of $\mathcal{L}$ is said to be regular if it satisfies the following condition:

$$(\forall x \in L)\,(x'' \to x \in F)\,. \tag{2.16}$$

Lemma 2.7 ([10]). *Let F be a filter of $\mathcal{L}$. Then the following assertions are equivalent:*

(1) *F is regular.*
(2) $(\forall x, y \in L)\,(x' \to y \in F \ \Rightarrow\ y' \to x \in F)\,.$

Definition 2.8 ([10]). A subset F of $\mathcal{L}$ is called an MV-filter of $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies:

$$(\forall x, y \in L)\,(y \to x \in F \;\Rightarrow\; ((x \to y) \to y) \to x \in F)\,. \tag{2.17}$$

Lemma 2.9 ([10]). *A filter F of $\mathcal{L}$ is an MV-filter of $\mathcal{L}$ if and only if it satisfies the condition:*

$$(\forall x, y \in L)\,(((x \to y) \to y) \to ((y \to x) \to x) \in F)\,. \tag{2.18}$$

3. Regular hesitant fuzzy filters

Let E be a reference set. A *hesitant fuzzy set* on E (see [7]) is defined in terms of a function $\mathcal{H}$ that when applied to E returns a subset of $[0,1]$, that is, $\mathcal{H} : E \to \mathscr{P}([0,1])$.

In what follows, we take a residuated lattice $\mathcal{L}$ as a reference set. For a hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ and $\tau \in \mathscr{P}([0,1])$, we consider a set

$$\mathcal{H}_\tau := \{x \in L \mid \tau \subseteq \mathcal{H}(x)\}$$

which is called the *τ-hesitant level set* on $\mathcal{L}$.

Definition 3.1 ([5]). A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is called a hesitant fuzzy filter of $\mathcal{L}$ if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.

Lemma 3.2 ([5]). *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a hesitant fuzzy filter of $\mathcal{L}$ if and only if the following assertions are valid.*

$$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; \mathcal{H}(x) \subseteq \mathcal{H}(y))\,, \tag{3.1}$$

$$(\forall x, y \in L)\,(\mathcal{H}(x) \cap \mathcal{H}(y) \subseteq \mathcal{H}(x \odot y))\,. \tag{3.2}$$

Lemma 3.3 ([5]). *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies*

$$(\forall x \in L)\,(\mathcal{H}(x) \subseteq \mathcal{H}(1))\,. \tag{3.3}$$

$$(\forall x, y \in L)\,(\mathcal{H}(x) \cap \mathcal{H}(x \to y) \subseteq \mathcal{H}(y))\,. \tag{3.4}$$

Definition 3.4. A hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ is said to be regular if it satisfies:

$$(\forall x \in L)\,(\mathcal{H}(x'' \to x) = \mathcal{H}(1))\,. \tag{3.5}$$

Note that the notion of regular hesitant fuzzy filters coincides with the notion of hesitant fuzzy filters in a regular residuated lattice.

Theorem 3.5. *For a hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$, the following assertions are equivalent:*

(1) *$\mathcal{H}$ is regular.*

(2) $(\forall x, y \in L)\,(\mathcal{H}(x' \to y') \subseteq \mathcal{H}(y \to x))$.
(3) $(\forall x, y \in L)\,(\mathcal{H}(x' \to y) \subseteq \mathcal{H}(y' \to x))$.

Proof. Assume that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ and let $x, y \in L$. Using (2.6) and (2.8), we have

$$x' \to y' \leq y'' \to x'' \leq y \to x''.$$

It follows from (2.7) and (2.6) that

$$\begin{aligned} x'' \to x &\leq (y \to x'') \to (y \to x) \\ &\leq (x' \to y') \to (y \to x). \end{aligned}$$

Hence, by (3.3), (3.5) and (3.4), we have

$$\begin{aligned} \mathcal{H}(x' \to y') &= \mathcal{H}(x' \to y') \cap \mathcal{H}(1) \\ &= \mathcal{H}(x' \to y') \cap \mathcal{H}(x'' \to x) \\ &\subseteq \mathcal{H}(x' \to y') \cap \mathcal{H}((x' \to y') \to (y \to x)) \\ &\subseteq \mathcal{H}(y \to x), \end{aligned}$$

and so the second assertion holds. Since $x' \to y \leq y' \to x''$, we have

$$x'' \to x \leq (y' \to x'') \to (y' \to x) \leq (x' \to y) \to (y' \to x)$$

by (2.7) and (2.6). It follows from (3.3), (3.5) and (3.4) that

$$\begin{aligned} \mathcal{H}(x' \to y) &= \mathcal{H}(x' \to y) \cap \mathcal{H}(1) \\ &= \mathcal{H}(x' \to y) \cap \mathcal{H}(x'' \to x) \\ &\subseteq \mathcal{H}(x' \to y) \cap \mathcal{H}((x' \to y) \to (y' \to x)) \\ &\subseteq \mathcal{H}(y' \to x). \end{aligned}$$

Hence the third condition is valid. Next, suppose that the second condition holds. The second condition together with the condition (2.8) induces

$$\mathcal{H}(1) = \mathcal{H}(x' \to x''') \subseteq \mathcal{H}(x'' \to x)$$

for all $x \in L$, and so $\mathcal{H}(x'' \to x) = \mathcal{H}(1)$. Hence $\mathcal{H}$ is regular. Finally, assume that the third condition is valid. Since $x' \to x' = 1$ for all $x \in L$, it follows from the third condition that $\mathcal{H}(1) = \mathcal{H}(x' \to x') \subseteq \mathcal{H}(x'' \to x)$, and that $\mathcal{H}(x'' \to x) = \mathcal{H}(1)$ by (3.3). Therefore $\mathcal{H}$ is regular. □

Theorem 3.6. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,(\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) \subseteq \mathcal{H}(y' \to x))\,. \tag{3.6}$$

Proof. Assume that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. Clearly the condition (3.3) is true. If we use the condition (3.4) and Theorem 3.5(3), then get

$$\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) \subseteq \mathcal{H}(x' \to y) \subseteq \mathcal{H}(y' \to x)$$

for all $x, y, z \in L$.

Conversely, suppose that $\mathcal{H}$ satisfies two conditions (3.3) and (3.6). Let $x, y \in L$. Since

$$x \to y = x \to (1 \to y) = x \to (0' \to y)$$

and

$$y'' = 1 \to y'' = 1 \to (y' \to 0),$$

it follows from (2.4), (3.3) and (3.6) that

$$\begin{aligned}
\mathcal{H}(x) \cap \mathcal{H}(x \to y) &= \mathcal{H}(x) \cap \mathcal{H}(x \to (0' \to y)) \\
&\subseteq \mathcal{H}(y' \to 0) \\
&= \mathcal{H}(y'') \\
&= \mathcal{H}(1) \cap \mathcal{H}(1 \to (y' \to 0)) \\
&\subseteq \mathcal{H}(0' \to y) \\
&= \mathcal{H}(1 \to y) = \mathcal{H}(y).
\end{aligned}$$

Therefore $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$ by Lemma 3.3. If we take $z := 1$ in (3.6) and use (2.4) and (3.3), then

$$\begin{aligned}
\mathcal{H}(y' \to x) &\supseteq \mathcal{H}(1) \cap \mathcal{H}(1 \to (x' \to y)) \\
&= \mathcal{H}(1 \to (x' \to y)) = \mathcal{H}(x' \to y).
\end{aligned}$$

Hence $\mathcal{H}$ is regular by Theorem 3.5. □

Theorem 3.7. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,(\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y')) \subseteq \mathcal{H}(y \to x))\,. \tag{3.7}$$

Proof. The proof is similar to the proof of Theorem 3.6. □

Theorem 3.8. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a regular hesitant fuzzy filter of $\mathcal{L}$ if and only if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is a regular filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.*

Proof. Assume that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. Let $\tau \in \mathscr{P}([0,1])$ be such that $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$. Let $x, y \in L$ be such that $x' \to y \in \mathcal{H}_\tau$. Then $\tau \subseteq \mathcal{H}(x' \to y) \subseteq \mathcal{H}(y' \to x)$ by Theorem 3.5, and so $y' \to x \in \mathcal{H}_\tau$. Hence $\mathcal{H}_\tau$ is regular by Lemma 2.7.

Conversely, suppose that $\mathcal{H}_\tau$ is a regular filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$, and thus $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$. For any $x, y \in L$, let $\mathcal{H}(x' \to y) = \delta$. Then $x' \to y \in \mathcal{H}_\delta$ which implies from Lemma 2.7 that $y' \to x \in \mathcal{H}_\delta$. Hence $\mathcal{H}(x' \to y) = \delta \subseteq \mathcal{H}(y' \to x)$, and so $\mathcal{H}$ is regular by Theorem 3.5. □

Theorem 3.9. *Every regular filter of $\mathcal{L}$ can be represented as a τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ for some $\tau \in \mathscr{P}([0,1]) \setminus \{\emptyset\}$ and a regular hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$.*

Proof. Let F be a regular filter of $\mathcal{L}$ and let $\mathcal{H}$ be a hesitant fuzzy set on $\mathcal{L}$ defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \tau & \text{if } x \in F, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}([0,1]) \setminus \{\emptyset\}$. Since $1 \in F$, we have $\mathcal{H}(x) \subseteq \tau = \mathcal{H}(1)$ for all $x \in L$. Let $x, y, z \in L$. If $z \in F$ and $z \to (x' \to y) \in F$, then $y' \to x \in F$ by Proposition 2.4 and Lemma 2.7. Hence $\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) = \tau = \mathcal{H}(y' \to x)$. Suppose that $z \notin F$ or $z \to (x' \to y) \notin F$. Then $\mathcal{H}(z) = \emptyset$ or $\mathcal{H}(x' \to y) = \emptyset$, and so $\mathcal{H}(z) \cap \mathcal{H}(z \to (x' \to y)) = \emptyset \subseteq \mathcal{H}(y' \to x)$. It follows from by Theorem 3.6 that $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$, and it is obvious that $F = \mathcal{H}_\tau$. This completes the proof. □

4. MV-hesitant fuzzy filters

Definition 4.1. A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is called an MV-hesitant fuzzy filter of $\mathcal{L}$ if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is an MV-filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.

Theorem 4.2. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$ if it is a hesitant fuzzy filter of $\mathcal{L}$ with the following additional condition:*

$$(\forall x, y \in L)\,(\mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x))\,. \tag{4.1}$$

Proof. If $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$, then $\mathcal{H}_\tau$ is an MV-filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Hence it is filter of $\mathcal{L}$, and so $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$. For any $x, y \in L$, let $\mathcal{H}(y \to x) = \delta$. Then $y \to x \in \mathcal{H}_\delta$, and so $((x \to y) \to y) \to x \in \mathcal{H}_\delta$ since $\mathcal{H}_\delta$ is an MV-filter of $\mathcal{L}$. Thus $\mathcal{H}(y \to x) = \delta \subseteq \mathcal{H}(((x \to y) \to y) \to x)$.

Conversely, suppose that $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$ which satisfies the condition (4.1). Let $\tau \in \mathscr{P}([0,1])$ be such that $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$. Let $x, y \in L$ be such that $y \to x \in \mathcal{H}_\tau$. Using the condition (4.1), we have

$$\tau \subseteq \mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x)$$

which implies that $((x \to y) \to y) \to x \in \mathcal{H}_\tau$. Therefore $\mathcal{H}_\tau$ is an MV-filter of $\mathcal{L}$, and thus $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.3. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,(\mathcal{H}(z) \cap \mathcal{H}(z \to (y \to x)) \subseteq \mathcal{H}(((x \to y) \to y) \to x))\,. \tag{4.2}$$

Proof. Assume that $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. Using (3.4) and (4.1), we have

$$\mathcal{H}(z) \cap \mathcal{H}(z \to (y \to x)) \subseteq \mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x)$$

for all $x, y \in L$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set on $\mathcal{L}$ which satisfies two conditions (3.3) and (4.2). Taking $y := 1$ in (4.2) and using (2.4) induces the condition (3.4). Hence $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$ by Lemma 3.3. If we take $z := 1$ in (4.2) and use (2.4) and (3.3), then we know that $\mathcal{H}$ satisfies the condition (4.1). Therefore $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.4. *Let $\mathcal{H}$ be a hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$ if and only if the following assertion is valid:*

$$(\forall x, y \in L)\,(\mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x)) = \mathcal{H}(1))\,. \tag{4.3}$$

Proof. Assume that $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$, and so $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. In particular, $\mathcal{H}_{\mathcal{H}(1)}$ is a filter of $\mathcal{L}$. Let $x, y \in L$ be such that $y \to x \in \mathcal{H}_{\mathcal{H}(1)}$. Then

$$\mathcal{H}(1) \subseteq \mathcal{H}(y \to x) \subseteq \mathcal{H}(((x \to y) \to y) \to x),$$

and so $((x \to y) \to y) \to x \in \mathcal{H}_{\mathcal{H}(1)}$. Therefore $\mathcal{H}_{\mathcal{H}(1)}$ is an MV-filter of $\mathcal{L}$, and thus

$$((x \to y) \to y) \to ((y \to x) \to x) \in \mathcal{H}_{\mathcal{H}(1)}$$

by Lemma 2.9. Hence $\mathcal{H}(1) \subseteq \mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x))$, which implies from (3.3) that $\mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x)) = \mathcal{H}(1)$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy filter of $\mathcal{L}$ that satisfies the condition (4.3). Using (3.3), (4.3), (2.5) and (3.4), we obtain

$$\begin{aligned}\mathcal{H}(y \to x) &= \mathcal{H}(y \to x) \cap \mathcal{H}(1) \\ &= \mathcal{H}(y \to x) \cap \mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x)) \\ &= \mathcal{H}(y \to x) \cap \mathcal{H}((y \to x) \to (((x \to y) \to y) \to x)) \\ &\subseteq \mathcal{H}(((x \to y) \to y) \to x).\end{aligned}$$

Therefore $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.5. *Every MV-hesitant fuzzy filter is a regular hesitant fuzzy filter.*

Proof. Let $\mathcal{H}$ be an MV-hesitant fuzzy filter of $\mathcal{L}$. If we take $y := 0$ in (4.1) and use (2.4), then

$$\mathcal{H}(1) = \mathcal{H}(0 \to x) \subseteq \mathcal{H}(((x \to 0) \to 0) \to x) = \mathcal{H}(x'' \to x)$$

and so $\mathcal{H}(x'' \to x) = \mathcal{H}(1)$ by (3.1). Therefore $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. □

The converse of Theorem 4.5 is not true in general as seen in the following example.

Example 4.6. Let $\mathcal{L} := [0, 1]$ (unit interval). For any $a, b \in L$, define

$$a \vee b = \max\{a, b\},\ a \wedge b = \min\{a, b\},$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ (1 - a) \vee b & \text{otherwise,} \end{cases} \quad \text{and } a \odot b = \begin{cases} 0 & \text{if } a + b \leq 1, \\ a \wedge b & \text{otherwise.} \end{cases}$$

Then $\mathcal{L} := (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice (see [10]). Let $\mathcal{H}$ be a hesitant fuzzy set on $\mathcal{L}$ defined by

$$\mathcal{H} : L \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} (x, 1] & \text{if } x \in [0.5, 1], \\ \emptyset & \text{otherwise,} \end{cases}$$

Then $\mathcal{H}$ is a regular hesitant fuzzy filter of $\mathcal{L}$. Let $F := (c, 1]$ for any $c \in L$. Note that if $c \in [0.5, 1]$ then F is a regular filter of $\mathcal{L}$. But, if $c \in (0.7, 1]$ then F is not an MV-filter of $\mathcal{L}$ since $0.4 \to 0.7 = 1 \in F$, but $((0.7 \to 0.4) \to 0.4) \to 0.7 = 0.7 \notin F$. Hence the hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ which is given as follows:

$$\mathcal{H} : L \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} U & \text{if } x \in F, \\ \emptyset & \text{otherwise,} \end{cases}$$

is a hesitant fuzzy filter of $\mathcal{L}$ which is regular. But, since $\mathcal{H}(0.4 \to 0.7) = \mathcal{H}(1) = U$ and $\mathcal{H}(((0.7 \to 0.4) \to 0.4) \to 0.7) = \mathcal{H}(0.7) = \emptyset$. Therefore $\mathcal{H}$ is not an MV-hesitant fuzzy filter of $\mathcal{L}$.

The following theorem shows that the converse of Theorem 4.5 is true in BL-algebras.

Theorem 4.7. *In a BL-algebra $\mathcal{L}$, the notion of an MV-hesitant fuzzy filter coincides with the notion of a regular hesitant fuzzy filter.*

Proof. Based on Theorem 4.5, it is sufficient to show that every regular hesitant fuzzy filter is an MV-hesitant fuzzy filter. Let $\mathcal{H}$ be a regular hesitant fuzzy filter of a BL-algebra $\mathcal{L}$ and let $x, y \in L$. Then $\mathcal{H}(x' \to y') \subseteq \mathcal{H}(y \to x)$ by Theorem 3.5. Since $y \to x \leq x' \to y'$, we have $\mathcal{H}(y \to x) \subseteq \mathcal{H}(x' \to y')$ by (3.1). Hence

$$\begin{aligned}
\mathcal{H}(y \to x) &= \mathcal{H}(x' \to y') = \mathcal{H}(x' \to (x' \to y')) \\
&= \mathcal{H}(x' \to (y' \odot (y' \to x'))) \\
&= \mathcal{H}(x' \to (y' \odot (x \to y''))) \\
&= \mathcal{H}((y' \odot (x \to y''))' \to x) \\
&= \mathcal{H}(((x \to y'') \to (y' \to 0)) \to x) \\
&= \mathcal{H}(((x \to y'') \to y'') \to x)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}(1) &= \mathcal{H}(y' \to y') = \mathcal{H}(y'' \to y) \\
&\subseteq \mathcal{H}((x \to y'') \to (x \to y)) \\
&\subseteq \mathcal{H}(((x \to y) \to y'') \to ((x \to y'') \to y'')) \\
&\subseteq \mathcal{H}((((x \to y'') \to y'') \to x) \to (((x \to y) \to y'') \to x)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{H}(y \to x) &= \mathcal{H}(y \to x) \cap \mathcal{H}(1) \\
&\subseteq \mathcal{H}(((x \to y'') \to y'') \to x) \\
&\quad \cap \mathcal{H}((((x \to y'') \to y'') \to x) \to (((x \to y) \to y'') \to x)) \\
&\subseteq \mathcal{H}((((x \to y) \to y'') \to x)) \\
&\subseteq \mathcal{H}(((x \to y) \to y) \to x).
\end{aligned}$$

Therefore $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

Definition 4.8. A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is called a Boolean hesitant fuzzy filter of $\mathcal{L}$ if it is a hesitant fuzzy filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x \in L)\,(\mathcal{H}(x \vee x') = \mathcal{H}(1))\,. \tag{4.4}$$

Theorem 4.9. *A hesitant fuzzy set $\mathcal{H}$ on $\mathcal{L}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$ if and only if the τ-hesitant level set $\mathcal{H}_\tau$ on $\mathcal{L}$ is a Boolean filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$.*

Proof. Suppose that $\mathcal{H}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$ and let $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$, and so $1 \in \mathcal{H}_\tau$, that is, $\tau \subseteq \mathcal{H}(1)$. It follows from (4.4) that $\tau \subseteq \mathcal{H}(1) = \mathcal{H}(x \vee x')$ for all $x \in L$. Hence $x \vee x' \in \mathcal{H}_\tau$ for all $x \in L$, and therefore $\mathcal{H}_\tau$ is a Boolean filter of $\mathcal{L}$.

Conversely assume that $\mathcal{H}_\tau$ is a Boolean filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}([0,1])$ with $\mathcal{H}_\tau \neq \emptyset$. Then $\mathcal{H}_\tau$ is a filter of $\mathcal{L}$, and so $\mathcal{H}$ is a hesitant fuzzy filter of $\mathcal{L}$. Note that $1 \in \mathcal{H}_{\mathcal{H}(1)}$. Since $\mathcal{H}_{\mathcal{H}(1)}$ is a Boolean filter of $\mathcal{L}$, we have $x \vee x' \in \mathcal{H}_{\mathcal{H}(1)}$ for all $x \in L$. Hence $\mathcal{H}(x \vee x') = \mathcal{H}(1)$, and therefore $\mathcal{H}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 4.10. (Extension property) *Let $\mathcal{H}$ and $\mathcal{G}$ be hesitant fuzzy filters of $\mathcal{L}$ satisfying two conditions:*

(1) $\mathcal{H}(1) = \mathcal{G}(1)$,
(2) $(\forall x \in L)\,(\mathcal{H}(x) \subseteq \mathcal{G}(x))$.

If $\mathcal{H}$ is an MV-hesitant fuzzy filter (resp., a Boolean hesitant fuzzy filter) of $\mathcal{L}$, then so is $\mathcal{G}$.

Proof. Assume that $\mathcal{H}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}(x \vee x') = \mathcal{H}(1)$ for all $x \in L$. Using two conditions, we have

$$\mathcal{G}(x \vee x') \supseteq \mathcal{H}(x \vee x') = \mathcal{H}(1) = \mathcal{G}(1) \tag{4.5}$$

for all $x \in L$. Combining (3.3) and (4.5) implies that $\mathcal{G}(x \vee x') = \mathcal{G}(1)$. Therefore $\mathcal{G}$ is a Boolean hesitant fuzzy filter of $\mathcal{L}$.

Now suppose that $\mathcal{H}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. Using Theorem 4.4, we have

$$\begin{aligned}\mathcal{G}(1) &= \mathcal{H}(1)\\ &= \mathcal{H}(((x \to y) \to y) \to ((y \to x) \to x))\\ &\subseteq \mathcal{G}(((x \to y) \to y) \to ((y \to x) \to x)),\end{aligned}$$

and so $\mathcal{G}(((x \to y) \to y) \to ((y \to x) \to x)) = \mathcal{G}(1)$ for all $x, y \in L$. It follows from Theorem 4.4 that $\mathcal{G}$ is an MV-hesitant fuzzy filter of $\mathcal{L}$. □

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Hyers-Ulam Stability of Set-Valued AQ-Functional Equations in Non-Archimedean Metric Spaces

Xian Sun, Zhanpeng Yang*, Hongqi Wang, Wenjuan Ren, Guangluan Xu and Kun Fu

Key Laboratory of Technology in Geo-spatial Information Processing and Application System, Institute of Electronics, Chinese Academy of Sciences, Beijing 100190, PR China

Abstract

In the paper, we introduce a notion of a non-Archimedean metric space endowed with the non-Archimedean Pompeiu-Hausdorff metric. Using the direct and fixed point methods, we study the Hyers-Ulam stability of set-valued AQ-functional equations in the framework of complete non-Archimedean metric spaces. We indeed present an interdisciplinary relations between the theory of set-valued mappings, the theory of non-Archimedean spaces and the stability theory of functional equations.

Keywords: Hyers-Ulam stability, non-Archimedean Pompeiu-Hausdorff metric, AQ-functional equation, non-Archimedean metric space, fixed point theorem

1 Introduction

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University [1]. The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [2]. Thereafter, this type of stability is called the Hyers–Ulam stability. In 1978, Rassias [3] provided a remarkable generalization of the Hyers–Ulam stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations [4, 5, 6]. Bota-Boriceanu and Petrusel [7], Shen [8], Popa [9], Xu [10, 11], and Rus [12, 13] discussed the Hyers–Ulam stability for operatorial equations and inclusions. Castro and Ramos [14], and Jung [15] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations.

In 1897, Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom:

*Corresponding author. Email: yzhp1980@163.com(Z. Yang), wiecas@sina.com(H. Wang), sunxian@mail.ie.ac.cn(X. Sun), wjren2011@mail.ie.ac.cn(W. Ren), gluanxu@mail.ie.ac.cn(G. Xu), fukun@mail.ie.ac.cn(K. Fu).

for all $x, y > 0$, there exists an integer n such that $x < ny$. It turned out that non-Archimedean spaces have many nice applications [16]. During the last three decades, theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings [17]. In [18], Huy obtained some Hyers-Ulam stability results concerning fixed point equations in non-Archimedean cone metric spaces. Using a fixed point approach, Brzdęk proved the Hyers-Ulam stability of a quite wide class of functional equations in a single variable [19]. Mirmostafaee gave some stability results of the Cauchy equation in the context of non-Archimedean fuzzy spaces [20].

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering papers by Aumann [21] and Debreu [22] were inspired by problems arising in control theory and mathematical economics. We can refer to the papers by Arrow [23] and McKenzie [24]. In particular, the stability of set-valued functional equations has been considered by many scholars. Chu proved the Hyers-Ulam stability of the generalized cubic set-valued functional equations in Banach spaces [25]. Stability of two types of cubic fuzzy set-valued functional equations was considered in [26]. Lee studied additive set-valued and quadratic set-valued functional equations in Banach spaces [27]. Notice that, the spaces discussed above are all Banach spaces. So far as we know, the work related with the stability of set-valued functions equation in non-Archimedean spaces needs to study.

We introduce a notion of a complete non-Archimedean metric space endowed with the non-Archimedean Pompeiu-Hausdorff distance, which is following [19] and modifying the definition of a complete metric space endowed with the Hausdorff distance [27]. Although many results in the classic normed space theory have corresponding non-Archimedean counterparts, our proofs of the results on the complete non-Archimedean metric space are different and require a new kind of intuition. In Section 3 and Section 4, we prove the stability of the set-valued AQ-functional equations in the framework of complete non-Archimedean metric spaces.

2 Preliminaries

We will give some definitions which will be used in this paper. Let $\mathbb{N}$ denote the set of positive integers and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. In a non-Archimedean metric space $(\mathcal{X}, d)$ the triangle inequality holds in the stronger form as follows

$$d(\mu, \nu) \leq \max\{d(\mu, w), d(w, \nu)\}, \ \mu, \nu, w \in \mathcal{X}. \tag{2.1}$$

The non-Archimedean space theory has many applications in the filed of superstrings, p-adic strings and quantum physics. A typical example of non-Archimedean metric spaces is the non-Archimedean normed space that can be described below.

Definition 1. *A mapping $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ in the field $\mathbb{K}$ is called a non-Archimedean valuation if it satisfies the following conditions*
(i) $|\mu| = 0$ *if and only if* $\mu = 0$*;*
(ii) $|\mu\nu| = |\mu||\nu|, \ \mu, \nu \in \mathbb{K}$*;*

(iii) $|\mu+\nu| \leq \max\{|\mu|, |\nu|\},\ \mu, \nu \in \mathbb{K}$.

The condition (iii) is called the strong triangle inequality. By (ii), it is easy to verify that $|\cdot|$ is a homomorphism of groups. In addition, we will always suppose that $|\cdot|$ is nontrivial, i.e., there exists an $\mu_0 \in \mathbb{K}$ such that $|\mu_0| \neq 0, 1$.

It is easy to show that $|n \cdot 1| \leq 1$ for $n \in \mathbb{N}$. If $|\mu| \neq |\nu|$ for some $\mu, \nu \in \mathbb{K}$, then (iii) can be sharpened into the equality $|\mu+\nu| = \max\{|\mu|, |\nu|\}$. Set $d(\mu, \nu) := |\mu - \nu|$. And the space $(\mathbb{K}, d)$ is a metric space. Denote the closed ball in $\mathbb{K}$ by $N_\epsilon(\mu) := \{\nu \in \mathbb{K} : |\nu - \mu| \leq \epsilon\}$ for $\mu \in \mathbb{K}$ and $\epsilon > 0$. Then they form a fundamental system of neighbourhoods of μ.

Definition 2. *A space $(\mathcal{X}, |\cdot|)$ is non-Archimedean if it is equipped with a non-Archimedean valuation $|\cdot|$ such that the corresponding metric space $\mathcal{X}$ is complete (i.e., every Cauchy sequence in $\mathcal{X}$ converges).*

Let $\mathcal{X}$ be a non-Archimedean space. We need the following definitions:
$2^{\mathcal{X}}$: the set of subsets of $\mathcal{X}$;
$C_b(\mathcal{X})$: the set of closed bounded subsets of $\mathcal{X}$;
$C_c(\mathcal{X})$: the set of closed convex subsets of $\mathcal{X}$;
$C_{cb}(\mathcal{X})$: the set of closed convex bounded subsets of $\mathcal{X}$.

Definition 3. *For any two nonempty subsets $A, B \in \mathcal{X}$, the (Minkowski) addition is defined as $A + B = \{w \in \mathcal{X} \mid w = \mu + \nu, \mu \in A, \nu \in B\}$ and the scalar multiplication as $\lambda A = \{w \in \mathcal{X} \mid w = \lambda\mu, \mu \in A\}$ for $\lambda \in \mathbb{R}$. Moreover, for $A, B \in C_c(\mathcal{X})$, $A \oplus B := \overline{A + B}$.*

By Definition 3, we can obtain the following two properties:
$(1)\lambda A + \lambda B = \lambda(A + B)$, $(2)(\lambda + \mu)A \subseteq \lambda A + \mu A$;
if A is convex, then $(\lambda + \mu)A = \lambda A + \mu A$ for all $\lambda\mu \geq 0$.

Definition 4. *Let $(\mathcal{X}, d)$ $(d(\mu, \nu) := |\mu - \nu|)$ be a non-Archimedean metric space. A gap function in $2^{\mathcal{X}}$ is defined as*

$$D_d : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to \mathbb{R}_+,\ D_d(A, B) = \inf\{d(\mu, \nu) \mid \mu \in A, \nu \in B\}.$$

In particular, $D_d(\mu, B) := D_d(\{\mu\}, B)$ for $\mu \in \mathcal{X}$.

Definition 5. *The non-Archimedean Pompeiu-Hausdorff distance H_d on 2^X is defined as*

$$H_d : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A, B) = \max\{\sup_{\mu \in A} D_d(\mu, B), \sup_{\nu \in B} D_d(\nu, A)\}.$$

By using Definition 5, we have the following properties of the non-Archimedean Pompeiu-Hausdorff distance.

Proposition 1. *For $A, A_1, B, B_1 \in C_{cb}(\mathcal{X})$ and $\lambda > 0$, the following properties hold*
(i) $H_d(A \oplus A_1, B \oplus B_1) \leq \max\{H_d(A, B), H_d(A_1, B_1)\}$;
(ii) $H_d(\lambda A, \lambda B) = \lambda H_d(A, B)$;
(iii) $H_d(A, A_1) = H_d(A \oplus B, A_1 \oplus B)$.

Then, $(C_{cb}(\mathcal{X}), H_d)$ is a complete non-Archimedean metric semigroup. Using the method given by Debreu [22], one can prove that $(C_{cb}(\mathcal{X}), H_d)$ can be isometrically embedded in a non-Archimedean normed space.

Now we consider that a set-valued function $f: \mathcal{X} \to C_{cb}(Y)$ satisfies the following additive-quadratic (AQ) set-valued functional equation,

$$f(kx+y) \oplus f(kx-y) = f(x+y) \oplus f(x-y) \oplus (k-1)[(k+2)f(x) \oplus kf(-x)], \tag{2.2}$$

for a fixed integer with $k \geq 2$. Eskandani et al. [32] and Xu et al. [31] have established the general solution and investigated the generalized Hyers-Ulam stability of (2.2) in quasi-β-normed spaces and non-Archimedean normed spaces.

The main purpose in our paper is to prove the generalized Hyers-Ulam stability of the set-valued AQ-functional equation (2.2) in complete non-Archimedean normed spaces using the direct and fixed point methods.

3 Stability of the Set-Valued Functional Equation: a Direct Method

Throughout this paper, we always suppose that $\mathcal{X}$ is a linear space over $\mathbb{Q}$ or a non-Archimedean field of characteristic different from 2 and k (i.e. $|2| \neq 0, |k| \neq 0$), and that $\mathcal{Y}$ is a complete non-Archimedean normed space over a non-Archimedean field of characteristic different from $2, 3$ and k (a fixed integer $k \geq 2$).

For convenience, we use the following abbreviation for a given function $f: \mathcal{X} \to C_{cb}(\mathcal{Y})$:

$$\mathcal{L}f(x,y) := H_d(f(kx+y) \oplus f(kx-y), f(x+y) \oplus f(x-y) \oplus (k-1)[(k+2)f(x) \oplus kf(-x)]). \tag{3.1}$$

Theorem 1. *Assume that $\varphi: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a mapping satisfying*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^n} = 0, \ x, y \in \mathcal{X}. \tag{3.2}$$

And suppose that for each $x \in \mathcal{X}$ the limit

$$\lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^j} : 0 \leq j < n \right\}, \tag{3.3}$$

denoted by $\tilde{\varphi}_a(x)$, exists. If $f: \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an odd mapping such that

$$\mathcal{L}f(x,y) \leq \varphi(x,y), \ x, y \in \mathcal{X}, \tag{3.4}$$

then there exists an additive mapping $A: X \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2k|} \tilde{\varphi}_a(x), \ x \in \mathcal{X}. \tag{3.5}$$

Moreover, if

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^j} : i \leq j < n+i \right\} = 0, \tag{3.6}$$

then A is the unique additive mapping satisfying (3.5).

Proof. Setting $y = 0$ in (3.4), we get

$$H_d(f(kx), kf(x)) \leq \frac{1}{|2|}\varphi(x, 0), x \in \mathcal{X}. \tag{3.7}$$

Replacing x by $k^{n-1}x$ in (3.7), we get

$$H_d\left(\frac{f(k^n x)}{k^n}, \frac{f(k^{n-1}x)}{k^{n-1}}\right) \leq \frac{1}{|2 \cdot k^n|}\varphi(k^{n-1}x, 0), x \in \mathcal{X}. \tag{3.8}$$

(3.8) and (3.2) imply that $\left\{\frac{f(k^n x)}{k^n}\right\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is complete, we conclude that the sequence $\left\{\frac{f(k^n x)}{k^n}\right\}$ is convergent. Let $A(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$. Using induction one can show that

$$H_d\left(\frac{f(k^n x)}{k^n}, f(x)\right) \leq \frac{1}{|2k|} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : 0 \leq j < n\right\}, x \in \mathcal{X}, n \in \mathbb{N}. \tag{3.9}$$

Letting $n \to \infty$ and using (3.3) we can get (3.5). (3.2) and (3.4) imply that

$$\mathcal{L}A(x, y) = \lim_{n\to\infty} \frac{1}{|k|^n}\mathcal{L}f(k^n x, k^n y) \leq \lim_{n\to\infty} \frac{1}{|k|^n}\varphi(k^n x, k^n y) = 0, x, y \in \mathcal{X}.$$

Therefore, the mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies (2.2). By Lemma 2.2 in [32], we get that the mapping A is additive. To prove the uniqueness property of A, let $A^{'}$ be another additive mapping satisfying (3.5). Then

$$\begin{aligned} H_d(A(x), A^{'}(x)) &\leq \lim_{i\to\infty} \frac{1}{|k|^i} H_d(A(k^i x), A^{'}(k^i x)) \\ &\leq \lim_{i\to\infty} \frac{1}{|k|^i} \max\{H_d(A(k^i x), f(k^i x)), H_d(A^{'}(k^i x), f(k^i x))\} \\ &\leq \frac{1}{|2k|} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \leq j < n + i\right\} \end{aligned} \tag{3.10}$$

for all $x \in \mathcal{X}$. If

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \leq j < n + i\right\} = 0, \tag{3.11}$$

then $A = A^{'}$. □

Corollary 1. *Assume that the function $\beta : [0, \infty) \to [0, \infty)$ satisfies*

(i) $\beta(|k|t) \leq \beta(|k|)\beta(t)$ *for all* $t > 0$;

(ii) $\beta(|k|) < |k|$.

Let $\delta > 0$. Assume that $\mathcal{X}$ is a normed space and that $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an odd mapping such that

$$\mathcal{L}f(x, y) \leq \delta[\beta(|x|) + \beta(|y|)]$$

for all $x, y \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2k|}\delta\beta(|x|), x \in \mathcal{X}.$$

Proof. If we set $\psi(x,y)=\delta[\beta(|x|)+\beta(|y|)]$ for all $x,y\in\mathcal{X}$, then we have

$$\lim_{n\to\infty}\frac{\psi(k^nx,k^ny)}{|k|^n}\leq\lim_{n\to\infty}\left(\frac{\beta(|k|)}{|k|}\right)^n\psi(x,y)=0\ (x,y\in\mathcal{X}),$$

$$\widetilde{\psi}_a(x)=\lim_{n\to\infty}\max\left\{\frac{\psi(k^jx,0)}{|k|^j}:0\leq j<n\right\}=\psi(x,0),$$

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\psi(k^jx,0)}{|k|^j}:i\leq j<n+i\right\}=\lim_{i\to\infty}\frac{\psi(k^ix,k^iy)}{|k|^i}=0.$$

By Theorem 1, we can get the corollary. □

Remark 1. The classical example of the function β is the mapping $\beta(t)=t^r$ for all $t\geq 0$, where $r>1$ with the further assumption that $|k|<1$. The assumption $|k|<1$ cannot be omitted.

Remark 2. A similar result can be formulated as Theorem 1 if we define the sequence $A(x):=\lim_{n\to\infty}k^nf(\frac{x}{k^n})$ under suitable conditions on the function $\psi(x)$.

Theorem 2. *Assume that a mapping $\varphi:X\times X\to[0,\infty)$ satisfies*

$$\lim_{n\to\infty}\frac{\varphi(k^nx,k^ny)}{|k|^{2n}}=0,x,y\in\mathcal{X}. \tag{3.12}$$

And suppose that for $x\in\mathcal{X}$ the limit

$$\lim_{n\to\infty}\max\left\{\frac{\varphi(k^jx,0)}{|k|^{2j}}:0\leq j<n\right\}, \tag{3.13}$$

denoted by $\tilde{\varphi}_q(x)$, exists. If an even mapping $f:X\to C_{cb}(\mathcal{Y})$ satisfies the following condition

$$\mathcal{L}f(x,y)\leq\varphi(x,y) \tag{3.14}$$

for all $x,y\in\mathcal{X}$, then there exists a quadratic mapping $Q:\mathcal{X}\to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x),Q(x))\leq\frac{1}{|2k^2|}\tilde{\varphi}_q(x),x\in\mathcal{X}. \tag{3.15}$$

Moreover, if

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(k^jx,0)}{|k|^{2j}}:i\leq j<n+i\right\}=0, \tag{3.16}$$

then the quadratic mapping Q is unique.

Proof. Setting $y=0$ in (3.14), we have

$$H_d(f(kx),k^2f(x))\leq\frac{1}{|2|}\varphi(x,0) \tag{3.17}$$

for all $x\in\mathcal{X}$. Replacing x by $k^{n-1}x$ in (3.17), we get

$$H_d\left(\frac{f(k^nx)}{k^{2n}},\frac{f(k^{n-1}x)}{k^{2(n-1)}}\right)\leq\frac{1}{|2\cdot k^{2n}|}\varphi(k^{n-1}x,0) \tag{3.18}$$

for all $x \in \mathcal{X}$. (3.18) and (3.12) imply that $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is complete, we can obtain that $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is convergent. Let $Q(x) = \lim_{n\to\infty} \frac{1}{k^{2n}} f(k^n x)$. Using induction one can show that

$$H_d\left(\frac{f(k^n x)}{k^{2n}}, f(x)\right) \leq \frac{1}{|2k^2|} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : 0 \leq j < n\right\} \tag{3.19}$$

for $x \in \mathcal{X}$ and $n \in \mathbb{N}$. By taking n to approach infinity in (3.19) and using (3.13) one can obtain (3.15). By (3.12) and (3.14), we get

$$\mathcal{L}Q(x,y) = \lim_{n\to\infty} \frac{1}{|k|^{2n}} \mathcal{L}f(k^n x, k^n y) \leq \lim_{n\to\infty} \frac{1}{|k|^{2n}} \varphi(k^n x, k^n y) = 0$$

for all $x, y \in \mathcal{X}$. Therefore, the mapping Q satisfies (2.2). By [[32], Lemma 2.1] we get that the mapping Q is quadratic. Let now

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \leq j < n+i\right\} = 0.$$

Assume that $Q^{'}$ is another quadratic mapping satisfying (3.15). Then

$$\begin{aligned} H_d(Q(x), Q^{'}(x)) &\leq \lim_{i\to\infty} \frac{1}{|k|^{2i}} H_d(Q(k^i x), Q^{'}(k^i x)) \\ &\leq \lim_{i\to\infty} \frac{1}{|k|^{2i}} \max\{H_d(Q(k^i x), f(k^i x)), H_d(Q^{'}(k^i x), f(k^i x))\} \\ &\leq \frac{1}{|2k^2|} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \leq j < n+i\right\} = 0 \end{aligned} \tag{3.20}$$

for all $x \in \mathcal{X}$. Hence $Q = Q^{'}$. □

Corollary 2. *Suppose that the function $\beta : [0,\infty) \to [0,\infty)$ satisfies the following conditions*

(i) *$\beta(|k|t) \leq \beta(|k|)\beta(t)$ for any $t > 0$;*

(ii) *$\beta(|k|) < |k|^2$.*

Let $\mathcal{X}$ be a normed space and $\delta > 0$. If the even mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$|\mathcal{L}f(x,y)| \leq \delta[\beta(|x|) + \beta(|y|)]$$

for $x, y \in \mathcal{X}$, then there exists a unique quadratic mapping $Q : X \to C_{cb}(Y)$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2k^2|} \delta\beta(|x|)$$

for $x \in \mathcal{X}$.

Proof. Setting $\varphi(x,y) = \delta[\beta(|x|) + \beta(|y|)]$ for all $x, y \in \mathcal{X}$, we can obtain

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} \leq \lim_{n\to\infty} \left(\frac{\beta(|k|)}{|k|^2}\right)^n \varphi(x,y) = 0 \ (x, y \in \mathcal{X}),$$

$$\widetilde{\varphi}_q(x) = \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : 0 \leq j < n\right\} = \varphi(x, 0),$$

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \leq j < n+i\right\} = \lim_{i\to\infty} \frac{\varphi(k^i x, k^i y)}{|k|^{2i}} = 0.$$

The corollary can be deduced by Theorem 2. □

Theorem 3. *Assume that the function $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfies*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} = 0, x, y \in \mathcal{X}.$$

Suppose that for each $x \in \mathcal{X}$ the limit

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : 0 \le j < n\right\},$$

denoted by $\tilde{\varphi}_a(x)$, and

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : 0 \le j < n\right\},$$

denoted by $\tilde{\varphi}_q(x)$, both exist. Let $f : \mathcal{X} \to C_{cb}(Y)$ be a mapping satisfying $\mathcal{L}f(x, y) \le \varphi(x, y)$ for all $x, y \in \mathcal{X}$. If f can be decomposed into an even and odd part, then there exist an additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x) \oplus Q(x)) \le \frac{1}{|4k|} \max\left\{\max\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x)\}, \frac{1}{|k|}\max\{\tilde{\varphi}_q(x), \tilde{\varphi}_q(-x)\}\right\} \tag{3.21}$$

for all $x \in \mathcal{X}$.

Moreover, if

$$\begin{aligned} &\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : i \le j < n + i\right\} \\ &\quad = \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : i \le j < n + i\right\} = 0, \end{aligned} \tag{3.22}$$

then A and Q are both unique.

Proof. Assume that $f(x) = f_e(x) \oplus f_o(x)$. We have for all $x, y \in \mathcal{X}$

$$\mathcal{L}f_o(x, y) \le \frac{1}{|2|}\max\{\varphi(x, y), \varphi(-x, -y)\},$$

$$\mathcal{L}f_e(x, y) \le \frac{1}{|2|}\max\{\varphi(x, y), \varphi(-x, -y)\}.$$

By Theorems 1 and 2, there exist an additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfying

$$H_d(f_o(x), A(x)) \le \frac{1}{|4k|}\max\{\widetilde{\varphi}_a(x), \widetilde{\varphi}_a(-x)\},$$

$$H_d(f_e(x), Q(x)) \le \frac{1}{|4k^2|}\max\{\widetilde{\varphi}_q(x), \widetilde{\varphi}_q(-x)\}$$

for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned} H_d(f(x), A(x) \oplus Q(x)) &\le \max\{H_d(f_o(x), A(x)), H_d(f_e(x), Q(x))\} \\ &\le \frac{1}{|4k|}\max\left\{\max\{\widetilde{\varphi}_a(x), \widetilde{\varphi}_a(-x)\}, \frac{1}{|k|}\max\{\widetilde{\varphi}_q(x), \widetilde{\varphi}_q(-x)\}\right\} \end{aligned} \tag{3.23}$$

for all $x \in \mathcal{X}$.

4 Stability of the Set-Valued Functional Equation: a Fixed Point Method

In this section, we establish the generalized Hyers-Ulam stability results for the mixed set-valued AQ-functional equation (2.2) in non-Archimedean Banach spaces by using the fixed point method introduced by Radu in [28].

Let Ω be a set. A mapping $\mathcal{D}: \Omega \times \Omega \to [0, \infty]$ is called a generalized metric on Ω if $\mathcal{D}$ satisfies the following three conditions:

(i) $\mathcal{D}(\mu, \nu) = 0$ if only if $\mu = \nu$;

(ii) $\mathcal{D}(\mu, \nu) = \mathcal{D}(\nu, \mu)$, $\nu, \mu \in \Omega$;

(iii) $\mathcal{D}(\nu, \mu) \leq \mathcal{D}(\nu, w) + \mathcal{D}(w, \mu)$, $\nu, \mu, w \in \Omega$.

For explicitly later use, we recall the following lemma proposed by Diaz and Margolis in [30].

Lemma 1 ([30]). *Assume that $(\Omega, \mathcal{D})$ is a complete generalized metric space, and that $\mathcal{T}: \Omega \to \Omega$ is a strictly contractive mapping with Lipschitz constant $L < 1$, that is*

$$\mathcal{D}(\mathcal{T}x, \mathcal{T}y) \leq L\mathcal{D}(x, y), x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$\mathcal{D}(\mathcal{T}^n x, \mathcal{T}^{n+1} x) = \infty, n \geq 0,$$

or there exists a non-negative integer n_0 such that

(1) *$\mathcal{D}(\mathcal{T}^n x, \mathcal{T}^{n+1} x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{\mathcal{T}^n(x)\}$ is converges to a fixed point y^*;*

(3) *y^* is the unique fixed point of $\mathcal{T}$ in the set $\Omega^* = \{y \in \Omega \mid \mathcal{D}(\mathcal{T}^{n_0} x, y) < \infty\}$;*

(4) *$\mathcal{D}(y, y^*) \leq \frac{1}{1-L}\mathcal{D}(y, \mathcal{T}y)$ for all $y \in \Omega^*$.*

Theorem 4. *Let $\varphi: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \leq |k| L\varphi(x, y) \tag{4.1}$$

for all $x, y \in \mathcal{X}$. If the odd mapping $f: \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}, \tag{4.2}$$

then there exists a unique additive mapping $A: \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2k|(1-L)}\varphi(x, 0) \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. In (4.2) setting $y = 0$, we have

$$H_d(f(kx), kf(x)) \leq \frac{1}{|2|}\varphi(x, 0) \tag{4.4}$$

for all $x \in \mathcal{X}$. Set $\Omega := \{g \mid g : \mathcal{X} \to C_{cb}(\mathcal{Y}),\ g(0) = \{0\}\}$, and we then introduce the generalized metric on Ω:

$$\mathcal{D}(g,h) = \inf\{C \in (0,\infty) \mid H_d(g(x),h(x)) \leq C\varphi(x,0), \forall x \in \mathcal{X}\}. \tag{4.5}$$

It is easy to show that $(\Omega, \mathcal{D})$ is a complete generalized metric space. We now define a mapping $\mathcal{T} : \Omega \to \Omega$ as

$$(\mathcal{T}g)(x) = \frac{1}{k}g(kx), \quad g \in \Omega, x \in \mathcal{X}. \tag{4.6}$$

Let $g, h \in \Omega$ and $C \in [0,\infty]$ be an arbitrary constant with $\mathcal{D}(g,h) < C$. The definition of $\mathcal{D}$ implies

$$H_d(g(x),h(x)) \leq C\varphi(x,0), \quad x \in \mathcal{X}. \tag{4.7}$$

By the last inequality, we can obtain

$$H_d\left(\frac{1}{k}g(kx), \frac{1}{k}h(kx)\right) \leq CL\varphi(x,0), \quad \forall x \in \mathcal{X}. \tag{4.8}$$

Hence, $\mathcal{D}(\mathcal{T}g, \mathcal{T}h) \leq L\mathcal{D}(g,h)$. It follows from (4.4) that $\mathcal{D}(\mathcal{T}f, f) \leq \frac{1}{|2k|} < \infty$. Therefore, by Lemma 1, the operator $\mathcal{T}$ has a unique fixed point $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ in the set $\Omega^* = \{g \in \Omega \mid \mathcal{D}(f,g) < \infty\}$ such that

$$A(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x) = \lim_{n\to\infty}\frac{1}{k^n}f(k^n x) \tag{4.9}$$

and $A(kx) = kA(x)$ for all $x \in X$. Also,

$$\mathcal{D}(A,f) \leq \frac{1}{1-L}\mathcal{D}(\mathcal{T}f,f) \leq \frac{1}{|2k|(1-L)}. \tag{4.10}$$

Then (4.3) holds for all $x \in \mathcal{X}$.

Now we will prove that A is additive. By (4.1), (4.2) and (4.9), we have

$$\mathcal{L}A(x,y) = \lim_{n\to\infty}\frac{1}{|k|^n}\mathcal{L}f(k^n x, k^n y) \leq \lim_{n\to\infty}\frac{1}{|k|^n}\varphi(k^n x, k^n y) \leq \lim_{n\to\infty} L^n\varphi(x,y) = 0$$

for all $x, y \in \mathcal{X}$. Therefore, it follows form [[32], Lemma 2.2] that A is an additive mapping. □

Corollary 3. *Let $\delta > 0,\ r > 1,\ |k| < 1$. If $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an odd mapping and satisfies the following condition*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X},$$

then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x),A(x)) \leq \frac{1}{|2|(|k| - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Proof. Set $\psi(x,y) = \delta(|x|^r + |y|^r)$ for all $x, y \in \mathcal{X}$. Then we can get the corollary by Theorem 4 by $L = |k|^{r-1} < 1$. □

The following example is a modification of the example of [31] and it shows that the assumption $|k| < 1$ cannot be omitted in Corollary 3.

Example 1. For a prime number $p > 2$ if the set-valued mapping $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ satisfies $f(x) = [0, x^3]$, then for $\delta = 1$ and $r = 3$

$$\mathcal{L}f(x,y) \leq \max\{|x|_p^3, |y|_p^3\} \leq |x|_p^3 + |y|_p^3, \quad x, y \in \mathbb{Q}_p.$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$H_d\left(\frac{1}{2^n} f(2^n x), \frac{1}{2^{n+1}} f(2^{n+1} x)\right) = |2^{2n}|_p |3|_p |x|_p^3 = |3|_p |x|_p^3$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Similar to Theorem 4, we have the following theorem.

Theorem 5. *Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function such that there exists an $0 < L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{|k|} \varphi(kx, ky) \tag{4.11}$$

for $x, y \in \mathcal{X}$. If the odd mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x,y) \leq \varphi(x,y), \quad x, y \in \mathcal{X}, \tag{4.12}$$

then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{L}{|2k|(1-L)} \varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.13}$$

Corollary 4. *Let $\delta > 0,\ 0 \leq r < 1$, and $|k| < 1$. If the odd set-valued mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r),$$

for all $x, y \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x)) \leq \frac{1}{|2|(|k|^r - |k|)} \delta |x|^r, x \in \mathcal{X}$$

Proof. Set $\varphi(x,y) = \delta(|x|^r + |y|^r)$ for $x, y \in \mathcal{X}$. The corollary can be obtained by using Theorem 5 with $L = |k|^{1-r} < 1$. □

We then give an example to show that the assumption $|k| < 1$ cannot be omitted in Corollary 4.

Example 2. For a prime number $p > 2$ if the set-valued function $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ satisfies $f(x) = [0, x^{\frac{1}{3}}]$, then for $\delta = 1$ and $r = \frac{1}{3}$,

$$\mathcal{L}f(x,y) \leq \max\{|x|_p^r, |y|_p^r\} \leq |x|_p^r + |y|_p^r, \quad x, y \in \mathbb{Q}_p.$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$H_d\left(2^n f(\frac{1}{2^n} x), 2^{n+1} f(\frac{1}{2^{n+1}} x)\right) = |2^{\frac{2n}{3}}|_p |1 - 2^{\frac{2}{3}}|_p |x|_p^{\frac{1}{3}} = |1 - 2^{\frac{2}{3}}|_p |x|_p^{\frac{1}{3}}.$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{2^n f(\frac{1}{2^n} x)\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Theorem 6. *Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \leq |k|^2 L\varphi(x, y). \tag{4.14}$$

for all $x, y \in \mathcal{X}$. If the even set-valued function $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}, \tag{4.15}$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2k^2|(1 - L)}\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.16}$$

Proof. Setting $y = 0$ in (4.15), we get

$$H_d(f(kx), k^2 f(x)) \leq \frac{1}{|2|}\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.17}$$

Let $\Omega := \{g \mid g : \mathcal{X} \to C_{cb}(\mathcal{Y})\}$, and we then introduce a generalized metric on Ω:

$$\mathcal{D}(g, h) = \inf\{C \in (0, \infty) \mid H_d(g(x), h(x)) \leq C\varphi(x, 0),\ \forall x \in \mathcal{X}\}. \tag{4.18}$$

It is easy to show that $\Omega := \{g \mid g : \mathcal{X} \to C_{cb}(\mathcal{Y})\}$ is a complete generalized metric space. Define an operator $\mathcal{T} : \Omega \to \Omega$ as

$$(\mathcal{T}g)(x) = \frac{1}{k^2}g(kx), \quad \forall g \in \Omega, x \in \mathcal{X}. \tag{4.19}$$

Let $g, h \in \Omega$ and $C \in [0, \infty]$ be an arbitrary constant with $\mathcal{D}(g, h) < C$. By the definition of the metric $\mathcal{D}$, we can get

$$H_d(g(x), h(x)) \leq C\varphi(x, 0), \quad x \in \mathcal{X}. \tag{4.20}$$

By the given hypothesis and the last inequality, we get

$$H_d\left(\frac{1}{k^2}g(kx), \frac{1}{k^2}h(kx)\right) \leq CL\varphi(x, 0),\ \forall x \in \mathcal{X}. \tag{4.21}$$

Hence, $\mathcal{D}(\mathcal{T}g, \mathcal{T}h) \leq L\mathcal{D}(g, h)$. (4.17) implies that $\mathcal{D}(\mathcal{T}f, f) \leq \frac{1}{|2k^2|} < \infty$. Therefore, by Lemma 1, $\mathcal{T}$ has a unique fixed point $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ in the set $\Omega^* = \{g \in \Omega \mid \mathcal{D}(f, g) < \infty\}$ such that

$$Q(x) := \lim_{n\to\infty}(\mathcal{T}^n f)(x) = \lim_{n\to\infty}\frac{1}{k^{2n}}f(k^n x), \tag{4.22}$$

and $Q(kx) = k^2 Q(x)$ for all $x \in \mathcal{X}$. Also,

$$\mathcal{D}(Q, f) \leq \frac{1}{1 - L}\mathcal{D}(\mathcal{T}f, f) \leq \frac{1}{|2k^2|(1 - L)}. \tag{4.23}$$

Then, (4.16) holds. By (4.14), (4.15) and (4.22) we obtain

$$\mathcal{L}Q(x, y) = \lim_{n\to\infty}\frac{1}{|k|^{2n}}\mathcal{L}f(k^n x, k^n y) \leq \lim_{n\to\infty}\frac{1}{|k|^{2n}}\varphi(k^n x, k^n y) \leq \lim_{n\to\infty}|k|^n\varphi(x, y) = 0$$

for all $x, y \in \mathcal{X}$. So by Lemma 2.1 in [32], we get that the mapping Q is quadratic. □

Corollary 5. *Let $\delta > 0,\ r > 2,\ |k| < 1$. If $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is an even set-valued mapping satisfying*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X},$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2|(|k|^2 - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Proof. Set $\varphi(x,y) = \delta(|x|^r + |y|^r)$ for all $x, y \in \mathcal{X}$. By Theorem 6 with $L = |k|^{r-2} < 1$, we can get the corollary. □

Example 3. For a prime number $p > 2$ if a set-valued mapping $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ satisfies $f(x) = [0, x^4]$, then for $\delta = 1$ and $r = 4$,

$$\mathcal{L}f(x,y) \leq \max\{|x|_p^4, |y|_p^4\} \leq |x|_p^4 + |y|_p^4, \quad x, y \in \mathbb{Q}_p.$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$H_d\left(\frac{1}{2^{2n}}f(2^n x), \frac{1}{2^{2(n+1)}}f(2^{n+1}x)\right) = |2^{2n}|_p|3|_p|x|_p^4 = |3|_p|x|_p^4.$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{\frac{1}{2^{2n}}f(2^n x)\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Similarly, we can obtain the following theorem.

Theorem 7. *Let $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping such that there exists an $0 < L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{|k|^2}\varphi\left(\frac{x}{k}, \frac{y}{k}\right), x, y \in \mathcal{X}. \tag{4.24}$$

If the even set-valued mapping $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ satisfies

$$\mathcal{L}f(x,y) \leq \varphi(x,y), \quad x, y \in \mathcal{X}, \tag{4.25}$$

then there exists a unique quadratic set-valued mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2k^2|(1-L)}\varphi(x,0), \quad x \in \mathcal{X}. \tag{4.26}$$

Corollary 6. *Let $\delta > 0,\ 0 \leq r < 2$ and $|k| < 1$. If the even mapping $f : X \to C_{cb}(Y)$ satisfies*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X},$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x)) \leq \frac{1}{|2|(|k|^2 - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

In Corollary 6 the assumption $|k| < 1$ cannot be omitted .

Example 4. For a prime number $p > 2$ if $f : \mathbb{Q}_p \to C_{cb}(\mathbb{Q}_p)$ is defined as $f(x) = [0, |2|_p]$, then for $\delta = 1, k = 2$ and $r = 0$,

$$\mathcal{L}f(x, y) = |12|_p \leq 1 = \delta \ (x, y \in \mathbb{Q}_p).$$

Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n, we have

$$H_d\left(2^n f(2^{-n}x), 2^{n+1} f(2^{-(n+1)}x)\right) = |2^{n+1}|_p = 1.$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{2^n f(2^{-n}x)\}$ is not a Cauchy sequence for $x \in \mathbb{Q}_p$.

Theorem 8. *Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \leq |k|^2 L\varphi(x, y), x, y \in \mathcal{X}. \tag{4.27}$$

Suppose that $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ is a set-valued mapping such that

$$\mathcal{L}f(x, y) \leq \varphi(x, y), \quad x, y \in \mathcal{X}. \tag{4.28}$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x) \oplus Q(x)) \leq \frac{1}{|4k^2|(1 - L)} \max\{\varphi(x, 0), \varphi(-x, 0)\}, \quad x \in \mathcal{X}. \tag{4.29}$$

Proof. Assume that $f(x) = f_e(x) \oplus f_o(x)$. Let $\psi(x, y) = \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}$, then by (4.27) and (4.28), we have

$$\psi(kx, ky) \leq |k|^2 L\psi(x, y) \leq |k|L\psi(x, y),$$

$$\mathcal{L}f_o(x, y) \leq \psi(x, y), \mathcal{L}f_e(x, y) \leq \psi(x, y).$$

Hence by Theorems 4 and 6, there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f_o(x), A(x)) \leq \frac{1}{|2k|(1 - L)} \psi(x, 0),$$

$$H_d(f_e(x), Q(x)) \leq \frac{1}{|2k^2|(1 - L)} \psi(x, 0)$$

for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned} H_d(f(x), A(x) \oplus Q(x)) &\leq \max\{H_d(f_o(x), A(x)), H_d(f_e(x), Q(x))\} \\ &\leq \max\left\{\frac{1}{|2k|(1 - L)} \psi(x, 0), \frac{1}{|2k^2|(1 - L)} \psi(x, 0)\right\} \\ &\leq \frac{1}{|4k^2|(1 - L)} \max\{\varphi(x, 0), \varphi(-x, 0)\} \end{aligned} \tag{4.30}$$

for all $x \in \mathcal{X}$. □

Corollary 7. *Let $\delta > 0,\ 2 < r,\ |k| < 1$ and $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ be a set-valued mapping for which*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X}.$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x) \oplus A(x)) \leq \frac{1}{|4|(|k|^2 - |k|^r)}\delta|x|^r, \quad x \in \mathcal{X}.$$

Similar to Theorem 8, one can obtain the following theorem.

Theorem 9. *Let $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{|k|}\varphi(kx, ky) \tag{4.31}$$

for all $x, y \in \mathcal{X}$. Let $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$ be a mapping such that

$$\mathcal{L}f(x,y) \leq \varphi(x,y), \quad x, y \in \mathcal{X}. \tag{4.32}$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), A(x) \oplus Q(x)) \leq \frac{1}{|4k^2|(1-L)} \max\{\varphi(x,0), \varphi(-x,0)\}, \quad x \in \mathcal{X}. \tag{4.33}$$

Corollary 8. *Let $\delta > 0,\ 0 \leq r < 1,\ |k| < 1$ and $f : \mathcal{X} \to C_{cb}(\mathcal{Y})$*

$$\mathcal{L}f(x,y) \leq \delta(|x|^r + |y|^r), \quad x, y \in \mathcal{X}.$$

If f can be decomposed into an even and odd part, then there exist a unique additive mapping $A : \mathcal{X} \to C_{cb}(\mathcal{Y})$ and a unique quadratic mapping $Q : \mathcal{X} \to C_{cb}(\mathcal{Y})$ such that

$$H_d(f(x), Q(x) \oplus A(x)) \leq \frac{1}{|4|(|k|^r - |k|)}\delta|x|^r, \quad x \in \mathcal{X}.$$

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Solution and stability of nonic functional equations in non-Archimedean normed spaces

John Michael Rassias[1], Seong Sik Kim[2] and Soo Hwan Kim[2]

[1] Pedagogical Department E.E., Section of Mathematics and Informatics,
National and Capodistrian University of Athens,
4, Agamemnonos St., Aghia Paraskevi, Athens 15342, Greece
E-mail: jrassias@primedu.uoa.gr; jrass@otenet.gr

[2] Department of Mathematics, Dongeui University,
Busan 614-714, Korea
E-mail: sskim@deu.ac.kr
E-mail: sh-kim@deu.ac.kr

Abstract: In this paper, we establish the general solution and stability of the following nonic functional equation:

$$f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)+126f(x+y)-126f(x)$$
$$+84f(x-y)-36f(x-2y)+9f(x-3y)-f(x-4y)=9!f(y),$$

where $9! = 362880$ in non-Archimedean normed spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. The first affirmative partial answer to Ulam's question was given by Hyers [11]. The result of Hyers by obtaining a unique linear mapping near an approximate additive mapping was generalized by Rassias [22]. The paper of Rassias has provided a lot of influence in the development of what we called the *generalized Hyers-Ulam-Rassias stability* of functional equations. In 1994, Găvruta then generalized the Rassias' result in [8] for unbounded Cauchy difference by a general control function. In 2003, Radu [20] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative. Cădariu and Radu [5] proposed a novel method for studying the stability of the Cauchy functional equation based on a fixed point result in generalized metric spaces. A great number of papers (see [2, 6, 9, 12, 13, 21, 24, 25, 28] and references therein for more detailed information) on the subject has been published, generalizing Ulam's problem and Hyer's theorem in various functional equations.

In 1897, Hensel [10] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis and introduced a normed space which does not have the Archimedean property. Let p be a prime number. For any nonzero rational number $x \in \mathbb{Q}$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a, b are integers divisible by p. Then the p-adic absolute value $|x|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the

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metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. Note that if $p > 2$, then $|2^n|_p = 1$ for each integer $n \in \mathbb{Z}$; however, $|2|_2 < 1$.

Let us recall some basic definitions and facts concerning non-Archimedean normed spaces [14]. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0, |rs| = |r||s|$, and $|r+s| \leq \max\{|r|,|s|\}$ for all $r, s \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. In the rest of the paper, let $|2| \neq 1$.

In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to [0,\infty)$ defined by

$$|x| = \begin{cases} 0, & x \in X; \\ 1, & \text{otherwise} \end{cases}$$

is a valuation which is called trivial, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings [27].

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$;

(NA2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}, x \in X$;

(NA3) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ (: the strong triangle inequality).

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*. It follows from (NA3) that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} \qquad \text{for} \quad n > m$$

and so a sequence $\{x_n\}$ is Cauchy in X if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. If every Cauchy sequence is convergent, the non-Archimedean normed space is said to be complete and called a *non-Archimedean Banach space*.

In 2005, Arriola and Beyer [1] investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ϵ such that $|f(x+y) - f(x) - f(y)| \leq \epsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive mapping $T : Q_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \epsilon$ for all $x \in \mathbb{Q}_p$. In 2007, Moslehian and Rassias [18] proved the Hyers-Ulam stability of functional equations in complete non-Archimedean normed spaces. Then a lot of papers on the stability of other many equations have been published in non-Archimedean normed spaces ([3, 4, 16, 17, 19, 23]).

Now, we consider the following nonic functional equation

$$\begin{aligned} f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) \\ + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!f(y). \end{aligned} \tag{1.1}$$

Then the function $f(x) = x^9$ is a solution of the functional equation (1.1). Thus, the functional equation (1.1) is called the *nonic functional equation* and every solution of the nonic functional equation is said to be a *nonic mapping*.

In this paper, using the direct and fixed point methods, we prove the generalized Hyers-Ulam stability of (1.1) in non-Archimedean normed spaces.

2. **Properties of the nonic functional equation (1.1)**

In this section, we investigate the properties and the general solution of the nonic functional equation (1.1).

Theorem 2.1. *Let X and Y be linear spaces. The nonic functional equation (1.1) has the following properties:*

(i) $f(0) = 0$.

(ii) $f(-x) = -f(x)$.

(iii) $f(2x) = 2^9 f(x)$.

Proof. (i) Replacing $x = 0, y = 0$ in (1.1), we get $f(0) = 0$.

(ii) Replacing $x = 0, y = x$ and $x = x, y = -x$ in (1.1) and adding the two resulting equations, we get $f(-x) = -f(x)$.

(iii) Replacing (x, y) with $(0, 2x)$ and $(5x, x)$ in (1.1), respectively, we get

$$f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x) = 0. \tag{2.1}$$

$$\begin{aligned} f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) \\ + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x) = 0. \end{aligned} \tag{2.2}$$

Subtracting the equations (2.1) and (2.2), we find

$$\begin{aligned} 9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) \\ + 36f(3x) - 362847f(2x) + 362881f(x) = 0. \end{aligned} \tag{2.3}$$

Replacing (x, y) with $(4x, x)$ in (1.1) and multiplying the resulting equation by 9, we get

$$\begin{aligned} 9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x) \\ + 756f(3x) - 324f(2x) - 3265839f(x) = 0. \end{aligned} \tag{2.4}$$

Subtracting the equations (2.3) and (2.4), we get

$$\begin{aligned} 37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) \\ - 720f(3x) - 362523f(2x) + 3628720f(x) = 0. \end{aligned} \tag{2.5}$$

Replacing (x, y) with $(3x, x)$ in (1.1) and multiplying the resulting equation by 37, we find

$$\begin{aligned} 37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 4662f(4x) \\ - 4662f(3x) + 3108f(2x) - 13427855f(x) = 0. \end{aligned} \tag{2.6}$$

Subtracting the equations (2.5) and (2.6), we arrive at

$$\begin{aligned} 93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) \\ + 3942f(3x) - 365631f(2x) + 17056575f(x) = 0. \end{aligned} \tag{2.7}$$

Replacing (x, y) with $(2x, x)$ in (1.1) and multiplying the resulting equation by 93, we get

$$\begin{aligned} 93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x) \\ - 11625f(2x) - 33740865f(x) = 0. \end{aligned} \tag{2.8}$$

Subtracting the equations (2.7) and (2.8) and then dividing by 2, we obtain

$$\begin{aligned} 81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) \\ - 177003f(2x) + 25398720f(x) = 0. \end{aligned} \tag{2.9}$$

Replacing (x, y) with (x, x) in (1.1) and multiplying the resulting equation by 81, we get

$$\begin{aligned} 81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) \\ + 9477f(2x) - 29400570f(x) = 0. \end{aligned} \tag{2.10}$$

Subtracting the equations (2.9) and (2.10), we arrive at

$$105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x) = 0. \tag{2.11}$$

Replacing (x, y) with $(0, x)$ in (1.1) and multiplying the resulting equation by 105, we get

$$105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x) = 0. \tag{2.12}$$

Subtracting the equations (2.11) and (2.12), we can obtain

$$-181440f(2x) + 92897280f(x) = 0,$$

which gives

$$f(2x) = 2^9 f(x). \tag{2.13}$$

This completes the proof. □

A function $A : X \to X$ is said to be *additive* if $A(x + y) = A(x) + A(y)$ for all $x, y \in X$. Let $n \in \mathbb{Z}^+$. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function A_n is called *symmetric* if $A_n(x_1, ..., x_n) = A_n(x_{\pi(1)}, ..., x_{\pi(n)})$ for every permutation $\{\pi(1), ..., \pi(n)\}$ of $1, ..., n$. If $A_n(x_1, ..., x_n)$ is an n-additive symmetric map, then $A_n(x_1, ...x_n)$, its diagonal is the function $A_n(x, ..., x)$ for $x \in X$ and denoted by $A^n(x)$. Evidently, $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$. Further the resulting function after substitution $x_1 = \cdots = x_l = x$ and $x_{l+1} = \cdots = x_n = y$ in $A_n(x_1, ..., x_n)$ will be denoted by $A^{l,n-l}(x, y)$.

A function $p : X \to Y$ is called a *generalized polynomial function of degree n* provided that there exist $A^n(x) = A^0 \in Y$ and i-additive symmetric function $A_i : X \to Y$ $(1 \leq i \leq n)$ such that

$$p(x) = \sum_{i=0} A^i(x)$$

for all $x \in X$ and $A^n \neq 0$. Let $f : X \to Y$. The difference operator Δ_h defined as follows:

$$\Delta_h f(x) = f(x + h) - f(x)$$

for $\in X$. In fact, a difference operator can be extended to an n-order difference operator in the usual composition way by induction. For each $h \in X$ and $n \in \mathbb{Z}^+ \cup \{0\}$, define

$$\Delta_h^{n+1} f(x) = \Delta_h \circ \Delta_h^n f(x)$$

with the convention $\Delta_h^0 f(x) = f(x)$ and $\Delta_h^1 f(x) = \Delta_h f(x)$. Furthermore, a more general difference operator which was used in the Fréchet functional equation, can be defined as

$$\Delta_{h_1,...,h_{n+1}} f(x) = \Delta_{n+1} \circ \cdots \circ \Delta_{h_1} f(x),$$

where $x, h_1, ..., h_{n+1} \in X$.

Lemma 2.2 (see [29]). *Let G be a commutative semigroup with identity, S be a symmetric group and n be a non-negative integer. Assume that the multiplication by $n!$ is bijective on S, that is, for every $b \in G$, the equation $n!a = b$ has a unique solution in S. Then the function $f : G \to S$ is a solution of Fréchet functional equation*

$$\Delta_{x_1,...,x_{n+1}} f(x_0) = 0$$

for all $x_0, x_1, ..., x_{n+1} \in G$ if and only if f is a polynomial of degree at most n.

Theorem 2.3. *A function $f : X \to Y$ is a solution of the functional equation (1.1) if and only if f is of the form $f(x) = A^9(x)$ for all $x \in X$, where $A^9(x)$ is the diagonal of the 9-additive symmetric map $A^9 : X^9 \to Y$.*

Proof. We can rewrite the functional equation (1.1) in the form

$$\begin{aligned} f(x) = \frac{1}{126}\big[f(x+5y) - f(x-4y)\big] - \frac{1}{14}\big[f(x+4y) - f(x-3y)\big] \\ + \frac{2}{7}\big[f(x+3y) - f(x-2y)\big] - \frac{2}{3}\big[f(x+2y) - f(x-y)\big] + f(x+y) - 2880f(y). \end{aligned} \tag{2.14}$$

It follows from Lemma 2.2 that f is a generalized polynomial function of degree at most 9 and so f is of the form

$$f(x) = \sum_{n=0}^{9} A^n(x) \tag{2.15}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^n(x)$ is the diagonal of the n-additive symmetric map $A^n : X^n \to Y$ for $n = 1, 2, ..., 9$. Since $f(0) = 0$ and f is odd, we have $A^0(x) = 0$ and $A^2(x) = A^4(x) = A^6(x) = A^8(x) = 0$. Thus, the expression (2.15) can be

$$f(x) = A^9(x) + A^7(x) + A^5(x) + A^3(x) + A^1(x)$$

for all $x \in X$. Since $f(2x) = 2^9 f(x)$ and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain

$$2^9\left(A^7(x) + A^5(x) + A^3(x) + A^1(x)\right) = 2^7A^7(x) + 2^5A^5(x) + 2^3A^3(x) + 2A^1(x).$$

Hence, we get $A^1(x) = -\frac{1}{255}(192A^7(x) + 240A^5(x) + 252A^3(x))$ and so $A^7(x) = A^5(x) = A^3(x) = A^1(x) = 0$ for all $x \in X$. Therefore, we obtain that $f(x) = A^9(x)$ for all $x \in X$.

Conversely, assume that $f(x) = A^9(x)$ for all $x \in X$, where $A^9(x)$ is the diagonal of the 9-additive symmetric map $A^9 : X^9 \to Y$. According to the definition of additive function, we get

$$A^n(x+y) = A^n(x) + A^n(y) + nA^{n-1,1}(x,y) + \frac{n(n-1)}{2}A^{n-2,2}(x,y) + \cdots + nA^{1,n-1}(x,y),$$

$A^n(rx) = r^n A^n(x)$ for all $n = 1, 2, ..., 9$ and $A^{s,t}(x, ry) = r^t A^{s,t}(x, y)$ $(s, t = 1, 2, ..., 8, s + t = 9)$ whenever $x, y \in X$ and $r \in \mathbb{Q}$. Letting the above equalities into (2.14), f satisfies the functional equation (1.1). This completes the proof of Theorem 2.3. □

3. Stability of nonic functional equations

In this section, we prove the generalized Hyers-Ulam stability of the nonic functional equation (1.1) in non-Archimedean normed spaces using the direct and fixed point methods. Throughout this section, we define the difference operator

$$\begin{aligned} \Delta f(x,y) =& f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) \\ &+ 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) - 9!f(y). \end{aligned}$$

3.1. Direct method

Theorem 3.1. *Let X be an additive semigroup and Y be a non-Archimedean Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2^9|^n} = 0 \tag{3.1}$$

for all $x, y \in X$, and the limit

$$\widetilde{\varphi}_d(x) \stackrel{\text{def}}{\equiv} \lim_{n\to\infty} \max\left\{\frac{\widetilde{\varphi}(2^j x)}{|2^9|^j} : 0 \le j < n\right\} \tag{3.2}$$

exists for all $x \in X$, where

$$\begin{aligned}\widetilde{\varphi}(x) = \frac{1}{|9!|}\max\{&\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x),\\ &|93|\varphi(2x, x), |162|\varphi(x, x), |210|\varphi(0, x)\}\end{aligned}$$

for all $x \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies

$$\|\Delta f(x, y)\| \le \varphi(x, y) \tag{3.3}$$

for all $x, y \in G$. Then there exists a nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{1}{|2^9|}\widetilde{\varphi}_d(x) \tag{3.4}$$

for all $x \in X$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{\frac{1}{|2^9|^k}\widetilde{\varphi}(2^k x) : j \le k < n + j\right\} = 0 \tag{3.5}$$

for all $x \in X$, then $\mathcal{N}$ is the unique nonic mapping satisfying (3.4).

Proof. Replacing (x, y) with $(0, 2x)$ in (3.3), we get

$$\|f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x)\| \le \varphi(0, 2x) \tag{3.6}$$

Replacing (x, y) with $(5x, x)$ in (3.3), we have

$$\begin{aligned}\|f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x)\\ + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x)\| \le \varphi(5x, x)\end{aligned} \tag{3.7}$$

for all $x \in X$. Subtracting (3.6) and (3.7), then

$$\begin{aligned}&\|9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x)\\ &\quad + 36f(3x) - 362847f(2x) + 362881f(x)\|\\ &\le \max\{\varphi(0, 2x), \varphi(5x, x)\}\end{aligned} \tag{3.8}$$

for all $x \in X$. Replacing (x, y) with $(4x, x)$ in (3.3), we get

$$\begin{aligned}\|9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x)\\ + 756f(3x) - 324f(2x) - 3265839f(x)\| \le |9|\varphi(4x, x)\end{aligned} \tag{3.9}$$

for all $x \in X$. Subtracting (3.8) and (3.9), we get

$$\begin{aligned}&\|37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x)\\ &- 720f(3x) - 362523f(2x) + 3628720f(x)\|\\ &\le \max\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x)\}\end{aligned} \tag{3.10}$$

for all $x \in X$. Replacing (x, y) with $(3x, x)$ in (3.3), we have

$$\begin{aligned}\|37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 4662f(4x)\\ - 4662f(3x) + 3108f(2x) - 13427855f(x)\| \leq |37|\varphi(3x, x)\end{aligned} \tag{3.11}$$

for all $x \in X$. Subtracting (3.10) and (3.11), we find

$$\begin{aligned}&\|93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) + 3942f(3x) - 365631f(2x) + 17056575f(x)\|\\ &\leq \max\{\varphi(0, 2x), \varphi(5x, x)), |9|\varphi(4x, x), |37|\varphi(3x, x)\}\end{aligned} \tag{3.12}$$

for all $x \in X$. Replacing (x, y) with $(2x, x)$ in (3.3), we get

$$\begin{aligned}\|93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x)\\ - 11625f(2x) - 33740865f(x)\| \leq |93|\varphi(2x, x)\end{aligned} \tag{3.13}$$

for all $x \in X$. Subtracting (3.12) and (3.13), we obtain

$$\begin{aligned}&\|81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) - 177003f(2x) + 25398720f(x)\|\\ &\leq \frac{1}{|2|}\max\Big\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x)\Big\}\end{aligned} \tag{3.14}$$

for all $x \in X$. Replacing (x, y) with (x, x) in (3.3), we get

$$\begin{aligned}\|81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x)\\ + 9477f(2x) - 29400570f(x)\| \leq |81|\varphi(x, x)\end{aligned} \tag{3.15}$$

for all $x \in X$. Subtracting (3.14) and (3.15), we find

$$\begin{aligned}&\|105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x)\|\\ &\leq \frac{1}{|2|}\max\Big\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x), |162|\varphi(x, x)\Big\}\end{aligned} \tag{3.16}$$

for all $x \in X$. Replacing (x, y) with $(0, x)$ in (3.16), we get

$$\|105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x)\| \leq |105|\varphi(0, x) \tag{3.17}$$

for all $x \in X$. Subtracting (3.16) and (3.17), we get

$$\begin{aligned}&\|181440f(2x) - 9289280f(x)\|\\ &\leq \frac{1}{|2|}\max\Big\{\varphi(0, 2x), \varphi(5x, x)), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x), |162|\varphi(x, x), |210|\varphi(0, x)\Big\}\end{aligned}$$

for all $x \in X$. Thus, we deduce that

$$\begin{aligned}&\|f(2x) - 2^9 f(x)\|\\ &\leq \frac{1}{|9!|}\max\Big\{\varphi(0, 2x), \varphi(5x, x), |9|\varphi(4x, x), |37|\varphi(3x, x), |93|\varphi(2x, x),\\ &\qquad |162|\varphi(x, x), |210|\varphi(0, x)\Big\}\\ &\overset{\text{def}}{=} \widetilde{\varphi}(x)\end{aligned} \tag{3.18}$$

for all $x \in X$. It follows from (3.18), we have

$$\Big\|\frac{f(2x)}{2^9} - f(x)\Big\| \leq \frac{1}{|2^9|}\widetilde{\varphi}(x) \tag{3.19}$$

for all $x \in X$. Replacing x by $2^n x$ in (3.19), then we get

$$\Big\|\frac{f(2^{n+1}x)}{2^{9(n+1)}} - \frac{f(2^n x)}{2^{9n}}\Big\| \leq \frac{1}{|2^9|^{n+1}}\widetilde{\varphi}(2^n x) \tag{3.20}$$

for all $x \in X$. From (3.20) and (3.1) the sequence $\{\frac{f(2^n x)}{2^{9n}}\}$ is a Cauchy sequence. Since X is a non-Archimedean Banach space, $\{\frac{f(2^n x)}{2^{9n}}\}$ is convergent. So, we can define the mapping $\mathcal{N} : X \to Y$ by $\mathcal{N}(x) = \lim_{n\to\infty}\{\frac{f(2^n x)}{2^{9n}}\}$ for all $x \in X$. By induction, we can show that

$$\begin{aligned}\Big\|\frac{f(2^n x)}{2^{9n}} - f(x)\Big\| &= \Big\|\sum_{j=0}^{n-1}\frac{f(2^{j+1}x)}{2^{9(j+1)}} - \frac{f(2^j x)}{2^{9j}}\Big\| \\ &\leq \frac{1}{|2^9|}\max\Big\{\frac{\widetilde{\varphi}(2^j x)}{|2^9|^j} : 0 \leq j < n\Big\}\end{aligned} \tag{3.21}$$

for all $x \in X$ and $n \in \mathbb{N}$. By taking $n \to \infty$ in (3.21) and using (3.2), we obtain the desired inequality (3.4).

It follows from (3.1) and (3.3) that

$$\|\Delta\mathcal{N}(x,y)\| \leq \lim_{n\to\infty}\frac{1}{|2^9|^n}\|\Delta f(2^n x, 2^n y)\| \leq \lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{|2^9|^n} = 0$$

for all $x, y \in X$. Then the mapping $\mathcal{N} : X \to Y$ satisfies (1.1), that is, $\mathcal{N}$ is the nonic mapping.

To prove the uniqueness property of $\mathcal{N}$, let $\mathcal{N}'$ be another nonic mapping satisfying (3.4). Then

$$\begin{aligned}\|\mathcal{N}(x) - \mathcal{N}'(x)\| &= \lim_{j\to\infty}\frac{1}{|2^9|^j}\|\mathcal{N}(2^j x) - \mathcal{N}'(2^j x)\| \\ &\leq \lim_{j\to\infty}\frac{1}{|2^9|^j}\max\{\|\mathcal{N}(2^j x) - f(2^j x)\|, \|f(2^j x) - \mathcal{N}'(2^j x)\|\} \\ &\leq \frac{1}{|2^9|}\lim_{j\to\infty}\lim_{n\to\infty}\max\Big\{\frac{\widetilde{\varphi}(2^k x)}{|2^9|^k} : j \leq k < n + j\Big\}\end{aligned}$$

for all $x \in X$. It follows from

$$\lim_{j\to\infty}\lim_{n\to\infty}\max\Big\{\frac{\widetilde{\varphi}(2^k x)}{|2^9|^k} : j \leq k < n + j\Big\} = 0$$

for all $x \in X$ that $\mathcal{N} = \mathcal{N}'$. This completes the proof.

Corollary 3.2. *Let $\rho : [0,\infty) \to [0,\infty)$ be a mapping satisfying*

$$\rho(|2|t) \leq \rho(|2|)\rho(t), \qquad \rho(|2|) < |2|^9$$

for all $t \geq 0$. Let $\theta > 0$ be real number and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying

$$\|\Delta f(x,y)\| \leq \theta(\rho(|x|) + \rho(|y|))$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \leq \frac{\theta}{|2^9|}\widetilde{\varphi}_d(x) \tag{3.22}$$

for all $x \in X$, where

$$\begin{aligned}\widetilde{\varphi}_d(x) = \frac{\rho(|x|)}{|9!|}\max\Big\{&\rho(|2|), (\rho(|5|)+1), |9|(\rho(|4|)+1), \\ &|37|(\rho(|3|)+1), |93|(\rho(|2|)+1), |324|, |210|\Big\}.\end{aligned}$$

Proof. If a mapping $\varphi : X \times X \to [0, \infty)$ is defined by $\varphi(x, y) = \theta(\rho(|x|) + \rho(|y|))$, then we would have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2^9|^n} = \lim_{n\to\infty} \Big(\frac{\rho(|2|)}{|2^9|}\Big)^n \varphi(x, y) = 0$$

for all $x, y \in X$ since $\rho(|2|) < |2|^9$, and

$$\widetilde{\varphi}_d(x) = \lim_{n\to\infty} \max\left\{\frac{\widetilde{\varphi}(2^j x)}{|2^9|^j} : 0 \le j < n\right\} = \widetilde{\varphi}(x)$$

exists for all $x \in X$, where

$$\begin{aligned}\widetilde{\varphi}(x) =& \frac{\rho(|x|)}{|9!|} \max\Big\{\rho(|2|), (\rho(|5|) + 1), |9|(\rho(|4|) + 1),\\ & |37|(\rho(|3|) + 1), |93|(\rho(|2|) + 1), |324|, |210|\Big\}\end{aligned}$$

for all $x \in X$. Also, we get

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{\frac{1}{|2|^{9k}} \widetilde{\varphi}(2^k x) : j \le k < n + j\right\} = \lim_{j\to\infty} \frac{1}{|2|^{9j}} \widetilde{\varphi}(2^j x) = 0$$

for all $x \in X$. By applying Theorem 3.1, we obtain the desired result. This completes the proof. □

3.2. Fixed point method

Next, we investigate the generalized Hyers-Ulam stability of the nonic functional equation (1.1) in non-Archimedean normed spaces using fixed point method. We recall the following fixed point theorem which was proved by Diaz and Margolis.

Theorem 3.3. [7] *Let (Ω, d) be a complete generalized metric space and $T : \Omega \to \Omega$ be a strictly contractive mapping with Lipshitz constant L. Then, for any $x \in \Omega$, either $d(T^n x, T^{n+1} x) = \infty$ for all nonnegative integers $n \ge 0$ or other exists a natural number n_o such that*

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \ge n_o$;
(ii) the sequence $\{T^n x\}$ is convergent to a fixed point $y*$ of T;
(iii) $y*$ is the unique fixed point of T in the set $\Omega_1 = \{y \in \Omega : d(T^{n_o} x, y) < \infty\}$;
(iv) $d(y, y*) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Omega_1$.

Theorem 3.4. *Assume that X is a non-Archimedean normed space and Y is a a non-Archimedean Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that there exists a constant $0 < L < 1$ with*

$$\varphi(2x, 2y) \le |2^9| L \varphi(x, y) \tag{3.23}$$

and $\lim_{n\to\infty} \frac{1}{2^{9n}} \varphi(2^n x, 2^n y) = 0$ for all $x, y \in X$. If $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying

$$\|\Delta f(x, y)\| \le \varphi(x, y) \tag{3.24}$$

for all $x, y \in X$, then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \le \frac{1}{|2^9|(1 - L)} \widetilde{\varphi}(x) \tag{3.25}$$

for all $x \in X$, where

$$\widetilde{\varphi}(x) = \frac{1}{|9!|}\max\Big\{\varphi(0,2x), \varphi(5x,x), |9|\varphi(4x,x), |37|\varphi(3x,x),$$
$$|93|\varphi(2x,x), |162|\varphi(x,x), |210|\varphi(0,x)\Big\}.$$

Proof. From (3.18) we obtain

$$\Big\|f(x) - \frac{f(2x)}{2^9}\Big\| \leq \frac{1}{|2^9|}\widetilde{\varphi}(x) \tag{3.26}$$

for all $x \in X$. Let Ω be a set of all mappings from X into Y and a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{\alpha \in [0,\infty) : \|g(x) - h(x)\| \leq \alpha\widetilde{\varphi}(x),\ \forall\, x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (Ω, d) is a generalized complete metric space.

Now, we consider the linear mapping $T : \Omega \to \Omega$ defined by

$$Tg(x) = \frac{g(2x)}{2^9}$$

for all $x \in X$. Let $f, g \in \Omega$ and $\alpha \in [0,\infty)$ be an arbitrary constant with $d(f,g) \leq \alpha$. Then $\|f(x) - g(x)\| \leq \alpha\widetilde{\varphi}(x)$ for all $x \in X$ and so

$$\|Tf(x) - Tg(x)\| \leq \frac{1}{|2^9|}\|f(2x) - g(2x)\| \leq \frac{\alpha}{|2^9|}\widetilde{\varphi}(2x) \leq \alpha L\widetilde{\varphi}(x)$$

for all $x \in X$. Thus, $d(f,g) \leq \alpha$ implies that $d(Tf,Tg) \leq L\alpha$. This means that

$$d(Tf,Tg) \leq Ld(f,g)$$

for all $f, g \in \Omega$. It follows from (3.18) that $d(Tf,f) \leq \frac{1}{|2^9|}$ for all $x \in X$. From the conditions (2) and (3) of Theorem 3.3, there exists a mapping $\mathcal{N} : X \to Y$, which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$\mathcal{N}(x) = \lim_{n\to\infty} \frac{1}{2^{9n}} f(2^n x)$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, \mathcal{N}) = 0$. Again, from the condition (4) of Theorem 3.3, we obtain

$$d(f,\mathcal{N}) \leq \frac{1}{1-L}d(Tf,f) \leq \frac{1}{|2^9|(1-L)},$$

which implies the inequality (3.25).

If we replace x by $2^n x$ and y by $2^n y$ in (3.3), then we obtain

$$\Big\|\frac{\Delta f(2^n x, 2^n y)}{2^{9n}}\Big\| \leq \frac{1}{|2^9|^n}\|\Delta f(2^n x, 2^n y)\| \leq \frac{1}{|2^9|^n}\varphi(2^n x, 2^n y) \tag{3.27}$$

for all $x, y \in X$. Taking as the limit $n \to \infty$ in (3.27), we deduce that $\Delta\mathcal{N}(x,y) = 0$ for all $x, y \in X$. That is, the function $\mathcal{N} : X \to Y$ is nonic, as desired. This completes the proof. □

Corollary 3.5. *Let X be a non-Archimedean normed space over $\mathbb{K}$ and Y be a non-Archimedean Banach space. Let $\theta > 0$ be real number and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\|\Delta f(x,y)\| \leq \theta$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \leq \frac{\theta}{|2240|(|2^9| - |2^p|)}. \tag{3.28}$$

Corollary 3.6. *Let $\theta, p > 0$ be real numbers with $0 \leq p < 9$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\|\Delta f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - \mathcal{N}(x)\| \leq \frac{\theta M_p}{|2^9| - |2^p|}\|x\|^p \tag{3.29}$$

for all $x \in X$, where

$$M_p = \frac{1}{|9!|} \max\Big\{|2|^p, |5|^p + 1, |9|(|4|^p + 1), |37|(|3|^p + 1), |93|(|2|^p + 1), |324|, |210|\Big\}.$$

Corollary 3.7. *Let $\theta, p, q > 0$ be real numbers with $p + q < 9$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\|\Delta f(x, y)\| \leq \theta\|x\|^p\|y\|^q$$

for all $x, y \in X$. Then there exists a unique nonic mapping $\mathcal{N} : X \to Y$ such that

$$\|f(x) - N(x)\| \leq \frac{\theta M_{p,q}}{|2^9 - 2^{p+q}|}\|x\|^{p+q} \tag{3.30}$$

for all $x \in X$, where

$$M_{p,q} = \frac{1}{|9!|} \max\Big\{|5|^p, |9||4|^p, |37||3|^p, |93||2|^p, |162|\Big\}$$

Remark. Let X be a normed space and Y be a Banach space in Theorem 3.5. Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that there exists a constant $0 < L < 1$ with

$$\varphi(2x, 2y) \leq 2^9 L\varphi(x, y)$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying $\|\Delta f(x, y)\| \leq \varphi(x, y)$ for all $x, y \in X$. Using the fixed point method, we can show that there exists a unique nonic mapping $\mathcal{N} : X \to Y$ satisfying (3.24) and

$$\|f(x) - \mathcal{N}(x)\| \leq \frac{1}{2^9(1 - L)}\widetilde{\varphi}(x) \tag{3.31}$$

where

$$\begin{aligned}\widetilde{\varphi}(x) = \frac{1}{9!}\Big(&\varphi(0, 2x) + \varphi(5x, x) + 9\varphi(4x, x) \\ &+ 37\varphi(3x, x) + 93\varphi(2x, x) + 162\varphi(x, x) + 210\varphi(0, x)\Big)\end{aligned}$$

for all $x \in X$ (see [15]).

It is easy to show that the approximation in non-Archimedean normed space (see (3.25)) is better than the approximation in (Archimedean) normed space (see (3.31)).

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Francesco Altomare
Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
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8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

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Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

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Approximation reduction for multi-granulation dual hesitant fuzzy rough sets

Yanping He [1], Lianglin Xiong[2*], Haidong Zhang [3†]

1. School of Electrical Engineering,
Northwest University for Nationalities,
Lanzhou, Gansu, 730030, P. R. China
2. School of Mathematics and Computer Science,
Yunnan Minzu University,
Kunming, Yunnan, 650500, P. R. China
3. School of Mathematics and Computer Science,
Northwest University for Nationalities,
Lanzhou, Gansu, 730030, P. R. China

Abstract

Multi-granulation dual hesitant fuzzy rough set is an extension of intuitionistic fuzzy multi-granulation rough sets and multi-granulation fuzzy rough sets. For further studying the theories and applications of multi-granulation dual hesitant fuzzy rough sets, in this paper, we mainly investigate reduction approaches of the multi-granulation dual hesitant fuzzy rough sets. We develop a reduction approach in multi-granulation dual hesitant fuzzy decision information systems based on multi-granulation dual hesitant fuzzy rough sets to eliminate redundant dual hesitant fuzzy granulations. And an example is provided to illustrate the validity of this approach.

Key words: Multi-granulation fuzzy rough set; Multi-granulation dual hesitant fuzzy rough set; Reduction approach

1 Introduction

Rough set theory, introduced by Pawlak [19, 20], is a new mathematical approach to cope with imprecision and uncertainty in data analysis, and can be regarded as a valid means of granular computing [21]. In Pawlak's rough set model, a key notion is equivalence

*Corresponding author. Address: School of Mathematics and Computer Science Yunnan Minzu University, Kunming, Yunnan, 650500, China. E-mail:lianglin_5318@126.com

†Corresponding author. Address: School of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou, Gansu, 730030, P.R.China. E-mail:lingdianstar@163.com

relation. However, the equivalence relation is a very stringent condition which may limit the application of rough sets. Therefore, by replacing the equivalence relation with other binary relations, such as fuzzy, interval-valued fuzzy, intuitionistic fuzzy, hesitant fuzzy and interval-valued hesitant fuzzy, and so on, lots of researchers have proposed many new rough sets model. For example, Dubois and Prade [4] initiated two rough set models which are called rough fuzzy sets and fuzzy rough sets. Furthermore, Wu et al. [31, 32] studied various generalized fuzzy approximation operators which are characterized by different sets of axioms. According to fuzzy rough sets in the sense of Nanda and Majumda [18], Jena and Ghosh [8] presented the concept of intuitionistic fuzzy rough sets which are not defined by an approximation space. By using a special type of intuitionistic fuzzy triangular norm *min*, Zhou and Wu [48] discussed various relation-based intuitionistic fuzzy rough approximation operators. Meanwhile, they [49] also investigated intuitionistic fuzzy rough approximations on one universe based on intuitionistic fuzzy implicators. In [46], Zhang et al. proposed a generalized interval-valued fuzzy rough set and applied it to decision making. Very recently, rough set theory has been developed into hesitant fuzzy environment and interval-valued hesitant fuzzy environment, and the results are, respectively, called hesitant fuzzy rough sets [41] and interval-valued hesitant fuzzy rough sets [45].

The generalization of Pawlak's rough set model has become a new research hotspot from the perspective of granular computing. Since Qian et al. [22] proposed multi-granulation rough set (MGRS) theory, lots of fruitful results about MGRS theory have been achieved. Qian et al. [23] proposed an incomplete multi-granulation rough set model by using multiple tolerance relations on the universe, and studied decision-theoretic rough sets based on multi-granulations [25]. She et al. [27] investigated topological structures of MGRSs. Yang et al. [42] extended Qian's MGRS model to fuzzy environment and explored a MGRS based on fuzzy relations. Along the lines of Qian's MGRSs, Xu et al. [36] initiated an ordered MGRS model. And they [34, 35] also proposed multi-granulation fuzzy rough sets based on multiple classical equivalence relations and multi-granulation fuzzy rough sets in a fuzzy tolerance approximation space. Through combining MGRSs and intuitionistic fuzzy rough sets, Huang et al. [7] proposed intuitionistic fuzzy MGRSs and gave a reduction approach of this model. Liu et al. [9, 10] presented covering fuzzy rough sets based on MGRSs. To handle data sets in the context of hybrid attributes, Lin et al. [11] introduced the neighborhood-based MGRSs, generalized the covering into multi-granulation environment and proposed the covering based on optimistic and pessimistic MGRSs [12]. More recently, Liang et al. [16] presented an efficient rough feature selection algorithm through a multi-granulation view. Yang et al. [43] proposed a test cost sensitive multi-granulation rough set model to take the test cost into consideration in both data mining and machine learning.

As one of the extensions of Zadeh's fuzzy set [50], hesitant fuzzy (HF) set theory, initiated by Torra [28, 29], permits the membership degree of an element to a set having

several possible values, and can express the hesitant information more comprehensively than other extensions of fuzzy set. Since the appearance of hesitant fuzzy set, it has attracted more and more scholars' attention. For example, Xu and Xia [33,37,38] discussed the aggregation operators, correlation measures, distance, and similarity measures for HF sets. Meanwhile, Chen et al. [2] gave correlation coefficients of HF sets and applied them to clustering analysis. Subsequently, Liao et al. [15] proposed novel correlation coefficients between hesitant fuzzy sets and and applied them to decision making. In [5], Farhadinia introduced information measures for HF sets. Rodrguez et al. [26] proposed a HF linguistic term set providing a more powerful form to represent decision makers' preferences in the decision making process. Liao and Xu [13,14] developed a hesitant fuzzy VIKOR method based on some new measures, and proposed some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment. Zhang and Wei [51] proposed an extension of VIKOR method based on hesitant fuzzy set in decision making problem.

Dual hesitant fuzzy (DHF) set, introduced by Zhu et al. [44], is a comprehensive set encompassing fuzzy sets, intuitionistic fuzzy sets [1], hesitant fuzzy sets, and fuzzy multisets [17] as special cases. By several possible values for the membership and nonmembership degrees, dual hesitant fuzzy sets are more objective than hesitant fuzzy sets to describe the vagueness of data or information. In recent years, many authors have investigated multiple attribute decision-making theories and methods under the dual hesitant fuzzy environment [3,6,30,40]. Very recently, the combination of dual hesitant fuzzy sets and other uncertainty theories is becoming a research hotspot. For example, by integrating rough set theory with dual hesitant fuzzy sets, Zhang et al. [47] proposed a single-granulation dual hesitant fuzzy rough set (SGDHFRS). Based on the SGDHFRSs, they presented the concept of multi-granulation dual hesitant fuzzy rough sets (MGDHFRSs) in which two types of this model are proposed: one is called the optimistic MGDHFRS; the other is called the pessimistic MGDHFRS. The relationships among the optimistic MGDHFRS, the pessimistic MGDHFRS and the SGDHFRS are then established. However, reduction approaches of the MGDHFRSs are not still be investigated. In order to develop the application of the MGDHFRSs, topological properties and reduction approaches on MGDHFRSs further need to be studied. The objective of this paper is mainly to focus on the study of reduction approaches of the MGDHFRSs.

The rest of the paper is organized as follows. The next section reviews some basic concepts considered in the study, such as HF sets, DHF sets and MGDHFRSs. In Section 3, we propose a reduction approach of MGDHFRSs to eliminate redundant DHF granulations by a numerical example. Finally, we conclude the paper in Section 4.

2 Preliminaries

2.1 Dual hesitant fuzzy sets

As an extension of hesitant fuzzy sets, dual hesitant fuzzy sets are defined by Zhu et al. [44] as follows:

Definition 2.1 ([44]) *Let U be a fixed set, a dual hesitant fuzzy set $\mathbb{D}$ on U is described as:*

$\mathbb{D} = \{< x, h_{\mathbb{D}}(x), g_{\mathbb{D}}(x) > | x \in U\}$,

in which $h_{\mathbb{D}}(x)$ and $g_{\mathbb{D}}(x)$ are two sets of some values in [0,1], denoting the possible membership degrees and non-membership degrees of the element $x \in U$ to the set $\mathbb{D}$ respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where for all $x \in U$, $\gamma \in h_{\mathbb{D}}(x), \eta \in g_{\mathbb{D}}(x)$, $\gamma^+ \in h^+_{\mathbb{D}}(x) = \cup_{\gamma \in h_{\mathbb{D}}(x)} max\{\gamma\}, \eta^+ \in g^+_{\mathbb{D}}(x) = \cup_{\eta \in g_{\mathbb{D}}(x)} max\{\eta\}$.

For convenience, the pair $d(x) = (h_{\mathbb{D}}(x), g_{\mathbb{D}}(x))$ is called a DHF element denoted by $d = (h, g)$. The set of all DHF sets on U is denoted by $DHF(U)$.

2.2 Multi-granulation dual hesitant fuzzy rough sets

In [47], Zhang et al. proposed a SGDHFRS by integrating rough set theory with dual hesitant fuzzy sets. First, they introduced a DHF relation as follows:

Definition 2.2 ([47]) *Let U, V be two nonempty and finite universes. A DHF subset $\mathbb{R}$ of the universe $U \times V$ is called a DHF relation from U to V, namely, $\mathbb{R}$ is given by*

$\mathbb{R} = \{< (x, y), h_{\mathbb{R}}(x, y), g_{\mathbb{R}}(x, y) > | (x, y) \in U \times V\}$,

where $h_{\mathbb{R}}, g_{\mathbb{R}} : U \times V \to [0, 1]$ are two sets of some values in [0,1], denoting the possible membership degrees and non-membership degrees of the relationships between x and y respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where for all $(x, y) \in U \times V$, $\gamma \in h_{\mathbb{R}}(x, y), \eta \in g_{\mathbb{R}}(x, y)$, $\gamma^+ \in h^+_{\mathbb{R}}(x, y) = \cup_{\gamma \in h_{\mathbb{R}}(x,y)} max\{\gamma\}, \eta^+ \in g^+_{\mathbb{R}}(x, y) = \cup_{\eta \in g_{\mathbb{R}}(x,y)} max\{\eta\}$.

In particular, if $U = V$, we call $\mathbb{R}$ a DHF relation on U. In what follows several special DHF relations are introduced as follows:

Definition 2.3 ([47]) *The DHF relation $\mathbb{R}$ from U to V is said to be serial if for each $x \in U$, there exists a $y \in V$ such that $h_{\mathbb{R}}(x, y) = \{1\}$ and $g_{\mathbb{R}}(x, y) = \{0\}$; $\mathbb{R}$ is said to be reflexive on U if $h_{\mathbb{R}}(x, x) = \{1\}$ and $g_{\mathbb{R}}(x, x) = \{0\}$ for all $x \in U$; $\mathbb{R}$ is referred to as a symmetric DHF relation on U if $h_{\mathbb{R}}(x, y) = h_{\mathbb{R}}(y, x)$ and $g_{\mathbb{R}}(x, y) = g_{\mathbb{R}}(y, x)$ for all $x, y \in U$.*

If a DHF relation $\mathbb{R}$ on U is reflexive and symmetric, it is called a DHF tolerance relation on U.

Based on the above DHF relation, lower and upper DHF approximation operators are defined as follows:

Definition 2.4 ([47]) *Let U be a nonempty and finite universes and $\mathbb{R}$ be a DHF tolerance relation on U. The pair $(U,\mathbb{R})$ is called a DHF tolerance approximation space. For any $\mathbb{A}\in DHF(U)$, the lower and upper approximations of $\mathbb{A}$ with respect to $(U,\mathbb{R})$, denoted by $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two DHF sets of U and are, respectively, defined as follows:*

$$\underline{\mathbb{R}}(\mathbb{A})=\{<x,h_{\underline{\mathbb{R}}(\mathbb{A})}(x),g_{\underline{\mathbb{R}}(\mathbb{A})}(x)>|x\in U\},\tag{1}$$

$$\overline{\mathbb{R}}(\mathbb{A})=\{<x,h_{\overline{\mathbb{R}}(\mathbb{A})}(x),g_{\overline{\mathbb{R}}(\mathbb{A})}(x)>|x\in U\},\tag{2}$$

where

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x)=\overline{\wedge}_{y_j\in U}\{g_{\mathbb{R}}(x,y_j)\veebar h_{\mathbb{A}}(y_j)\},\ g_{\underline{\mathbb{R}}(\mathbb{A})}(x)=\veebar_{y_j\in U}\{h_{\mathbb{R}}(x,y_j)\ \overline{\wedge}\ g_{\mathbb{A}}(y_j)\};$$

$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x)=\veebar_{y_j\in U}\{h_{\mathbb{R}}(x,y_j)\ \overline{\wedge}\ h_{\mathbb{A}}(y_j)\},\ g_{\overline{\mathbb{R}}(\mathbb{A})}(x)=\overline{\wedge}_{y_j\in U}\{g_{\mathbb{R}}(x,y_j)\veebar g_{\mathbb{A}}(y_j)\}.$$

$\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ are, respectively, called the single-granulation lower and upper approximations of $\mathbb{A}$ with respect to $(U,\mathbb{R})$. The pair $(\underline{\mathbb{R}}(\mathbb{A}),\overline{\mathbb{R}}(\mathbb{A}))$ is called a SGDHFRS of $\mathbb{A}$ with respect to $(U,\mathbb{R})$, and $\underline{\mathbb{R}},\overline{\mathbb{R}}:DHF(U)\to DHF(U)$ are referred to as single granulation lower and upper DHF rough approximation operators, respectively.

Based on the SGDHFRSs, Zhang et al. [47] presented two MGDHFRS models: one is called the optimistic MGDHFRS; the other is called the pessimistic MGDHFRS.

Definition 2.5 ([47]) *Let U be a nonempty and finite universe of discourse and $\mathbb{R}_i(1\le i\le m)$ be m DHF tolerance relations on U; the pair $(U,\{\mathbb{R}_i|1\le i\le m\})$ is called the DHF tolerance approximation space. For any $\mathbb{A}\in DHF(U)$, the optimistic multi-granulation dual hesitant fuzzy lower and upper approximations of $\mathbb{A}$ with respect to $(U,\{\mathbb{R}_i|1\le i\le m\})$, denoted by $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$ and $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$, are two DHF sets and are, respectively, defined as follows:*

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})=\{<x,h_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x),g_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x)>|x\in U\},\tag{3}$$

$$\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})=\{<x,h_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x),g_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x)>|x\in U\},\tag{4}$$

where

$h_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x) = \underline{\vee}_{i=1}^{m}\overline{\wedge}_{y_j\in U}\{g_{\mathbb{R}_i}(x,y_j)\,\underline{\vee}\,h_{\mathbb{A}}(y_j)\}$,
$g_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x) = \overline{\wedge}_{i=1}^{m}\underline{\vee}_{y_j\in U}\{h_{\mathbb{R}_i}(x,y_j)\,\overline{\wedge}\,g_{\mathbb{A}}(y_j)\}$,
$h_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x) = \overline{\wedge}_{i=1}^{m}\underline{\vee}_{y_j\in U}\{h_{\mathbb{R}_i}(x,y_j)\,\overline{\wedge}\,h_{\mathbb{A}}(y_j)\}$,
$g_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})}(x) = \underline{\vee}_{i=1}^{m}\overline{\wedge}_{y_j\in U}\{g_{\mathbb{R}_i}(x,y_j)\,\underline{\vee}\,g_{\mathbb{A}}(y_j)\}$.

The pair $(\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A}), \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A}))$ *is called an optimistic MGDHFRS of* $\mathbb{A}$ *with respect to* $(U,\{\mathbb{R}_i|1\le i\le m\})$. *If* $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$, *then* $\mathbb{A}$ *is referred to as optimistic-definable in* $(U,\{\mathbb{R}_i|1\le i\le m\})$; *otherwise,* $\mathbb{A}$ *is referred to as optimistic-undefinable in* $(U,\{\mathbb{R}_i|1\le i\le m\})$.

Definition 2.6 ([47]) *Let* $(U,\{\mathbb{R}_i|1\le i\le m\})$ *be the DHF tolerance approximation space. For any* $\mathbb{A}\in DHF(U)$, *the pessimistic multi-granulation dual hesitant fuzzy lower and upper approximations of* $\mathbb{A}$ *with respect to* $(U,\{\mathbb{R}_i|1\le i\le m\})$, *denoted by* $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$ *and* $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$, *are two DHF sets and are, respectively, defined as follows:*

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A}) = \{< x, h_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x), g_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x) > |x\in U\}, \tag{5}$$

$$\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A}) = \{< x, h_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x), g_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x) > |x\in U\}, \tag{6}$$

where

$h_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x) = \overline{\wedge}_{i=1}^{m}\overline{\wedge}_{y_j\in U}\{g_{\mathbb{R}_i}(x,y_j)\,\underline{\vee}\,h_{\mathbb{A}}(y_j)\}$,
$g_{\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x) = \underline{\vee}_{i=1}^{m}\underline{\vee}_{y_j\in U}\{h_{\mathbb{R}_i}(x,y_j)\,\overline{\wedge}\,g_{\mathbb{A}}(y_j)\}$,
$h_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x) = \underline{\vee}_{i=1}^{m}\underline{\vee}_{y_j\in U}\{h_{\mathbb{R}_i}(x,y_j)\,\overline{\wedge}\,h_{\mathbb{A}}(y_j)\}$,
$g_{\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})}(x) = \overline{\wedge}_{i=1}^{m}\overline{\wedge}_{y_j\in U}\{g_{\mathbb{R}_i}(x,y_j)\,\underline{\vee}\,g_{\mathbb{A}}(y_j)\}$.

The pair $(\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A}), \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A}))$ *is called a pessimistic MGDHFRS of* $\mathbb{A}$ *with respect to* $(U,\{\mathbb{R}_i|1\le i\le m\})$. *If* $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$, *then* $\mathbb{A}$ *is referred to as pessimistic-definable in* $(U,\{\mathbb{R}_i|1\le i\le m\})$; *otherwise,* $\mathbb{A}$ *is referred to as pessimistic-undefinable in* $(U,\{\mathbb{R}_i|1\le i\le m\})$.

Then, Zhang et al. [47] established the relationships among the optimistic MGDHFRS, the pessimistic MGDHFRS and the SGDHFRS.

Theorem 2.7 ([47]) *Let* U *be a nonempty and finite universe of discourse and* $\mathbb{R}_i(1\le i\le m)$ *be* m *DHF tolerance relations on* U. *For any* $\mathbb{A}\in DHF(U)$, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$ *and* $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$ *are the optimistic multi-granulation dual hesitant fuzzy lower and upper approximations of* $\mathbb{A}$ *with respect to* $(U,\{\mathbb{R}_i|1\le i\le m\})$, *respectively. Then,*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})=\Cup_{i=1}^{m}\underline{\mathbb{R}_i}(\mathbb{A})$,

(2) $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})=\Cap_{i=1}^{m}\overline{\mathbb{R}_i}(\mathbb{A})$.

Theorem 2.8 ([47]) *Let U be a nonempty and finite universe of discourse and $\mathbb{R}_i(1\leq i\leq m)$ be m DHF tolerance relations on U. For any $\mathbb{A}\in DHF(U)$, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$ and $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$ are the pessimistic multi-granulation DHF lower and upper approximations of $\mathbb{A}$ with respect to $(U,\{\mathbb{R}_i|1\leq i\leq m\})$, respectively. Then,*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})=\Cap_{i=1}^{m}\underline{\mathbb{R}_i}(\mathbb{A})$,

(2) $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})=\Cup_{i=1}^{m}\overline{\mathbb{R}_i}(\mathbb{A})$.

3 Approximation reduction approach in multi-granulation DHF decision information system

In this section, we establish a practical reduction approach in multi-granulation DHF decision information system (MGDHFDIS) based on the MGDHFRS model. The objective of reduction is to obtain a smallest subset of DHF relations that may preserve consistence of MGDHFDIS.

Definition 3.1 *Let apr=$(U,\mathbb{R}=\{\mathbb{R}_i|1\leq i\leq m\})$ be the DHF tolerance approximation space, $\mathbb{A}$ be the DHF set and $\underline{\mathbb{R}}^{O},\overline{\mathbb{R}}^{O},\underline{\mathbb{R}}^{P},\overline{\mathbb{R}}^{P}\subseteq\mathbb{R}$.*

(1) If $\underline{\sum_{\mathbb{R}_i\in\underline{\mathbb{R}}^{O}}\mathbb{R}_i}^{O}(\mathbb{A})=\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$, then $\underline{\mathbb{R}}^{O}$ is referred to as a consistent optimistic lower approximation of apr. If $\underline{\mathbb{R}}^{O}$ is a consistent optimistic lower approximation, and no proper subset of $\underline{\mathbb{R}}^{O}$ is a consistent optimistic lower approximation, then $\underline{\mathbb{R}}^{O}$ is called an optimistic lower approximation reduct of apr.

(2) If $\underline{\sum_{\mathbb{R}_i\in\underline{\mathbb{R}}^{P}}\mathbb{R}_i}^{P}(\mathbb{A})=\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$, then $\underline{\mathbb{R}}^{P}$ is referred to as a consistent pessimistic lower approximation of apr. If $\underline{\mathbb{R}}^{P}$ is a consistent pessimistic lower approximation, and no proper subset of $\underline{\mathbb{R}}^{P}$ is a consistent pessimistic lower approximation, then $\underline{\mathbb{R}}^{P}$ is called a pessimistic lower approximation reduct of apr.

(3) If $\overline{\sum_{\mathbb{R}_i\in\overline{\mathbb{R}}^{O}}\mathbb{R}_i}^{O}(\mathbb{A})=\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{O}(\mathbb{A})$, then $\overline{\mathbb{R}}^{O}$ is referred to as a consistent optimistic upper approximation of apr. If $\overline{\mathbb{R}}^{O}$ is a consistent optimistic upper approximation set, and no proper subset of $\overline{\mathbb{R}}^{O}$ is a consistent optimistic upper approximation, then $\overline{\mathbb{R}}^{O}$ is called an optimistic upper approximation reduct of apr.

(4) If $\overline{\sum_{\mathbb{R}_i\in\overline{\mathbb{R}}^{P}}\mathbb{R}_i}^{P}(\mathbb{A})=\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{P}(\mathbb{A})$, then $\overline{\mathbb{R}}^{P}$ is referred to as a consistent pessimistic upper approximation of apr. If $\overline{\mathbb{R}}^{P}$ is a consistent pessimistic upper approximation, and no proper subset of $\overline{\mathbb{R}}^{P}$ is a consistent pessimistic upper approximation, then $\overline{\mathbb{R}}^{P}$ is called a pessimistic upper approximation reduct of apr.

Table 1: DHF relation $\mathbb{R}_1$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.4,0.5}, {0.2,0.4})	({0.2,0.3}, {0.5,0.7})	({0.6,0.8}, {0.1,0.2})
x_2	({0.1,0.2}, {0.7,0.8})	({1,1}, {0,0})	({0.2,0.3}, {0.6,0.7})	({0.4,0.5}, {0.5,0.5})
x_3	({0.2,0.2}, {0.6,0.7})	({0.5,0.8}, {0.2,0.2})	({1,1}, {0,0})	({0.3,0.4}, {0.5,0.6})
x_4	({0.3,0.5}, {0.4,0.5})	({0.4,0.5}, {0.2,0.4})	({0.2,0.3}, {0.5,0.7})	({1,1}, {0,0})

Table 2: DHF relation $\mathbb{R}_2$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.3,0.5}, {0.2,0.5})	({0.2,0.2}, {0.7,0.8})	({0.4,0.5}, {0.3,0.5})
x_2	({0.2,0.2}, {0.6,0.8})	({1,1}, {0,0})	({0.4,0.6}, {0.3,0.4})	({0.2,0.5}, {0.3,0.5})
x_3	({0.1,0.3}, {0.5,0.6})	({0.4,0.5}, {0.3,0.4})	({1,1}, {0,0})	({0.1,0.2}, {0.7,0.8})
x_4	({0.2,0.5}, {0.3,0.5})	({0.1,0.1}, {0.8,0.9})	({0.5,0.6}, {0.3,0.4})	({1,1}, {0,0})

From Definition 3.1, we see that the lower approximation reduct is the smallest subset of $\mathbb{R} = \{\mathbb{R}_i | 1 \leq i \leq m\}$ which preserves the lower approximations of all DHF sets in U. And so is for the upper approximation reduct.

Definition 3.2 *A multi-granulation DHF decision information system is a quads* $S = (U, \{\mathbb{R}_i | 1 \leq i \leq m\}, D, V)$, *where* U *is a nonempty and finite universe;* $\{\mathbb{R}_i | 1 \leq i \leq m\}$ *is a set of* m *DHF relations on* U*;* D *is a nonempty and finite set of decision attributes;* $V = \{g(x,d) | x \in U, d \in D\}$ *is a set of the relationships between* U *and* D*, and* $g(x,d)$ *is a DHF element denoted as* $g(x,d) = (h_d(x), g_d(x))$*. We call* $g(x,d)$ *the decision DHF value of* x *under decision attribute* d*.*

Example 3.3 *A MGDHFDIS can be described as follows:* $U = \{x_1, x_2, x_3, x_4\}$*;* $\mathbb{R}_i (1 \leq i \leq 5)$ *are DHF relations on* U *shown as Tables 1-5;* $D = \{d_1, d_2\}$*;* $V = \{g(x_i, d_j) | x_i \in U, d_j \in D\}$*, where* $g(x_1, d_1) = (\{0.3, 0.4\}, \{0.2, 0.6\})$*,* $g(x_2, d_1) = (\{0.3, 0.5\}, \{0.4, 0.5\})$*,* $g(x_3, d_1) = (\{0.3, 0.7\}, \{0.2, 0.3\})$*,* $g(x_4, d_1) = (\{0.2, 0.3\}, \{0.6, 0.7\})$*,* $g(x_1, d_2) = (\{0.5, 0.7\}, \{0.2, 0.3\})$*,* $g(x_2, d_2) = (\{0.5, 0.6\}, \{0.2, 0.4\})$*,* $g(x_3, d_2) = (\{0.2, 0.4\}, \{0.3, 0.6\})$ *and* $g(x_4, d_2) = (\{0, 0.1\}, \{0.8, 0.9\})$.

On the basis of Definition 3.1, approximation reducts in MGDHFDIS based on the MGDHFRS model are defined as follows.

Definition 3.4 *Let a MGDHFDIS* $= (U, \{\mathbb{R}_i | 1 \le i \le m\}, D, V)$, *where* $U = \{x_1, x_2, \cdots, x_n\}$ *and* $D = \{d_1, d_2, \cdots, d_v\}$, $D_j = \{< x_i, g(x_i, d_j) > | x_i \in U, d_j \in D\} \in DHF(U)$.

(1) For all $j(1 \le j \le v)$, *if* $\underline{\sum_{\mathbb{R}_i \in \underline{\mathbb{R}}^O} \mathbb{R}_i}^O(D_j) = \underline{\sum_{i=1}^m \mathbb{R}_i}^O(D_j)$, *then* $\underline{\mathbb{R}}^O$ *is referred to as a consistent optimistic lower approximation of MGDHFDIS. If* $\underline{\mathbb{R}}^O$ *is a consistent optimistic lower approximation, and no proper subset of* $\underline{\mathbb{R}}^O$ *is a consistent optimistic lower approximation, then* $\underline{\mathbb{R}}^O$ *is called an optimistic lower approximation reduct of MGDHFDIS.*

(2) For all $j(1 \le j \le v)$, *if* $\underline{\sum_{\mathbb{R}_i \in \underline{\mathbb{R}}^P} \mathbb{R}_i}^P(D_j) = \underline{\sum_{i=1}^m \mathbb{R}_i}^P(D_j)$, *then* $\underline{\mathbb{R}}^P$ *is referred to as a consistent pessimistic lower approximation of MGDHFDIS. If* $\underline{\mathbb{R}}^P$ *is a consistent pessimistic lower approximation, and no proper subset of* $\underline{\mathbb{R}}^P$ *is a consistent pessimistic lower approximation, then* $\underline{\mathbb{R}}^P$ *is called a pessimistic lower approximation reduct of MGDHFDIS.*

(3) For all $j(1 \le j \le v)$, *if* $\overline{\sum_{\mathbb{R}_i \in \overline{\mathbb{R}}^O} \mathbb{R}_i}^O(D_j) = \overline{\sum_{i=1}^m \mathbb{R}_i}^O(D_j)$, *then* $\overline{\mathbb{R}}^O$ *is referred to as a consistent optimistic upper approximation of MGDHFDIS. If* $\overline{\mathbb{R}}^O$ *is a consistent optimistic upper approximation, and no proper subset of* $\overline{\mathbb{R}}^O$ *is a consistent optimistic upper approximation, then* $\overline{\mathbb{R}}^O$ *is called an optimistic upper approximation reduct of MGDHFDIS.*

(4) For all $j(1 \le j \le v)$, *if* $\overline{\sum_{\mathbb{R}_i \in \overline{\mathbb{R}}^P} \mathbb{R}_i}^P(D_j) = \overline{\sum_{i=1}^m \mathbb{R}_i}^P(D_j)$, *then* $\overline{\mathbb{R}}^P$ *is referred to as a consistent pessimistic upper approximation of MGDHFDIS. If* $\overline{\mathbb{R}}^P$ *is a consistent pessimistic upper approximation, and no proper subset of* $\overline{\mathbb{R}}^P$ *is a consistent pessimistic upper approximation, then* $\overline{\mathbb{R}}^P$ *is called a pessimistic upper approximation reduct of MGDHFDIS.*

In order to obtain the optimistic and pessimistic approximation reducts of MGDHFDIS, we introduce the concepts of DHF vectors and DHF matrices. In the text that follows, without loss of generality, we suppose that the first HF elements in all the DHF elements have the same length k, and the second HF elements in all the DHF elements have the same length l.

Definition 3.5 *Let* $n-$*dimensional vector* $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T$, *where* $\alpha_i = (h_i, g_i)(1 \le i \le n)$ *are* n *DHF elements. Then we call* $\vec{\alpha}$ *a* $n-$*dimensional DHF vector. If* $M_{nm} = (\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_m)$, *where* $\vec{\alpha}_j(1 \le j \le m)$ *are* m $n-$*dimensional DHF vectors, then we call* M_{nm} *a* $n \times m$ *DHF matrix. Specially, a* $n-$*dimensional DHF vector can be viewed as a* $n \times 1$ *DHF matrix.*

Based on Definition 3.5, a MGDHFDIS can be described as multiple DHF matrices (DHF relation matrices) and vectors (called decision DHF vectors). For example, by using DHF relation matrices and decision DHF vectors, the MGDHFDIS in Example 3.3 can be described as follows:

$$M_{\mathbb{R}_1} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.4,0.5\},\{0.2,0.4\}) & (\{0.2,0.3\},\{0.5,0.7\}) & (\{0.6,0.8\},\{0.1,0.2\}) \\ (\{0.1,0.2\},\{0.7,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.2,0.3\},\{0.6,0.7\}) & (\{0.4,0.5\},\{0.5,0.5\}) \\ (\{0.2,0.2\},\{0.6,0.7\}) & (\{0.5,0.8\},\{0.2,0.2\}) & (\{1,1\},\{0,0\}) & (\{0.3,0.4\},\{0.5,0.6\}) \\ (\{0.3,0.5\},\{0.4,0.5\}) & (\{0.4,0.5\},\{0.2,0.4\}) & (\{0.2,0.3\},\{0.5,0.7\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

Table 3: DHF relation $\mathbb{R}_3$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.2,0.4}, {0.5,0.6})	({0.1,0.2}, {0.6,0.8})	({0.3,0.4}, {0.4,0.6})
x_2	({0.1,0.2}, {0.7,0.8})	({1,1}, {0,0})	({0.5,0.6}, {0.2,0.3})	({0.3,0.5}, {0.4,0.4})
x_3	({0.0,0.5}, {0.4,0.4})	({0.1,0.2}, {0.7,0.8})	({1,1}, {0,0})	({0.7,0.9}, {0.1,0.1})
x_4	({0.3,0.4}, {0.5,0.6})	({0.2,0.3}, {0.5,0.7})	({0.3,0.6}, {0.3,0.4})	({1,1}, {0,0})

$$M_{\mathbb{R}_2} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.3,0.5\},\{0.2,0.5\}) & (\{0.2,0.2\},\{0.7,0.8\}) & (\{0.4,0.5\},\{0.3,0.5\}) \\ (\{0.2,0.2\},\{0.6,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.4,0.6\},\{0.3,0.4\}) & (\{0.2,0.5\},\{0.3,0.5\}) \\ (\{0.1,0.3\},\{0.5,0.6\}) & (\{0.4,0.5\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) & (\{0.1,0.2\},\{0.7,0.8\}) \\ (\{0.2,0.5\},\{0.3,0.5\}) & (\{0.1,0.1\},\{0.8,0.9\}) & (\{0.5,0.6\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

$$M_{\mathbb{R}_3} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.2,0.4\},\{0.5,0.6\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{0.3,0.4\},\{0.4,0.6\}) \\ (\{0.1,0.2\},\{0.7,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.5,0.6\},\{0.2,0.3\}) & (\{0.3,0.5\},\{0.4,0.4\}) \\ (\{0.0,0.5\},\{0.4,0.4\}) & (\{0.1,0.2\},\{0.7,0.8\}) & (\{1,1\},\{0,0\}) & (\{0.7,0.9\},\{0.1,0.1\}) \\ (\{0.3,0.4\},\{0.5,0.6\}) & (\{0.2,0.3\},\{0.5,0.7\}) & (\{0.3,0.6\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

$$M_{\mathbb{R}_4} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.4,0.6\},\{0.3,0.4\}) & (\{0.6,0.7\},\{0.3,0.3\}) & (\{0.8,0.9\},\{0.1,0.1\}) \\ (\{0.1,0.2\},\{0.7,0.7\}) & (\{1,1\},\{0,0\}) & (\{0.5,0.6\},\{0.2,0.3\}) & (\{0.2,0.3\},\{0.6,0.7\}) \\ (\{0.3,0.4\},\{0.5,0.6\}) & (\{0.5,0.5\},\{0.3,0.4\}) & (\{1,1\},\{0,0\}) & (\{0.3,0.4\},\{0.6,0.6\}) \\ (\{0.4,0.5\},\{0.3,0.5\}) & (\{0.0,0.2\},\{0.7,0.8\}) & (\{0.1,0.4\},\{0.5,0.5\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

$$M_{\mathbb{R}_5} = \begin{pmatrix} (\{1,1\},\{0,0\}) & (\{0.5,0.5\},\{0.4,0.5\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{0.2,0.3\},\{0.5,0.6\}) \\ (\{0.1,0.1\},\{0.8,0.9\}) & (\{1,1\},\{0,0\}) & (\{0.6,0.7\},\{0.2,0.3\}) & (\{0.2,0.3\},\{0.6,0.7\}) \\ (\{0.0,0.3\},\{0.6,0.7\}) & (\{0.2,0.5\},\{0.4,0.5\}) & (\{1,1\},\{0,0\}) & (\{0.2,0.2\},\{0.6,0.7\}) \\ (\{0.4,0.5\},\{0.3,0.4\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{0.1,0.2\},\{0.6,0.8\}) & (\{1,1\},\{0,0\}) \end{pmatrix}$$

and decision DHF vectors:

$\mathbb{D}_1 = (\{\{0.3,0.4\},\{0.2,0.6\}\},\{\{0.3,0.5\},\{0.4,0.5\}\},\{\{0.3,0.7\},\{0.2,0.3\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T$,

$\mathbb{D}_2 = (\{\{0.5,0.7\},\{0.2,0.3\}\},\{\{0.5,0.6\},\{0.2,0.4\}\},\{\{0.2,0.4\},\{0.3,0.6\}\},\{\{0,0.1\},\{0.8,0.9\}\})^T$.

Now, the union, intersection and complement of two DHF vectors and matrices can be defined as follows:

Definition 3.6 *Let* $\vec{\alpha}_1 = (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1n})^T$ *and* $\vec{\alpha}_2 = (\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2n})^T$ *be two* $n-$*dimensional DHF vectors, where* $\alpha_{1i} = (h_{1i}, g_{1i})$ *and* $\alpha_{2i} = (h_{2i}, g_{2i})(1 \le i \le n)$ *are DHF elements. Assume that* $M_1 = (\vec{\alpha}_{11}, \vec{\alpha}_{12}, \cdots, \vec{\alpha}_{1m})$ *and* $M_2 = (\vec{\alpha}_{21}, \vec{\alpha}_{22}, \cdots, \vec{\alpha}_{2m})$ *be two* $n \times m$ *DHF matrices, where* $\vec{\alpha}_{1j}$ *and* $\vec{\alpha}_{2j}(1 \le j \le m)$ *are* $n-$*dimensional DHF vectors; then,*

(1) $\vec{\alpha}_1 \Cup \vec{\alpha}_2 = (\alpha_{11} \veebar \alpha_{21}, \alpha_{12} \veebar \alpha_{22}, \cdots, \alpha_{1n} \veebar \alpha_{2n})^T$,

where $\alpha_{1i} \veebar \alpha_{2i} = \{(\{h_{1i}^{\sigma(s)} \vee h_{2i}^{\sigma(s)}\}, \{g_{1i}^{\sigma(t)} \wedge g_{2i}^{\sigma(t)}\}) | 1 \le s \le k, 1 \le t \le l\}(1 \le i \le n)$;

(2) $\vec{\alpha}_1 \Cap \vec{\alpha}_2 = (\alpha_{11} \barwedge \alpha_{21}, \alpha_{12} \barwedge \alpha_{22}, \cdots, \alpha_{1n} \barwedge \alpha_{2n})^T$,
where $\alpha_{1i} \barwedge \alpha_{2i} = \{(\{h_{1i}^{\sigma(s)} \wedge h_{2i}^{\sigma(s)}\}, \{g_{1i}^{\sigma(t)} \vee g_{2i}^{\sigma(t)}\}) | 1 \le s \le k, 1 \le t \le l\} (1 \le i \le n)$;
(3) The complementary vector of $\vec{\alpha}_1$ *is denoted as*
$(\vec{\alpha}_1)^c = (\sim \alpha_{11}, \sim \alpha_{12}, \cdots, \sim \alpha_{1n})^T$,
where $\sim \alpha_{1i} = \{(\{g_{1i}^{\sigma(t)}\}, \{h_{1i}^{\sigma(s)}\}) | 1 \le s \le k, 1 \le t \le l\} (1 \le i \le n)$;
(4) $M_1 \Cup M_2 = (\vec{\alpha}_{11} \Cup \vec{\alpha}_{21}, \vec{\alpha}_{12} \Cup \vec{\alpha}_{22}, \cdots, \vec{\alpha}_{1m} \Cup \vec{\alpha}_{2m})$;
(5) $M_1 \Cap M_2 = (\vec{\alpha}_{11} \Cap \vec{\alpha}_{21}, \vec{\alpha}_{12} \Cap \vec{\alpha}_{22}, \cdots, \vec{\alpha}_{1m} \Cap \vec{\alpha}_{2m})$;
(6) The complementary matrix of M_1 *is denoted as*
${M_1}^c = ((\vec{\alpha}_{11})^c, (\vec{\alpha}_{12})^c, \cdots, (\vec{\alpha}_{1m})^c)^T$.

In the following we introduce the product operation of DHF matrices.

Definition 3.7 *Let* P *and* Q *be two DHF matrices, and*

$$P = \begin{pmatrix} (\overline{p}_{11}, \underline{p}_{11}) & (\overline{p}_{12}, \underline{p}_{12}) & \cdots & (\overline{p}_{1w}, \underline{p}_{1w}) \\ (\overline{p}_{21}, \underline{p}_{21}) & (\overline{p}_{22}, \underline{p}_{22}) & \cdots & (\overline{p}_{2w}, \underline{p}_{2w}) \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{p}_{m1}, \underline{p}_{m1}) & (\overline{p}_{m2}, \underline{p}_{m2}) & \cdots & (\overline{p}_{mw}, \underline{p}_{mw}) \end{pmatrix},$$

$$Q = \begin{pmatrix} (\overline{q}_{11}, \underline{q}_{11}) & (\overline{q}_{12}, \underline{q}_{12}) & \cdots & (\overline{q}_{1n}, \underline{q}_{1n}) \\ (\overline{q}_{21}, \underline{q}_{21}) & (\overline{q}_{22}, \underline{q}_{22}) & \cdots & (\overline{q}_{2n}, \underline{q}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{q}_{w1}, \underline{q}_{w1}) & (\overline{q}_{w2}, \underline{q}_{w2}) & \cdots & (\overline{q}_{wn}, \underline{q}_{wn}) \end{pmatrix}.$$

Then, the product of P *and* Q *is a* $m \times n$ *DHF matrix, denoted as follows:*
$M = P \circ Q = \left((\overline{r}_{ij}, \underline{r}_{ij})\right)_{1 \le i \le m, 1 \le j \le n}$,
where
$\overline{r}_{ij} = \underline{\vee}_{1 \le u \le w} \{\overline{p}_{iu} \barwedge \overline{q}_{uj}\} = \{\bigvee\limits_{1 \le u \le w} (\overline{p}_{iu}^{\sigma(s)} \wedge \overline{q}_{uj}^{\sigma(s)}) | 1 \le s \le k\}$,
$\underline{r}_{ij} = \overline{\wedge}_{1 \le u \le w} \{\underline{p}_{iu} \underline{\vee} \underline{q}_{uj}\} = \{\bigwedge\limits_{1 \le u \le w} (\underline{p}_{iu}^{\sigma(t)} \vee \underline{q}_{uj}^{\sigma(t)}) | 1 \le t \le l\}$.

In the following discussions, for convenience, we don't distinguish between DHF vectors and DHF sets on U.

Theorem 3.8 *Let* $\mathbb{R}$ *be the DHF relation on* U, $M_{\mathbb{R}}$ *be DHF matrix of* $\mathbb{R}$ *and* $\mathbb{A} \in DHF(U)$*; then*
(1) $\underline{\mathbb{R}}(\mathbb{A}) = (M_{\mathbb{R}} \circ \mathbb{A}^c)^c$,
(2) $\overline{\mathbb{R}}(\mathbb{A}) = M_{\mathbb{R}} \circ \mathbb{A}$,
where $\underline{\mathbb{R}}(\mathbb{A})$ *and* $\overline{\mathbb{R}}(\mathbb{A})$ *are the single-granulation lower and upper approximations defined in Definition 2.4.*

Proof. It can be easily verified from Definitions 3.7 and 2.4. □

According to Theorems 3.8, 2.7 and 2.8, we conclude that the following theorem holds.

Table 4: DHF relation $\mathbb{R}_4$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.4,0.6}, {0.3,0.4})	({0.6,0.7}, {0.3,0.3})	({0.8,0.9}, {0.1,0.1})
x_2	({0.1,0.2}, {0.7,0.7})	({1,1}, {0,0})	({0.5,0.6}, {0.2,0.3})	({0.2,0.3}, {0.6,0.7})
x_3	({0.3,0.4}, {0.5,0.6})	({0.5,0.5}, {0.3,0.4})	({1,1}, {0,0})	({0.3,0.4}, {0.6,0.6})
x_4	({0.4,0.5}, {0.3,0.5})	({0.0,0.2}, {0.7,0.8})	({0.1,0.4}, {0.5,0.5})	({1,1}, {0,0})

Table 5: DHF relation $\mathbb{R}_5$ in Example 3.3

$U \times U$	x_1	x_2	x_3	x_4
x_1	({1,1}, {0,0})	({0.5,0.5}, {0.4,0.5})	({0.1,0.2}, {0.6,0.8})	({0.2,0.3}, {0.5,0.6})
x_2	({0.1,0.1}, {0.8,0.9})	({1,1}, {0,0})	({0.6,0.7}, {0.2,0.3})	({0.2,0.3}, {0.6,0.7})
x_3	({0.0,0.3}, {0.6,0.7})	({0.2,0.5}, {0.4,0.5})	({1,1}, {0,0})	({0.2,0.2}, {0.6,0.7})
x_4	({0.4,0.5}, {0.3,0.4})	({0.1,0.2}, {0.6,0.8})	({0.1,0.2}, {0.6,0.8})	({1,1}, {0,0})

Theorem 3.9 *Let $\mathbb{R}_i(1 \le i \le m)$ be m DHF relations on U, $M_{\mathbb{R}_i}$ be the DHF relation matrices of $\mathbb{R}_i(1 \le i \le m)$ and $\mathbb{A} \in DHF(U)$; then*

(1) $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A}) = \biguplus_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A}^c)^c$,

(2) $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{O}(\mathbb{A}) = \biguplus_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A})$;

(3) $\underline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A}) = \biguplus_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A}^c)^c$,

(4) $\overline{\sum_{i=1}^{m} \mathbb{R}_i}^{P}(\mathbb{A}) = \biguplus_{i=1}^{m}(M_{\mathbb{R}_i} \circ \mathbb{A})$.

Example 3.10 *(Continued from Example 3.3) According to Theorem 3.8(2), we have*

$$\begin{aligned}\overline{\mathbb{R}_1}(\mathbb{D}_1) = M_{\mathbb{R}_1} \circ \mathbb{D}_1 =&((\{0.3,0.5\},\{0.2,0.5\}),(\{0.3,0.5\},\{0.4,0.5\}),\\ &(\{0.3,0.7\},\{0.2,0.3\}),(\{0.3,0.5\},\{0.4,0.5\}))^T,\end{aligned}$$

$$\begin{aligned}\overline{\mathbb{R}_2}(\mathbb{D}_1) = M_{\mathbb{R}_2} \circ \mathbb{D}_1 =&((\{0.3,0.5\},\{0.2,0.5\}),(\{0.3,0.6\},\{0.3,0.4\}),\\ &(\{0.3,0.7\},\{0.2,0.3\}),(\{0.3,0.6\},\{0.3,0.4\}))^T,\end{aligned}$$

$$\begin{aligned}\overline{\mathbb{R}_3}(\mathbb{D}_1) = M_{\mathbb{R}_3} \circ \mathbb{D}_1 =&((\{0.3,0.4\},\{0.2,0.6\}),(\{0.3,0.6\},\{0.2,0.3\}),\\ &(\{0.3,0.7\},\{0.2,0.3\}),(\{0.3,0.6\},\{0.3,0.4\}))^T,\end{aligned}$$

$$\overline{\mathbb{R}_4}(\mathbb{D}_1) = M_{\mathbb{R}_4} \circ \mathbb{D}_1 =((\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.6\}, \{0.2, 0.3\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.4\}, \{0.3, 0.5\}))^T,$$

$$\overline{\mathbb{R}_5}(\mathbb{D}_1) = M_{\mathbb{R}_4} \circ \mathbb{D}_1 =((\{0.3, 0.5\}, \{0.2, 0.5\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.4\}, \{0.3, 0.6\}))^T.$$

Then by Theorem 3.9(2) and (4), we obtain

$$\overline{\sum_{i=1}^{5} \mathbb{R}_i}^{O} (\mathbb{D}_1) =((\{0.3, 0.4\}, \{0.2, 0.6\}), (\{0.3, 0.5\}, \{0.4, 0.5\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.4\}, \{0.4, 0.6\}))^T,$$

and

$$\overline{\sum_{i=1}^{5} \mathbb{R}_i}^{P} (\mathbb{D}_1) =((\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.7\}, \{0.2, 0.3\}), (\{0.3, 0.6\}, \{0.3, 0.4\}))^T.$$

Similarly, we have

$$\overline{\mathbb{R}_1}(\mathbb{D}_2) = M_{\mathbb{R}_1} \circ \mathbb{D}_2 =((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.4, 0.5\}, \{0.2, 0.4\}))^T,$$

$$\overline{\mathbb{R}_2}(\mathbb{D}_2) = M_{\mathbb{R}_2} \circ \mathbb{D}_2 =((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.4, 0.5\}, \{0.3, 0.4\}), (\{0.2, 0.5\}, \{0.3, 0.5\}))^T,$$

$$\overline{\mathbb{R}_3}(\mathbb{D}_2) = M_{\mathbb{R}_3} \circ \mathbb{D}_2 =((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.2, 0.5\}, \{0.3, 0.4\}), (\{0.3, 0.4\}, \{0.3, 0.6\}))^T,$$

$$\overline{\mathbb{R}_4}(\mathbb{D}_2) = M_{\mathbb{R}_4} \circ \mathbb{D}_2 =((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.5, 0.5\}, \{0.3, 0.4\}), (\{0.4, 0.5\}, \{0.3, 0.5\}))^T,$$

$$\overline{\mathbb{R}_5}(\mathbb{D}_2) = M_{\mathbb{R}_4} \circ \mathbb{D}_2 =((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.2, 0.5\}, \{0.3, 0.5\}), (\{0.4, 0.5\}, \{0.3, 0.4\}))^T.$$

Then

$$\overline{\sum_{i=1}^{5} \mathbb{R}_i}^{O} (\mathbb{D}_2) =((\{0.5, 0.7\}, \{0.2, 0.3\}), (\{0.5, 0.6\}, \{0.2, 0.4\}), (\{0.2, 0.5\}, \{0.3, 0.5\}), (\{0.2, 0.4\}, \{0.3, 0.6\}))^T,$$

and

$$\overline{\sum_{i=1}^{5}\mathbb{R}_i}^{P}(\mathbb{D}_2) =(\{\{0.5,0.7\},\{0.2,0.3\}\},\{\{0.5,0.6\},\{0.2,0.4\}\},$$
$$\{\{0.5,0.6\},\{0.2,0.4\}\},\{\{0.4,0.5\},\{0.2,0.4\}\})^T.$$

According to Theorem 3.8(1), we have

$$\underline{\mathbb{R}_1}(\mathbb{D}_1) = (M_{\mathbb{R}_1}\circ {\mathbb{D}_1}^c)^c =(\{\{0.2,0.3\},\{0.6,0.7\}\},\{\{0.3,0.5\},\{0.4,0.5\}\},$$
$$\{\{0.3,0.5\},\{0.4,0.5\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T,$$

$$\underline{\mathbb{R}_2}(\mathbb{D}_1) = (M_{\mathbb{R}_2}\circ {\mathbb{D}_1}^c)^c =(\{\{0.3,0.4\},\{0.4,0.6\}\},\{\{0.3,0.5\},\{0.4,0.5\}\},$$
$$\{\{0.3,0.5\},\{0.4,0.5\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T,$$

$$\underline{\mathbb{R}_3}(\mathbb{D}_1) = (M_{\mathbb{R}_3}\circ {\mathbb{D}_1}^c)^c =(\{\{0.3,0.4\},\{0.3,0.6\}\},\{\{0.3,0.4\},\{0.4,0.5\}\},$$
$$\{\{0.2,0.3\},\{0.6,0.7\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T,$$

$$\underline{\mathbb{R}_4}(\mathbb{D}_1) = (M_{\mathbb{R}_4}\circ {\mathbb{D}_1}^c)^c =(\{\{0.2,0.3\},\{0.6,0.7\}\},\{\{0.3,0.5\},\{0.4,0.5\}\},$$
$$\{\{0.3,0.5\},\{0.4,0.5\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T,$$

$$\underline{\mathbb{R}_5}(\mathbb{D}_1) = (M_{\mathbb{R}_5}\circ {\mathbb{D}_1}^c)^c =(\{\{0.3,0.4\},\{0.4,0.6\}\},\{\{0.3,0.5\},\{0.4,0.5\}\},$$
$$\{\{0.3,0.5\},\{0.2,0.5\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T,$$

Then according to Theorem 3.9(1) and (3), we obtain

$$\underline{\sum_{i=1}^{5}\mathbb{R}_i}^{O}(\mathbb{D}_1) =(\{\{0.3,0.4\},\{0.3,0.6\}\},\{\{0.3,0.5\},\{0.4,0.5\}\},$$
$$\{\{0.3,0.5\},\{0.2,0.5\}\},\{\{0.2,0.3\},\{0.6,0.7\}\})^T,$$

and

$$\underline{\sum_{i=1}^{5}\mathbb{R}_i}^{P}(\mathbb{D}_1) =(\{\{0.2,0.3\},\{0.6,0.7\}\},\{\{0.3,0.4\},\{0.4,0.5\}\},$$
$$\{\{0.2,0.3\},\{0.6,0.7\}\},\{\{0.2,0.3\},\{0.6,0.7\}\}).$$

Similarly, we have

$$\underline{\mathbb{R}_1}(\mathbb{D}_2) = (M_{\mathbb{R}_1}\circ {\mathbb{D}_2}^c)^c =(\{\{0.1,0.2\},\{0.6,0.8\}\},\{\{0.5,0.5\},\{0.4,0.5\}\},$$
$$\{\{0.2,0.4\},\{0.3,0.6\}\},\{\{0.0,0.1\},\{0.8,0.9\}\})^T,$$

$$\underline{\mathbb{R}_2}(\mathbb{D}_2) = (M_{\mathbb{R}_2} \circ {\mathbb{D}_2}^c)^c = (\{\{0.3, 0.5\}, \{0.4, 0.5\}\}, \{\{0.3, 0.4\}, \{0.3, 0.6\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.0, 0.1\}, \{0.8, 0.9\}\})^T,$$

$$\underline{\mathbb{R}_3}(\mathbb{D}_2) = (M_{\mathbb{R}_3} \circ {\mathbb{D}_2}^c)^c = (\{\{0.4, 0.6\}, \{0.3, 0.4\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.1, 0.1\}, \{0.7, 0.9\}\}, \{\{0.0, 0.1\}, \{0.8, 0.9\}\})^T,$$

$$\underline{\mathbb{R}_4}(\mathbb{D}_2) = (M_{\mathbb{R}_4} \circ {\mathbb{D}_2}^c)^c = (\{\{0.1, 0.1\}, \{0.8, 0.9\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.0, 0.1\}, \{0.8, 0.9\}\})^T,$$

$$\underline{\mathbb{R}_5}(\mathbb{D}_2) = (M_{\mathbb{R}_5} \circ {\mathbb{D}_2}^c)^c = (\{\{0.5, 0.6\}, \{0.2, 0.4\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.0, 0.1\}, \{0.8, 0.9\}\})^T.$$

Then

$$\underline{\sum_{i=1}^{5} \mathbb{R}_i}^{O}(\mathbb{D}_2) = (\{\{0.5, 0.6\}, \{0.2, 0.4\}\}, \{\{0.5, 0.5\}, \{0.3, 0.5\}\}, \{\{0.2, 0.4\}, \{0.3, 0.6\}\}, \{\{0.0, 0.1\}, \{0.8, 0.9\}\})^T,$$

and

$$\underline{\sum_{i=1}^{5} \mathbb{R}_i}^{P}(\mathbb{D}_2) = (\{\{0.1, 0.1\}, \{0.8, 0.9\}\}, \{\{0.2, 0.4\}, \{0.4, 0.6\}\}, \{\{0.1, 0.1\}, \{0.7, 0.9\}\}, \{\{0.0, 0.1\}, \{0.8, 0.9\}\}).$$

It is well known that a discernibility function is a key notion to reduction algorithms in rough set theory. Therefore, by constructing the discernibility functions, we present a practical method to determine the optimistic and pessimistic approximation reducts of MGDHFDIS.

Definition 3.11 *Let* $MGDHFDIS = (U, \{\mathbb{R}_j | 1 \le j \le m\}, D = \{d_i | 1 \le i \le v\}, V)$, $|U| = n$ *and* $D_i (1 \le i \le v)$ *be decision vectors. Denote*

$$\underline{\sum_{j=1}^{m} \mathbb{R}_j}^{O}(\mathbb{D}_i) = (\underline{o}_{i1}, \underline{o}_{i2}, \cdots, \underline{o}_{in})(1 \le i \le v),$$

where $\underline{o}_{iu} = \{(\{\underline{o}_{iu}^{\sigma(s)}\}, \{\underline{o}_{iu}^{\sigma(t)}\}) | 1 \le s \le k, 1 \le t \le l\} (1 \le u \le n)$;

$$\overline{\sum_{j=1}^{m} \mathbb{R}_j}^{O}(\mathbb{D}_i) = (\overline{o}_{i1}, \overline{o}_{i2}, \cdots, \overline{o}_{in})(1 \le i \le v),$$

where $\overline{o}_{iu} = \{(\{\overline{o}_{iu}^{\sigma(s)}\}, \{\overline{o}_{iu}^{\sigma(t)}\})|1 \le s \le k, 1 \le t \le l\}(1 \le u \le n)$;

$$\underline{\sum_{j=1}^{m} \mathbb{R}_j}^{P}(\mathbb{D}_i) = (\underline{p}_{i1}, \underline{p}_{i2}, \cdots, \underline{p}_{in})(1 \le i \le v),$$

where $\underline{p}_{iu} = \{(\{\underline{p}_{iu}^{\sigma(s)}\}, \{\underline{p}_{iu}^{\sigma(t)}\})|1 \le s \le k, 1 \le t \le l\}(1 \le u \le n)$;

$$\overline{\sum_{j=1}^{m} \mathbb{R}_j}^{P}(\mathbb{D}_i) = (\overline{p}_{i1}, \overline{p}_{i2}, \cdots, \overline{p}_{in})(1 \le i \le v),$$

where $\overline{p}_{iu} = \{(\{\overline{p}_{iu}^{\sigma(s)}\}, \{\overline{p}_{iu}^{\sigma(t)}\})|1 \le s \le k, 1 \le t \le l\}(1 \le u \le n)$;

$$\underline{\mathbb{R}_j}(\mathbb{D}_i) = (\underline{r}_{ij1}, \underline{r}_{ij2}, \cdots, \underline{r}_{ijn})(1 \le i \le v, 1 \le j \le m),$$

where $\underline{r}_{iju} = \{(\{\underline{r}_{iju}^{\sigma(s)}\}, \{\underline{r}_{iju}^{\sigma(t)}\})|1 \le s \le k, 1 \le t \le l\}(1 \le u \le n)$;

$$\overline{\mathbb{R}_j}(\mathbb{D}_i) = (\overline{r}_{ij1}, \overline{r}_{ij2}, \cdots, \overline{r}_{ijn})(1 \le i \le v, 1 \le j \le m),$$

where $\overline{r}_{iju} = \{(\{\overline{r}_{iju}^{\sigma(s)}\}, \{\overline{r}_{iju}^{\sigma(t)}\})|1 \le s \le k, 1 \le t \le l\}(1 \le u \le n)$.

Then, the optimistic lower approximation discernibility function of MGDHFDIS is

$$\underline{f}^O = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\underline{r}_{iju}^{\sigma(s)} = \underline{o}_{iu}^{\sigma(s)}, 1 \le j \le m} \mathbb{R}_j \bigwedge \bigvee_{\underline{r}_{iju}^{\sigma(t)} = \underline{o}_{iu}^{\sigma(t)}, 1 \le j \le m} \mathbb{R}_j \Big);$$

the optimistic upper approximation discernibility function of MGDHFDIS is

$$\overline{f}^O = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\overline{r}_{iju}^{\sigma(s)} = \overline{o}_{iu}^{\sigma(s)}, 1 \le j \le m} \mathbb{R}_j \bigwedge \bigvee_{\underline{r}_{iju}^{\sigma(t)} = \underline{o}_{iu}^{\sigma(t)}, 1 \le j \le m} \mathbb{R}_j \Big);$$

the pessimistic lower approximation discernibility function of MGDHFDIS is

$$\underline{f}^P = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\underline{r}_{iju}^{\sigma(s)} = \underline{p}_{iu}^{\sigma(s)}, 1 \le j \le m} \mathbb{R}_j \bigwedge \bigvee_{\underline{r}_{iju}^{\sigma(t)} = \underline{p}_{iu}^{\sigma(t)}, 1 \le j \le m} \mathbb{R}_j \Big);$$

the pessimistic upper approximation discernibility function of MGDHFDIS is

$$\overline{f}^P = \bigwedge_{i=1}^{v} \bigwedge_{u=1}^{n} \bigwedge_{s=1}^{k} \bigwedge_{t=1}^{l} \Big(\bigvee_{\overline{r}_{iju}^{\sigma(s)} = \overline{p}_{iu}^{\sigma(s)}, 1 \le j \le m} \mathbb{R}_j \bigwedge \bigvee_{\overline{r}_{iju}^{\sigma(t)} = \overline{p}_{iu}^{\sigma(t)}, 1 \le j \le m} \mathbb{R}_j \Big).$$

According to Definitions 3.11 and 3.4, we can easily obtain the following theorem.

Theorem 3.12 *Let* $MGDHFDIS = (U, \{\mathbb{R}_j|1 \le j \le m\}, D = \{d_i|1 \le i \le v\}, V)$, $|U| = n$. *We can convert the approximation discernibility functions* $\underline{f}^O$, $\overline{f}^O$, $\underline{f}^P$ *and* $\overline{f}^P$ *of MGDHFDIS into their disjunction forms* $\underline{f}^O = \bigvee_{\alpha=1}^{\alpha_1} (\bigwedge_{\beta=1}^{\beta_1} \mathbb{R}_{\alpha\beta 1})$, $\overline{f}^O = \bigvee_{\alpha=1}^{\alpha_2} (\bigwedge_{\beta=1}^{\beta_2} \mathbb{R}_{\alpha\beta 2})$, $\underline{f}^P = \bigvee_{\alpha=1}^{\alpha_3} (\bigwedge_{\beta=1}^{\beta_3} \mathbb{R}_{\alpha\beta 3})$, *and* $\overline{f}^P = \bigvee_{\alpha=1}^{\alpha_4} (\bigwedge_{\beta=1}^{\beta_4} \mathbb{R}_{\alpha\beta 4})$, *respectively. Then,* $\underline{B}_\alpha^O = \{\mathbb{R}_{\alpha\beta 1}|\beta =$

$1,2,\cdots,\beta_1\}(\alpha=1,2,\cdots,\alpha_1)$, $\overline{B}_{\alpha}^{O}=\{\mathbb{R}_{\alpha\beta 2}|\beta=1,2,\cdots,\beta_2\}(\alpha=1,2,\cdots,\alpha_2)$, $\underline{B}_{\alpha}^{P}=\{\mathbb{R}_{\alpha\beta 3}|\beta=1,2,\cdots,\beta_3\}(\alpha=1,2,\cdots,\alpha_3)$, *and* $\overline{B}_{\alpha}^{P}=\{\mathbb{R}_{\alpha\beta 4}|\beta=1,2,\cdots,\beta_4\}(\alpha=1,2,\cdots,\alpha_4)$ *are the optimistic lower upper, and pessimistic lower and upper approximation reductions of MGDHFDIS, respectively.*

From Theorem 3.12, we see that all the approximation reducts of MGDHFDIS can be obtained through using the discernibility functions defined in Definition 3.11.

Example 3.13 *(Continued from Example 3.10) From Definition 3.11, we obtain*

$\underline{f}^{O}=((\mathbb{R}_2\vee\mathbb{R}_3\vee\mathbb{R}_5)\wedge\mathbb{R}_3\wedge\mathbb{R}_5)\wedge(\mathbb{R}_1\wedge\mathbb{R}_5)=\mathbb{R}_1\wedge\mathbb{R}_3\wedge\mathbb{R}_5$,

$\overline{f}^{O}=(\mathbb{R}_3\wedge\mathbb{R}_1\wedge((\mathbb{R}_4\vee\mathbb{R}_5)\wedge\mathbb{R}_1\wedge\mathbb{R}_5))\wedge(\mathbb{R}_2\wedge\mathbb{R}_3\wedge\mathbb{R}_5)=\mathbb{R}_1\wedge\mathbb{R}_2\wedge\mathbb{R}_3\wedge\mathbb{R}_5$,

$\underline{f}^{P}=(\mathbb{R}_4\wedge\mathbb{R}_3)\wedge(\mathbb{R}_4\wedge(\mathbb{R}_3\vee\mathbb{R}_4\vee\mathbb{R}_5)\wedge\mathbb{R}_1\wedge(\mathbb{R}_2\vee\mathbb{R}_3\vee\mathbb{R}_4\vee\mathbb{R}_5)\wedge\mathbb{R}_3)=\mathbb{R}_1\wedge\mathbb{R}_3\wedge\mathbb{R}_4$, *and*

$\overline{f}^{P}=(\mathbb{R}_4\wedge\mathbb{R}_5\wedge(\mathbb{R}_2\vee\mathbb{R}_3))\wedge\mathbb{R}_1=(\mathbb{R}_1\wedge\mathbb{R}_2\wedge\mathbb{R}_4\wedge\mathbb{R}_5)\vee(\mathbb{R}_1\wedge\mathbb{R}_3\wedge\mathbb{R}_4\wedge\mathbb{R}_5)$.

Hence, by virtue of Theorem 3.12, we draw the conclusion that the optimistic lower approximation reducts of MGDHFDIS are $\{\mathbb{R}_1,\mathbb{R}_3,\mathbb{R}_5\}$;

The optimistic upper approximation reducts of MGDHFDIS are $\{\mathbb{R}_1,\mathbb{R}_2,\mathbb{R}_3,\mathbb{R}_5\}$;

The pessimistic lower approximation reducts of MGDHFDIS are $\{\mathbb{R}_1,\mathbb{R}_3,\mathbb{R}_4\}$;

The pessimistic upper approximation reducts of MGDHFDIS are $\{\mathbb{R}_1,\mathbb{R}_2,\mathbb{R}_4,\mathbb{R}_5\}$ *and* $\{\mathbb{R}_1,\mathbb{R}_3,\mathbb{R}_4,\mathbb{R}_5\}$.

4 Conclusion

As two new mathematical approaches to cope with imprecision and uncertainty in data analysis, DHF sets and MGRS theory have their own advantages. Considering the facts, Zhang et al. [47] proposed a MGDHFRS by combining DHF sets and MGRS theory which includes many existing MGRS models as special types, such as MGRSs [22], MGFRSs in a fuzzy tolerance approximation space [34] and IFMGRSs [7]. Since the MGDHFRS includes both ingredients of DHF sets and MGRSs, it is more effective and flexible than both DHF sets and MGRSs to handle imprecise and imperfect information. In this study, in order to further investigate the applications of MGDHFRSs, we present a reduction method in MGDHFDIS based on MGDHFRSs. An example is also provided to illustrate the validity of this method. Generally, this reduction approach based on discernibility functions can be extended to other various rough set models in the context of defining discernibility functions.

In the future, topological structures of the MGDHFRSs are the main research direction considered by our group. Moreover, it is important and interesting to further investigate the applications of the MGDHFRSs.

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THE FEKETE-SZEGÖ PROBLEM FOR SOME CLASSES OF ANALYTIC FUNCTIONS

ADAM LECKO, BOGUMIŁA KOWALCZYK, OH SANG KWON AND NAK EUN CHO

Abstract. Given an analytic standardly normalized function g in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, by $\mathcal{C}(g)$ will be denoted the class of analytic standardly normalized function f such that

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D},$$

for some $\delta \in (-\pi/2, \pi/2)$. For the class $\mathcal{C}(g)$ the Fekete-Szegö problem is examined.

1. Introduction

In [3] Fekete and Szegö found the maximum value of the coefficient functional

$$\Phi_\lambda(f) := \left|a_3 - \lambda a_2^2\right|, \quad \lambda \in [0,1],$$

over the class $\mathcal{S}$ of univalent functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

By applying the Loewner method they proved that

$$\max_{f\in\mathcal{S}} \Phi_\lambda(f) = \begin{cases} 1 + 2\exp\left(-2\lambda/(1-\lambda)\right), & \lambda \in [0,1), \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating $\max_{f\in\mathcal{F}} \Phi_\lambda(f)$ for various compact subclasses $\mathcal{F}$ of the class $\mathcal{A}$ of all analytic functions f in $\mathbb{D}$ of the form (1.1), as well as for λ being an arbitrary real or complex number, was considered by many authors (see e.g., [8], [12], [23], [14], [10], [20], [13], [2]).

Let $\mathcal{S}^*$ denote the class of *starlike* functions, i.e., $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

Let $\mathcal{S}^c$ denote the class of *convex* functions, i.e., $f \in \mathcal{S}^c$ if $f \in \mathcal{A}$ and

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{A}$, let $\mathcal{C}_\delta(g)$ denote the class of all functions $f \in \mathcal{A}$ such that

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

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For $g \in \mathcal{A}$ let

$$\mathcal{C}(g) := \bigcup_{\delta \in (-\pi/2, \pi/2)} \mathcal{C}_\delta(g)$$

and for $\delta \in (-\pi/2, \pi/2)$ let

$$\mathcal{C}_\delta := \bigcup_{g \in \mathcal{A}} \mathcal{C}_\delta(g).$$

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, functions in $\mathcal{C}_\delta(g)$ and in $\mathcal{C}(g)$ are called *close-to-convex with argument δ with respect to g* and *close-to-convex with respect to g*, respectively. For $\delta \in (-\pi/2, \pi/2)$ let

$$\mathcal{C}_\delta^* := \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g), \quad \mathcal{C}_\delta^c := \bigcup_{g \in \mathcal{S}^c} \mathcal{C}_\delta(g).$$

Let

$$\mathcal{C}^* := \bigcup_{\delta \in (-\pi/2, \pi/2)} \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g)$$

denote the class of *close-to-convex* functions and let

$$\mathcal{C}^c := \bigcup_{\delta \in (-\pi/2, \pi/2)} \bigcup_{g \in \mathcal{S}^c} \mathcal{C}_\delta(g).$$

For details on close-to-convex functions see [22, pp. 184-185], [11], [7] (cf. [6, Vol. II, pp. 1-11]). The class $\mathcal{C}_0^c$ was considered in [1].

For the whole class $\mathcal{C}^*$ of close-to-convex functions, the sharp bound of the Fekete-Szegö functional on $\mathbb{R}$ was calculated by Koepf in [14] who extended the earlier result for the class $\mathcal{C}_0^*$ due to Keogh and Merkes [12], namely, he proved that

$$\max_{f \in \mathcal{C}^*} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0^*} \Phi_\lambda(f) = \begin{cases} |3 - 4\lambda|, & \lambda \in (-\infty, 1/3] \cup [1, +\infty), \\ 1/3 + 4/(9\lambda), & \lambda \in [1/3, 2/3], \\ 1, & \lambda \in [2/3, 1]. \end{cases} \tag{1.3}$$

For the class $\mathcal{C}^c$ of close-to-convex functions with respect to convex functions, the sharp bound of the Fekete-Szegö functional on the interval $[0, 1]$ was studied by Srivastava, Mishra and Das in [25], who extended the earlier result for the class $\mathcal{C}_0^c$ due to Abdel-Gawad and Thomas [1]. By Theorem 3 of [1], Theorems 1 to 4 of [25] and by observation in Section 2 of the paper [18], the following result holds:

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0^c} \Phi_\lambda(f) = \begin{cases} 5/3 - 9\lambda/4, & \lambda \in [0, 2/9] \\ 2/3 + 1/(9\lambda), & \lambda \in [2/9, 2/3], \end{cases} \tag{1.4}$$

and

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) \leqq 5/6, \quad \lambda \in (2/3, 1]. \tag{1.5}$$

Given $\alpha \in [0, 1]$, let

$$g_\alpha(z) := \frac{z}{(1 - \alpha z)^2}, \quad z \in \mathbb{D},$$

and

$$h_\alpha(z) := \frac{z}{1 - \alpha z}, \quad z \in \mathbb{D}.$$

The corresponding classes $\mathcal{C}(g_\alpha)$ and $\mathcal{C}(h_\alpha)$ are defined, respectively, by the following conditions:

$$\operatorname{Re}\left\{ \mathrm{e}^{\mathrm{i}\delta} (1 - \alpha z)^2 f'(z) \right\} > 0, \quad z \in \mathbb{D}, \tag{1.6}$$

and

$$\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}(1-\alpha z)f'(z)\right\}>0,\quad z\in\mathbb{D}, \tag{1.7}$$

where $\delta\in(-\pi/2,\pi/2)$.

The upper bound on the Fekete-Szegö functional for the class $\mathcal{C}(g_\alpha)$ was obtained in [15], where it was shown that

$$\begin{aligned}&\max_{f\in\mathcal{C}(g_\alpha)}\Phi_\lambda(f)\\ &\leq\begin{cases}\left|\dfrac{2}{3}+\dfrac{4}{3}\alpha+\alpha^2-(1+\alpha)^2\lambda\right|, & \lambda\in\mathbb{R}\setminus(\tau_1(\alpha),\tau_2(\alpha)),\\ \dfrac{2}{3}+\alpha^2\left(\dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|}+|1-\lambda|\right), & \lambda\in[\tau_1(\alpha),\tau_2(\alpha)],\end{cases}\end{aligned} \tag{1.8}$$

where

$$\tau_1(\alpha):=\frac{2\alpha}{3(1+\alpha)},\quad \tau_2(\alpha):=\frac{2(2+\alpha)}{3(1+\alpha)}.$$

As it is well known, the Koebe function $k:=g_1$ and the function $h:=h_1$ are extremal for various computational problems in the class $\mathcal{S}^*$ of starlike and in the class $\mathcal{S}^c$ of convex functions, respectively. The Fekete-Szegö problem was separately considered for the class $\mathcal{C}(k)$ in [16] and for the class $\mathcal{C}(h)$ in [17], i.e., when $\alpha:=1$ in (1.6) and (1.7). Setting $\alpha:=1$ into (1.8) we get the result for the class $\mathcal{C}(k)$.

For $\alpha:=0$ the condition (1.6) as well as (1.7) is of the form

$$\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}f'(z)\right\}>0,\quad z\in\mathbb{D}. \tag{1.9}$$

Functions f having such a property are called *of bounded turning with argument* δ and form the class $\mathcal{C}_\delta(h)$ denoted usually as $\mathcal{P}'(\delta)$, and further the class $\mathcal{P}'$ of functions called *of bounded turning* (cf. [6, Vol. I, p. 101]). On the other hand, the condition (1.7) is known as a famous criterium of univalence due to Noshiro [21] and Warschawski [27] (cf. [6, p. 88]). By setting $\alpha:=0$ into (1.8) we get the following result published, among other results, in [10, Theorem 2.3]:

$$\max_{f\in\mathcal{P}'}\Phi_\lambda(f)=\frac{2}{3}.$$

In this paper we unify mentioned results proving the Fekete-Szegö inequality for the class $\mathcal{C}(g)$ with $g\in\mathcal{A}$ such that

$$|g''(0)|\leq 4.$$

2. Main result

By $\mathcal{P}$ we denote the class of all analytic functions p in $\mathbb{D}$ of the form

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n,\quad z\in\mathbb{D}, \tag{2.1}$$

having a positive real part in $\mathbb{D}$. Let

$$L(z):=\frac{1+z}{1-z},\quad z\in\mathbb{D}.$$

For each $\varepsilon\in\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$ let

$$L_\varepsilon(z):=L(\varepsilon z),\quad z\in\mathbb{D}.$$

Clearly $L_\varepsilon\in\mathcal{P}$ for every $\varepsilon\in\mathbb{T}$.

The inequalities (2.2) and (2.3) below are well known. They can be found in [24, pp. 41 and 166].

Lemma 2.1. *If $p \in \mathcal{P}$ is of the form* (2.1)*, then*

$$|c_n| \leq 2, \quad n \in \mathbb{N}, \tag{2.2}$$

and

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}. \tag{2.3}$$

Both inequalities are sharp. The equality in (2.2) *holds for every function $L_\varepsilon \in \mathcal{P}$, $\varepsilon \in \mathbb{T}$. The equality in* (2.3) *holds for every function*

$$\begin{aligned} p_{t,\theta}(z) &:= tL\left(\mathrm{e}^{\mathrm{i}\theta}z\right) + (1-t)L\left(\mathrm{e}^{2\mathrm{i}\theta}z^2\right) \\ &= 1 + 2t\mathrm{e}^{\mathrm{i}\theta}z + 2\mathrm{e}^{2\mathrm{i}\theta}z^2 + \ldots, \quad z \in \mathbb{D}, \end{aligned}$$

where $t \in [0,1]$ and $\theta \in \mathbb{R}$.

Now we prove the main theorem of this paper. The idea of the proof is based on the Koepf's method [14] of calculating Φ_λ for close-to-convex functions with λ restricted to the interval $(1/2, 2/3)$. However, we apply it homogenously for the class $\mathcal{C}(g)$ for all real λ in the same manner as in [15] and [16]. Also the Laguerre's rule of counting zeros of polynomials in an interval is the key tool in the proof.

We recall shortly the Laguerre's rule of counting zeros of polynomials in an interval (see [19], [9], [26, pp. 19-20]). Given a real polynomial

$$Q(u) = a_0u^n + a_1u^{n-1} + \cdots + a_{n-1}u + a_n \tag{2.4}$$

consider a finite sequence (q_k), $k = 0, 1, \ldots, n$, of polynomials of the form

$$q_k(u) = \sum_{j=0}^{k} a_j u^{k-j}. \tag{2.5}$$

For each $u_0 \in \mathbb{R}$ let $N(Q; u_0)$ denote the number of sign changes in the sequence $(q_k(u_0))$, $k = 0, 1, \ldots, n$. Given an interval $I \subset \mathbb{R}$, denote by $Z(Q; I)$ the number of zeros of Q in I counted with their orders. Then the following theorem due to Laguerre holds.

Theorem 2.2. *If $a < b$, $Q(a) \neq 0$ and $Q(b) \neq 0$, then $Z(Q; [a, b]) = N(Q; a) - N(Q; b)$ or $N(Q; a) - N(Q; b) - Z(Q; [a, b])$ is an even positive integer.*

Note that $q_k(0) = a_k$ and $q_k(1) = \sum_{j=0}^{k} a_j$. Thus in the case of the interval $[0, 1]$ Theorem 2.2 reduces to the following useful corollary.

Corollary 2.3. *If $Q(0) \neq 0$ and $Q(1) \neq 0$, then $Z(Q; [0, 1]) = N(Q; 0) - N(Q; 1)$ or $N(Q; 0) - N(Q; 1) - Z(Q; [0, 1])$ is an even positive integer, where $N(Q; 0)$ and $N(Q; 1)$ are the numbers of sign changes in the sequence of polynomial coefficients (a_k) and in the sequence of sums $\left(\sum_{j=0}^{k} a_j\right)$, where $k = 0, 1, \ldots, n$, respectively.*

The main theorem of the paper is

Theorem 2.4. *If $g \in \mathcal{A}$ is of the form*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2.6}$$

with

$$|b_2| \leq 2, \tag{2.7}$$

then

$$\begin{aligned} &\max_{f\in\mathcal{C}(g)} \Phi_\lambda(f) \\ &\leq \begin{cases} \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + (1+|b_2|)\left|\frac{2}{3} - \lambda\right|, & \lambda \in \mathbb{R}\setminus[\tau_1(|b_2|),\tau_2(|b_2|)], \\ \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{(2-3\lambda)^2|b_2|^2}{12(2-|2-3\lambda|)} + \frac{2}{3}, & \lambda \in [\tau_1(|b_2|),\tau_2(|b_2|)], \end{cases} \end{aligned} \tag{2.8}$$

where

$$\tau_1(|b_2|) := \frac{2|b_2|}{3(|b_2|+2)}, \quad \tau_2(|b_2|) := \frac{2(|b_2|+4)}{3(|b_2|+2)}. \tag{2.9}$$

Proof. Let $g \in \mathcal{A}$ be of the form (2.6) and $f \in \mathcal{C}(g)$ be of the form (1.1). Observe that $f \in \mathcal{C}(g)$ if and only if

$$zf'(z) = \mathrm{e}^{-\mathrm{i}\delta} g(z)\left(p(z)\cos\delta + \mathrm{i}\sin\delta\right), \quad z \in \mathbb{D}, \tag{2.10}$$

for some $\delta \in (-\pi/2, \pi/2)$ and $p \in \mathcal{P}$. Setting the series (1.1), (1.3) and (2.1) into (2.10) by comparing coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}\left(c_1\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + b_2\right), \\ a_3 &= \frac{1}{3}\left(c_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + c_1 b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + b_3\right). \end{aligned} \tag{2.11}$$

Let $\lambda \in \mathbb{R}$. Using (2.3) from the above we have

$$\begin{aligned} \Phi_\lambda(f) &= \left|a_3 - \lambda a_2^2\right| \\ &= \left|\frac{1}{3}c_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \frac{1}{3}c_1 b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \frac{1}{3}b_3\right. \\ &\quad \left. -\frac{1}{4}\lambda\left(c_1^2\mathrm{e}^{-2\mathrm{i}\delta}\cos^2\delta + 2c_1 b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + b_2^2\right)\right| \\ &= \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2 + \frac{1}{3}\left(c_2 - \frac{c_1^2}{2}\right)\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right. \\ &\quad \left. +\frac{1}{6}c_1^2\left(1 - \frac{3}{2}\lambda\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right)\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \left(\frac{1}{3} - \frac{1}{2}\lambda\right)c_1 b_2\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right| \\ &\leq \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{1}{3}\left(2 - \frac{|c_1|^2}{2}\right)\cos\delta + \frac{|c_1|^2}{6}\left|1 - \frac{3}{2}\lambda\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right|\cos\delta \\ &\quad + \left|\frac{1}{3} - \frac{1}{2}\lambda\right||c_1||b_2|\cos\delta \\ &= \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \left(\frac{2}{3} + \frac{|c_1|^2}{6}\left(\sqrt{1 - \left(3\lambda - \frac{9}{4}\lambda^2\right)\cos^2\delta} - 1\right)\right. \\ &\quad \left. + \left|\frac{1}{3} - \frac{1}{2}\lambda\right||c_1||b_2|\right)\cos\delta. \end{aligned} \tag{2.12}$$

Set $x := |c_1|$ and $y := \cos\delta$. Clearly, $y \in (0,1]$ and, in view of (2.2), $x \in [0,2]$. It is convenient to use in further computation $\gamma := 2 - 3\lambda$ instead of λ. For $\gamma \in \mathbb{R}$ let

$$s_\gamma(y) := \sqrt{1 - \left(1 - \frac{1}{4}\gamma^2\right) y^2}, \quad y \in [0,1].$$

By the assumption (2.7), set $|b_2| := 2\alpha$, where $\alpha \in [0,1]$. Set $R := [0,2] \times [0,1]$. For $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ define

$$F_{\alpha,\gamma}(x,y) := \left(4 + x^2\left(s_\gamma(y) - 1\right) + 2\alpha|\gamma|x\right) y, \quad (x,y) \in R.$$

Hence and from (2.12) we have

$$\max_{f \in \mathcal{C}(g)} \Phi_\lambda(f) \le \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{1}{6}\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y). \tag{2.13}$$

Now for $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ we find the maximum value of $F_{\alpha,\gamma}$ on the rectangle R.

1. In the corners of R we have

$$\begin{aligned} F_{\alpha,\gamma}(0,0) &= F_{\alpha,\gamma}(2,0) = 0, \\ F_{\alpha,\gamma}(0,1) &= 4, \\ F_{\alpha,\gamma}(2,1) &= 2(1+2\alpha)|\gamma|. \end{aligned} \tag{2.14}$$

2. For $x = 0$ and $y \in (0,1)$ we have a linear function and for $x \in (0,2)$ and $y = 0$ we have a constant function.

3. For $x \in (0,2)$ and $y = 1$, let

$$G_{\alpha,\gamma}(x) := F_{\alpha,\gamma}(x,1) = \frac{1}{2}\left(|\gamma| - 2\right)x^2 + 2\alpha|\gamma|x + 4.$$

For $|\gamma| = 2$ we get the linear functions, so let $|\gamma| \neq 2$. Then $G'_{\alpha,\gamma}(x) = 0$ if and only if

$$x = \frac{2\alpha|\gamma|}{2 - |\gamma|} =: x_{\alpha,\gamma}.$$

Thus $x_{\alpha,\gamma} \in (0,2)$ if and only if

$$\alpha \neq 0 \wedge 0 < \frac{\alpha|\gamma|}{2 - |\gamma|} < 1. \tag{2.15}$$

The left-hand inequality in (2.15) holds if and only if

$$\alpha \neq 0 \wedge 0 < |\gamma| < 2. \tag{2.16}$$

We can write the right-hand inequality (2.15) as

$$\frac{(1+\alpha)|\gamma| - 2}{2 - |\gamma|} < 0$$

and, in view of (2.16), it holds when $|\gamma| < 2/(1+\alpha)$. But $2/(1+\alpha) < 2$ for $\alpha \in (0,1]$, so this with (2.16) yields that $x_{\alpha,\gamma} \in (0,2)$ if and only if

$$\alpha \neq 0 \wedge 0 < |\gamma| < \frac{2}{1+\alpha}. \tag{2.17}$$

Thus the function $G_{\alpha,\gamma}$ has a critical point in $(0,2)$, namely, $x_{\alpha,\gamma}$ as the unique one, if and only if (2.17) holds. Moreover we have

$$F_{\alpha,\gamma}(x_{\alpha,\gamma},1) = G_{\alpha,\gamma}\left(x_{\alpha,\gamma}\right) = \frac{2\alpha^2\gamma^2}{2 - |\gamma|} + 4. \tag{2.18}$$

4. For $x = 2$ and $y \in (0, 1)$, let

$$H_{\alpha,\gamma}(y) := F_{\alpha,\gamma}(2, y) = 4\left(ys_\gamma(y) + \alpha|\gamma|y\right).$$

For $|\gamma| = 2$ we have the linear functions evidently, so let $|\gamma| \neq 2$. Note first that

$$s_\gamma(y) > 0, \quad y \in (0, 1), \tag{2.19}$$

since the equation $s_\gamma(y) = 0,\ y \in (0, 1)$, equivalently written as

$$(4 - \gamma^2)y^2 = 4, \quad y \in (0, 1), \tag{2.20}$$

has no solution. Indeed, as $y^2 > 0$, we have $|\gamma| < 2$. But from (2.20) we obtain

$$y^2 = \frac{4}{4 - \gamma^2} > 1,$$

which is a contradiction. Thus (2.20) has no solution, so (2.19) holds. Taking into account (2.19) we have

$$ys'_\gamma(y) = \frac{-\left(1 - \frac{1}{4}\gamma^2\right)y^2}{\sqrt{1 - \left(1 - \frac{1}{4}\gamma^2\right)y^2}} = \frac{s_\gamma^2(y) - 1}{s_\gamma(y)}, \quad y \in (0, 1). \tag{2.21}$$

Using (2.21) we get

$$H'_{\alpha,\gamma}(y) = 4\left(s_\gamma(y) + \frac{s_\gamma^2(y) - 1}{s_\gamma(y)} + \alpha|\gamma|\right), \quad y \in (0, 1).$$

Hence

$$H'_{\alpha,\gamma}(y) = 0 \tag{2.22}$$

if and only if

$$2s_\gamma^2(y) + \alpha|\gamma|s_\gamma(y) - 1 = 0,$$

i.e., in view of (2.19) if and only if

$$s_\gamma(y) = \frac{-\alpha|\gamma| + \sqrt{8 + \alpha^2\gamma^2}}{4} =: s_{\alpha,\gamma}. \tag{2.23}$$

As $|\gamma| \neq 2$, so from the above we get the equation

$$y^2 = \frac{4 - \alpha^2\gamma^2 + \alpha|\gamma|\sqrt{8 + \alpha^2\gamma^2}}{2(4 - \gamma^2)}. \tag{2.24}$$

Thus the solution of the equation (2.24), and hence of (2.22), exists if and only if

$$0 < \frac{4 - \alpha^2\gamma^2 + \alpha|\gamma|\sqrt{8 + \alpha^2\gamma^2}}{2(4 - \gamma^2)} < 1. \tag{2.25}$$

Let $|\gamma| < 2$. The left-hand inequality in (2.25) is clearly true since $4 - \alpha^2\gamma^2 > 0$. Write the right-hand inequality in (2.25) equivalently as

$$\alpha|\gamma|\sqrt{8 + \alpha^2\gamma^2} < 4 - (2 - \alpha^2)\gamma^2. \tag{2.26}$$

The above inequality can hold only when

$$|\gamma| < \frac{2}{\sqrt{2 - \alpha^2}}. \tag{2.27}$$

But $2/\sqrt{2-\alpha^2} \leq 2$, so squaring (2.26) and reducing we equivalently have

$$(1-\alpha^2)\gamma^4 - 4\gamma^2 + 4 > 0. \tag{2.28}$$

Let $\alpha = 1$. Then, taking into account (2.27), the inequality (2.28) holds if and only if $|\gamma| < 1$. Let $\alpha \in [0,1)$. Then (2.28) holds if and only if

$$|\gamma| > \sqrt{\frac{2}{1-\alpha}} \quad \text{or} \quad |\gamma| < \sqrt{\frac{2}{1+\alpha}}.$$

Hence, from (2.27) and by the fact that for $\alpha \in [0,1)$,

$$\sqrt{\frac{2}{1+\alpha}} \leq \frac{2}{\sqrt{2-\alpha^2}} \leq \sqrt{\frac{2}{1-\alpha}},$$

we see that (2.28) and, consequently, (2.25) holds if and only if

$$|\gamma| < \sqrt{\frac{2}{1+\alpha}}. \tag{2.29}$$

In this way, we proved that for $\alpha \in [0,1]$, the inequality (2.28), so (2.25) holds if and only if (2.29) holds.

Let $|\gamma| > 2$. Then the left-hand inequality in (2.25) holds if and only if

$$\alpha|\gamma|\sqrt{8+\alpha^2\gamma^2} < \alpha^2\gamma^2 - 4. \tag{2.30}$$

Note that $\alpha^2\gamma^2 - 4 \leq 0$ for $|\gamma| \leq 2/\alpha$, so then (2.30) is false. Assume that $|\gamma| > 2/\alpha$. Squaring (2.30), after reducing, we get $|\gamma| < 1/\alpha$, which contradicts the assumption.

Thus we proved that the function $H_{\alpha,\gamma}$ has a critical point in $(0,1)$, namely,

$$y = \sqrt{\frac{4-\alpha^2\gamma^2+\alpha|\gamma|\sqrt{\alpha^2\gamma^2+8}}{2(4-\gamma^2)}} =: y_{\alpha,\gamma},$$

as the unique solution of (2.24), if and only if (2.29) holds. Moreover,

$$\begin{aligned} F_{\alpha,\gamma}(2, y_{\alpha,\gamma}) &= H_{\alpha,\gamma}(y_{\alpha,\gamma}) \\ &= \sqrt{\frac{4-\alpha^2\gamma^2+\alpha|\gamma|\sqrt{8+\alpha^2\gamma^2}}{2(4-\gamma^2)}} \left(\sqrt{8+\alpha^2\gamma^2} + 3\alpha|\gamma|\right). \end{aligned} \tag{2.31}$$

5. We will prove that for each $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ the function $F_{\alpha,\gamma}$ has no critical point in $(0,2)\times(0,1)$.

We have

$$\frac{\partial F_{\alpha,\gamma}}{\partial x} = 0$$

if and only if

$$y\left(x\left(s_\gamma(y)-1\right)+\alpha|\gamma|\right) = 0,$$

and since $y \neq 0$ and $x \neq 0$, if and only if

$$s_\gamma(y) = 1 - \frac{\alpha|\gamma|}{x}, \quad y \in (0,1). \tag{2.32}$$

Observe first that $\gamma \neq 0$ because if $\gamma = 0$, then the equation (2.32) reduces to $s_0(y) = 1,\ y \in (0,1)$, which has no solution in $(0,1)$.

If $\alpha = 0$, then the equation (2.32) reduces to $s_\gamma(y) = 1,\ y \in (0,1)$, which is satisfied if and only if $|\gamma| = 2$ and $y \in (0,1)$ is any. On the other hand, if $|\gamma| = 2$, then the equation (2.32) is satisfied for $\alpha = 0$ only.

Since $x > 0$, by comparing (2.32) and (2.19), we additionally see that $x > \alpha|\gamma|$.

Thus the solution of (2.32) can exist only when

$$(\alpha = 0 \wedge |\gamma| = 2) \vee (\alpha \neq 0 \wedge \gamma \neq 0 \wedge |\gamma| \neq 2 \wedge x > \alpha|\gamma|). \tag{2.33}$$

Squaring then (2.32) we obtain

$$s_\gamma^2(y) - 1 = -\frac{2\alpha|\gamma|}{x} + \frac{\alpha^2\gamma^2}{x^2}. \tag{2.34}$$

Since by (2.19), $s_\gamma(y) \neq 0$ for $y \in (0,1)$, taking into account (2.21) we have

$$\frac{\partial F_{\alpha,\gamma}}{\partial y} = 4 + x^2\left(s_\gamma(y) - 1\right) + 2\alpha|\gamma|x + \frac{(s_\gamma^2(y) - 1)x^2}{s_\gamma(y)}.$$

Thus, by using (2.32) and (2.34), we have

$$\frac{\partial F_{\alpha,\gamma}}{\partial y} = 0$$

if and only if

$$4 + x^2\left(-\frac{\alpha|\gamma|}{x}\right) + 2\alpha|\gamma|x + \frac{\left(-\frac{2\alpha|\gamma|}{x} + \frac{\alpha^2\gamma^2}{x^2}\right)x^2}{1 - \frac{\alpha|\gamma|}{x}} = 0,$$

and after simplifying, if and only if

$$4 + \alpha|\gamma|x + \frac{-2\alpha|\gamma|x^2 + \alpha^2\gamma^2 x}{x - \alpha|\gamma|} = 0.$$

Thus

$$\alpha|\gamma|x^2 - 4x + 4\alpha|\gamma| = 0, \quad x \in (0,2). \tag{2.35}$$

Note first that for $\alpha = 0$ the equation (2.35) has no solution. Let $\alpha \neq 0$. From (2.33), $\gamma \neq 0$. Then the discriminant $\Delta = 16(1 - \alpha^2\gamma^2) \geq 0$ if and only if $0 < |\gamma| \leq 1/\alpha$. Note that $\Delta = 0$ if and only if $|\gamma| = 1/\alpha$, and then the equation (2.35) has no solution. Thus the equation (2.35) has no root when $|\gamma| \geq 1/\alpha$. Consequently, for $\alpha = 0$ and $\gamma \in \mathbb{R}$ as well as for $\alpha \in (0,1]$ and $|\gamma| \geq 1/\alpha$ the function $F_{\alpha,\gamma}$ has no critical point in $(0,2) \times (0,1)$.

Thus by (2.33) we consider

$$\alpha \neq 0 \wedge |\gamma| \neq 2 \wedge 0 < |\gamma| < 1/\alpha \wedge x > \alpha|\gamma|. \tag{2.36}$$

Solving now (2.35) we have

$$x = \frac{2 - 2\sqrt{1 - \alpha^2\gamma^2}}{\alpha|\gamma|} =: x_{1;\alpha,\gamma}, \quad x = \frac{2 + 2\sqrt{1 - \alpha^2\gamma^2}}{\alpha|\gamma|} =: x_{2;\alpha,\gamma}.$$

Since $x_{2;\alpha,\gamma} > 0$ and $x_{1;\alpha,\gamma}x_{2;\alpha,\gamma} = 4$, so we immediately see that $0 < x_{1;\alpha,\gamma} < 2 < x_{2;\alpha,\gamma}$. Thus $x_{2;\alpha,\gamma} \notin (0,2)$ and it remains to consider $x_{1;\alpha,\gamma}$.

Observe that $x_{1;\alpha,\gamma} > \alpha|\gamma|$. Indeed, this follows from the fact that the inequality

$$\frac{2 - 2\sqrt{1 - \alpha^2\gamma^2}}{\alpha|\gamma|} > \alpha|\gamma|$$

is equivalent to

$$2 - \alpha^2\gamma^2 > 2\sqrt{1 - \alpha^2\gamma^2},$$

which is evidently true for $0 < |\gamma| < 1/\alpha$.

Setting $x := x_{1;\alpha,\gamma}$ into (2.34) we have

$$s_\gamma^2(y) - 1 = -\frac{2\alpha|\gamma|}{x_{1;\alpha,\gamma}} + \frac{\alpha^2\gamma^2}{x_{1;\alpha,\gamma}^2}.$$

Hence

$$\begin{aligned} y^2 &= \frac{\dfrac{2\alpha|\gamma|}{x_{1;\alpha,\gamma}} - \dfrac{\alpha^2\gamma^2}{x_{1;\alpha,\gamma}^2}}{1 - \dfrac{1}{4}\gamma^2} = \frac{2\alpha|\gamma|x_{1;\alpha,\gamma} - \alpha^2\gamma^2}{x_{1;\alpha,\gamma}^2\left(1 - \dfrac{1}{4}\gamma^2\right)} \\ &= \frac{\left(4 - \alpha^2\gamma^2 - 4\sqrt{1-\alpha^2\gamma^2}\right)\alpha^2\gamma^2}{\left(1 - \sqrt{1-\alpha^2\gamma^2}\right)^2(4-\gamma^2)}. \end{aligned} \tag{2.37}$$

A solution in $(0,1)$ of (2.37) exists if and only if

$$0 < \frac{\left(4 - \alpha^2\gamma^2 - 4\sqrt{1-\alpha^2\gamma^2}\right)\alpha^2\gamma^2}{\left(1 - \sqrt{1-\alpha^2\gamma^2}\right)^2(4-\gamma^2)} < 1. \tag{2.38}$$

By (2.36) consider

$$\alpha \neq 0 \wedge |\gamma| \neq 2 \wedge 0 < |\gamma| < \frac{1}{\alpha}. \tag{2.39}$$

We will prove that then the condition (2.38) is false.

(A) Suppose that $2 < |\gamma| < 1/\alpha$. Since, as easy to check, the left-hand side of the inequality

$$4 - \alpha^2\gamma^2 > 4\sqrt{1-\alpha^2\gamma^2} \tag{2.40}$$

is positive, by squaring and computing, we equivalently get the inequality

$$\alpha^2\gamma^2 + 8 > 0,$$

which is true. Hence and by the fact that $4 - \gamma^2 < 0$ we see that the left-hand inequality in (2.38) is false.

(B) By (2.39) it remains to consider

$$\alpha \neq 0 \wedge 0 < |\gamma| < \frac{1}{\alpha} \leq 2.$$

(a) As in Part (A), we prove that (2.40) holds. Hence and by the fact that $4 - \gamma^2 > 0$ we see that the left-hand inequality in (2.38) holds.

(b) Since $4 - \gamma^2 > 0$, write the right-hand inequality in (2.38) as

$$\left(4 - \alpha^2\gamma^2 - 4\sqrt{1-\alpha^2\gamma^2}\right)\alpha^2\gamma^2 < \left(1 - \sqrt{1-\alpha^2\gamma^2}\right)^2(4-\gamma^2)$$

and, after computing, equivalently as

$$\left(8 - 2(1+2\alpha^2)\gamma^2\right)\sqrt{1-\alpha^2\gamma^2} < (\alpha^4+\alpha^2)\gamma^4 - 2(1+4\alpha^2)\gamma^2 + 8. \tag{2.41}$$

We will show that (2.41) is false. To verify it, we will prove that the inequality

$$s_\alpha(t) \geq r_\alpha(t), \quad t \in \left(0, 1/\alpha^2\right), \tag{2.42}$$

holds, where

$$s_\alpha(t) := \left(8 - 2(1+2\alpha^2)t\right)\sqrt{1-\alpha^2 t}, \quad t \in \left[0, 1/\alpha^2\right],$$

and

$$r_\alpha(t) := (\alpha^4+\alpha^2)t^2 - 2(1+4\alpha^2)t + 8, \quad t \in \left[0, 1/\alpha^2\right].$$

Then substituting $t := \gamma^2$ into (2.42), we get the true inequality which shows that (2.41) is false.

Let

$$w_\alpha(t) := s_\alpha^2(t) - r_\alpha^2(t), \quad t \in \left[0, 1/\alpha^2\right].$$

Thus after computing we have

$$w_\alpha(t) = \alpha^4 t^3\left(4 - (1+\alpha^2)^2 t\right), \quad t \in \left[0, 1/\alpha^2\right]. \tag{2.43}$$

Note that $w_\alpha(t) = 0$ if and only if

$$t = 0 \vee t = \frac{4}{(1+\alpha^2)^2} =: t_\alpha,$$

since, as easy to check, $t_\alpha \in \left[0, 1/\alpha^2\right]$ for $\alpha \in (0,1]$.

Let $\alpha := 1$. Then $t_1 = 1$ and by (2.43),

$$w_1(t) = (s_1(t) - r_1(t))(s_1(t) + r_1(t)) = 4t^3(1-t) > 0, \quad t \in (0,1).$$

Hence and from the fact that

$$s_1(0) + r_1(0) = 16 > 0,$$

it follows that

$$s_1(t) - r_1(t) > 0, \quad t \in (0,1),$$

which confirms (2.42).

Let now $\alpha \in (0,1)$. Then by (2.43),

$$w_\alpha(t) = (s_\alpha(t) - r_\alpha(t))(s_\alpha(t) + r_\alpha(t)) > 0, \quad t \in (0, t_\alpha), \tag{2.44}$$

and

$$w_\alpha(t) = (s_\alpha(t) - r_\alpha(t))(s_\alpha(t) + r_\alpha(t)) < 0, \quad t \in \left(t_\alpha, 1/\alpha^2\right). \tag{2.45}$$

Since

$$s_\alpha(0) + r_\alpha(0) = 16 > 0,$$

from (2.44) it follows that

$$s_\alpha(t) - r_\alpha(t) > 0, \quad t \in (0, t_\alpha). \tag{2.46}$$

Similarly, since

$$s_\alpha\left(\frac{1}{\alpha^2}\right) + r_\alpha\left(\frac{1}{\alpha^2}\right) = 1 - \frac{1}{\alpha^2} < 0,$$

from (2.45) it follows that

$$s_\alpha(t) - r_\alpha(t) > 0, \quad t \in \left(t_\alpha, 1/\alpha^2\right). \tag{2.47}$$

Thus from (2.46), (2.47) and by the continuity of the functions s_α and r_α at $t := t_\alpha$, we have

$$s_\alpha(t) - r_\alpha(t) \geq 0, \quad t \in \left(0, 1/\alpha^2\right),$$

which confirms (2.42).

Thus, taking into account Parts (A) and (B), we proved that (2.41) is false, so the condition (2.38) does not hold and, therefore the equation (2.37) has no solution in $(0,1)$.

In this way, the proof that for $\alpha \in [0,1]$ and $\gamma \in \mathbb{R}$ the function $F_{\alpha,\gamma}$ has no critical point in $(0,2)\times(0,1)$ is finished.

6. Now we calculate the maximum value of $F_{\alpha,\gamma}$ in R, which, as was shown, is attained on the boundary of R. Let $\alpha \in [0,1]$.

(A) $|\gamma| \geq 2/(1+\alpha)$. Taking into account Part 3 with (2.17) and Part 4 with (2.29), we see that the maximum value of $F_{\alpha,\gamma}$ is attained in a corner of R. Thus by (2.14) it suffices to compare the following values:

$$0, \quad 4, \quad 2(1+2\alpha)|\gamma|. \tag{2.48}$$

Since, for $|\gamma| \geq 2/(1+\alpha)$,

$$2(1+2\alpha)|\gamma| \geq \frac{4+8\alpha}{1+\alpha} \geq 4,$$

so from (2.48) we have

$$\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y) = F_{\alpha,\gamma}(2,1) = 2(1+2\alpha)|\gamma|. \tag{2.49}$$

(B) $\sqrt{2/(1+\alpha)} \leq |\gamma| < 2/(\alpha+1)$. Taking into account Part 4 with (2.17) and Part 5 with (2.29), we see that the maximum value of $F_{\alpha,\gamma}$ is attained in a corner of R or in $(x_{\alpha,\gamma}, 1)$. Thus we compare all values (2.48) and, by (2.18), the value

$$F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1) = \frac{2\alpha^2\gamma^2}{2-|\gamma|} + 4.$$

Observe that

$$\frac{2\alpha^2\gamma^2}{2-|\gamma|} + 4 \geq 2(1+2\alpha)|\gamma|.$$

Indeed, since $|\gamma| < 2/(\alpha+1) \leq 2$, the above after computing is equivalent to the inequality

$$\left((1+\alpha)^2|\gamma| - 2\right)^2 \geq 0,$$

which clearly holds. Thus

$$\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y) = F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1) = \frac{2\alpha^2\gamma^2}{2-|\gamma|} + 4. \tag{2.50}$$

(C) $\gamma = 0$. Taking into account Part 3 with (2.17) and Part 4 with (2.29), the maximum value of $F_{\alpha,0}$ is attained in a corner of R or in the point $(2, y_{\alpha,0}) = \left(2, 1/\sqrt{2}\right)$. Thus, by comparing all values (2.48) for $\gamma := 0$ and, by (2.31), the value

$$F_{\alpha,0}(2, y_{\alpha,0}) = 2,$$

we have

$$\max_{(x,y)\in R} F_{\alpha,0}(x,y) = F_{\alpha,0}(0,1) = 4. \tag{2.51}$$

(D) $0 < |\gamma| < \sqrt{2/(\alpha+1)}$. Then we compare all values (2.48) and, by (2.18) and (2.31), $F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1)$ and $F_{\alpha,\gamma}(2, y_{\alpha,\gamma})$. We will show that the value $F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1)$ is the largest one.

(D_1) Since $|\gamma| < \sqrt{2/(\alpha+1)} < 2$ for $\alpha \in [0,1]$, so $\alpha^2\gamma^2/(2-|\gamma|) \geq 0$ and therefore

$$F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1) \geq 4.$$

Moreover, repeating arguments of Part (B), we see that

$$F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1) \geq 2(1+2\alpha)|\gamma|.$$

(D_2) Thus it remains to prove that

$$F_{\alpha,\gamma}(x_{\alpha,\gamma}, 1) \geq F_{\alpha,\gamma}(2, y_{\alpha,\gamma}) \tag{2.52}$$

i.e., in view of (2.18) and (2.31), that

$$\begin{aligned}&\frac{2\alpha^2\gamma^2}{2-|\gamma|}+4\\ &\geq\sqrt{\frac{4-\alpha^2\gamma^2+\alpha|\gamma|\sqrt{8+\alpha^2\gamma^2}}{2(4-\gamma^2)}}\left(\sqrt{8+\alpha^2\gamma^2}+3\alpha|\gamma|\right).\end{aligned}\tag{2.53}$$

As $|\gamma|<2$, so both sides of (2.53) are positive. Thus squaring (2.53) and computing we equivalently have

$$\begin{aligned}&\alpha^4|\gamma|^5+\left(6\alpha^4-8\alpha^2\right)\gamma^4+\left(20\alpha^2+8\right)|\gamma|^3-\left(8\alpha^2+16\right)\gamma^2-24|\gamma|+48\\ &\geq\alpha|\gamma|(2-|\gamma|)\left(\alpha^2\gamma^2+8\right)^{3/2}.\end{aligned}\tag{2.54}$$

To verify that (2.54) holds, we will show that

$$Q_\alpha(u)\geq S_\alpha(u),\quad u\in[0,u_\alpha],\tag{2.55}$$

where $u_\alpha:=\sqrt{2/(\alpha+1)}$,

$$\begin{aligned}Q_\alpha(u)\ :=\ &\alpha^4u^5+\left(6\alpha^4-8\alpha^2\right)u^4+\left(20\alpha^2+8\right)u^3\\ &-\left(8\alpha^2+16\right)u^2-24u+48,\quad u\in[0,u_\alpha],\end{aligned}$$

and

$$S_\alpha(u):=\alpha u(2-u)\left(\alpha^2u^2+8\right)^{3/2},\quad u\in[0,u_\alpha].$$

(1°) $\alpha=0$. Then $u_0=\sqrt{2}$ and the inequality (2.55) reduces to

$$Q_0(u)=u^3-2u^2-3u+6=\left(u^2-3\right)(u-2)>0=S_0(u),\quad u\in\left[0,\sqrt{2}\right],$$

which is true. Thus (2.55) holds, which confirms (2.54).

(2°) $\alpha=1$. Then $u_1=1$ and the inequality (2.55) reduces to

$$\begin{aligned}&u^5-2u^4+28u^3-24u^2-24u+48\\ &\geq u(2-u)\left(u^2+8\right)^{3/2},\quad u\in[0,1].\end{aligned}\tag{2.56}$$

Since

$$\begin{aligned}u^5-2u^4+28u^3-24u^2-24u+48\ &\geq\ u^5-2u^3+28u^3-24-24+48\\ &=\ u^5+26u^3\geq0,\quad u\in[0,1],\end{aligned}$$

so both sides of (2.56) are nonnegative. Thus squaring (2.56) and computing we equivalently get the inequality

$$(u-1)^2\left(2u^6+32u^4+40u^3-92u^2+144u+144\right)\geq0,\quad u\in[0,1],$$

which clearly holds. To see this, replace u^2 by u. Thus (2.55) holds, which confirms (2.54).

(3°) $\alpha\in(0,1)$. Define

$$V_\alpha(u):=Q_\alpha^2(u)-S_\alpha^2(u),\quad u\in[0,u_\alpha].$$

We will show that

$$V_\alpha(u)>0,\quad u\in[0,u_\alpha],\tag{2.57}$$

i.e., that

$$(Q_\alpha(u)-S_\alpha(u))(Q_\alpha(u)+S_\alpha(u))>0,\quad u\in[0,u_\alpha].\tag{2.58}$$

Further, taking into account that Q_α and S_α are continuous functions with

$$Q_\alpha(0)-S_\alpha(0)=48>0,$$

from (2.58) we deduce that

$$Q_\alpha(u) - S_\alpha(u) > 0, \quad u \in [0, u_\alpha],$$

i.e., that (2.55) holds.

Now we prove that (2.57) holds, i.e., that the following inequality holds:

$$\begin{aligned} &V_\alpha(u) \\ &= \left(\alpha^4 u^5 + \left(6\alpha^4 - 8\alpha^2\right) u^4 + \left(20\alpha^2 + 8\right) u^3 - \left(8\alpha^2 + 16\right) u^2 - 24u + 48\right)^2 \\ &-\alpha^2 u^2 (2-u)^2 \left(\alpha^2 u^2 + 8\right)^3 > 0, \quad u \in [0, u_\alpha], \end{aligned}$$

which after computation is equivalent to

$$\begin{aligned} V_\alpha(u) &= (\alpha^8 - \alpha^6)u^9 + (2\alpha^8 - 5\alpha^6 + 5\alpha^4)u^8 \\ &+ (20\alpha^6 - 16\alpha^4 - 8\alpha^2)u^7 + (-12\alpha^6 + 6\alpha^4 + 36\alpha^2 + 4)u^6 \\ &+ (16\alpha^4 - 24\alpha^2 - 16)u^5 - (8\alpha^4 + 124\alpha^2 + 8)u^4 + (272\alpha^2 + 96)u^3 \\ &- (176\alpha^2 + 60)u^2 - 144u + 144 =: \sum_{j=0}^{9} a_j u^{9-j} > 0, \quad u \in [0, u_\alpha]. \end{aligned}$$

As in (2.5), let (q_k), $k = 0, 1, \ldots, 9$, be a sequence of polynomials of the form

$$q_k(u) = \sum_{j=0}^{k} a_j u^{k-j}, \quad u \in [0, u_\alpha],$$

corresponding to the polynomial $Q := V_\alpha$ in (2.4) for Laguerre's rule.

(a) Now we check the signs of the elements of the sequence $(q_k(0))$, i.e., of the sequence (a_k) for $k = 0, 1, \ldots, 9$. A simple computing shows that for $\alpha \in (0, 1)$ we have

$$\begin{aligned} q_0(0) &= \alpha^6(\alpha^2 - 1) < 0, \\ q_1(0) &= \alpha^4(2\alpha^4 - 5\alpha^2 + 5) > 0, \\ q_2(0) &= 4\alpha^2(5\alpha^4 - 4\alpha^2 - 2) < 0, \\ q_3(0) &= 2(-6\alpha^6 + 3\alpha^4 + 18\alpha^2 + 2) > 0, \\ q_4(0) &= 8(2\alpha^4 - 3\alpha^2 - 2) < 0, \\ q_5(0) &= -4(2\alpha^4 + 31\alpha^2 + 2) < 0, \\ q_6(0) &= 272\alpha^2 + 96 > 0, \\ q_7(0) &= -176\alpha^2 - 60 < 0, \\ q_8(0) &= -144 < 0, \\ q_9(0) &= 144 > 0. \end{aligned}$$

Hence

$$N(V_\alpha; 0) = 7, \quad \alpha \in (0, 1). \tag{2.59}$$

(b) Now we check the signs of the elements of the sequence $(q_k(u_\alpha))$ for $k = 0, 1, \ldots, 9$.

(i) $k = 0$. We have

$$q_0(u) = \alpha^6(\alpha^2 - 1), \quad u \in [0, u_\alpha].$$

Thus

$$q_0(u_\alpha) < 0, \quad \alpha \in (0, 1). \tag{2.60}$$

(ii) $k = 1$. We have

$$q_1(u) = \alpha^4 \left((\alpha^4 - \alpha^2)u + 2\alpha^4 - 5\alpha^2 + 5\right), \quad u \in [0, u_\alpha].$$

We will show that

$$q_1(u_\alpha) > 0, \quad \alpha \in (0, 1), \tag{2.61}$$

i.e., after computing that

$$-(2\alpha^4 - 5\alpha^2 + 5)\sqrt{\alpha + 1} < (\alpha^4 - \alpha^2)\sqrt{2}, \quad \alpha \in (0, 1). \tag{2.62}$$

Observe that since both sides of (2.62) are negative, after squaring and computing we equivalently get

$$4\alpha^8 - 2\alpha^7 - 18\alpha^6 + 2\alpha^5 + 43\alpha^4 - 50\alpha^2 + 25 > 0, \quad \alpha \in (0, 1). \tag{2.63}$$

To verify that (2.63) holds, we will show that

$$w(t) > 0, \quad t \in [0, 1], \tag{2.64}$$

where

$$\begin{aligned} w(t) &:= 4t^8 - 2t^7 - 18t^6 + 2t^5 + 43t^4 - 50t^2 + 25 \\ &=: \sum_{j=0}^{8} b_j t^{8-j}, \quad t \in [0, 1]. \end{aligned}$$

Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k), and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 8$, equal 4, i.e., $N(w; 0) = N(w; 1) = 4$. Applying Corollary 2.3, we see that the polynomial w has no zero in the interval $(0, 1)$ and, since $w(0) = 25 > 0$, so (2.64) and, consequently, (2.63) holds. Thus (2.61) is confirmed.

(iii) $k = 2$. We have

$$q_2(u) = (\alpha^8 - \alpha^6)u^2 + (2\alpha^8 - 5\alpha^6 + 5\alpha^4)u + 20\alpha^6 - 16\alpha^4 - 8\alpha^2, \quad u \in [0, u_\alpha].$$

We will show that

$$q_2(u_\alpha) < 0, \quad \alpha \in (0, 1), \tag{2.65}$$

i.e., after computing that

$$\begin{aligned} &\sqrt{2}\alpha^2(2\alpha^4 - 5\alpha^2 + 5) \\ &< (-2\alpha^5 - 18\alpha^4 + 16\alpha^2 + 8)\sqrt{\alpha + 1}, \quad \alpha \in (0, 1). \end{aligned} \tag{2.66}$$

It is easily seen that the left-hand side of (2.66) is positive and since

$$\begin{aligned} -2\alpha^5 - 18\alpha^4 + 16\alpha^2 + 8 &\geq -2\alpha^2 - 14\alpha^2 - 4\alpha^4 + 16\alpha^2 + 8 \\ &= -4\alpha^4 + 8 > 0, \quad \alpha \in (0, 1), \end{aligned} \tag{2.67}$$

so is the right-hand side of (2.66). Thus squaring (2.66) and computing we equivalently get

$$\begin{aligned} &-4\alpha^{12} + 2\alpha^{11} + 58\alpha^{10} + 198\alpha^9 + 85\alpha^8 - 320\alpha^7 - 254\alpha^6 \\ &-32\alpha^5 - 41\alpha^4 + 128\alpha^3 + 128\alpha^2 + 32\alpha + 32 > 0. \end{aligned} \tag{2.68}$$

To verify that (2.68) holds, we will show that

$$w(t) > 0, \quad t \in [0, 1], \tag{2.69}$$

where

$$\begin{aligned} w(t) &:= -4t^{12}+2t^{11}+58t^{10}+198t^9+85t^8-320t^7-254t^6 \\ &\quad -32t^5-41t^4+128t^3+128t^2+32t+32 =: \sum_{j=0}^{12} b_j t^{12-j}, \quad t\in[0,1]. \end{aligned}$$

Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k) and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k=0,1,\ldots,12$, equal 3, i.e., $N(w;0)=N(w;1)=3$. Applying Corollary 2.3, we see that the polynomial w has no zero in the interval $(0,1)$ and, since $w(0)=32>0$, so (2.69) and, consequently, (2.68) holds. Thus (2.65) is confirmed.

(iv) $k=3$. We have

$$\begin{aligned} q_3(u) &= (\alpha^8-\alpha^6)u^3+(2\alpha^8-5\alpha^6+5\alpha^4)u^2+(20\alpha^6-16\alpha^4-8\alpha^2)u \\ &\quad -12\alpha^6+6\alpha^4+36\alpha^2+4, \quad u\in[0,u_\alpha]. \end{aligned} \tag{2.70}$$

We will show that

$$q_3(u_\alpha)>0, \quad \alpha\in(0,1), \tag{2.71}$$

i.e., after computing that

$$\begin{aligned} &\alpha^2(\alpha+1)(-\alpha^5-9\alpha^4+8\alpha^2+4)\sqrt{2} \\ &< \left(2\alpha^8-6\alpha^7-11\alpha^6+3\alpha^5+8\alpha^4+18\alpha^3+18\alpha^2\right. \\ &\quad \left.+2\alpha+2\right)\sqrt{\alpha+1}, \quad \alpha\in(0,1). \end{aligned} \tag{2.72}$$

Since for $\alpha\in(0,1)$,

$$-\alpha^5-9\alpha^4+8\alpha^2+4\geq -2\alpha^4+4>0$$

and

$$\begin{aligned} &-6\alpha^7-11\alpha^6+18\alpha^3+18\alpha^2\geq -6\alpha^3-11\alpha^2+18\alpha^3+18\alpha^2 \\ &=12\alpha^3+7\alpha^2>0, \end{aligned}$$

so both sides of (2.72) are positive. Thus squaring (2.72) and computing we get equivalently

$$\begin{aligned} &4\alpha^{17}-22\alpha^{16}-72\alpha^{15}-100\alpha^{14}-67\alpha^{13}+217\alpha^{12}+223\alpha^{11} \\ &-531\alpha^{10}-748\alpha^9-24\alpha^8+652\alpha^7+1112\alpha^6+1056\alpha^5 \\ &+540\alpha^4+220\alpha^3+84\alpha^2+12\alpha+4>0. \end{aligned} \tag{2.73}$$

Observe now that the left-hand side of (2.73) is greater or equal to

$$\begin{aligned} &4\alpha^{17}-22\alpha^{12}-72\alpha^{12}-100\alpha^{12}-67\alpha^{11}+217\alpha^{12}+223\alpha^{11} \\ &-531\alpha^7-748\alpha^6-24\alpha^6+652\alpha^7+1112\alpha^6+1056\alpha^5 \\ &+540\alpha^4+220\alpha^3+84\alpha^2+12\alpha+4 \\ &=4\alpha^{17}+23\alpha^{12}+156\alpha^{11}+121\alpha^7+340\alpha^6+1056\alpha^5 \\ &+540\alpha^4+220\alpha^3+84\alpha^2+12\alpha+4 \end{aligned}$$

which is clearly positive for $\alpha\in(0,1)$. Thus (2.73) holds, which confirms (2.71).

(v) $k=4$. We have

$$\begin{aligned} q_4(u) &= (\alpha^8-\alpha^6)u^4+(2\alpha^8-5\alpha^6+5\alpha^4)u^3+(20\alpha^6-16\alpha^4-8\alpha^2)u^2 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u \\ &\quad +16\alpha^4-24\alpha^2-16, \quad u\in[0,u_\alpha]. \end{aligned} \tag{2.74}$$

We will show that there exists a unique $\alpha \in (0,1)$ such that

$$q_4(u_\alpha) = 0, \tag{2.75}$$

i.e., after computing, a unique $\alpha \in (0,1)$ such that

$$\begin{aligned} &\left(2\alpha^8 - 6\alpha^7 - 11\alpha^6 + 3\alpha^5 + 8\alpha^4 + 18\alpha^3 + 18\alpha^2 + 2\alpha + 2\right)\sqrt{2} \\ &= \left(-2\alpha^7 - 18\alpha^6 - 8\alpha^5 + 8\alpha^4 + 12\alpha^3 + 20\alpha^2 + 8\alpha + 8\right)\sqrt{\alpha+1}. \end{aligned} \tag{2.76}$$

To verify that (2.76) holds, we will show that the equation

$$r(t) = s(t) \tag{2.77}$$

has a unique solution in (0,1), where for $t \in [0,1]$,

$$\begin{aligned} r(t) &:= \left(2t^8 - 6t^7 - 11t^6 + 3t^5 + 8t^4 + 18t^3 + 18t^2 + 2t + 2\right)\sqrt{2}, \\ s(t) &:= \left(-2t^7 - 18t^6 - 8t^5 + 8t^4 + 12t^3 + 20t^2 + 8t + 8\right)\sqrt{t+1}. \end{aligned}$$

Define

$$w(t) := s^2(t) - r^2(t), \quad t \in [0,1].$$

Thus

$$\begin{aligned} w(t) &= 4t^{16} - 26t^{15} - 46t^{14} - 70t^{13} - 189t^{12} - 82t^{11} \\ &\quad +145t^{10} + 204t^9 + 424t^8 + 528t^7 + 316t^6 + 92t^5 \\ &\quad -188t^4 - 304t^3 - 180t^2 - 88t - 28 =: \sum_{j=0}^{16} b_j t^{16-j}, \quad t \in [0,1]. \end{aligned}$$

Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k) and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 16$, equal 3 and 2, respectively, i.e., $N(w;0) = 3$ and $N(w;1) = 2$. Thus applying Corollary 2.3, we see that the equation

$$w(t) = (s(t) - r(t))\,(s(t) + r(t)) = 0 \tag{2.78}$$

has a unique zero $t =: t_0$. Since $w(0) = -28$ and $w(1) = 512$, so $t_0 \in (0,1)$. Observe that for $t \in [0,1]$ we have

$$\begin{aligned} \frac{r(t)}{\sqrt{2}} &\geq 2t^8 - 6t^4 - 11t^3 + 3t^5 + 8t^4 + 18t^3 + 18t^2 + 2t + 2 \\ &= 2t^8 + 3t^5 + 2t^4 + 7t^3 + 18t^2 + 2t + 2 > 0 \end{aligned}$$

and

$$\begin{aligned} \frac{s(t)}{\sqrt{t+1}} &\geq -2t^4 - 18t^2 - 8t^3 + 8t^4 + 12t^3 + 20t^2 + 8t + 8 \\ &= 6t^4 + 4t^3 + 2t^2 + 8t + 8 > 0. \end{aligned}$$

Hence $r(t) > 0$ and $s(t) > 0$ for $t \in [0,1]$. Thus from (2.78) it follows that $r(t_0) = s(t_0)$. Consequently, the equation (2.77) has a unique solution in $(0,1)$, namely, $t = t_0$. Thus, (2.76) so (2.75) holds with $\alpha := t_0$.

Moreover, since for $\alpha = 1$ we have $u_1 = 1$ and $q_4(u_1) = q_4(1) = 8 > 0$, we deduce that

$$q_4(u_\alpha) < 0, \quad \alpha \in (0, \alpha_0), \tag{2.79}$$

and

$$q_4(u_\alpha) > 0, \quad \alpha \in (\alpha_0, 1). \tag{2.80}$$

(vi) $k=5$. We have

$$\begin{aligned} q_5(u) &= (\alpha^8-\alpha^6)u^5+(2\alpha^8-5\alpha^6+5\alpha^4)u^4+(20\alpha^6-16\alpha^4-8\alpha^2)u^3 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^2+(16\alpha^4-24\alpha^2-16)u \\ &\quad -8\alpha^4-124\alpha^2-8, \quad u\in[0,u_\alpha]. \end{aligned}$$

We will show that

$$q_5(u_\alpha)<0, \quad \alpha\in(0,1), \tag{2.81}$$

i.e., after computing that

$$\begin{aligned} &\left(\alpha^8+10\alpha^7+13\alpha^6-10\alpha^4-16\alpha^3-14\alpha^2-8\alpha-4\right)\sqrt{2} \\ &<\alpha(-2\alpha^7+6\alpha^6+13\alpha^5+\alpha^4+25\alpha^3+44\alpha^2 \\ &\quad +15\alpha+2)\sqrt{\alpha+1}, \quad \alpha\in(0,1). \end{aligned} \tag{2.82}$$

Clearly, for $\alpha\in(0,1)$ we have

$$\alpha^8+10\alpha^7+13\alpha^6-10\alpha^4-16\alpha^3-14\alpha^2-8\alpha-4<0,$$

so the left-hand side of (2.82) is negative. But the right-hand side of (2.82) is clearly positive. In this way, (2.82) holds, which confirms (2.81).

(vii) $k=6$. We have

$$\begin{aligned} q_6(u) &= (\alpha^8-\alpha^6)u^6+(2\alpha^8-5\alpha^6+5\alpha^4)u^5+(20\alpha^6-16\alpha^4-8\alpha^2)u^4 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^3+(16\alpha^4-24\alpha^2-16)u^2 \\ &\quad -(8\alpha^4+124\alpha^2+8)u+272\alpha^2+96, \quad u\in[0,u_\alpha]. \end{aligned}$$

We will show that

$$q_6(u_\alpha)>0, \quad \alpha\in(0,1), \tag{2.83}$$

i.e., after computing that

$$\begin{aligned} &\alpha(-2\alpha^7+6\alpha^6+13\alpha^5+\alpha^4+25\alpha^3+44\alpha^2+15\alpha+2)\sqrt{2} \\ &<\left(2\alpha^7+18\alpha^6+8\alpha^5+60\alpha^4+124\alpha^3+72\alpha^2\right. \\ &\quad \left.+40\alpha+16\right)\sqrt{\alpha+1}, \quad \alpha\in(0,1). \end{aligned} \tag{2.84}$$

Both sides of (2.84) are positive evidently. Thus squaring (2.84) and computing we get equivalently the inequality

$$\begin{aligned} &4\alpha^{16}-26\alpha^{15}-54\alpha^{14}-62\alpha^{13}-361\alpha^{12}-1474\alpha^{11} \\ &-3097\alpha^{10}-5658\alpha^9-11809\alpha^8-19102\alpha^7-21382\alpha^6 \\ &-18548\alpha^5-12975\alpha^4-6756\alpha^3-2588\alpha^2-768\alpha-128<0, \end{aligned}$$

which is clearly true for $\alpha\in(0,1)$. Thus (2.83) is confirmed.

(viii) $k=7$. We have

$$\begin{aligned} q_7(u) &= (\alpha^8-\alpha^6)u^7+(2\alpha^8-5\alpha^6+5\alpha^4)u^6+(20\alpha^6-16\alpha^4-8\alpha^2)u^5 \\ &\quad +(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^4+(16\alpha^4-24\alpha^2-16)u^3 \\ &\quad -(8\alpha^4+124\alpha^2+8)u^2+(272\alpha^2+96)u-176\alpha^2-60, \quad u\in[0,u_\alpha]. \end{aligned}$$

We will show that

$$q_7(u_\alpha)>0, \quad \alpha\in(0,1), \tag{2.85}$$

i.e., after computing that

$$\begin{aligned}&\left(-4\alpha^8+12\alpha^7+26\alpha^6+46\alpha^5+182\alpha^4+235\alpha^3\right.\\&\left.+119\alpha^2+49\alpha+15\right)\sqrt{\alpha+1}\\&<\left(2\alpha^8+20\alpha^7+26\alpha^6+68\alpha^5+184\alpha^4+196\alpha^3\right.\\&\left.+112\alpha^2+56\alpha+16\right)\sqrt{2},\quad \alpha\in(0,1).\end{aligned}\tag{2.86}$$

Both sides of (2.86) are positive evidently. Thus squaring (2.86) and computing we equivalently get

$$\begin{aligned}&(\alpha^2-1)(-16\alpha^{15}+88\alpha^{14}+304\alpha^{13}+904\alpha^{12}+2348\alpha^{11}+3964\alpha^{10}\\&+4560\alpha^9+1228\alpha^8-9016\alpha^7-22876\alpha^6-30417\alpha^5-25691\alpha^4\\&-14838\alpha^3-6286\alpha^2-1889\alpha-287)>0,\quad \alpha\in(0,1).\end{aligned}\tag{2.87}$$

Since for $\alpha\in(0,1)$ we have

$$\begin{aligned}&-16\alpha^{15}+88\alpha^{14}+304\alpha^{13}+904\alpha^{12}+2348\alpha^{11}+3964\alpha^{10}\\&+4560\alpha^9+1228\alpha^8-9016\alpha^7-22876\alpha^6-30417\alpha^5\\&-25691\alpha^4-14838\alpha^3-6286\alpha^2-1889\alpha-287\\&\le-16\alpha^{15}+88\alpha^7+304\alpha^6+904\alpha^5+2348\alpha^4+3964\alpha^3\\&+4560\alpha^2+1228\alpha-9016\alpha^7-22876\alpha^6-30417\alpha^5\\&-25691\alpha^4-14838\alpha^3-6286\alpha^2-1889\alpha-287\\&=-16\alpha^{15}-8928\alpha^7-22572\alpha^6-29513\alpha^5-23343\alpha^4\\&-10874\alpha^3-1726\alpha^2-661\alpha-287<0,\end{aligned}$$

so (2.87) holds. Thus (2.85) is confirmed.

(ix) $k=8$. We have

$$\begin{aligned}q_8(u)\;=\;&(\alpha^8-\alpha^6)u^8+(2\alpha^8-5\alpha^6+5\alpha^4)u^7+(20\alpha^6-16\alpha^4-8\alpha^2)u^6\\&+(-12\alpha^6+6\alpha^4+36\alpha^2+4)u^5+(16\alpha^4-24\alpha^2-16)u^4\\&-(8\alpha^4+124\alpha^2+8)u^3+(272\alpha^2+96)u^2\\&-(176\alpha^2+60)u-144,\quad u\in[0,u_\alpha].\end{aligned}$$

We will show that

$$q_8(u_\alpha)<0,\quad \alpha\in(0,1),\tag{2.88}$$

i.e., after computing that

$$r(\alpha)<s(\alpha),\quad \alpha\in(0,1),\tag{2.89}$$

where

$$\begin{aligned}r(\alpha)\;:=\;&\left(4\alpha^7+36\alpha^6+16\alpha^5+120\alpha^4+212\alpha^3+36\alpha^2\right.\\&\left.-28\alpha-4\right)\sqrt{\alpha+1},\quad \alpha\in(0,1),\end{aligned}$$

and

$$\begin{aligned}s(\alpha)\;:=\;&\left(-4\alpha^8+12\alpha^7+26\alpha^6+46\alpha^5+182\alpha^4+235\alpha^3+119\alpha^2\right.\\&\left.+49\alpha+15\right)\sqrt{2},\quad \alpha\in(0,1).\end{aligned}$$

We see at once that

$$s(\alpha)>0,\quad \alpha\in(0,1).\tag{2.90}$$

It is easily seen that for $\alpha \in (0,1)$,

$$\begin{aligned} s^2(\alpha) - r^2(\alpha) &= (s(\alpha) - r(\alpha))\,(s(\alpha) + r(\alpha)) \\ &= 16\alpha^{16} - 104\alpha^{15} - 216\alpha^{14} - 600\alpha^{13} - 1444\alpha^{12} - 1472\alpha^{11} \\ &\quad +1276\alpha^{10} + 9956\alpha^9 + 25220\alpha^8 + 48204\alpha^7 + 73421\alpha^6 \\ &\quad +76706\alpha^5 + 50275\alpha^4 + 20320\alpha^3 + 5611\alpha^2 + 1350\alpha + 217 > 0. \end{aligned}$$

Hence either

$$s(\alpha) - r(\alpha) > 0, \quad s(\alpha) + r(\alpha) > 0, \quad \alpha \in (0,1), \tag{2.91}$$

or

$$s(\alpha) - r(\alpha) < 0, \quad s(\alpha) + r(\alpha) < 0, \quad \alpha \in (0,1). \tag{2.92}$$

Supposing that (2.92) holds, we see that then $s(\alpha) < 0$ for $\alpha \in (0,1)$. However this contradicts (2.90). Thus (2.91) holds so (2.88) is confirmed.

(x) $k = 9$. We have

$$q_9(u) = V_\alpha(u), \quad u \in [0, u_\alpha].$$

We will show that

$$q_9(u_\alpha) > 0, \quad \alpha \in (0,1), \tag{2.93}$$

i.e., after computing that

$$\begin{aligned} &\big(-4\alpha^8 + 12\alpha^7 + 26\alpha^6 + 46\alpha^5 + 164\alpha^4 + 163\alpha^3 + 11\alpha^2 \\ &-23\alpha - 3)\sqrt{\alpha+1} \\ &< \big(2\alpha^8 + 20\alpha^7 + 26\alpha^6 + 68\alpha^5 + 166\alpha^4 + 124\alpha^3 + 4\alpha^2 \\ &-16\alpha - 2)\sqrt{2}, \quad \alpha \in (0,1). \end{aligned} \tag{2.94}$$

To verify that (2.94) holds, we will show that

$$r(t) < s(t), \quad t \in (0,1), \tag{2.95}$$

where

$$\begin{aligned} r(t) &:= \big(-4t^8 + 12t^7 + 26t^6 + 46t^5 + 164t^4 + 163t^3 + 11t^2 \\ &\quad -23t - 3)\sqrt{t+1}, \quad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} s(t) &:= \big(2t^8 + 20t^7 + 26t^6 + 68t^5 + 166t^4 + 124t^3 + 4t^2 \\ &\quad -16t - 2)\sqrt{2}, \quad t \in [0,1]. \end{aligned}$$

Let

$$w(t) := s^2(t) - r^2(t), \quad t \in [0,1].$$

Thus after computing we have

$$w(t) = (t-1)^2(t+1)\,(2t+1)\,v(t), \tag{2.96}$$

where

$$\begin{aligned} v(t) &:= -8t^{13} + 40t^{12} + 168t^{11} + 556t^{10} + 1464t^9 + \\ &\quad +2776t^8 + 4148t^7 + 4220t^6 + 2358t^5 + +455t^4 \\ &\quad -176t^3 - 106t^2 - 18t - 1 =: \sum_{j=0}^{13} b_j t^{k-j}, \quad t \in [0,1]. \end{aligned} \tag{2.97}$$

We use now Corollary 2.3 since $v(0) = -1 \neq 0$ and $v(1) = 15876 \neq 0$. Note that the numbers of sign changes in the sequence of polynomial coefficients (b_k) and in the sequence of sums $\left(\sum_{j=0}^{k} b_j\right)$, where $k = 0, 1, \ldots, 13$, equal 2 and 1, respectively, i.e., $N(v;0) = 2$ and $N(v;1) = 1$. Applying Corollary 2.3, we see that the polynomial v has the unique zero $t =: t_0 \in (0,1)$. Moreover t_0 is the zero of order 1. Hence and from (2.96) it follows that t_0 is the unique zero of order 1 of w in $(0,1)$, also. Since

$$w(0) = v(0) = -1 < 0,$$

so from (2.96) and by the continuity of the function w we have

$$w(t) = (s(t) - r(t))(s(t) + r(t)) < 0, \quad t \in (0, t_0) \tag{2.98}$$

and

$$w(t) = (s(t) - r(t))(s(t) + r(t)) > 0, \quad t \in (t_0, 1). \tag{2.99}$$

Since

$$s(0) + r(0) = -2\sqrt{2} - 3 < 0,$$

from (2.98) it follows that

$$s(t) - r(t) > 0, \quad t \in (0, t_0), \tag{2.100}$$

and

$$s(t) + r(t) < 0, \quad t \in (0, t_0). \tag{2.101}$$

Similarly, since

$$s(1) + r(1) = 784\sqrt{2} > 0,$$

from (2.99) it follows that

$$s(t) - r(t) > 0, \quad t \in (t_0, 1), \tag{2.102}$$

and

$$s(t) + r(t) > 0, \quad t \in (t_0, 1). \tag{2.103}$$

Thus from (2.100) and (2.102) we have

$$s(t) - r(t) > 0, \quad t \in (0,1) \setminus \{t_0\}. \tag{2.104}$$

Moreover, from (2.101) and (2.103) is follows that

$$s(t_0) + r(t_0) = 0. \tag{2.105}$$

The continuity of the function $s - r$ and (2.104) yield

$$s(t_0) - r(t_0) \geq 0. \tag{2.106}$$

Suppose now that

$$s(t_0) - r(t_0) = 0. \tag{2.107}$$

Hence and from (2.105) we have

$$s(t_0) = r(t_0) = 0.$$

Thus

$$\frac{r(t)}{\sqrt{t+1}} = (t - t_0)\varrho(t), \quad t \in [0,1],$$

and

$$\frac{s(t)}{\sqrt{2}} = (t - t_0)\sigma(t), \quad t \in [0,1],$$

where ϱ and σ are some polynomials in $[0,1]$. Hence

$$w(t) = s^2(t) - r^2(t) = (t-t_0)^2\left(2\sigma^2(t) - (t+1)\varrho^2(t)\right), \quad t \in [0,1],$$

which yields a contradiction since, as was shown, t_0 is the unique zero of order 1 of w in $(0,1)$. Thus the strong inequality in (2.106) holds, which together with (2.104) finishes the proof of (2.95). In this way (2.93) is confirmed.

Summarizing, from (2.60), (2.61), (2.65), (2.71), (2.75), (2.79), (2.80), (2.81), (2.83), (2.85), (2.88) and (2.93) it follows that for three cases, namely, for $\alpha \in (0,\alpha_0)$, $\alpha := \alpha_0$ and $\alpha \in (\alpha_0,1)$, where α_0 is the unique root of the equation (2.75), we have

$$N(V_\alpha; u_\alpha) = 7.$$

Hence, by (2.59) and by Corollary 2.3 we conclude that for each $\alpha \in (0,1)$ the polynomial V_α has no zero in $(0,u_\alpha)$, and since $V_\alpha(0) = 144 > 0$, so (2.57) holds. Thus (2.55) is confirmed, which finishes the proof of the inequality (2.52).

Summarizing, taking into account (2.49)-(2.52), we proved that

$$\max_{(x,y)\in R} F_{\alpha,\gamma}(x,y) = \begin{cases} 2(1+2\alpha)|\gamma|, & |\gamma| \geq \dfrac{2}{1+\alpha}, \\ \dfrac{2\alpha^2\gamma^2}{2-|\gamma|} + 4, & |\gamma| \leq \dfrac{2}{1+\alpha}. \end{cases}$$

Finally, substituting $\gamma = 2-3\lambda$ and $\alpha = |b_2|/2$, the above and (2.13) yield (2.8). □

***Remark* 2.5.** Since the condition (2.7), i.e., the inequality $|b_2| \leq 2$ holds for $g \in \mathcal{S}$, Theorem 2.4 is true for the class $\mathcal{C}(g)$, where g is in $\mathcal{S}$.

Now we recall the result for the class $\mathcal{C}(g_\alpha)$ proved in [15].

Theorem 2.6. *Let $\alpha \in [0,1]$. Then*

$$\begin{aligned} &\max_{f\in\mathcal{C}(g_\alpha)} \Phi_\lambda(f) \\ &\leq \begin{cases} \left|\dfrac{2}{3} + \dfrac{4}{3}\alpha + \alpha^2 - (1+\alpha)^2\lambda\right|, & \lambda \in \mathbb{R}\setminus(\tau'(\alpha),\tau''(\alpha)), \\ \alpha^2\left(|1-\lambda| + \dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|}\right) + \dfrac{2}{3}, & \lambda \in [\tau'(\alpha),\tau''(\alpha)], \end{cases} \end{aligned} \tag{2.108}$$

where

$$\tau'(\alpha) := \frac{2\alpha}{3(1+\alpha)}, \quad \tau''(\alpha) := \frac{2(2+\alpha)}{3(1+\alpha)}.$$

Proof. Let $\alpha \in [0,1]$. Since

$$g_\alpha(z) = \frac{z}{(1-\alpha z)^2} = \sum_{n=1}^{\infty} n\alpha^{n-1}z^n, \quad z \in \mathbb{D},$$

so

$$b_2 = 2\alpha, \quad b_3 = 3\alpha^2. \tag{2.109}$$

Then in view of (2.9) we have

$$\tau_1(|b_2|) = \frac{2|b_2|}{3(|b_2|+2)} = \frac{2\alpha}{3(1+\alpha)} =: \tau'(\alpha)$$

and

$$\tau_2(|b_2|) = \frac{2(|b_2|+4)}{3(|b_2|+2)} = \frac{2(2+\alpha)}{3(1+\alpha)} =: \tau''(\alpha).$$

Now for $\lambda \in \mathbb{R} \setminus [\tau'(\alpha), \tau''(\alpha)]$ by using (2.109) the inequality (2.8) is of the form

$$\begin{aligned} \max_{f \in \mathcal{C}(g_\alpha)} \Phi_\lambda(f) &\leq \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + (1+|b_2|)\left|\frac{2}{3} - \lambda\right| \\ &= \alpha^2|1-\lambda| + (1+2\alpha)\left|\frac{2}{3} - \lambda\right| \\ &= \left|\frac{2}{3} + \frac{4}{3}\alpha + \alpha^2 - (1+\alpha)^2\lambda\right| \end{aligned}$$

and for $\lambda \in [\tau'(\alpha), \tau''(\alpha)]$ of the form

$$\begin{aligned} \max_{f \in \mathcal{C}(g_\alpha)} \Phi_\lambda(f) &\leq \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{(2-3\lambda)^2|b_2|^2}{12(2-|2-3\lambda|)} + \frac{2}{3} \\ &= \alpha^2\left(|1-\lambda| + \frac{1}{3}\cdot\frac{(2-3\lambda)^2}{2-|2-3\lambda|}\right) + \frac{2}{3}. \end{aligned}$$

□

For $\alpha := 1$ we get the following result proved in [16].

Corollary 2.7.

$$\begin{aligned} &\max_{f \in \mathcal{C}(k)} \Phi_\lambda(f) \\ &\leq \begin{cases} |3-4\lambda|, & \lambda \in (-\infty, 1/3] \cup [1, +\infty), \\ \dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|} + |1-\lambda| + \dfrac{2}{3}, & \lambda \in [1/3, 1]. \end{cases} \end{aligned} \tag{2.110}$$

Let now formulate the result for the class $\mathcal{C}(h_\alpha)$.

Theorem 2.8. *Let $\alpha \in [0,1]$. Then*

$$\begin{aligned} &\max_{f \in \mathcal{C}(h_\alpha)} \Phi_\lambda(f) \\ &\leq \begin{cases} \alpha^2\left|\dfrac{1}{3} - \dfrac{1}{4}\lambda\right| + (1+\alpha)\left|\dfrac{2}{3} - \lambda\right|, & \lambda \in \mathbb{R} \setminus [\tau'(\alpha), \tau''(\alpha)], \\ \alpha^2\left(\dfrac{1}{12}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|} + \left|\dfrac{1}{3} - \dfrac{1}{4}\lambda\right|\right) + \dfrac{2}{3}, & \lambda \in [\tau'(\alpha), \tau''(\alpha)], \end{cases} \end{aligned} \tag{2.111}$$

where

$$\tau'(\alpha) := \frac{2\alpha}{3(2+\alpha)}, \quad \tau''(\alpha) := \frac{2(4+\alpha)}{3(2+\alpha)}.$$

Proof. Let $\alpha \in [0,1]$. Since

$$h_\alpha(z) = \frac{z}{1-\alpha z} = \sum_{n=1}^{\infty} \alpha^{n-1} z^n, \quad z \in \mathbb{D},$$

so

$$b_2 = \alpha, \quad b_3 = \alpha^2. \tag{2.112}$$

Then in view of (2.9) we have

$$\tau_1(|b_2|) = \frac{2|b_2|}{3(|b_2|+2)} = \frac{2\alpha}{3(2+\alpha)} =: \tau'(\alpha)$$

and

$$\tau_2(|b_2|) = \frac{2(|b_2|+4)}{3(|b_2|+2)} = \frac{2(4+\alpha)}{3(2+\alpha)} =: \tau''(\alpha).$$

Now for $\lambda \in \mathbb{R} \setminus [\tau'(\alpha), \tau''(\alpha)]$ by using (2.112) the inequality (2.8) is of the form

$$\begin{aligned} \max_{f\in\mathcal{C}(h_\alpha)} \Phi_\lambda(f) &\leq \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + (1+|b_2|)\left|\frac{2}{3}-\lambda\right| \\ &= \alpha^2\left|\frac{1}{3}-\frac{1}{4}\lambda\right| + (1+\alpha)\left|\frac{2}{3}-\lambda\right| \end{aligned}$$

and for $\lambda \in [\tau'(\alpha), \tau''(\alpha)]$ of the form

$$\begin{aligned} \max_{f\in\mathcal{C}(h_\alpha)} \Phi_\lambda(f) &\leq \left|\frac{1}{3}b_3 - \frac{1}{4}\lambda b_2^2\right| + \frac{(2-3\lambda)^2|b_2|^2}{12(2-|2-3\lambda|)} + \frac{2}{3} \\ &= \alpha^2\left(\left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{(2-3\lambda)^2}{12(2-|2-3\lambda|)}\right) + \frac{2}{3}. \end{aligned}$$

Thus (2.111) was proved. □

For $\alpha := 1$ we get the following result proved in [17].

Theorem 2.9.

$$\begin{aligned} &\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \\ &\leq \begin{cases} \left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{2}{3}|2-3\lambda|, & \lambda \in (-\infty, 2/9] \cup [10/9, +\infty), \\ \frac{1}{12}\cdot\frac{(2-3\lambda)^2}{2-|2-3\lambda|} + \left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{2}{3}, & \lambda \in [2/9, 10/9]. \end{cases} \end{aligned} \tag{2.113}$$

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Department of Complex Analysis, University of Warmia and Mazury in Olsztyn, Słoneczna 54, 10-710, Olsztyn, Poland, E-mail: *alecko@matman.uwm.edu.pl*

Department of Complex Analysis, University of Warmia and Mazury in Olsztyn, Słoneczna 54, 10-710, Olsztyn, Poland, E-mail: *b.kowalczyk@matman.uwm.edu.pl*

Department of Mathematics, Kyungsung University, Busan 608-736, Korea, E-mail: *oskwon@ks.ac.kr*

Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea, E-mail: *necho@pknu.ac.kr*

On Mild Solution of Abstract Neutral Fractional Order Impulsive Differential Equations with Infinite Delay

A. Anguraj[1], S. Kanjanadevi[1] and D. Baleanu[2,3*]

[1]PSG College of Arts & Science, Coimbatore- 641 014, Tamil Nadu, India

[2]Department of Mathematics,Cankaya University, Ankara, Balgat 06530, Turkey

[3]Institute of Space Sciences, Magurele-Bucharest, Romania

Abstract

We prove the existence and uniqueness of fractional neutral impulsive differential equations with infinite delay via contraction mapping principle and fixed point technique for condensing map. We use the resolvent operator technique for integral equations to make the mild solution of the problem more appropriate.

Keywords: Fractional differential equations, Fractional order impulsive conditions, Neutral differential equations, Infinite delay, Resolvent operators.

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1 Introduction

In recent years, a significant number of the investigates managed the possibility of the fractional differential equations in different areas of engineering and science disciplines, for example, rheology, viscoelasticity, biomedical, control theory, porous media. Fractional differential equations give an incredible mathematical model for real world phenomena, in which the fractional rate of progress relies on upon the impact of the hereditary effects and describing the long memory of the systems. For detailed investication of the fractional differential equations, we read [4, 13, 19, 23].

The hypothesis of partial neutral integro-differential equations with infinite delay have been utilized for displaying the advancement of physical systems, in which the reaction

*Corresponding author. E-mail:
angurajpsg@yahoo.com(A. Anguraj), kanjanadevimaths@gmail.com(S. Kanjanadevi) and dumitru@cankaya.edu.tr(D. Baleanu).

depends on the present and previous history of the system. This sort of equations emerge in the theory of heat conduction in material with fading memory [18]. Since we consider the infinite delay, we use the notion is phase space which acts as an essential part in the study of qualitative theory of delay equations. This idea was presented by Hale and Kato in [7].

Study of impulsive differential equations turn into an essential field of research because of their various applications. The purpose behind this applicability emerges from the way that, numerous real world processes and phenomena which are subjected amid their improvement to short-term external impacts can be demostrated as impulsive differential equations with non-integer order and which cannot be depicted by using classical differential equations [15]. For more subtle elements of fractional impulsive differential equations, see [5, 6, 16, 21].

Similar results for integer order derivative for abstract neutral functional differential equations with impulsive condition was studied by [2, 8, 12]. The work on fractional neutral impulsive differential equations with infinite delay are carried out by [3, 22]. In [14] N. Kosmatov studied the fractional order initial value problems with fractional impulses by the contribution of Caputo and Riemann-Liouville derivatives.

Hernández et al. [9], examined that the concepts of mild solutions utilized as a part of a few late writing on abstract fractional differential equations are not suitable. In [9], he consider the more appropriate mild solution of the abstract fractional differential equations with time by means of resolvent operator for integral equations [20]. The same idea was used by some authors to show the existence of fractional differential equations without impulse, see [1, 10]. But in our best of knowledge this resolvent operator concept was not used in the fractional impulsive differential equations of order lies in $(1, 2)$. Note that the order of integration determines the shape of the memory function.

Impulsive fractional differential equations is constructed with either the lower bound as the corresponding impulses or the lower bound as zero at each impulses. Here we construct the solution of fractional order impulsive Cauchy problem involving Caputo derivative with lower bound as zero. That is the different solutions keeping in each impulses the lower bound as zero. This will improve the characterization of the memory property of the factional derivative.

Motivations of the study in [9, 14] and the applications of fractional order derivative give rise in this present article. Here we prove the existence and uniqueness theorems of mild solutions for fractional neutral impulsive differential equations with infinite delay given by

$$^{c}D^{\alpha}_{0^+}(u(t) + q(t, u_t)) = \mathscr{A}u(t) + p(t, u_t),\ t \neq t_{\mathfrak{k}},\ t \in \mathcal{J} := [0, a], \tag{1.1}$$

$$^{c}D^{\beta}_{0^+}u(t^+_{\mathfrak{k}}) - {}^{c}D^{\beta}_{0^+}u(t^-_{\mathfrak{k}}) = I_{\mathfrak{k}}(u_{t_{\mathfrak{k}}}),\ \mathfrak{k} = 1, \cdots, m, \tag{1.2}$$

$$u(0) = \phi \in \mathscr{B}_A,\ \ u'(0) = z \in \mathscr{E}, \tag{1.3}$$

where $0 < \beta < 1$ and $\alpha \in (1, 2)$. Here $\mathscr{A}$ is the infinitesimal generator of a cosine operator family $\{\mathfrak{C}(t)\}_{t\geq 0}$ on a Banach space $\mathscr{E}$. The memory function $u_t : (-\infty, 0] \to \mathscr{E},\ u_t(\sigma) = u(t + \sigma),\ \sigma \leq 0$, associated with some suitable abstract phase space $\mathscr{B}_A$, $0 = t_0 < t_1 < \cdots < t_{m+1} = a$ are pre-fixed values and the appropriate functions $p, q : \mathcal{J} \times \mathscr{B}_A \to \mathscr{E},\ I_{\mathfrak{k}} : \mathscr{B}_A \to \mathscr{E}, \mathfrak{k} = 1, \cdots, m$, which are defined later.

We derive the mild solution of (1.1)-(1.3) by resolvent operator technique. The existence results of fractional neutral impulsive differential equations with infinite delay via fixed point technique for condensing map and the uniqueness of the problem is verified by using contraction mapping principle.

2 Preliminaries

Let the space $\mathcal{L}(\mathscr{E},\mathscr{E}')$ is the set of all bounded linear operators from Banach space $\mathscr{E}$ into Banach space $\mathscr{E}'$ provided with the norm $\|\cdot\|_{\mathcal{L}(\mathscr{E},\mathscr{E}')}$. Here the domain $\mathscr{D}(\mathscr{A})$, takes the norm $\|u\|_{\mathscr{D}(\mathscr{A})} = \|u\| + \|\mathscr{A}u\|$. Further more, $\mathcal{B}_r(u,\mathscr{E})$ symbolizes the closed ball having center at u and distance r in $\mathscr{E}$.

The class of all continuous functions from $\mathcal{J}$ into $\mathscr{E}$ is referred by $C(\mathcal{J};\mathscr{E})$ with the sup-norm $\|\cdot\|_{C(\mathcal{J};\mathscr{E})}$. Likewise $C^\gamma(\mathcal{J};\mathscr{E}), 0 < \gamma < 1$ is the set of all γ–Hölder $\mathscr{E}$-valued continuous functions from $\mathcal{J}$ into $\mathscr{E}$ provided with $\|u\|_{C^\gamma(\mathcal{J};\mathscr{E})} = \|u\|_{C(\mathcal{J};\mathscr{E})} + [|u|]_{C^\gamma(\mathcal{J};\mathscr{E})}$, where $[|u|]_{C^\gamma(\mathcal{J};\mathscr{E})} = \sup_{t\neq s,t,s\in\mathcal{J}} \frac{\|u(t)-u(s)\|_{\mathscr{E}}}{(t-s)^\gamma}$.

Now, we present the piece-wise continuous space $PC(\mathscr{E})$ which is framed by set of all the functions $u : \mathcal{J} \to \mathscr{E}$ such that the function $u(\cdot)$ is continuous at $t \neq t_\mathfrak{t}$, $u(t_\mathfrak{t}^+)$ and $u(t_\mathfrak{t}^-) = u(t_\mathfrak{t})$ exists for every $\mathfrak{t} = 1, 2, \cdots, m$. We can easily seen that it is a Banach space concerning the norm $\|u\|_{PC(\mathscr{E})} = \sup_{t\in\mathcal{J}} \|u(t)\|_{\mathscr{E}}$.

We consider the phase space $(\mathscr{B}_A, \|\cdot\|_{\mathscr{B}_A})$, is a linear space of function u_t mapping from $(-\infty, 0]$ into $\mathscr{E}$ with respect to the seminorm $\|\cdot\|_{\mathscr{B}_A}$, which is previously addressed in Hino et al., [11] to examine the infinite delay problem. We assume the space $\mathscr{B}_A$ meets the axioms given below:

(1) If $u : (-\infty, \nu+a] \to \mathscr{E}$, $\nu \in \mathbb{R}, a > 0$ such that $u_\nu \in \mathscr{B}_A$, and $u|_{[\nu,\nu+a]} \in PC([\nu,\nu+a];\mathscr{E})$, then the subsequent conditions hold for all $t \in [\nu, \nu + a)$

(i) $u_t \in \mathscr{B}_A$.

(ii) $\|u(t)\|_{\mathscr{E}} \leq \mathfrak{H}\|u\|_{\mathscr{B}_A}$

(iii) $\|u_t\|_{\mathscr{B}_A} \leq \mathfrak{K}(t-\nu)\sup\{\|u(s)\|_{\mathscr{E}} : \nu \leq s \leq t\} + \mathfrak{M}(t-\nu)\|u_\nu\|_{\mathscr{B}_A}$, where $\mathfrak{M}, \mathfrak{K} : [0,\infty) \to [1,\infty)$, is locally bounded and continuous respectively; $\mathfrak{H} > 0$ is a constant. $\mathfrak{K}, \mathfrak{H}, \mathfrak{M}$ are independent of $u(\cdot)$.

(2) The phase space $\mathscr{B}_A$ is complete.

We know that the Caputo fractional derivative of a function u of order $\alpha > 0$ defined as follows:

$$^cD^\alpha_{0^+}u(t) = I^{n-\alpha}_{0^+}D^n u(t), \quad n = \lceil\alpha\rceil,$$

where $I^\alpha_{0^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$. Also, in general the Caputo derivative $^cD^\alpha_{0^+}$ is a left inverse of $I^\alpha_{0^+}$ but not a right inverse, i.e., we have $^cD^\alpha_{0^+}I^\alpha_{0^+}u(t) = u(t)$, and $I^\alpha_{0^+}\,^cD^\alpha_{0^+}u(t) = u(t) - u(0) - tu'(0)$, for $0 < \alpha < 2$.

Next, we consider that the Volterra integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{A} u(s) ds + p(t),\ t \in \mathcal{J}, \tag{2.1}$$

has a corresponding resolvent operator $\{S(t)\}_{t\geq 0}$ on $\mathscr{E}$, see [9] and p in $C(\mathcal{J};\mathscr{E})$. More detailed explanations about resolvent operator for integral equations one can refer [20]. The definition of mild solution for the integral equation (2.1) by utilizing the concept presented in [20] is given in [9].

Definition 2.1. *[9, Definition 1.2] A function u in the space $C(\mathcal{J};\mathscr{E})$ is called a mild solution of* (2.1) *on $\mathcal{J}$, if $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$ in $C(\mathcal{J};\mathscr{D}(\mathscr{A}))$ and*

$$u(t) = \frac{\mathscr{A}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + p(t),\ t \in \mathcal{J}.$$

Definition 2.2. *[20, Definition 1.4] A resolvent operator $S(t)$ for equation (2.1) is said to be differentiable, if $S(\cdot)u \in \mathcal{W}^{1,1}([0,\infty);\mathscr{E})$ for every $u \in \mathscr{D}(\mathscr{A})$ and there is $\varphi \in L^1_{loc}([0,\infty))$ with $\|S'(t)u\| \leq \varphi(t)\|u\|_{\mathscr{D}(\mathscr{A})}$, a.e. on $[0,\infty)$, for every $u \in \mathscr{D}(\mathscr{A})$.*

Lemma 2.1. *[9, Lemma 1.1] Suppose (2.1) admits a differentiable resolvent $S(t)$ and if $p \in C(\mathcal{J};\mathscr{D}(\mathscr{A}))$, then*

$$u(t) = p(t) + \int_0^t S'(t-s) p(s) ds,\ t \in \mathcal{J},$$

is said to be a mild solution of (2.1).

Now, our point is to present the concept of mild solution for equation (1.1) to (1.3). In this way, we first identify that if $u(\cdot)$ is a solution of (1.1)-(1.3), then one can estimate the corresponding integral equation given by

$$\begin{aligned} u(t) &= \phi(0) + q(0,\phi) + (z+\xi)t - q(t,u_t) + \Gamma(2-\beta) \sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t-t_\mathfrak{k}) I_\mathfrak{k}(u_{t_\mathfrak{k}}) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{A} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s,u_s) ds,\ t \in \mathcal{J}, \end{aligned} \tag{2.2}$$

where $\frac{d}{dt} q(t,u_t)|_{t=0} = \xi$, ξ is independent of u.

Motivated by Definition 2.1 and the representation (2.2), we introduce the following definition.

Definition 2.3. *A function $u : (-\infty, a] \to \mathscr{E}$ is a mild solution of (1.1)-(1.3), if $u(0) = \phi$, $u'(0) = z$, $u|_{\mathcal{J}} \in PC(\mathscr{E})$, $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \in \mathscr{D}(\mathscr{A})$, $\forall\ t \in \mathcal{J}$, and*

$$\begin{aligned} u(t) &= \phi(0) + q(0,\phi) + (z+\xi)t - q(t,u_t) + \Gamma(2-\beta) \sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t-t_\mathfrak{k}) I_\mathfrak{k}(u_{t_\mathfrak{k}}) \\ &+ \frac{\mathscr{A}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s,u_s) ds,\ t \in \mathcal{J}, \end{aligned}$$

where $\frac{d}{dt} q(t,u_t)|_{t=0} = \xi$, ξ is independent of u.

3 Existence and Uniqueness Results

Now, we will make the subsequent hypotheses:

(H1) $p : \mathcal{J} \times \mathscr{B}_A \to \mathscr{D}(\mathscr{A})$ is continuous function and let $L_p \in C(\mathcal{J}; \mathbb{R})$ such that $\|p(t, \varpi_1) - p(t, \varpi_2)\|_{\mathscr{D}(\mathscr{A})} \leq L_p(t)\|\varpi_1 - \varpi_2\|_{\mathscr{B}_A},\ t \in \mathcal{J}, \varpi_1, \varpi_2 \in \mathscr{B}_A$.

(H2) The function m_p belongs to $C(\mathcal{J}; \mathbb{R})$ and a non-decreasing function $W : [0, +\infty) \to (0, +\infty)$ such that $\|p(t, \varpi)\|_{\mathscr{D}(\mathscr{A})} \leq m_p(t)W(\|\varpi\|_{\mathscr{B}_A})$, $t \in \mathcal{J}, \varpi \in \mathscr{B}_A$.

(H3) $q : \mathcal{J} \times \mathscr{B}_A \to \mathscr{D}(\mathscr{A})$ is continuous function and $L_q \in C(\mathcal{J}; \mathbb{R})$ with $\|q(t, \varpi_1) - q(t, \varpi_2)\|_{\mathscr{D}(\mathscr{A})} \leq L_q(t)\|\varpi_1 - \varpi_2\|_{\mathscr{B}_A},\ t \in \mathcal{J}, \varpi_1, \varpi_2 \in \mathscr{B}_A$.

(H4) $C_1 > 0$, and $C_2 > 0$ such that $\|q(t, \varpi)\|_{\mathscr{B}_A} \leq C_1\|\varpi\|_{\mathscr{B}_A} + C_2,\ t \in \mathcal{J}, \varpi \in \mathscr{B}_A$.

(H5) $I_\mathfrak{k} : \mathscr{B}_A \to \mathscr{D}(\mathscr{A})$ are continuous functions and let positive constants $L_\mathfrak{k}$ such that $\|I_\mathfrak{k}(\varpi_1) - I_\mathfrak{k}(\varpi_2)\|_{\mathscr{D}(\mathscr{A})} \leq L_\mathfrak{k}\|\varpi_1 - \varpi_2\|_{\mathscr{B}_A},\ \mathfrak{k} = 1, 2, \cdots, m, \varpi_1, \varpi_2 \in \mathscr{B}_A$.

(H6) Let $d^1_\mathfrak{k}, > 0$ and $d^1_\mathfrak{k} > 0$ such that $\|I_\mathfrak{k}(\varpi)\| \leq d^1_\mathfrak{k}\|\varpi\| + d^2_\mathfrak{k}$ for all $\mathfrak{k} = 1, 2, \cdots, m,\ \varpi \in \mathscr{B}_A$.

From Lemma 2.1 we note the subsequent Proposition,

Proposition 3.1. *Suppose equation (2.2) admits a differential resolvent operator* $\{S(t)\}_{t\geq 0}$ *and if* $p, q \in C(\mathcal{J} \times \mathscr{B}_A; \mathscr{D}(\mathscr{A})), I_\mathfrak{k} \in C(\mathscr{B}_A; \mathscr{D}(\mathscr{A}))$, *then*

$$u(t) = \phi(0) + q(0, \phi) + (z + \xi)t - q(t, u_t) + \Gamma(2 - \beta) \sum_{0<t_\mathfrak{k}<t} t^{\beta-1}_\mathfrak{k}(t - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}})$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} p(s, u_s)ds + \int_0^t S'(t - s)\left(\phi(0) + q(0, \phi) + (z + \xi)s - q(s, u_s)\right.$$
$$\left. +\Gamma(2 - \beta) \sum_{0<t_\mathfrak{k}<s} t^{\beta-1}_\mathfrak{k}(s - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}}) + \frac{1}{\Gamma(\alpha)} \int_0^s (s - \tau)^{\alpha-1} p(\tau, u_\tau)d\tau\right) ds,\ t \in \mathcal{J},$$

is called a mild solution of the problem (1.1)-(1.3).

Let a function $x : (-\infty, a] \to \mathscr{E}$ be defined by $x_0 = \phi$ and $x(t) = \phi(0) + \int_0^t S'(t - s)\phi(0)ds$ for all $t \in \mathcal{J}$. It is easily say that $\|x_t\| \leq (\Re_a \mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}$, where $\mathfrak{M}_a = \sup_{t\in\mathcal{J}} \mathfrak{M}(t),\ \Re_a = \sup_{t\in\mathcal{J}} \Re(t)$.

Theorem 3.1. *Assume that* $(H1), (H3)$ *and* $(H5)$ *are satisfied, and if*

$$\Re_a(\|L_q(t)\| + \Gamma(2 - \beta)a \sum_{0<t_\mathfrak{k}<a} t^{\beta-1}_\mathfrak{k} L_\mathfrak{k} + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|L_p(t)\|)(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) < 1.$$

Then (1.1)-(1.3) has a unique mild solution.

Proof. Let the space $\mathscr{Z}(a) = \{u : (-\infty, a] \to \mathscr{E} : u_0, u|_{\mathcal{J}} \in PC(\mathscr{E})\}$ endowed with the sup-norm. Now by Proposition 3.1, we consider the operator $\mathfrak{T} : \mathscr{Z}(a) \to \mathscr{Z}(a)$ by

$$\mathfrak{T}u(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ q(0,\phi) + (z+\xi)t - q(t, u_t + x_t) + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s, u_s + x_s)ds + \int_0^t S'(t-s)\,(q(0,\phi) + (z+\xi)s \\ -q(s, u_s + x_s) + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) \\ +\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}p(\tau, u_\tau + x_\tau)d\tau\Big)\,ds, t \in \mathcal{J}. \end{cases}$$

It is easily seen that $\|u_t + x_t\|_{\mathscr{B}_A} \leq \mathfrak{K}_a\|u\|_t + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}$, where $\|u\|_t = \sup_{0\leq s\leq t}\|u(s)\|$.

Let $u \in \mathscr{Z}(a)$ and from the assumption $(H1), (H3)$ and $(H5)$, we get that

$$\begin{aligned} &\int_0^t \|S'(t-s)(q(0,\phi) + (z+\xi)s - q(s, u_s + x_s) + \Gamma(2-\beta) \\ &\quad\times\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}p(\tau, u_\tau + x_\tau)d\tau)\Big\|\,ds \\ &\leq (\|q(0,\phi)\| + a\|z+\xi\| + \|q(s, u_s + x_s)\| + \Gamma(2-\beta)a \\ &\quad\times\sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1}\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|p(\tau, u_\tau + x_\tau)\|\Big)\|\varphi\|_{L^1(\mathcal{J};\mathbb{R})} \end{aligned}$$

which follows that $s \to S'(t-s)(q(0,\phi) + (z+\xi)s - q(s, u_s + x_s) + \Gamma(2-\beta)\sum_{0<t_{k<s}} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}p(\tau, u_\tau + x_\tau)d\tau)$ is integrable on $[0, t]$, $\forall t \in \mathcal{J}$. Then, the operator $\mathfrak{T}$ is well defined and $\mathfrak{T}$ have the values in $\mathscr{Z}(a)$.

Now, for u and v in $\mathscr{Z}(a)$ and $t \in \mathcal{J}$, we get

$$\begin{aligned} \|\mathfrak{T}u(t) - \mathfrak{T}v(t)\| \leq\ & \|q(t, u_t + x_t) - q(t, v_t + x_t)\| + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t - t_\mathfrak{k}) \\ & \times\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\ & +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|p(s, u_s) - p(s, v_s)\|ds \\ & +\int_0^t \varphi(t-s)\,(\|q(s, u_s + x_s) - q(s, v_s + x_s)\| \\ & +\Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k}))\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\ & +\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|p(\tau, u_\tau) - p(\tau, v_\tau)\|d\tau\Big)ds \end{aligned}$$

$$\leq \left(\|L_q(t)\|_{C(\mathcal{J};\mathbb{R})} + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k} + \frac{a^\alpha}{\alpha\Gamma(\alpha)} \|L_p(t)\|_{C(\mathcal{J};\mathbb{R})} \right) (1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) \|u_t - v_t\|_{\mathscr{B}_A}$$

$$\leq \left[\mathfrak{K}_a \left(\|L_q(t)\|_{C(\mathcal{J};\mathbb{R})} + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k} + \frac{a^\alpha}{\alpha\Gamma(\alpha)} \|L_p(t)\|_{C(\mathcal{J};\mathbb{R})} \right) (1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) \right] \|u - v\|_t.$$

Then $\mathfrak{T}$ is a contraction map and has a fixed point $u(\cdot)$ of $\mathfrak{T}$. Thus, we determine that $u(\cdot)$ is a unique mild solution of (1.1)-(1.3). □

Theorem 3.2. *Let $S(t)$ be compact for all $t \geq 0$, $(H2) - (H6)$ are satisfied and if*

$$\mathfrak{K}_a \left(C_1 + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} d_\mathfrak{k}^1 + \frac{a^\alpha}{\alpha\Gamma(\alpha)} \|m_p(t)\| \liminf_{r\to\infty} \frac{W(r)}{r} \right) < 1,$$

$$\mathfrak{K}_a (\|L_q(t)\| + \Gamma(2-\beta)a \sum_{0<t_{k<a}} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k})(1 + \|\varphi\|) < 1.$$

Then (1.1)-(1.3) has a mild solution.

Proof. Take $r > 0$, such that

$$\left(C_1 \|\phi\|_{\mathscr{B}_A} + 2C_2 + \left(C_1 + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} d_\mathfrak{k}^1 \right) (\mathfrak{K}_a r + (\mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) \right.$$

$$+ \mathfrak{M}_a) \|\phi\|_{\mathscr{B}_A}) + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} d_\mathfrak{k}^2 + \|z + \xi\| a + \frac{a^\alpha}{\alpha\Gamma(\alpha)} \|m_p(t)\|$$

$$\left. \times W(\mathfrak{K}_a r + (\mathfrak{K}_a \mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a) \|\phi\|_{\mathscr{B}_A}) \right) (1 + \|\varphi\|) \leq s,$$

for all $s \geq r$.

Let the operator $\mathfrak{T} : \mathcal{B}_r(0, \mathscr{Z}(a)) \to \mathscr{Z}(a)$ be defined likewise considered in the previous Theorem 3.1, and in a similar manner we can easy to see that $\mathfrak{T}$ is well defined. Now, our aim to show that $\mathfrak{T} : \mathcal{B}_r(0, \mathscr{Z}(a)) \to \mathcal{B}_r(0, \mathscr{Z}(a))$ is a condensing map.

The subsequent steps shows the remaining proof.

Step 1. $\mathfrak{T}$ has values in $\mathcal{B}_r(0, \mathscr{Z}(a))$, i.e., $\mathfrak{T}\mathcal{B}_r(0, \mathscr{Z}(a)) \subset \mathcal{B}_r(0, \mathscr{Z}(a))$.

Let $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$ and $t \in \mathcal{J}$, then

$$\begin{aligned}
\|\mathfrak{T}u(t)\| &\leq \|q(0,\phi)\| + \|q(t, u_t + x_t)\| + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t-t_\mathfrak{k})\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\
&\quad + \|z+\xi\|t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|p(s, u_s + x_s)\|ds \\
&\quad + \int_0^t \|S'(t-s)\|\left(\|q(0,\phi)\| + \|q(s, u_s + x_s)\| + \|z+\xi\|s\right. \\
&\quad + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s-t_\mathfrak{k})\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|p(\tau, u_\tau + x_\tau)\|d\tau\right)ds \\
&\leq \left(C_1\|\phi\|_{\mathscr{B}_A} + 2C_2 + C_1\|u_t + x_t\|_{\mathscr{B}_A} + \Gamma(2-\beta)a\sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1}(d_\mathfrak{k}^1\|u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}\|_{\mathscr{B}_A}\right. \\
&\quad \left. + d_\mathfrak{k}^2) + \|z+\xi\|a + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|m_p(t)\|W(\|u_t + x_t\|_{\mathscr{B}_A})\right)(1 + \|\varphi\|) \\
&\leq \left(C_1\|\phi\|_{\mathscr{B}_A} + 2C_2 + \left(C_1 + \Gamma(2-\beta)a\sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1}d_\mathfrak{k}^1\right)r^*\right. \\
&\quad \left. + \Gamma(2-\beta)a\sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1}d_\mathfrak{k}^2 + \|z+\xi\|a + \frac{a^\alpha}{\alpha\Gamma(\alpha)}\|m_p(t)\|W(r^*)\right)(1 + \|\varphi\|)
\end{aligned}$$

where $r^* = \mathfrak{K}_a r + (\mathfrak{K}_a\mathfrak{H}(1 + \|\varphi\|_{L^1(\mathcal{J};\mathbb{R})}) + \mathfrak{M}_a)\|\phi\|_{\mathscr{B}_A}$.

This implies that $\|\mathfrak{T}u(t)\| \leq r$, i.e., $\mathfrak{T}u \in \mathcal{B}_r(0, \mathscr{Z}(a))$ and $\mathfrak{T}\mathcal{B}_r(0, \mathscr{Z}(a)) \subset \mathcal{B}_r(0, \mathscr{Z}(a))$.

The remainder of the proof continuing with the decomposition operator $\mathfrak{T} = \sum_{i=1}^3 \mathfrak{T}_i$, where

$$\begin{aligned}
\mathfrak{T}_1u(t) &= q(0,\phi) + (z+\xi)t - q(t, u_t + x_t) + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t-t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) \\
&\quad + \int_0^t S'(t-s)\left(q(0,\phi) + (z+\xi)s - q(s, u_s + x_s)\right. \\
&\quad \left. + \Gamma(2-\beta)\sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s-t_\mathfrak{k})I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\right)ds \\
\mathfrak{T}_2u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s, u_s + x_s)ds \\
\mathfrak{T}_3u(t) &= \int_0^t S'(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}p(\tau, u_\tau + x_\tau)d\tau ds
\end{aligned}$$

Step 2. $\mathfrak{T}_1$ is a contraction map on $\mathcal{B}_r(0, \mathscr{Z}(a))$.

Let $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$.

$$\begin{aligned}
\|\mathfrak{T}_1 u(t) - \mathfrak{T}_1 v(t)\| &\leq \|q(t, u_t + x_t) - q(t, v_t + x_t)\| \\
&\quad + \Gamma(2-\beta) \sum_{0<t_\mathfrak{k}<t} t_\mathfrak{k}^{\beta-1}(t - t_\mathfrak{k})\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\| \\
&\quad + \int_0^t \|S'(t-s)\|(\|q(s, u_s + x_s) - q(s, v_s + x_s)\| \\
&\quad + \Gamma(2-\beta) \sum_{0<t_\mathfrak{k}<s} t_\mathfrak{k}^{\beta-1}(s - t_\mathfrak{k})\|I_\mathfrak{k}(u_{t_\mathfrak{k}} + x_{t_\mathfrak{k}}) - I_\mathfrak{k}(v_{t_\mathfrak{k}} + x_{t_\mathfrak{k}})\|) \\
&\leq (\|L_q(t)\| + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k})\|u_t - v_t\|_{\mathscr{B}_A}(1 + \|\varphi\|) \\
&\leq \aleph_a(\|L_q(t)\| + \Gamma(2-\beta)a \sum_{0<t_\mathfrak{k}<a} t_\mathfrak{k}^{\beta-1} L_\mathfrak{k})(1 + \|\varphi\|)\|u - v\|_t.
\end{aligned}$$

Hence, $\mathfrak{T}_1$ is a contraction map on $\mathcal{B}_r(0, \mathscr{Z}(a))$.

Step 3. $\mathfrak{T}_2$ is a completely continuous map.

It is easy to see that the map $\mathfrak{T}_2$ is continuous, since the function f is continuous.

Next, we only we need to prove that $\mathfrak{T}_2$ is a compact operator.

Let $0 < \epsilon < t \leq a$, $u \in \mathscr{Z}(a)$. From the mean value theorem for the Bochner integral (see [17, Lemma II.1.3]), we have that

$$\begin{aligned}
\mathfrak{T}_2 u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^\epsilon (t-s)^{\alpha-1} p(s, u_s + x_s) ds + \frac{1}{\Gamma(\alpha)} \int_\epsilon^t (t-s)^{\alpha-1} p(s, u_s + x_s) ds \\
&\in \mathcal{B}_{\frac{Q\epsilon^\alpha}{\alpha\Gamma(\alpha)}}(0, \mathscr{E}) + \frac{(t-\epsilon)}{\Gamma(\alpha)} \overline{co}(\{(t-s)^{\alpha-1} p(s, u_s + x_s) : s \in [\epsilon, t]\})
\end{aligned}$$

where $Q = \|m_p(t)\|W(r^*)$, and the notion $co(U)$ refers the convex hull of the set U.

Since from [9, Lemma 2.2], the map i_c is compact and $p \in C(\mathcal{J} \times \mathscr{B}_A; \mathscr{D}(\mathscr{A}))$, from the above inclusion we find that $\mathfrak{T}_2\mathcal{B}_r(0, \mathscr{Z}(a)) = \{\mathfrak{T}_2 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\} \subset C_\epsilon + K_\epsilon$, where K_ϵ is compact and $\text{diam}(C_\epsilon) = \frac{Q\epsilon^\alpha}{\alpha\Gamma(\alpha)} \to 0$ as $\epsilon \to 0$. This proves that the set $\mathfrak{T}_2\mathcal{B}_r(0, \mathscr{Z}(a))$ is relatively compact in space $\mathscr{E}$ for all t in $\mathcal{J}$.

Consider $l > 0,\ 0 \leq t < a$ such that $0 \leq t + l \leq a$, and for $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$,

$$\begin{aligned}
\|\mathfrak{T}_2 u(t+l) - \mathfrak{T}_2 u(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t ((t-s)^{\alpha-1} - (t+l-s)^{\alpha-1})\|p(s, u_s + x_s)\|_{\mathscr{D}(\mathscr{A})} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+l} (t+l-s)^{\alpha-1}\|p(s, u_s + x_s)\|_{\mathscr{D}(\mathscr{A})} ds \\
&\leq \frac{2l^\alpha}{\alpha\Gamma(\alpha)}\|p(s, u_s + x_s)\|_{\mathscr{D}(\mathscr{A})} \\
&\leq \frac{2Ql^\alpha}{\alpha\Gamma(\alpha)}
\end{aligned}$$

which implies that $\mathfrak{T}_2\mathcal{B}_r(0, \mathscr{Z}(a))$ is equicontinuous.

Hence from the above results $\mathfrak{T}_2$ is completely continuous.

Step 4. The operator $\mathfrak{T}_3$ is completely continuous.

Let $t \in [0, a)$ and consider $\mathscr{P}(s) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s, u_s + x_s) ds$. For $u \in \mathcal{B}_r(0, \mathscr{Z}(a))$ and there exist $\epsilon > 0$, we take $l \in (0, \epsilon)$ such that $t + l \leq a$, and from [10, Lemma 2.2],

$$\begin{aligned} &\| \mathfrak{T}_3 u(t+l) - \mathfrak{T}_3 u(t) \| \\ &\leq \int_0^l \| S'(t+l-s)\mathscr{P}(s) \| ds + \int_0^t \| S'(s)(\mathscr{P}(t-s+l) - \mathscr{P}(t-s)) \| ds \\ &\leq \int_0^l \varphi(t+l-s) \| \mathscr{P}(s) \|_{\mathscr{D}(\mathscr{A})} ds + \int_0^t \varphi(s)[|\mathscr{P}|]_{C^\alpha(\mathcal{J};\mathscr{D}(\mathscr{A}))} l^\alpha ds \\ &\leq \frac{2Q}{\alpha\Gamma(\alpha)} \left(a^\alpha \int_0^l \varphi(t+l-s) ds + l^\alpha \| \varphi \|_{L^1(\mathcal{J})} \right) \end{aligned}$$

which proves that $\mathfrak{T}_3\mathcal{B}_r(0, \mathscr{Z}(a))$ is right equicontinuous at t in $[0, a)$. The above discussion allow us to show that $\mathfrak{T}_3\mathcal{B}_r(0, \mathscr{Z}(a))$ is left equicontinuous at t in the interval $(0, a]$. From this argument we say that $\mathfrak{T}_3\mathcal{B}_r(0, \mathscr{Z}(a))$ is equicontinuous.

In this sequel we finally prove that $\{\mathfrak{T}_3 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\}$ is relatively compact in $\mathscr{E}$, $\forall\ t \in (0, a]$.

Take $0 < t \leq a$ and $Q_1 = \|m_p(t)\| W(r^*) \|\varphi\|_{L^1([0,\epsilon])}$. The set $V = \{\mathscr{P}(s) : s \in \mathcal{J}, u \in \mathcal{B}_r(0, \mathscr{Z}(a))\}$ is relatively compact in $\mathscr{E}$, since from the previous Step 2. If u belongs to $\mathcal{B}_r(0, \mathscr{Z}(a))$, by using the concept in [17, Lemma II.1.3], we get

$$\begin{aligned} \mathfrak{T}_3 u(t) &= \int_0^\epsilon S'(t-s)\mathscr{P}(s) ds + \int_\epsilon^t S'(t-s)\mathscr{P}(s) ds \\ &\in \mathcal{B}_{\frac{Q_1 a^\alpha}{\alpha\Gamma(\alpha)}}(0, \mathscr{E}) + (t-\epsilon)\overline{co(\{S'(s)y : s \in [\epsilon, t], y \in \overline{V}\})} \end{aligned}$$

and hence, $\{\mathfrak{T}_3 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\} \subset \mathcal{B}_{\frac{Q_1 a^\alpha}{\alpha\Gamma(\alpha)}}(0, \mathscr{E}) + K_\epsilon$, where K_ϵ is compact and $\frac{Q_1 a^\alpha}{\alpha\Gamma(\alpha)} \to 0$ as $\epsilon \to 0$. This proves that $\{\mathfrak{T}_3 u(t) : u \in \mathcal{B}_r(0, \mathscr{Z}(a))\}$ is relatively compact in $\mathscr{E}$. Hence we finally conclude that $\mathfrak{T}_3$ is completely continuous.

From the above steps we can say that the operator $\mathfrak{T} : \mathcal{B}_r(0, \mathscr{Z}(a)) \to \mathcal{B}_r(0, \mathscr{Z}(a))$ is a condensing map. Then the existence results follows from [17, Theorem IV.3.2]. □

4 Application

We look at the following partial fractional impulsive neutral differential equations with infinite delay of the form

$$
\begin{aligned}
&{}^{c}D^{\alpha}_{0^+}\left(v(t,\eta)+\int_{-\infty}^{t}\int_{0}^{\pi}a(t-s,\zeta,\eta)v(s,\zeta)d\zeta ds\right)\\
&\quad=\frac{\partial^2}{\partial\eta^2}v(t,\eta)+\int_{-\infty}^{t}d(t,t-s,\eta,v(s,\eta))ds,\ (t,\eta)\in\mathcal{J}\times[0,\pi], \qquad (4.1)\\
&v(t,0)=v(t,\pi)=0,\ t\in\mathcal{J}, \qquad (4.2)\\
&v(\tau,\eta)=\phi(\tau,\eta),\ 0\le\eta\le\pi,\ \tau\le 0, \qquad (4.3)\\
&{}^{c}D^{\beta}_{0^+}v(t_{\mathfrak{k}}^{+})(\eta)-{}^{c}D^{\beta}_{0^+}v(t_{\mathfrak{k}})(\eta)=\int_{-\infty}^{t_{\mathfrak{k}}}e_{\mathfrak{k}}(t_{\mathfrak{k}}-s)v(s,\eta)ds, \qquad (4.4)
\end{aligned}
$$

where $0<\beta<1, 1<\alpha<2$, and $\phi\in\mathscr{B}_A=\mathcal{PC}_0\times L^2(g,\mathscr{E})$. Assume that $d:\mathbb{R}^4\to\mathbb{R}$, $e_{\mathfrak{k}}:\mathbb{R}\to\mathbb{R}$, $a:\mathbb{R}^3\to\mathbb{R}$ are continuous functions and $\frac{\partial^{\alpha}a(s,\zeta,\eta)}{\partial\eta^{\alpha}}$ exists. $0<t_1<\cdots<t_m<a$ are prefixed numbers.

Let the space $\mathscr{E}=L^2([0,\pi])$. Let $\mathscr{A}:\mathscr{D}(\mathscr{A})\subset\mathscr{E}\to\mathscr{E}$ be defined by $\mathscr{A}u=u''$ with $\mathscr{D}(\mathscr{A})$ consist of set of all u and u'' in $\mathscr{E}$ such that $u(0)=u(\pi)=0$. If $\{\mathfrak{C}(t)\}_{t\ge0}$ is a strongly continuous cosine family on $\mathscr{E}$, then $\mathscr{A}$ is its infinitesimal generator. The well known result that the associated sine operator $\mathfrak{S}(t)$ is compact for every $t\in\mathbb{R}$ and hence $(\lambda-\mathscr{A})^{-1}$ is compact for every λ belongs to $\rho(\mathscr{A})$.

Consider

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathscr{A}u(s)ds,\quad s\ge0,$$

have an analytic resolvent $\{S(t)\}_{t\ge0}$ on $\mathscr{E}$ given by

$$S(t)=\begin{cases}\frac{1}{2\pi i}\int_{\Gamma_{\varrho,\upsilon}}e^{\lambda t}\lambda^{\alpha-1}(\lambda^{\alpha}-\mathscr{A})^{-1}d\lambda, & t>0,\\ I, & t=0,\end{cases}$$

with $\Gamma_{\varrho,\upsilon}$ consisting of the rays $\{\varrho e^{i\upsilon}:\varrho\ge0\}$ and $\{\varrho e^{-i\upsilon}:\varrho\ge0\}$. Here $\Gamma_{\varrho,\upsilon},\ \upsilon\in(\pi,\frac{\pi}{2})$, is a contour, [20, Example II.2.1].

To represent the equations (4.1)-(4.4) in the form of (1.1)-(1.3) by

$$q(t,\varsigma)(\eta)=\int_{-\infty}^{0}\int_{0}^{\pi}a(s,\zeta,\eta)\varsigma(s,\zeta)d\zeta ds,$$

$$p(t,\varsigma)(\eta)=\int_{-\infty}^{0}d(t,s,\eta,\varsigma(s,\eta))ds$$

$$I_{\mathfrak{k}}(\varsigma)(\eta)=\int_{-\infty}^{0}e_{\mathfrak{k}}(s)\varsigma(s,\eta)ds$$

Therefore, under the appropriate conditions on the functions $a, d, e_{\mathfrak{k}}$, the mild solution exists for partial fractional impulsive problem (4.1)-(4.4) in view of Theorem 3.2 and uniqueness results exists from Theorem 3.1.

Conclusion

In this work we consider the fractional neutral infinite delay differential equations with fractional impulsive conditions involving Caputo derivative of order lies in the interval (1,2). To improve the characterization of the memory property of the fractional derivative, we consider the lower bound at each impulse as zero. We use resolvent operator to derive the mild solutions in order to make it as more appropriate.

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The solution to matrix inequality $AXB + (AXB)^* \geqslant C$ and its applications

Xifu Liu[a]*, Guangdong Wu[b]

[a]*School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China*
[b]*School of tourism and urban management, Jiangxi University of Finance & Economics, Nanchang, 330013, China*

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Abstract

In this paper, firstly, we study the solution to linear matrix inequality $AXB + (AXB)^* \geqslant C$ for Hermitian matrix C. Furthermore, for the applications, we derive the representations for the common Re-nnd solution to equations $AX = C$ and $XB = D$, and the Re-nnd $\{1, 3, 4\}$-inverse for square matrix.

Keywords: Matrix inequality, Re-nnd solution, Re-nnd generalized inverse

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1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$, $\mathbb{C}_H^m$ denote the set of all $m \times m$ Hermitian matrices, $\mathbb{U}_n$ denote the set of all $n \times n$ unitary matrices. For $A \in \mathbb{C}^{m\times n}$, its range space, rank and conjugate transpose will be denoted by $R(A)$, $r(A)$ and A^* respectively. $i_+(A)$ and $i_-(A)$ denote the numbers of the positive and negative eigenvalues of a Hermitian matrix A counted with multiplicities, respectively. The identity matrix of order n is denoted by I_n.

For a matrix $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ is defined to be the unique solution of the four Penrose equations [1]

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

For convenience, we denote $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

The Hermitian part of $A \in \mathbb{C}^{m\times m}$ is defined by $H(A) = \frac{1}{2}(A + A^*)$. We say that A is Re-nnd (Re-nonnegative definite) if $H(A) \geqslant 0$ and A is Re-pd (Re-positive definite) if $H(A) > 0$. Let $A_{re}^{(i,j,\cdots,k)}$ be the

*Corresponding author. *E-mail addresses*: liuxifu211@hotmail.com (X. Liu).

Re-nnd $\{i, j, \cdots, k\}$-inverse of square matrix A. Recently, some researches on Re-nnd solution and Re-nnd generalized inverse were done by several authors [2-7].

The Löwner partial ordering is one of the most basic concepts for characterizing relations between two Hermitian matrices. A challenging research topic on Hermitian matrices is to solve linear matrix inequalities (LMIs) induced from the Löwner partial ordering, such as

$$AXB + (AXB)^* \geqslant C, \tag{1.1}$$

$$AXB + (AXB)^* \leqslant (>, <)C, \quad AX + (AX)^* \geqslant (\leqslant, >, <)C, \quad AXA^* \geqslant (\leqslant, >, <)C.$$

In this article, we consider the matrix inequality (1.1), where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}^m_H$ are given, $X \in \mathbb{C}^{n\times p}$ is variable matrix.

Newly, some special cases of (1.1) were considered by several authors, such as: the case that C is nonnegative definite matrix [8], the case $B = I_m$ [9], the case that block matrix $\begin{pmatrix} A & B^* \end{pmatrix}$ is full row rank [10]. Researches on other linear matrix inequalities can be found in [8, 11]. For the applications, (1.1) can be used to establish the general forms of Re-nnd solution of matrix equation $AXB = C$ [10], and the solution of matrix equation $AXA^* = B$ (or $AX = B$) subject matrix inequality constraint $CXC^* \geqslant D$ [12, 13], and the Re-nnd inverses $A^{(1,2,i)}_{re}$, $A^{(1,i)}_{re}$ $(i = 3, 4)$ of square matrix [3, 4, 10]. In [2, 6], the authors provided some necessary and sufficient conditions for the existence of common Re-nnd and Re-pd solutions to $AX = C$ and $XB = D$, however, the general solutions are still unsolved.

We are, therefore, motivated to focus our research interest on (1.1) without any restrictions on matrices A, B, C.

It is well known that (1.1) can equivalently be written as

$$AXB + (AXB)^* = C + VV^* \tag{1.2}$$

for some V. Tian and Rosen [8] shown that equation (1.2) is solvable for X if and only if VV^* satisfies

$$E_G VV^* = -E_G C, \quad E_A VV^* E_A = -E_A C E_A, \quad F_B VV^* F_B = -F_B C F_B, \tag{1.3}$$

where $G = \begin{pmatrix} A & B^* \end{pmatrix}$.

This paper is organized as follows. In section 2, firstly, we establish some necessary and sufficient conditions for the solvability of matrix inequality (1.1), secondly, we derive a general form for VV^*, finally, we present a general solution of X to matrix inequality (1.1). Furthermore, for the applications, we provide the explicit expressions for the common Re-nnd solution to equations $AX = C$ and $XB = D$, and the Re-nnd generalized inverse $A^{(1,3,4)}$ of square matrix A.

Before proceeding to the next section, we list some useful results which will facilitate the proof of our theorems.

Lemma 1.1. ([14]) Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given. Then

$$\begin{aligned}
\max_{X\in\mathbb{C}^{n\times m}} i_\pm[A-BXC-(BXC)^*] &= \min\left\{i_\pm(M_1),\ i_\pm(M_2)\right\},\\
\min_{X\in\mathbb{C}^{n\times m}} i_\pm[A-BXC-(BXC)^*] &= r\begin{pmatrix} A & B & C^* \end{pmatrix} + \max\left\{i_\pm(M_1)-r(N_1),\ i_\pm(M_2)-r(N_2)\right\},
\end{aligned}$$

where

$$M_1=\begin{pmatrix} A & B\\ B^* & 0\end{pmatrix},\ M_2=\begin{pmatrix} A & C^*\\ C & 0\end{pmatrix},\ N_1=\begin{pmatrix} A & B & C^*\\ B^* & 0 & 0\end{pmatrix},\ N_2=\begin{pmatrix} A & B & C^*\\ C & 0 & 0\end{pmatrix}.$$

Lemma 1.2. ([14]) Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, and denote $M=\begin{pmatrix} A & B\\ B^* & 0\end{pmatrix}$. Then

$$i_\pm(M) = r(B) + i_\pm(E_BAE_B).$$

Lemma 1.3. ([15]) Let $A, B \in \mathbb{C}^{m\times n}$ be given. Then the matrix equation $AXX^* = B$ has a solution for XX^* if and only if $R(B) \subseteq R(A)$, $AB^* \geqslant 0$ and $r(AB^*) = r(B)$. In this case, the general solution can be written in the following parametric form

$$XX^* = B^*(AB^*)^\dagger B + F_AWW^*F_A,$$

where $W \in \mathbb{C}^{n\times n}$ is arbitrary.

Lemma 1.4. ([16]) Given matrices $A, B, C, D \in \mathbb{C}^{p\times n}$. The matrix equations $AXX^*A^* = BB^*$ and $CXX^*C^* = DD^*$ have a common Hermitian nonnegative-definite solution if and only if $R(B) \subseteq R(A)$ (*or* $r\begin{pmatrix} A & B\end{pmatrix} = r(A)$) and there exists $T \in \mathbb{U}_n$ such that

$$E_{CF_A}(DT - CA^\dagger B) = 0. \tag{1.4}$$

If a common Hermitian nonnegative-definite solution exists, then a representation of the general common Hermitian nonnegative-definite solution is XX^* with

$$X = A^\dagger B + F_A(CF_A)^\dagger(DT - CA^\dagger B) + F_AF_{CF_A}Z,$$

where $Z \in \mathbb{C}^{n\times n}$ is arbitrary and $T \in \mathbb{U}_n$ is a parameter matrix satisfying (1.4).

Lemma 1.5. ([8]) Let $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}_H^m$ are given. Then the matrix equation $AXB + (AXB)^* = C$ has a solution $X \in \mathbb{C}^{p\times q}$ if and only if

$$\begin{pmatrix} A & B^*\end{pmatrix}\begin{pmatrix} A & B^*\end{pmatrix}^\dagger C = C,\ E_ACE_A = 0,\ F_BCF_B = 0.$$

In this case, the general solution can be written as

$$X = \frac{1}{2}(X_1 + X_2^*),$$

where X_1 and X_2 are general solutions of the equation $AX_1B + B^*X_2A^* = C$.

Lemma 1.6. ([17]) Let $A_1 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{p\times k}$, $A_2 \in \mathbb{C}^{m\times l}$, $B_2 \in \mathbb{C}^{q\times k}$ and $C \in \mathbb{C}^{m\times k}$ be known and $X_1 \in \mathbb{C}^{n\times p}$, $X_2 \in \mathbb{C}^{l\times q}$ unknown; $M = E_{A_1}A_2$, $N = B_2F_{B_1}$, $S = A_2F_M$. Then the following statements are equivalent:

(i) The system $A_1X_1B_1 + A_2X_2B_2 = C$ is solvable;

(ii) The following rank equalities are satisfied,

$$r\begin{pmatrix} A_1 & C \\ 0 & B_2 \end{pmatrix} = r\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix},\ r\begin{pmatrix} A_2 & C \\ 0 & B_1 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ 0 & B_1 \end{pmatrix},$$
$$r\begin{pmatrix} C & A_1 & A_2 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix},\ r\begin{pmatrix} B_1 \\ B_2 \\ C \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

In this case, the general solution can be expressed as

$$\begin{aligned} X_1 &= A_1^\dagger CB_1^\dagger - A_1^\dagger A_2M^\dagger E_{A_1}CB_1^\dagger - A_1^\dagger SA_2^\dagger CF_{B_1}N^\dagger B_2B_1^\dagger - A_1^\dagger SVE_NB_2B_1^\dagger + F_{A_1}U + ZE_{B_1}, \\ X_2 &= M^\dagger E_{A_1}CB_2^\dagger + F_MS^\dagger SA_2^\dagger CF_{B_1}N^\dagger + F_M(V - S^\dagger SVNN^\dagger) + WE_{B_2}, \end{aligned}$$

where U, V, W and Z are arbitrary matrices over complex field with appropriate sizes.

Lemma 1.7. ([8]) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then

$$r\begin{pmatrix} A & B \end{pmatrix} = r(A) + r(E_AB),\ r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A),$$
$$r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = r(B) + r(C) + r(E_BAF_C).$$

Lemma 1.8. ([9]) Let $A, C \in \mathbb{C}^{n\times m}$. There exists a Re-nnd solution to equation $AX = C$ if and only if $R(C) \subseteq R(A)$, AC^* is Re-nnd. There exists a Re-pd solution to equation $AX = C$ if and only if $R(C) \subseteq R(A)$, $i_+(AC^* + CA^*) = r(A)$.

2 Main results

In this section, our purpose is to investigate the solution to the linear matrix inequality (1.1), and then apply our result to establish the general expressions for the common Re-nnd solution to $AX = C$ and $XB = D$, and the Re-nnd $\{1,3,4\}$-inverse for square matrix A.

First, we come to establish some necessary and sufficient conditions for the solvability of matrix inequality (1.1).

Theorem 2.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. Then the following statements are equivalent:

(1) Matrix inequality (1.1) is solvable;

(2) $E_ACE_A \leqslant 0$, $F_BCF_B \leqslant 0$, and

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) = r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix},\ r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) = r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix};$$

(3) $r(E_GCE_A) = r(E_GCF_B) = r(E_GC)$, $E_ACE_A \leqslant 0$ and $F_BCF_B \leqslant 0$.

Proof. Note that (1.1) can be rewritten as $C - AXB - (AXB)^* \leqslant 0$. So, (1.1) is solvable if and only if

$$\min_X i_+[C - AXB - (AXB)^*] = 0.$$

Applying Lemma 1.1, we get

$$\begin{aligned}
&\min_X i_+[C - AXB - (AXB)^*] \\
=\ & r\begin{pmatrix} C & A & B^* \end{pmatrix} + \max\left\{ i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, i_+\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix} \right\} \\
=\ & r\begin{pmatrix} C & A & B^* \end{pmatrix} + \max\left\{ r(A) + i_+(E_ACE_A) - r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \right. \\
& \left. r(B) + i_+(F_BCF_B) - r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix} \right\}. \qquad (2.1)
\end{aligned}$$

Letting the right hand side of (2.1) be zero yields

$$\begin{aligned}
r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) + i_+(E_ACE_A) &= r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \\
r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) + i_+(F_BCF_B) &= r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix},
\end{aligned}$$

which are equivalent to

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) = r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \qquad (2.2)$$

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) = r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix}, \qquad (2.3)$$

$$E_ACE_A \leqslant 0 \quad and \quad F_BCF_B \leqslant 0.$$

Applying Lemma 1.7 to (2.2) and (2.3) yields $r(E_GCE_A) = r(E_GC)$ and $r(E_GCF_B) = r(E_GC)$ respectively. Thus, the proof is complete. □

Next, we present some properties for matrices A, B and C which satisfy the conditions in Theorem 2.1.

Corollary 2.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}^{m}_{H}$ be given, denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. If the conditions in the statement (2) or (3) of Theorem 2.1 are satisfied, then the following hold,

$$r(E_GCE_G) = r(E_GC) \ \ or \ \ R(E_GCE_G) = R(E_GC), \ \ E_GCE_G \leqslant 0, \tag{2.4}$$

$$R[E_ACE_G(E_GCE_G)^{\dagger}E_GCE_A - E_ACE_A] \subseteq R(E_AGG^{\dagger}), \tag{2.5}$$

$$E_ACE_G(E_GCE_G)^{\dagger}E_GCE_A - E_ACE_A \geqslant 0, \tag{2.6}$$

$$R[F_BCE_G(E_GCE_G)^{\dagger}E_GCF_B - F_BCF_B] \subseteq R(F_BGG^{\dagger}), \tag{2.7}$$

$$F_BCE_G(E_GCE_G)^{\dagger}E_GCF_B - F_BCF_B \geqslant 0. \tag{2.8}$$

Proof. It follows from the two rank equalities of statement (2) in Theorem 2.1 that

$$R\left(\begin{pmatrix} C & A & B^* \end{pmatrix}^*\right) \cap R\left(\begin{pmatrix} A^* & 0 & 0 \end{pmatrix}^*\right) = \emptyset, \ \ R\left(\begin{pmatrix} C & A & B^* \end{pmatrix}^*\right) \cap R\left(\begin{pmatrix} B & 0 & 0 \end{pmatrix}^*\right) = \emptyset.$$

Hence,

$$R\left(\begin{pmatrix} C & A & B^* \end{pmatrix}^*\right) \cap R\left(\begin{pmatrix} A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix}^*\right) = \emptyset,$$

which means that

$$r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = r\begin{pmatrix} C & A & B^* \end{pmatrix} + r\begin{pmatrix} A & B^* \end{pmatrix}.$$

By Lemma 1.7, we have

$$r(E_GCE_G) + 2r(G) = r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = r\begin{pmatrix} C & A & B^* \end{pmatrix} + r\begin{pmatrix} A & B^* \end{pmatrix} = r(E_GC) + 2r(G),$$

so,

$$r(E_GCE_G) = r(E_GC) \ \ or \ \ R(E_GCE_G) = R(E_GC).$$

On the other hand, it follows from Lemma 1.2 that

$$\begin{aligned} i_+(E_GCE_G) + r(G) &= i_+\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = i_+\begin{pmatrix} F_BCF_B & F_BA \\ A^*F_B & 0 \end{pmatrix} + r(B) \\ &= r(B) + r(F_BA) + i_+(E_{F_BA}F_BCF_BE_{F_BA}) \\ &= r(B) + r(F_BA) = r(G), \quad (F_BCF_B \leqslant 0 \ is \ used) \end{aligned}$$

means that $i_+(E_GCE_G) = 0$ or $E_GCE_G \leqslant 0$. Then (2.4) holds.

Furthermore, applying Lemma 1.7, and elementary block matrix operations, we get

$$r(E_AGG^{\dagger}) = r(E_AG) = r\begin{pmatrix} A & G \end{pmatrix} - r(A) = r(G) - r(A),$$

and

$$\begin{aligned}
& r\left(\begin{array}{cc} E_AGG^\dagger & E_ACE_G(E_GCE_G)^\dagger E_GCE_A - E_ACE_A \end{array}\right) \\
= \; & r\left(\begin{array}{cc} E_AG & E_ACE_G(E_GCE_G)^\dagger E_GCE_A - E_ACE_A \end{array}\right) \\
= \; & r\left(\begin{array}{ccc} A & G & CE_G(E_GCE_G)^\dagger E_GCE_A - CE_A \end{array}\right) - r(A) \\
= \; & r\left(\begin{array}{cc} G & CE_G(E_GCE_G)^\dagger E_GCE_A - CE_A \end{array}\right) - r(A) \\
= \; & r[E_GCE_G(E_GCE_G)^\dagger E_GCE_A - E_GCE_A] + r(G) - r(A) \\
= \; & r(G) - r(A) = r(E_AGG^\dagger). \quad (R(E_GCE_G) = R(E_GC) \ \textit{is used})
\end{aligned}$$

Thus, (2.5) is evident.

By Lemma 1.2 and (2.4), one can compute that

$$\begin{aligned}
& i_-[E_ACE_G(E_GCE_G)^\dagger E_GCE_A - E_ACE_A] \\
= \; & i_+[E_ACE_A - E_ACE_G(E_GCE_G)^\dagger E_GCE_A] = i_+\left(\begin{array}{cc} E_GCE_G & E_GCE_A \\ E_ACE_G & E_ACE_A \end{array}\right) - i_+(E_GCE_G) \\
= \; & i_+\left\{\left(\begin{array}{cc} E_G & 0 \\ 0 & E_A \end{array}\right)\left(\begin{array}{cc} C & C \\ C & C \end{array}\right)\left(\begin{array}{cc} E_G & 0 \\ 0 & E_A \end{array}\right)\right\} = i_+\left(\begin{array}{cccc} C & C & G & 0 \\ C & C & 0 & A \\ G^* & 0 & 0 & 0 \\ 0 & A^* & 0 & 0 \end{array}\right) - r(G) - r(A) \\
= \; & i_+\left(\begin{array}{cccc} C & 0 & G & 0 \\ 0 & 0 & -G & A \\ G^* & -G^* & 0 & 0 \\ 0 & A^* & 0 & 0 \end{array}\right) - r(G) - r(A) = i_+\left(\begin{array}{cccc} C & 0 & 0 & A \\ 0 & 0 & -G & A \\ 0 & -G^* & 0 & 0 \\ A^* & A^* & 0 & 0 \end{array}\right) - r(G) - r(A) \\
= \; & i_+\left(\begin{array}{ccccc} C & 0 & 0 & 0 & A \\ 0 & 0 & -A & -B^* & A \\ 0 & -A^* & 0 & 0 & 0 \\ 0 & -B & 0 & 0 & 0 \\ A^* & A^* & 0 & 0 & 0 \end{array}\right) - r(G) - r(A) \\
= \; & i_+\left(\begin{array}{ccccc} C & 0 & 0 & 0 & A \\ 0 & 0 & -A & -B^* & 0 \\ 0 & -A^* & 0 & 0 & 0 \\ 0 & -B & 0 & 0 & 0 \\ A^* & 0 & 0 & 0 & 0 \end{array}\right) - r(G) - r(A) \\
= \; & i_+\left(\begin{array}{cccc} C & 0 & 0 & A \\ 0 & 0 & -G & 0 \\ 0 & -G^* & 0 & 0 \\ A^* & 0 & 0 & 0 \end{array}\right) - r(G) - r(A) = i_+(E_ACE_A) = 0,
\end{aligned}$$

which is equivalent to (2.6).

Similarly, (2.7) and (2.8) can be proved. □

When the matrices A, B and C satisfy the conditions in Theorem 2.1, then, by Lemma 1.3 and Lemma 1.4, we get the solution of VV^* to (1.3).

Lemma 2.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix. Denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$, $P = E_A GG^\dagger$, $Q = F_B GG^\dagger F_P$, and

$$H_1 = E_A CE_G(E_G CE_G)^\dagger E_G CE_A - E_A CE_A,\ \ H_2 = F_B CE_G(E_G CE_G)^\dagger E_G CF_B - F_B CF_B.$$

Suppose that the conditions in the statement (2) or (3) of Theorem 2.1 are satisfied, then equations in (1.3) have a common solution for VV^*, which can be written as

$$VV^* = -CE_G(E_G CE_G)^\dagger E_G C + GG^\dagger WW^* GG^\dagger, \tag{2.9}$$

where

$$W = P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - F_B P^\dagger H_1^{\frac{1}{2}}) + F_P F_Q Z, \tag{2.10}$$

with $T \in \mathbb{U}_m$ and $Z \in \mathbb{C}^{m\times m}$ are arbitrary.

Proof. In view of Lemma 1.3 and Corollary 2.1, we know that $E_G VV^* = -E_G C$ is solvable, and the solution of VV^* can be formed by

$$VV^* = -CE_G(E_G CE_G)^\dagger E_G C + GG^\dagger WW^* GG^\dagger, \tag{2.11}$$

where $W \in \mathbb{C}^{m\times m}$ is arbitrary. Substituting VV^* into the last two equations in (1.3) produces

$$E_A GG^\dagger WW^* GG^\dagger E_A = E_A CE_G(E_G CE_G)^\dagger E_G CE_A - E_A CE_A \triangleq H_1, \tag{2.12}$$

$$F_B GG^\dagger WW^* GG^\dagger F_B = F_B CE_G(E_G CE_G)^\dagger E_G CF_B - F_B CF_B \triangleq H_2. \tag{2.13}$$

Corollary 2.1 shows that both (2.12) and (2.13) are consistent. Next, we come to prove that (2.12) and (2.13) have a common Hermitian nonnegative-definite solution WW^*.

By Lemma 1.4, the matrix equations (2.12) and (2.13) have a common Hermitian nonnegative-definite solution if and only if there exists $T \in \mathbb{U}_m$ such that

$$E_{F_B GG^\dagger F_{E_A GG^\dagger}}(H_2^{\frac{1}{2}}T - F_B GG^\dagger(E_A GG^\dagger)^\dagger H_1^{\frac{1}{2}}) = 0. \tag{2.14}$$

It follows from Lemma 1.7 that

$$\begin{aligned} r\left(F_B GG^\dagger F_{E_A GG^\dagger}\right) &= r\begin{pmatrix} F_B GG^\dagger \\ E_A GG^\dagger \end{pmatrix} - r(E_A GG^\dagger) = r\begin{pmatrix} F_B G \\ E_A G \end{pmatrix} - r(E_A G) \\ &= r\begin{pmatrix} B^* & 0 & G \\ 0 & A & G \end{pmatrix} - r(A) - r(B) - [r\begin{pmatrix} A & G \end{pmatrix} - r(A)] \\ &= r(G) - r(B) = r(F_B GG^\dagger), \end{aligned}$$

i.e., $R\left(F_B GG^\dagger F_{E_A GG^\dagger}\right) = R(F_B GG^\dagger)$, therefore $E_{F_B GG^\dagger F_{E_A GG^\dagger}} = E_{F_B GG^\dagger}$ and

$$E_{F_B GG^\dagger F_{E_A GG^\dagger}} F_B GG^\dagger(E_A GG^\dagger)^\dagger H_1^{\frac{1}{2}} = 0.$$

Applying Lemma 1.7 again and (2.4), we have

$$\begin{aligned} r[E_{F_BGG^\dagger}H_2] &= r\begin{pmatrix} F_BGG^\dagger & H_2 \end{pmatrix} - r(F_BGG^\dagger) = r\begin{pmatrix} F_BG & H_2 \end{pmatrix} - r(F_BG) \\ &= r\begin{pmatrix} B^* & G & CE_G(E_GCE_G)^\dagger E_GCF_B - CF_B \end{pmatrix} - r(B) - r(F_BG) \\ &= r\begin{pmatrix} G & CE_G(E_GCE_G)^\dagger E_GCF_B - CF_B \end{pmatrix} - r(G) \\ &= r[E_GCE_G(E_GCE_G)^\dagger E_GCF_B - E_GCF_B] = 0, \end{aligned}$$

means that $E_{F_BGG^\dagger}H_2 = 0$, i.e., $E_{F_BGG^\dagger F_{E_AGG^\dagger}}H_2^{\frac{1}{2}} = 0$. Hence, (2.14) holds for any $T \in \mathbb{U}_m$, and there exists a common Hermitian nonnegative-definite solution to (2.12) and (2.13). By Lemma 1.4, the common Hermitian nonnegative-definite solution is WW^* with

$$\begin{aligned} W &= P^\dagger H_1^{\frac{1}{2}} + F_PQ^\dagger(H_2^{\frac{1}{2}}T - F_BGG^\dagger P^\dagger H_1^{\frac{1}{2}}) + F_PF_QZ \\ &= P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - F_BP^\dagger H_1^{\frac{1}{2}}) + F_PF_QZ, \end{aligned} \tag{2.15}$$

where $T \in \mathbb{U}_m$ and $Z \in \mathbb{C}^{m\times m}$ are arbitrary.

Substituting (2.15) into (2.14) yields (2.9). □

Combining Theorem 2.1 and Lemma 2.1, we can deduce the following result.

Theorem 2.2. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, and suppose that matrix inequality (1.1) is solvable. Then, a general solution to (1.1) can be expressed as

$$X = \frac{1}{2}(X_1 + X_2^*), \tag{2.16}$$

where

$$\begin{aligned} X_1 &= A^\dagger(C+VV^*)B^\dagger - A^\dagger B^*M^\dagger(C+VV^*)B^\dagger - A^\dagger S(B^*)^\dagger(C+VV^*)N^\dagger A^*B^\dagger \\ &\quad -A^\dagger SY_1E_NA^*B^\dagger + F_AY_2 + Y_3E_B, \end{aligned} \tag{2.17}$$

$$X_2 = M^\dagger(C+VV^*)(A^*)^\dagger + S^\dagger S(B^*)^\dagger(C+VV^*)N^\dagger + F_M(Y_1 - S^\dagger SY_1NN^\dagger) + Y_4F_A, \tag{2.18}$$

with VV^* is given by (2.9), $M = E_AB^*$, $N = A^*F_B$, $S = B^*F_M$, and Y_i $(i = 1, 2, 3, 4)$ are arbitrary matrices over complex field with appropriate sizes.

Proof. Since the matrix inequality (1.1) is equivalent to (1.2), where VV^* is given by (2.9). In view of Lemma 1.5, the general solution to (1.2) can be written as

$$X = \frac{1}{2}(X_1 + X_2^*),$$

where X_1 and X_2 are general solutions of the equation

$$AX_1B + B^*X_2A^* = C + VV^*. \tag{2.19}$$

It follows from (1.3) and Lemma 1.6 that (2.19) is solvable, and

$$\begin{aligned} X_1 &= A^\dagger(C+VV^*)B^\dagger - A^\dagger B^* M^\dagger E_A(C+VV^*)B^\dagger - A^\dagger S(B^*)^\dagger(C+VV^*)F_B N^\dagger A^* B^\dagger \\ &\quad -A^\dagger S Y_1 E_N A^* B^\dagger + F_A Y_2 + Y_3 E_B, \\ X_2 &= M^\dagger E_A(C+VV^*)(A^*)^\dagger + F_M S^\dagger S(B^*)^\dagger(C+VV^*)F_B N^\dagger + F_M(Y_1 - S^\dagger S Y_1 NN^\dagger) + Y_4 F_A, \end{aligned}$$

where $M = E_A B^*$, $N = A^* F_B$, $S = B^* F_M$, and $Y_i, (i = 1, 2, 3, 4)$ are arbitrary matrices over complex field with appropriate sizes. Together with $M^\dagger E_A = M^\dagger$, $F_B N^\dagger = N^\dagger$ and $F_M S^\dagger = S^\dagger$, then (2.17) and (2.18) are followed. □

In [2], the author presented some sufficient and necessary conditions for the existence of common Re-nnd solution to $AX = C$ and $XB = D$, however, the general solution has not been established by now. Next, we restudy this problem, and derive its general solution.

Theorem 2.3. Let $A, C \in \mathbb{C}^{n\times m}$, and $B, D \in \mathbb{C}^{m\times n}$, suppose that both $AX = C$ and $XB = D$ have a a Re-nnd solution. If the pair of equations have a common solution (*i.e.*, $AD = CB$), then there exists a common Re-nnd solution if and only if

$$r\begin{pmatrix} A & C \\ B^* & -D^* \end{pmatrix} = r\begin{pmatrix} A & CA^* \\ B^* & -D^*A^* \end{pmatrix} = r\begin{pmatrix} A & CB \\ B^* & -D^*B \end{pmatrix}. \tag{2.20}$$

In this case, a general common Re-nnd solution can be written as

$$X = A^\dagger C + F_A D B^\dagger + \frac{1}{2}(\tilde{Y}_1 + \tilde{Y}_2^*), \tag{2.21}$$

where,

$$\begin{aligned} \tilde{Y}_1 &= F_A(\tilde{C}+VV^*)E_B - F_A M^\dagger(\tilde{C}+VV^*)E_B - E_B F_M(\tilde{C}+VV^*)N^\dagger E_B - E_B F_M Z_1 E_N F_A, \\ \tilde{Y}_2 &= M^\dagger(\tilde{C}+VV^*)F_A + S^\dagger S(\tilde{C}+VV^*)N^\dagger + E_B F_M(Z_1 - S^\dagger S Z_1 NN^\dagger)F_A, \\ VV^* &= -\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C} + GG^\dagger WW^* GG^\dagger, \\ W &= P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - BB^\dagger P^\dagger H_1^{\frac{1}{2}}) + F_P F_Q Z, \\ H_1 &= A^\dagger A\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}A^\dagger A - A^\dagger A\tilde{C}A^\dagger A, \\ H_2 &= BB^\dagger\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}BB^\dagger - BB^\dagger\tilde{C}BB^\dagger. \end{aligned}$$

with $\tilde{C} = -[(A^\dagger C + F_A D B^\dagger) + (A^\dagger C + F_A D B^\dagger)^*]$, $G = \begin{pmatrix} F_A & E_B \end{pmatrix}$, $M = A^\dagger A E_B$, $N = F_A BB^\dagger$, $S = E_B F_M$, $P = A^\dagger AGG^\dagger$, $Q = BB^\dagger GG^\dagger F_P$, $T \in \mathbb{U}_m$ and $Z, Z_1 \in \mathbb{C}^{m\times m}$ are arbitrary.

Proof. The rank equality (2.20) was obtained by [Theorem 2.1, 2]. Furthermore, by [Lemma 1.1, 2], a

general common solution to $AX = C$ and $XB = D$ can be expressed as

$$X = A^{\dagger}C + F_A DB^{\dagger} + F_A Y E_B, \tag{2.22}$$

where $Y \in \mathbb{C}^{m\times m}$ is arbitrary. Therefore, there exists a common Re-nnd solution X if and only if $X + X^* \geqslant 0$ for some Y, i.e.,

$$F_A Y E_B + (F_A Y E_B)^* \geqslant -[(A^{\dagger}C + F_A DB^{\dagger}) + (A^{\dagger}C + F_A DB^{\dagger})^*] \triangleq \tilde{C} \tag{2.23}$$

is solvable. Applying Theorem 2.2 to (2.23) yields

$$Y = \frac{1}{2}(Y_1 + Y_2^*), \tag{2.24}$$

where

$$\begin{aligned} Y_1 &= F_A(\tilde{C} + VV^*)E_B - F_A E_B M^{\dagger}(\tilde{C} + VV^*)E_B - F_A S E_B(\tilde{C} + VV^*)N^{\dagger}F_A E_B \\ &\quad -F_A S Z_1 E_N F_A E_B + A^{\dagger}AZ_2 + Z_3 BB^{\dagger}, \\ Y_2 &= M^{\dagger}(\tilde{C} + VV^*)F_A + S^{\dagger}S E_B(\tilde{C} + VV^*)N^{\dagger} + F_M(Z_1 - S^{\dagger}S Z_1 NN^{\dagger}) + Z_4 A^{\dagger}A, \end{aligned}$$

with $M = A^{\dagger}AE_B$, $N = F_A BB^{\dagger}$, $S = E_B F_M$, and $Z_i, (i = 1, 2, 3, 4)$ are arbitrary matrices over complex field with appropriate sizes. Together with $F_A S = E_B F_M$, $F_M E_B = E_B F_M$, we have

$$\begin{aligned} F_A Y_1 E_B &= F_A(\tilde{C} + VV^*)E_B - F_A E_B M^{\dagger}(\tilde{C} + VV^*)E_B - F_A S E_B(\tilde{C} + VV^*)N^{\dagger}F_A E_B - F_A S Z_1 E_N F_A E_B \\ &= F_A(\tilde{C} + VV^*)E_B - F_A M^{\dagger}(\tilde{C} + VV^*)E_B - E_B F_M(\tilde{C} + VV^*)N^{\dagger}E_B - E_B F_M Z_1 E_N F_A, \\ E_B Y_2 F_A &= E_B M^{\dagger}(\tilde{C} + VV^*)F_A + E_B S^{\dagger}S E_B(\tilde{C} + VV^*)N^{\dagger}F_A + E_B F_M(Z_1 - S^{\dagger}S Z_1 NN^{\dagger})F_A \\ &= M^{\dagger}(\tilde{C} + VV^*)F_A + S^{\dagger}S(\tilde{C} + VV^*)N^{\dagger} + E_B F_M(Z_1 - S^{\dagger}S Z_1 NN^{\dagger})F_A. \end{aligned}$$

Denote $\tilde{Y}_1 = F_A Y_1 E_B$ and $\tilde{Y}_2 = E_B Y_2 F_A$. Combining (2.22) and (2.24) produces (2.21). □

Since the Re-nnd generalized inverse $A^{(1,3,4)}$ can be regarded as the common Re-nnd solution of $A^*AX = A^*$ and $XAA^* = A^*$, where $A \in \mathbb{C}^{m\times m}$, therefore, by Theorem 2.3, we have the following result.

Theorem 2.4. Let $A \in \mathbb{C}^{m\times m}$. Then there exists a Re-nnd generalized inverse $A^{(1,3,4)}$ if and only if A^*A^2, A^2A^* are Re-nnd, and

$$r\begin{pmatrix} A^*A & A^* \\ AA^* & -A \end{pmatrix} = r\begin{pmatrix} A^*A & (A^*)^2 \\ AA^* & -AA^* \end{pmatrix} = r\begin{pmatrix} A^*A & A^*A \\ AA^* & -A^2 \end{pmatrix}. \tag{2.25}$$

In this case, a general Re-nnd generalized inverse $A^{(1,3,4)}$ can be written as

$$A_{re}^{(1,3,4)} = A^{\dagger} + \frac{1}{2}(\tilde{Y}_1 + \tilde{Y}_2^*),$$

where,

$$\begin{aligned}
\tilde{Y}_1 &= F_A(\tilde{C}+VV^*)E_A - F_AM^\dagger(\tilde{C}+VV^*)E_A - E_AF_M(\tilde{C}+VV^*)N^\dagger E_A - E_AF_MZ_1E_NF_A,\\
\tilde{Y}_2 &= M^\dagger(\tilde{C}+VV^*)F_A + S^\dagger S(\tilde{C}+VV^*)N^\dagger + E_AF_M(Z_1 - S^\dagger SZ_1NN^\dagger)F_A,\\
VV^* &= -\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C} + GG^\dagger WW^*GG^\dagger,\\
W &= P^\dagger H_1^{\frac{1}{2}} + Q^\dagger(H_2^{\frac{1}{2}}T - AA^\dagger P^\dagger H_1^{\frac{1}{2}}) + F_PF_QZ,\\
H_1 &= A^\dagger A\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}A^\dagger A - A^\dagger A\tilde{C}A^\dagger A,\\
H_2 &= AA^\dagger\tilde{C}E_G(E_G\tilde{C}E_G)^\dagger E_G\tilde{C}AA^\dagger - AA^\dagger\tilde{C}AA^\dagger.
\end{aligned}$$

with $\tilde{C} = -[A^\dagger + (A^\dagger)^*]$, $G = \begin{pmatrix} F_A & E_A \end{pmatrix}$, $M = A^\dagger AE_A$, $N = F_AAA^\dagger$, $S = E_AF_M$, $P = A^\dagger AGG^\dagger$, $Q = AA^\dagger GG^\dagger F_P$, $T \in \mathbb{U}_m$ and $Z, Z_1 \in \mathbb{C}^{m\times m}$ are arbitrary.

Proof. In view of Lemma 1.8, $A^*AX = A^*$ and $XAA^* = A^*$ have Re-nnd solution if and only if A^*A^2 and A^2A^* are Re-nnd respectively. Moreover, by Theorem 2.3, these two equations have a common Re-nnd solution if and only if

$$r\begin{pmatrix} A^*A & A^* \\ AA^* & -A \end{pmatrix} = r\begin{pmatrix} A^*A & (A^*)^2A \\ AA^* & -AA^*A \end{pmatrix} = r\begin{pmatrix} A^*A & A^*AA^* \\ AA^* & -A^2A^* \end{pmatrix},$$

which is equivalent to (2.25). The formula of $A_{re}^{(1,3,4)}$ follows directly by (2.21). The proof is complete. □

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Existence and Stability Results for Quaternion Fuzzy Fractional Differential Equations

Zhanpeng Yang*, Wenjuan Ren

Institute of Electronics, Chinese Academy of Sciences, Beijing 100080,PR China

Abstract

We consider the initial value problem of quaternion fuzzy fractional differential equations in the generalized regular fuzzy function space. And we propose a notion of the disturbed fuzzy Dirac operator. By using the associate space method and fixed point theorem, a sufficient condition for the existence and stability of the solution of the initial value problem is given.

Keywords: quaternion-valued grades of membership, quaternion fuzzy fractional differential equation, associate space, generalized regular function, Hyers–Ulam stability

1 Introduction

The notion of fuzzy complex number was first proposed by Buckley in [1]. In [2], Tamir et al. pointed out the limitations of the mixed fuzzy and crisp definition of [3] and generalized it by allowing a fuzzy phase term. As illustrated with examples in [2], the advantage of this augmented definition of complex fuzzy sets is its ability to accommodate fuzzy cycles. In order to extent fuzzy complex number, the concept of the fuzzy quaternion number was introduced by Moura et al., who in [4] discuss some concepts such as their arithmetic properties, infimum, supremum, distance, and so on. The quaternion membership function was given by a mapping $u : \mathbb{H} \to [0, 1]$ such that

$$u(a + bi + cj + dk) = \min\{\bar{A}(a), \bar{B}(b), \bar{C}(c), \bar{D}(d)\},$$

where $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are all real fuzzy numbers. Yang et al. proposed a different definition of quaternion fuzzy sets and discussed entailed results which parallel those of regular fuzzy numbers in [5].

The study on fractional differential equations has been rapidly advancing in recent years. Fractional equations have received increasing attentions [6, 7, 8, 9, 10, 11]. Recently, Agarwal et al. considered a differential equation of fractional order with uncertainty and presented the concept of solution [12]. They considered the Riemann-Liouville differentiability which was a combination of Hukuhara difference and Riemann-Liouville derivative. The shortcomings of applications of Hukuhara difference was discussed in [13] by Bede and Gal. The results on existence and uniqueness of the solution were later established in [14, 15, 16, 17], and in [18, 19]. Salahshour et

*Corresponding author. Email: zhanpengyang@mail.ie.ac.cn(Z.P. Yang), iecasrwj@163.com(W.J. Ren).

al. applied fuzzy Laplace transforms to solve fuzzy differential equations [20, 21]. The numerical solution of the fuzzy differential equation was obtained in [22, 23, 24]. Furthermore, Malinowski introduced random fuzzy fractional integral equations-theoretical [25].

The study of stability problems for functional equations is related to a question of Ulam [26] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [27]. Some authors then considered the stability of the fuzzy difference and functional equations [28, 29, 30, 31]. In this paper, we consider existence and stability of the solution for quaternion fuzzy fractional differential equations. By the associate space method and fixed point theorem, we given a sufficient condition of the Hyers–Ulam stability for quaternion fuzzy fractional differential equations. Moreover, We provide a way of incorporating such the theory of fuzzy fractional differential equations into quaternionic analysis.

2 Notation and Basic results

Let $P_K(\mathbb{R}^3)$ denote the set of all nonempty convex compact subsets of $\mathbb{R}^3$. The Hausdorff metric for $A,\ B \in P_K(\mathbb{R}^3)$ is defines by

$$d(A,B) = \inf\{\varepsilon \mid A \subset N(B,\varepsilon) \text{ and } B \subset N(A,\varepsilon)\},$$

where $N(A,\varepsilon) = \{x \in \mathbb{R}^3 \mid \|x-y\| < \varepsilon \text{ for some } y \in A\}$.

Throughout this paper, we put $\Lambda := \{0,1,2,3\}$ and denote by $e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k$, where $i,\ j,\ k$ are units of the real quaternion algebra $\mathbb{H}$.

In [5], Yang et al. considered quaternion fuzzy sets on $\mathbb{R}^3$, i.e., quaternion grades of membership.

Definition 1. [5] *The quaternion membership function f is defined by*

$$f(V,x) = e_0 f_0(V) + e_1 f_1(x) + e_2 f_2(x) + e_3 f_3(x),$$

where V is to be interpreted as a set in a fuzzy set of sets and x as an element of V.

In particluar, for $x \in \mathbb{R}^3$, we have

$$f(x) = f_0(x)e_0 + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3,$$

where $f_0, f_1, f_2, f_3 : \mathbb{R}^3 \to [0,1]$. Denote f by (f_0, f_1, f_2, f_3). The $\overline{r} = (r_0, r_1, r_2, r_3)$-level sets for $f = (f_0, f_1, f_2, f_3)$ is defined by

$$[f]^{\overline{r}} = [f_0]^{r_0} \cap [f_1]^{r_1} \cap [f_2]^{r_2} \cap [f_3]^{r_3}. \tag{2.1}$$

Denote $\mathcal{F}^n$ the set of all $\nu : \mathbb{R}^n \to [0,1]$ satisfying all of the following conditions:
(i) ν is normal, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $\nu(x_0) = 1$;
(ii) ν is fuzzy convex, i.e., for all $t_1, t_2 \in \mathbb{R}^n, \lambda \in [0,1]$:

$$\nu\left(\lambda t_1 + (1-\lambda)t_2\right) \geq \min\{\nu(t_1), \nu(t_2)\};$$

(iii) ν is upper semi-continuous;
(iv) $[\nu]^0$ is compact.

Moreover, we define $\hat{\mathcal{F}}^{4n}$ as follows:

$$\hat{\mathcal{F}}^{4n} = \{(\nu_0, \nu_1, \nu_2, \nu_3) \in \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n | \\ \exists t_0,\ s.t.,\ v_l(t_0) = 1, l \in \Lambda\}.$$

Then, for $\nu = (\nu_0, \nu_1, \nu_2, \nu_3) \in \hat{\mathcal{F}}^{4n}$, $[f]^{\bar{\alpha}} = \cap_{l\in\Lambda}[\nu_l]^{\alpha_l} \in P_K(\mathbb{R}^3)$ for all $\alpha_l \in [0,1], l \in \Lambda$.

For $f, g \in \hat{\mathcal{F}}^{4n}$, where $f = (f_0, f_1, f_2, f_3)$ and $g = (g_0, g_1, g_2, g_3)$, and λ is a scalar, let

$$\begin{aligned} f + g &= (f_0 + g_0, f_1 + g_1, f_2 + g_2, f_3 + g_3), \\ \lambda f &= (\lambda f_0, \lambda f_1, \lambda f_2, \lambda f_3). \end{aligned}$$

Let us define $D : \mathcal{F}^n \times \mathcal{F}^n \to [0, \infty)$ by

$$D(\nu_1, \nu_2) = \sup\{d([\nu_1]^r, [\nu_2]^r) \mid r \in [0,1]\}, \tag{2.2}$$

where d is the Hausdorff metric. $(\mathcal{F}^n, D)$ is a metric space which can be embedded isomorphically as a cone in a Banach space [32]. However, D is not a suitable metric for our space of interest, $\hat{\mathcal{F}}^{4n}$, as we quickly see that linearity is violated. Instead, let us consider the product metric D' on $\mathcal{F}^{4n} = \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n \times \mathcal{F}^n$. For $f = (f_0, f_1, f_2, f_3) \in \mathcal{F}^{4n}$ and $g = (g_0, g_1, g_2, g_3) \in \mathcal{F}^{4n}$, we define $D' : \mathcal{F}^{4n} \times \mathcal{F}^{4n} \to [0, \infty)$ by the relation

$$\begin{aligned} D'(f, g) &= D'((f_0, f_1, f_2, f_3), (g_0, g_1, g_2, g_3)) \\ &= \max_{l\in\Lambda}\{D(f_l, g_l)\}. \end{aligned} \tag{2.3}$$

Then, D' is a linearity preserving metric for $\mathcal{F}^{4n}$. Since $\hat{\mathcal{F}}^{4n} \subset \mathcal{F}^{4n}$, D' is also a metric for $\hat{\mathcal{F}}^{4n}$. Hence, $(\hat{\mathcal{F}}^{4n}, D')$ is a complete metric space. Now, as $(\hat{\mathcal{F}}^{4n}, D')$ is a metric space and D' preserves linearity, by the Arens-Eells theorem [33] there exists an embedding $\hat{\mathcal{F}}^{4n} \hookrightarrow \mathcal{B}$ where $\mathcal{B}$ is a Banach space. The zero element on $\hat{\mathcal{F}}^{4n}$ then reads $\hat{0}_4(x) = (\hat{0}(x), \hat{0}(x), \hat{0}(x), \hat{0}(x)) \in \mathcal{F}^{4n}$.

We define strongly generalized differentiability as in [13] in terms of the generalize Hukuhara difference. For $x, y \in \hat{\mathcal{F}}^{4n}$, if there exists $z \in \hat{\mathcal{F}}^{4n}$ such that $x = z + y$ or $y = x + (-1)z$, we write $x \ominus y = z$ and call z the difference of x and y.

A fuzzy-valued function f defined in the bounded, simply connected domain $\Omega \subset \mathbb{R}^3$ is a mapping $f : \Omega \to \hat{\mathcal{F}}^{4n}$, and f can be represented in a form $f = \sum_{j=0}^{3} e_j f_j(x)$. Its conjugate $\bar{f}$ is defined by

$$\bar{f} = e_0 f_0(x) \ominus \sum_{j=1}^{3} e_j f_j(x),$$

where $f_j(x)$ are continuous fuzzy-valued functions in $x = (x_1, x_2, x_3) \in \Omega$.

Definition 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain. We call a mapping $F : \Omega \to \hat{\mathcal{F}}^{4n}$ strongly generalized partial derivative at $x = (x_1, x_2, x_3) \in \Omega$ if there exists some $\frac{\partial F}{\partial x_i} \in \hat{\mathcal{F}}^{4n}$ such that*

(i) there exists the differences $F(\cdot, x_i + h, \cdot) \ominus F(\cdot, x_i, \cdot)$,
$F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i - h, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i} = \lim_{h \to 0^+} \frac{F(\cdot, x_i + h, \cdot) \ominus F(\cdot, x_i, \cdot)}{h} = \lim_{h \to 0^+} \frac{F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i - h, \cdot)}{h}, \tag{2.4}$$

or

(ii) there exists the differences $F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i + h, \cdot)$, $F(\cdot, x_i - h, \cdot) \ominus F(\cdot, x_i, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i} = \lim_{h \to 0^+} \frac{F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i + h, \cdot)}{-h} = \lim_{h \to 0^+} \frac{F(\cdot, x_i - h, \cdot) \ominus F(\cdot, x_i, \cdot)}{-h}, \tag{2.5}$$

or

(iii) there exists the differences $F(\cdot, x_i + h, \cdot) \ominus F(\cdot, x_i, \cdot)$, $F(\cdot, x_i - h, \cdot) \ominus F(\cdot, x_i, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i} = \lim_{h \to 0^+} \frac{F(\cdot, x_i + h, \cdot) \ominus F(\cdot, x_i, \cdot)}{h} = \lim_{h \to 0^+} \frac{F(\cdot, x_i - h, \cdot) \ominus F(\cdot, x_i, \cdot)}{-h}, \tag{2.6}$$

or

(iv) there exists the differences $F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i + h, \cdot)$, $F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i - h, \cdot)$ *and*

$$\frac{\partial F}{\partial x_i} = \lim_{h \to 0^+} \frac{F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i + h, \cdot)}{-h} = \lim_{h \to 0^+} \frac{F(\cdot, x_i, \cdot) \ominus F(\cdot, x_i - h, \cdot)}{h}. \tag{2.7}$$

In general, we have the following results on the connection between the strongly generalized partial derivative of F and its endpoint function F_l^α and F_r^α.

Let $F : \Omega \to \hat{\mathcal{F}}^{4n}$ be a quaternion fuzzy function. If F is strongly generalized partial derivative at $x \in \Omega$, then we have the following case:

If F is strongly generalized partial derivative at $x \in \Omega$ in (i), then, for each $\alpha_i \in [0,1]$, F_{il} and F_{ir} are strongly generalized partial derivative functions at x and

$$\left[\frac{\partial F}{\partial x_i}\right]^\alpha = \left[\left(\frac{\partial F}{\partial x_i}\right)_l^\alpha, \left(\frac{\partial F}{\partial x_i}\right)_r^\alpha\right],$$

where

$$\left(\frac{\partial F}{\partial x_i}\right)_l^\alpha = \left[\left(\frac{\partial F}{\partial x_i}\right)_{0l}^{\alpha_0}, \left(\frac{\partial F}{\partial x_i}\right)_{1l}^{\alpha_1}, \left(\frac{\partial F}{\partial x_i}\right)_{2l}^{\alpha_2}, \left(\frac{\partial F}{\partial x_i}\right)_{3l}^{\alpha_3}\right] \tag{2.8}$$

and

$$\left(\frac{\partial F}{\partial x_i}\right)_r^\alpha = \left[\left(\frac{\partial F}{\partial x_i}\right)_{0r}^{\alpha_0}, \left(\frac{\partial F}{\partial x_i}\right)_{1r}^{\alpha_1}, \left(\frac{\partial F}{\partial x_i}\right)_{2r}^{\alpha_2}, \left(\frac{\partial F}{\partial x_i}\right)_{3r}^{\alpha_3}\right]. \tag{2.9}$$

Definition 3. *Let* $F : \Omega \to \hat{\mathcal{F}}^{4n}$ *be a continuous mapping. The fuzzy Riemann-Liouville integral of* F *is defined by*

$$(I_{0^+}^\beta F)(x) = \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i - \tau)^{\beta - 1} F(\cdot, \tau, \cdot) d\tau, \tag{2.10}$$

where $x \in \Omega, x_i > 0, 0 < \beta < 1$.

Then, the Riemann-Liouville integral of a quaternion fuzzy-valued function F can be expressed as follow:

$$(I_{0^+}^\beta F^\alpha)(x) = [(I_{0^+}^\beta F_l^\alpha)(x), (I_{0^+}^\beta F_r^\alpha)(x)],$$

where

$$(I^{\beta}_{0^+}F^{\alpha}_{l})(x)=\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-\tau)^{\beta-1}F^{\alpha}_{l}(\cdot,\tau,\cdot)\mathrm{d}\tau$$

and

$$(I^{\beta}_{0^+}F^{\alpha}_{r})(x)=\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-\tau)^{\beta-1}F^{\alpha}_{r}(\cdot,\tau,\cdot)\mathrm{d}\tau.$$

Definition 4. *The fuzzy Riemann-Liouville fractional derivatives of order $n-1<\beta<n$ for fuzzy-valued function F is defined by (provided it exists)*

$$({}^{RL}D^{\beta}_{0^+}F)(x)=\frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}F(\cdot,\tau,\cdot)\,d\tau. \tag{2.11}$$

Similarly, we have

$$({}^{RL}D^{\beta}_{0^+}F^{\alpha})(x)=[({}^{RL}D^{\beta}_{0^+}F^{\alpha}_{l})(x),({}^{RL}D^{\beta}_{0^+}F^{\alpha}_{r})(x)],$$

where $({}^{RL}D^{\beta}_{0^+}F^{\alpha}_{l})(x)=$

$$\frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}F^{\alpha}_{l}(\cdot,\tau,\cdot)\mathrm{d}\tau$$

and $({}^{RL}D^{\beta}_{0^+}F^{\alpha}_{r})(x)=$

$$\frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}F^{\alpha}_{r}(\cdot,\tau,\cdot)\mathrm{d}\tau.$$

Definition 5. *The fuzzy Caputo derivative of F for $n-1<\beta<n$ and $x\in\Omega$ is denoted by $({}^{C}D^{\beta}_{0+}F)(x)$ (provided it exists) and defined by*

$$({}^{C}D^{\beta}_{0^+}F)(x)=\frac{1}{\Gamma(n-\beta)}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}\frac{\partial^n}{\partial\tau^n}F(\cdot,\tau,\cdot)\,d\tau. \tag{2.12}$$

Then,

$$({}^{C}D^{\beta}_{0^+}F^{\alpha})(x)=[({}^{C}D^{\beta}_{0^+}F^{\alpha}_{l})(x),({}^{C}D^{\beta}_{0^+}F^{\alpha}_{r})(x)],$$

where

$$({}^{C}D^{\beta}_{0^+}F^{\alpha}_{l})(x)=\frac{1}{\Gamma(n-\beta)}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}\frac{\partial^n}{\partial\tau^n}F^{\alpha}_{l}(\cdot,\tau,\cdot)\mathrm{d}\tau$$

and

$$({}^{C}D^{\beta}_{0^+}F^{\alpha}_{r})(x)=\frac{1}{\Gamma(n-\beta)}\int_0^{x_i}(x_i-\tau)^{n-\beta-1}\frac{\partial^n}{\partial\tau^n}F^{\alpha}_{r}(\cdot,\tau,\cdot)\mathrm{d}\tau.$$

Now let us introduce the fuzzy Dirac operator as

$$D=\sum_{k=1}^{3}e_k\frac{\partial}{\partial x_k}.$$

The fuzzy Dirac operator acts on f as follows

$$Df=\sum_{k=1,j=0}^{3}e_ke_j\frac{\partial f_j}{\partial x_k}.$$

Definition 6. *The disturbed fuzzy Dirac operator is the operator which is defined by*

$$D_\beta u = Du + \beta D,$$

where β is a real number.

Definition 7. *A fuzzy function $u : \Omega \to \hat{\mathcal{F}}^{4n}$ is called a generalized regular fuzzy function if it satisfies $D_\beta u = \hat{0}_4$.*

Definition 8. *Let $L(t, x, u)$ be a first order differential operator depending on t, x, u and the first order derivative $\frac{\partial u}{\partial x_j}$, while $l(t, x, u)$ is a differential operator on the time t. Then L is called "associated" to l if L transforms solutions of $lu = \hat{0}_4$ into solutions of the same equation for fixed t, i.e. $lu = \hat{0}_4$ implies $l[Lu] = \hat{0}_4$.*

If $A : \mathcal{Y} \to \mathcal{X}$ is an operator, let us consider the fixed point equation

$$x = A(x), \quad x \in \mathcal{Y} \tag{2.13}$$

and the inequation

$$d(y, A(y)) \leq \epsilon. \tag{2.14}$$

Definition 9. *The equation* (2.13) *is called generalized Hyers–Ulam stable if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution y^* of* (2.14) *there exists a solution x^* of the fixed point equation* (2.13) *such that*

$$d(y^*, x^*) \leq \psi(\epsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}^+$, the equation (2.13) *is said to be Hyers–Ulam stable.*

3 Main results

In this section, we consider the initial value problem

$$\begin{cases} {}^C D^\alpha_{0^+t} u = \sum_{j=1}^{3} A^{(j)} \dfrac{\partial u}{\partial x_j} + Bu + C := L(u), \\ u(0, x) = \varphi(x), \end{cases} \tag{3.1}$$

where $x = (x_1, x_2, x_3) \in \Omega$ and Ω is a bounded, simply connected domain in $\mathbb{R}^3$; $t \in [0, T]$ is the time variable; ${}^C D^\alpha_{0^+t}$ is the Caputo fractional derivative of t; $u = u(t, x)$ is quaternion fuzzy-valued functions defined in $[0, T] \times \Omega$. $A^{(j)} = A^{(j)}(t, x)$; $B = B(t, x)$ and $C = C(t, x)$ are quaternion-valued functions defined in $[0, T] \times \Omega$. The initial function $\varphi(x)$ is a generalized regular fuzzy function.

It is easy to show that solutions of the initial value problem are fixed points of the operator

$$B(u) := u(t, x) = \varphi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} L(u) \mathrm{d}\tau. \tag{3.2}$$

In order to use fixed points theorem, we have to estimate the integro-differential operator on the right-hand side of (3.2). That is a little bit difficult because the integrand contains derivative with the spacelike variables x_j. But we can estimation it by using the following two properties of the associated function space:

(i) The operator maps the space into itself. Here we use the concept "associated pair" [34, 35, 36].

(ii) For the element of the associated space one has an "interior estimate" [36], that is, the norm (metric) of the derivative with respect to spacelike variables of the element of the associated space can be estimated by the norm of the element.

For our subsequent results, we need the following hypotheses.

(H1) $A_0^{(1)} = A_3^{(2)} = -A_2^{(3)}$,
$A_1^{(1)} = A_2^{(2)} = -A_3^{(3)}$,
$A_2^{(1)} = -A_1^{(2)} = A_0^{(3)}$,
$A_3^{(1)} = -A_0^{(2)} = -A_1^{(3)}$;

(H2) $(DA^{(1)} + \beta A^{(1)} - 2B_1e_0)e_1 = (DA^{(2)} + \beta A^{(2)} - 2B_2e_0)e_2 =$
$(DA^{(3)} + \beta A^{(3)} - 2B_3e_0)e_3$;

(H3) $\beta DA^{(1)}e_1 + 2\beta^2 \sum_{j=1}^{3} A_j^{(1)}e_je_1 + 2\beta^2 A_1^{(1)}e_0 + DB + 2\beta(B_2e_2 + B_3e_3) = 0$;

(H4) $D_\beta C = DC + \beta C = 0$ for each $t \in [0, T]$.

Theorem 1. *Assume that $A^{(j)}(t,x)(j = 1, 2, 3), B(t,x)$ and $C(t,x)$ are all quaternion-valued function for $t \in [0, T]$. The operator L is associated with the operator D_β if hypotheses* (H1)–(H4) *are satisfied.*

According to Definition 8, we can obtain that the operator L is associated with the operator D_β, if $D_\beta u = \hat{0}_4$ implies $D_\beta(Lu) = \hat{0}_4$. Here, we omit the proof.

To solve the initial value problem (3.1) we need the interior estimate of generalized fuzzy regular functions.

Theorem 2. *Let $\Omega_{s_1} \subset \Omega_{s_2}$ and $\overline{\Omega}_{s_2} \subset \Omega$. Let $m\Omega$ denote the finite measure of $\Omega \subset \mathbb{R}^n$ and u be a generalized fuzzy regular function. We obtain the interior estimate of generalized regular functions*

$$\begin{aligned} D^{'}\left(\frac{\partial u}{\partial x_i}, \hat{0}_4\right) &\leq \frac{\beta^2(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} D^{'}(u, \hat{0}_4) \\ &= \eta D^{'}(u, \hat{0}_4). \end{aligned} \tag{3.3}$$

Proof. Assume that u is a quaternion-valued function. By Theorem 5 in [37], we have

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{s_1} \leq \frac{\beta^2(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} \|u\|_{s_2} = \eta\|u\|_{s_2}. \tag{3.4}$$

Now, for a generalized fuzzy regular function u, we consider its endpoint function u_l^α and u_r^α. It easy to see that u_l^α and u_r^α are also generalized regular functions. Then, we obtain their interior estimate as follows:

$$\left\| \frac{\partial u_l^\alpha}{\partial x_i} \right\|_{s_1} \leq \frac{\beta^2 (\frac{3m\Omega}{4\pi})^{\frac{1}{3}} [3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} \|u_l^\alpha\|_{s_2} \tag{3.5}$$

and

$$\left\| \frac{\partial u_r^\alpha}{\partial x_i} \right\|_{s_1} \leq \frac{\beta^2 (\frac{3m\Omega}{4\pi})^{\frac{1}{3}} [3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} \|u_r^\alpha\|_{s_2}. \tag{3.6}$$

Moreover, we can obtain

$$\begin{aligned} & D^{'}\left(\frac{\partial u}{\partial x_i}, \hat{0}_4\right) = \sup_{0\leq\alpha\leq 1}\left\{ d\left(\left[\frac{\partial u}{\partial x_i}\right]^\alpha, \hat{(0)}_4 \right)\right\} \\ & = \sup_{0\leq\alpha\leq 1}\left\{ d\left(\left[\left(\frac{\partial u}{\partial x_i}\right)_l^\alpha, \left(\frac{\partial u}{\partial x_i}\right)_r^\alpha\right], \hat{(0)}_4 \right)\right\} \\ & \leq \frac{\beta^2 (\frac{3m\Omega}{4\pi})^{\frac{1}{3}} [3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s_1}, \partial\Omega_{s_2})} \\ & \sup_{0\leq\alpha\leq 1}\left\{ d([u_l^\alpha, u_r^\alpha], \hat{(0)}_4)\right\} = \eta D^{'}(u, \hat{0}_4). \end{aligned} \tag{3.7}$$

This concludes the proof. □

Theorem 3. *Assume that L satisfies the hypotheses of Theorem 1 and assume that φ is an arbitrary generalized fuzzy regular function. The initial value problem* (3.1) *is solvable in the conical domain $M_\sigma = \{(t,x) : x \in \Omega, 0 \leq t \leq \sigma \cdot dist(x, \partial\Omega)\}$($\sigma$ is small enough). The solution $u(t,x)$ is also generalized fuzzy regular function for each t. Moreover, the fixed point equation $u = B(u)$ is Hyers–Ulam stable.*

Proof. To prove this, we know that the solution of the differential equation (3.1) must satisfy the Volterra equation

$$B(u) := u(t,x) = \varphi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} L(u) \mathrm{d}\tau. \tag{3.8}$$

We then proof that the operator B has a fixed point. It is easy to see that B maps $C([0,T]\times\Omega, E^*)$ to itself. Moreover, we have

$$\begin{aligned} & D^{'}(B(u) \ominus B(v), \hat{0}_4) = D^{'}(\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} L(u) \mathrm{d}\tau, \\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} L(v) \mathrm{d}\tau) \\ & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (\sum_{j=1}^{3} A^{(j)} \frac{\partial}{\partial x_j} D^{'}(u \ominus v, \hat{0}_4) + B D^{'}(u \ominus v, \hat{0}_4) \mathrm{d}\tau \\ & \leq \frac{1}{\Gamma(\alpha)} (M + 3\eta N) D^{'}(u \ominus v, \hat{0}_4) \int_0^t (t-\tau)^{\alpha-1} \mathrm{d}\tau \\ & = \frac{1}{\Gamma(\alpha+1)} (M + 3\eta N) t^\alpha D^{'}(u \ominus v, \hat{0}_4) \\ & := \gamma D^{'}(u \ominus v, \hat{0}_4), \end{aligned} \tag{3.9}$$

where $M = \|B\|$, $N = \max_{j=1,2,3}\{\|A^{(j)}\|\}$.

We may then choose a number $\tau > 0$ such that

$$\gamma = \frac{1}{\Gamma(\alpha+1)}(M + 3\eta N)\tau^{\alpha} < 1.$$

Then in the domain $M_\sigma = \{(t,x) : x \in \Omega, 0 \le t \le \sigma \cdot dist(x, \partial\Omega) \le \tau\}$, B is a contraction mapping. Thus, by the Banach's fixed point theorem, we obtain the desired uniqueness of the solution of the differential equation. Theorem 2.10 in [38] implies that the operator B is a c-weakly Picard operator with the positive constant $c = \frac{1}{1-\gamma}$ and the fixed point equation $u = B(u)$ is Hyers–Ulam stable.

Moreover, the solution $u(t,x)$ belongs to the associated space for each t. The solution $u(t,x)$ is also generalized regular. □

Competing interests

The author declares to have no competing interests.

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Set-valued quadratic ρ-functional inequalities

Choonkil Park and Jung Rye Lee*

Abstract. In this paper, we introduce set-valued quadratic ρ-functional inequalities and prove the Hyers-Ulam stability of the set-valued quadratic ρ-functional inequalities by using the fixed point method.

1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering paper by Aumann [5] and Debreu [14] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [3], McKenzie [27], the momographs by Hindenbrand [20], Aubin and Frankowska [4], Castaing and Valadier [8], Klein and Thompson [25] and the survey by Hess [19].

The stability problem of functional equations originated from a question of Ulam [53] concerning the stability of group homomorphisms. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [42] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [52] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [13] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation

$$2f(x+y)+2f(x-y)=f(2x)+f(2y)$$

is called a *Jensen quadratic functional equation*. In particular, every solution of the Jensen quadratic functional equation is said to be a *Jensen quadratic mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 17, 18, 22, 23], [39]–[41], [43]–[51], [54, 55]).

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

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*Corresponding author.

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Let (X,d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx,Ty) \leq Ld(x,y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [9, 15] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 11, 29, 34, 35, 38]).

Let Y be a Banach space. We define the following:
2^Y : the set of all subsets of Y;
$C_b(Y)$: the set of all closed bounded subsets of Y;
$C_c(Y)$: the set of all closed convex subsets of Y;
$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.
On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$\begin{aligned} d(x,C) &= \inf\{\|x - y\| : y \in C\}, \qquad x \in Y, \\ s(x^*, C) &= \sup\{\langle x^*, x\rangle : x \in C\}, \qquad x^* \in Y^*. \end{aligned}$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h(C,C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \qquad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.2. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*
(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;
(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [8]). Debreu [14] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

Lemma 1.3. [14] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*
(a) $j(A \oplus B) = j(A) + j(B)$;
(b) $j(\lambda A) = \lambda j(A)$;
(c) $h(A, B) = \|j(A) - j(B)\|_u$;
(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$
for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [7]). In this case, the Debreu integral of f in Ω is the unique element $(D)\int_\Omega f d\nu \in C_{cb}(Y)$ such tha $j((D)\int_\Omega f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6], [30]–[33], [36, 37]).

Using the fixed point method, we prove the Hyers-Ulam stability of the following set-valued quadratic ρ-functional inequalities

$$h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \leq \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \tag{1.1}$$

and

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \rho \cdot h\left(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)\right). \tag{1.2}$$

Throughout this paper, let X be a real vector space and Y a real Banach space.

2. Stability of the set-valued quadratic ρ-functional inequality (1.1)

Throughout this section, assume that ρ is a positive real number less than $\frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued quadratic ρ-functional inequality (1.1).

Definition 2.1. Let $f : X \to C_{cb}(Y)$. The quadratic set-valued functional equation is defined by

$$f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y)$$

for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is called a *quadratic set-valued mapping*.

Definition 2.2. [26] Let $f : X \to C_{cb}(Y)$. The Jensen quadratic set-valued functional equation is defined by

$$2f(x+y) \oplus 2f(x-y) = f(2x) \oplus f(2y)$$

for all $x, y \in X$. Every solution of the Jensen quadratic set-valued functional equation is called a *Jensen quadratic set-valued mapping*.

Lemma 2.3. *Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \leq \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to (C_{cb}(Y), h)$ is a quadratic set-valued mapping.

Proof. Letting $y = x$ in (2.1), we get $h(f(2x), 4f(x)) = 0$ for all $x \in X$. Thus $f(2x) = 4f(x)$ and so

$$\begin{aligned} h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) &\leq \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \\ &= \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), 4f(x) \oplus 4f(y)\right) \\ &= 2\rho \cdot h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \end{aligned}$$

for all $x, y \in X$. Since $\rho < \frac{1}{2}$, $h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) = 0$ and so

$$f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y)$$

for all $x, y \in X$. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi\left(2x, 2y\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \leq \rho \cdot h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) + \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \leq \frac{L}{4 - 4L}\varphi(x, x) \tag{2.3}$$

for all $x \in X$.

Proof. Let $y = x$ in (2.2). Since $f(x)$ is convex, we get

$$h(f(2x), 4f(x)) \leq \varphi(x, x) \tag{2.4}$$

and so

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{4}\varphi\left(x, x\right) \tag{2.5}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ h(g(x), f(x)) \leq \mu\varphi(x, x), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16, Theorem 2.4] and [28, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g, f) = \varepsilon$. Then

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, x)$$

for all $x \in X$. Hence

$$h(Jg(x), Jf(x)) \quad = \quad h\left(4g\left(\frac{x}{2}\right), 4f\left(\frac{x}{2}\right)\right) = 4h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \leq L\varphi(x, x)$$

for all $x \in X$. So $d(g, f) = \varepsilon$ implies that $d(Jg, Jf) \leq L\varepsilon$. This means that

$$d(Jg, Jf) \leq Ld(g, f)$$

for all $g, f \in S$.

It follows from (2.5) that $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$h(f(x), Q(x)) \leq \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned}
&h\left(4^n f\left(\frac{x+y}{2^n}\right) \oplus 4^n f\left(\frac{x-y}{2^n}\right), 2 \cdot 4^n f\left(\frac{x}{2^n}\right) \oplus 2 \cdot 4^n f\left(\frac{y}{2^n}\right)\right) \\
&\leq \rho \cdot h\left(2 \cdot 4^n f\left(\frac{x+y}{2^n}\right) \oplus 2 \cdot 4^n f\left(\frac{x-y}{2^n}\right), 4^n f\left(\frac{x}{2^{n-1}}\right) \oplus 4^n f\left(\frac{y}{2^{n-1}}\right)\right) + 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&\leq \rho \cdot h\left(2 \cdot 4^n f\left(\frac{x+y}{2^n}\right) \oplus 2 \cdot 4^n f\left(\frac{x-y}{2^n}\right), 4^n f\left(\frac{x}{2^{n-1}}\right) \oplus 4^n f\left(\frac{y}{2^{n-1}}\right)\right) + L^n \varphi(x, y)
\end{aligned}$$

and so

$$h(Q(x+y)\oplus Q(x-y),2Q(x)\oplus 2Q(y))\leq \rho\cdot h\left(2Q(x+y)\oplus 2Q(x-y),Q(2x)\oplus Q(2y)\right)$$

for all $x,y\in X$. By Lemma 2.3, $Q(x+y)\oplus Q(x-y)=2Q(x)\oplus 2Q(y)$, as desired. □

Corollary 2.5. *Let $p>2$ and $\theta\geq 0$ be real numbers, and let X be a real normed space. Suppose that $f:X\to (C_{cb}(Y),h)$ is a mapping satisfying $f(0)=\{0\}$ and*

$$\begin{aligned} h(f(x+y)\oplus f(x-y),2f(x)\oplus 2f(y)) &\leq \rho\cdot h\left(2f(x+y)\oplus 2f(x-y),f(2x)\oplus f(2y)\right)\\ &+ \theta(||x||^p+||y||^p) \end{aligned} \tag{2.7}$$

for all $x,y\in X$. Then there exists a unique quadratic set-valued mapping $Q:X\to Y$ satisfying

$$h(f(x),Q(x))\leq \frac{2\theta}{2^p-4}||x||^p$$

for all $x\in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$$

for all $x,y\in X$. Then we can choose $L=2^{2-p}$ and we get the desired result. □

Theorem 2.6. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x,y\in X$. Suppose that $f:X\to(C_{cb}(Y),h)$ is a mapping satisfying $f(0)=\{0\}$ and (2.2). Then there exists a unique quadratic set-valued mapping $Q:X\to(C_{cb}(Y),h)$ such that

$$h(f(x),Q(x))\leq \frac{1}{4-4L}\varphi(x,x)$$

for all $x\in X$.

Proof. It follows from (2.4) that

$$h\left(f(x),\frac{1}{2}f\left(2x\right)\right)\leq \frac{1}{4}\varphi\left(x,x\right)$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Corollary 2.7. *Let $2>p>0$ and $\theta\geq 0$ be real numbers, and let X be a real normed space. Suppose that $f:X\to(C_{cb}(Y),h)$ is a mapping satisfying $f(0)=\{0\}$ and (2.7). Then there exists a unique quadratic set-valued mapping $Q:X\to Y$ satisfying*

$$h(f(x),Q(x))\leq \frac{2\theta}{4-2^p}||x||^p$$

for all $x\in X$.

Proof. The proof follows from Theorem 2.6 by taking

$$\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$$

for all $x,y\in X$. Then we can choose $L=2^{p-2}$ and we get the desired result. □

3. Stability of the set-valued quadratic ρ-functional inequality (1.2)

Throughout this section, assume that ρ is a positive real number less than 2.

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued quadratic ρ-functional inequality (1.2).

Lemma 3.1. *Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \rho \cdot h\left(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to (C_{cb}(Y), h)$ is a Jensen quadratic set-valued mapping.

Proof. Letting $y = 0$ in (3.1), we get $h(4f(x), f(2x)) = 0$ for all $x \in X$. Thus $f(2x) = 4f(x)$ and so

$$\begin{aligned} 2h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) &= h(2f(x+y) \oplus 2f(x-y), 4f(x) \oplus 4f(y)) \\ &= h\left(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)\right) \\ &\leq \rho \cdot h\left(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)\right) \end{aligned}$$

for all $x, y \in X$. Since $\rho < 2$, $h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) = 0$ and so

$$2f(x+y) \oplus 2f(x-y) = f(2x) \oplus f(2y)$$

for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi\left(2x, 2y\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \rho \cdot h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) + \varphi(x, y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \leq \frac{L}{4 - 4L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.2). Since $f(x)$ is convex, we get

$$h(f(2x), 4f(x)) \leq \varphi(x, 0) \tag{3.3}$$

and

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{4}\varphi(x, 0) \tag{3.4}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ h(g(x), f(x)) \leq \mu\varphi(x, 0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16, Theorem 2.4] and [28, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.4, one can show that

$$d(Jg, Jf) \leq Ld(g, f)$$

for all $g, f \in S$.

It follows from (3.4) that $d(f, Jf) \leq \frac{L}{4}$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Corollary 3.3. *Let $p > 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$\begin{aligned} h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) &\leq \rho \cdot h(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \\ &+ \theta(||x||^p + ||y||^p) \end{aligned} \tag{3.5}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to Y$ satisfying

$$h(f(x), Q(x)) \leq \frac{\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.2). Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \leq \frac{1}{4 - 4L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$h\left(f(x), \frac{1}{4}f\left(2x\right)\right) \leq \frac{1}{4}\varphi\left(x, 0\right)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.4 and 3.2. □

Corollary 3.5. *Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.5). Then there exists a unique Jensen quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h(f(x), Q(x)) \leq \frac{\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

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Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

Jung Rye Lee
Department of Mathematics, Daejin University, Kyunggi 11159, Republic of Korea
E-mail address: jrlee@daejin.ac.kr

APPROXIMATE TERNARY QUADRATIC 3-DERIVATIONS ON TERNARY BANACH ALGEBRAS AND C^*-TERNARY RINGS

HOSSEIN PIRI*, SHAGHAYEGH ASLANI, VAHID KESHAVARZ, THEMISTOCLES M. RASSIAS, CHOONKIL PARK* AND YOUNG SUN PARK*

Abstract. In the current article, we use a fixed point alternative theorem to establish the Hyers-Ulam stability and also the superstability of a ternary quadratic 3-derivation on ternary Banach algebras and C^*-ternary rings.

1. Introduction and preliminaries

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [22]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [25] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (see [1, 35]). The comments on physical applications of ternary structures can be found in ([6, 12, 17, 18, 26, 27, 31]).

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q). A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation? If the problem accepts a unique solution, we say the equation is stable. Also, if every approximately solution is an exact solution of it, we say the functional equation is superstable (see [3]). The first stability problem concerning group homomorphisms was raised by Ulam [34] and partially solved by Hyers [20]. In [29], Rassias [16] generalized the result of Hyers for approximately linear mappings. Gajda [15] answered the question for another case of linear mapping, which was rased by Rassias. The stability problems of various functional equations have been extensively investigated by a number of authors (see [13, 14, 21]).

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called quadratic functional equation. In addition, every solution of the above equation is said to be a quadratic mapping. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [33] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Later, Czerwik [7] proved the Cauchy-Rassias stability of the quadratic functional equation. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (see [4, 9, 11, 23, 28]). As it is extensively discussed in [30], the full description of a physical system S implies the knowledge of three basic ingredients: the set of

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*Corresponding authors.

the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally the set of the observables were considered to be a C^*-algebra [19]. In many applications, however, this was shown not to be the most convenient choice, and so the C^*-algebra was replaced by a von Neumann algebra. This is because the role of the representation turns out to be crucial, mainly when long range interactions are involved. Here we used a different algebraic structure.

A ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is linear in the outer variables, conjugate linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u]v\Big] = \Big[[x, y, z], u, v\Big]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$. A C^*-ternary ring is a complex Banach space, A equipped with a ternary product which is associative and linear in the outer variables, conjugate linear in the middle variable, and $\|[x, x, x]\| = \|x\|^3$ (see [37]).

If a C^*-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with the operation $xoy := [x, e, y], \quad x^* := [e, x, e]$ is a unital C^*-algebra. Conversely, if (A, o) is a unital C^*-algebra, then $[x, y, z] := xoy^*oz$ makes A into a C^*-ternary ring.

Recently, Shagholi et al. [32] proved the stability of ternary quadratic derivations on ternary Banach algebras. Moslehian investigated the stability and the superstability of ternary derivations on C^*-ternary rings [24]. Xu et al. [36] used the fixed point alternative (Theorem 4.2 of current article) to establish the Hyers-Ulam stability of the general mixed additive-cubic functional equation, where functions map a linear space into a complete quasi fuzzy p-normed space. The Hyers-Ulam stability of an additive-cubic-quartic functional equation in NAN -spaces was also proved by using the mentioned theorem in [2].

In this article, we prove the Hyers-Ulam stability and the superstability of ternary quadratic 3-derivations on ternary Banach algebras and C^*-ternary rings associated with the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ using the fixed point theorem.

2. **Stability of ternary quadratic 3-derivations**

Throughout this article, for a ternary Banach algebra (or C^*-ternary ring) A , we denote

$$\overbrace{A \times A \times \cdots \times A}^{n-times}$$

by A^n.

Definition 2.1. *Let A be a ternary Banach algebra or C^*-ternary ring. Then a mapping $D : A \to A$ is called a ternary quadratic 3-derivation if it is a quadratic mapping that satisfies*

$$\begin{aligned}
&D\Big(\Big[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]\Big]\Big) \\
&= \Big[D([x_1, x_2, x_3]), [y_1, y_2, y_3], \Big[[y_1^*, y_2^*, y_3^*], [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\
&+ \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[D([y_1^*, y_2^*, y_3^*]), [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\
&+ \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[[y_1^*, y_2^*, y_3^*], [y_1, y_2, y_3], D([z_1, z_2, z_3])\Big]\Big]
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

It was proved in [10] that for the vector spaces X and Y and a fixed positive integer k, the mapping $f : X \to Y$ is quadratic if and only if the following equality holds:

$$2f\left(\frac{kx+ky}{2}\right)+2f\left(\frac{kx-ky}{2}\right)=k^2f(x)+k^2f(y)$$

for all $x, y \in X$. Also, we can show that f is quadratic if and only if for a fixed positive integer k, we have

$$f(kx+ky)+f(kx-ky)=2k^2f(x)+2k^2f(y)$$

for all $x, y \in X$. Before proceeding to the main results, to achieve our aim, we need the following known fixed point theorem which has been proved in [8].

Theorem 2.2. *Suppose that (Ω, d) is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d(J^n x, J^{n+1}x)=\infty, \quad \forall n \geq 0,$$

or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

In the following theorem, we prove the Hyers-Ulam stability of ternary quadratic 3-derivation on C^*-ternary rings.

Theorem 2.3. *Let A be a C^*-ternary ring, $f : A \to A$ be a mapping with $f(0) = 0$, and also let $\varphi : A^{11} \to [0, \infty)$ be a function such that*

$$\left\|2f\left(\frac{\mu a+\mu b}{2}\right)+2f\left(\frac{\mu a-\mu b}{2}\right)-\mu^2(f(a)+f(b))\right\| \leq \varphi(a,b,0,0,0,0,0,0,0,0,0), \tag{1}$$

$$\begin{aligned}&\Big\|f\Big(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\Big)\\&\quad-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\&\quad\leq \varphi(0,0,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3)\end{aligned} \tag{2}$$

for all $\mu \in T = \{\lambda \in C : |\lambda| = 1\}$ and all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. If there exists a constant $M \in (0, 1)$ such that

$$\begin{aligned}&\varphi(2a,2b,2x_1,2x_2,2x_3,2y_1,2y_2,2y_3,2z_1,2z_2,2z_3)\\&\quad\leq 4M\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3)\end{aligned} \tag{3}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$, then there exists a unique ternary quadratic 3-derivation $D : A \to A$ such that

$$\|f(a)-D(a)\| \leq \frac{M}{1-M}\psi(a) \tag{4}$$

for all $a \in A$, where $\psi(a) = \varphi(a, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Proof. It follows from (3) that

$$\lim_{j\to\infty} \frac{\varphi(2^j a, 2^j b, 2^j x_1, 2^j x_2, 2^j x_3, 2^j y_1, 2^j y_2, 2^j y_3, 2^j z_1, 2^j z_2, 2^j z_3)}{4^j} = 0 \tag{5}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Putting $\mu = 1, b = 0$ and replacing a by $2a$ in (1), we have

$$\Big\|4f(a) - f(2a)\Big\| \leq \psi(2a) \leq \psi(2a) \leq 4M\psi(a)$$

and so

$$\left\|f(a) - \frac{1}{4}f(2a)\right\| \leq M\psi(a) \tag{6}$$

for all $a \in A$. We consider the set $\Omega := \{h : A \to A \,|h(0) = 0\}$ and introduce the generalized metric on X as follows:

$$d(h_1, h_2) := inf\{K \in (0, \infty) : \|h_1(a) - h_2(a)\| \leq K\psi(a), \forall a \in A\},$$

if there exists such a constant K, and $d(h_1, h_2) = \infty$, otherwise. One can show that (Ω, d) is a complete metric space. We now show that $J : \Omega \to \Omega$ by

$$J(h)(a) = \frac{1}{4}h(2a) \tag{7}$$

for all $a \in A$. Given $h_1, h_2 \in \Omega$, let $K \in R^+$ an arbitrary constant with $d(h_1, h_2) \leq K$, that is,

$$d(h_1(a), h_2(a)) \leq K\psi(a) \tag{8}$$

for all $a \in A$. Substituting a by $2a$ in (8) and using (3) and (7), we have

$$\|(Jh_1)(a) - (Jh_2)(a)\| = \frac{1}{4}\|h_1(2a) - h_2(2a)\| \leq \frac{1}{4}K\psi(2a) \leq KM\psi(a)$$

for all $a \in A$ and thus $d(Jh_1, Jh_2) \leq KM$. Therefore, we conclude that $d(Jh_1, Jh_2) \leq Md(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. It follows from (6) that

$$d(Jf, f) \leq M. \tag{9}$$

By Theorem 2.2, the sequence $\{J^n f\}$ converges to a unique fixed point $D : A \to A$ in the set $\Omega_1 = \{h \in \Omega, d(f, h) < \infty\}$, i.e.,

$$\lim_{n\to\infty} \frac{2^n a}{4^n} = D(a), \tag{10}$$

for all $a \in A$. By Theorem 2.2 and (9), we have

$$d(f, D) \leq \frac{d(Jf, f)}{1 - M} \leq \frac{M}{1 - M}.$$

The last inequality shows that (4) holds for all $a \in A$. Replace $2^n a$ and $2^n b$ by a and b, respectively. Now, dividing both sides of the resulting inequality by 2^n, and letting n goes to infinity, we obtain

$$2D\left(\frac{\mu a + \mu b}{2}\right) + 2D\left(\frac{\mu a - \mu b}{2}\right) = \mu^2(D(a) + D(b)) \tag{11}$$

for all $a, b \in A$ and $\mu \in T$. Putting $\mu = 1$ in (11), we have

$$2D\left(\frac{a+b}{2}\right) + 2D\left(\frac{a-b}{2}\right) = D(a) + D(b)$$

for all $a, b \in A$. Hence D is a quadratic mapping by [33, Proposition 1]. So it follows from the definition of D, (2), (5) and (10) that

$$\begin{aligned}
&D\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&\quad-\Big[D([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[D([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],D([z_1,z_2,z_3])\Big]\Big]\\
&=\lim_{n\to\infty}\left(\frac{1}{4^{9n}}f\left(\Big[[2^nx_1,2^nx_2,2^nx_3],[2^ny_1,2^ny_2,2^ny_3],[2^nz_1,2^nz_2,2^nz_3]\Big]\right)\right.\\
&\quad-\Big[\frac{1}{4^{3n}}f\left([2^nx_1,2^nx_2,2^nx_3]\right),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[\frac{1}{4^{3n}}f([2^ny_1^*,2^ny_2^*,2^ny_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad\left.-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],\frac{1}{4^{3n}}f([2^nz_1,2^nz_2,2^nz_3])\Big]\Big]\right)\\
&\le\lim_{n\to\infty}\frac{1}{4^{9n}}\varphi(0,0,2^nx_1,2^nx_2,2^nx_3,2^ny_1,2^ny_2,2^ny_3,2^nz_1,2^nz_2,2^nz_3)=0
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$ and so

$$\begin{aligned}
&D\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&=\Big[D([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[D([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],D([z_1,z_2,z_3])\Big]\Big],
\end{aligned}$$

which means that D is a ternary quadratic 3-derivation. □

Corollary 2.4. *Let p, θ be nonnegative real numbers such that $p < 2$ and let f be a mapping on a C^*-ternary ring A with $f(0) = 0$ and*

$$\left\|2f\left(\frac{\mu a+\mu b}{2}\right)+2f\left(\frac{\mu a-\mu b}{2}\right)-\mu^2(f(a)+f(b))\right\| \le \theta(\|a\|^p+\|b\|\|^p),$$

$$\begin{aligned}
&\Big\|f\Big(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\Big)\\
&\quad-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\
&\quad\le\theta(\|x_1\|^p+\|x_2\|^p+\|x_3\|^p+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)
\end{aligned}$$

for all $\mu\in T$ and all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Then there exists a unique ternary quadratic 3-derivation $D:A\to A$ satisfying

$$\|f(a)-D(a)\|\le\frac{2^p\theta}{4-2^p}\|a\|^p$$

for all $a\in A$.

Proof. The result follows from Theorem 2.3 by putting

$$\begin{aligned}
\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3):=\theta(\|a\|^p+\|b\|\|^p+\|x_1\|^p+\|x_2\|^p+\|x_3\|^p\\
+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)
\end{aligned}$$

for all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. □

Now, we establish the superstability of ternary quadratic 3-derivations on C^*-ternary rings as follows:

Corollary 2.5. *Let p,θ be nonnegative real numbers such that $11p<2$ and let f be a mapping on a C^*-ternary ring A with $f(0)=0$ and*

$$\Big\|2f\left(\frac{\mu a+\mu b}{2}\right)+2f\left(\frac{\mu a-\mu b}{2}\right)-\mu^2(f(a)+f(b))\Big\|\le\theta\cdot\|a\|^p\cdot\|b\|^p, \tag{12}$$

$$\begin{aligned}
&\Big\|f\Big(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\Big)\\
&\quad-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\
&\quad\le\theta\cdot\|x_1\|^p\cdot\|x_2\|^p\cdot\|x_3\|^p\cdot\|y_1\|^p\cdot\|y_2\|^p\cdot\|y_3\|^p\cdot\|z_1\|^p\cdot\|z_2\|^p\cdot\|z_3\|^p
\end{aligned}$$

for all $\mu\in T$ and all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Then f is a ternary quadratic 3-derivation on A.

Proof. Putting $a=b=0$ in (12), we get $f(0)=0$. Now, if we put $b=0$, $\mu=1$ and replace a by $2a$ in (12), then we have $f(2a)=4f(a)$ for all $a\in A$. It is easy to see by induction that

$f(2^n a) = 4^n f(a)$, and so $f(a) = \frac{f(2^n a)}{4^n}$ for all $a \in A$ and $n \in N$. It follows from Theorem 2.3 that f is a quadratic mapping. Putting

$$\varphi(a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$$

$$:= \theta \cdot \|a\|^p \cdot \|b\|^p \cdot \|x_1\|^p \cdot \|x_2\|^p \cdot \|x_3\|^p \cdot \|y_1\|^p \cdot \|y_2\|^p \cdot \|y_3\|^p \cdot \|z_1\|^p \cdot \|z_2\|^p \cdot \|z_3\|^p$$

in Theorem 2.3, we can obtain the desired result. □

Theorem 2.6. *Let A be a ternary Banach algebra, and let $f : A \to A$ be a mapping with $f(0) = 0$, and also let $\varphi : A^5 \to [0, \infty)$ be a function such that*

$$\left\| f(\mu a + \mu b) + f(\mu a - \mu b) - 2\mu^2 (f(a) + f(b)) \right\| \leq \varphi(a, b, 0, 0, 0, 0, 0, 0, 0, 0, 0) \tag{13}$$

$$\begin{aligned} &\Big\| f\Big(\Big[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]\Big]\Big) \\ &\quad - \Big[f([x_1, x_2, x_3]), [y_1, y_2, y_3], \Big[[y_1^*, y_2^*, y_3^*], [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\ &\quad - \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[f([y_1^*, y_2^*, y_3^*]), [z_1^*, z_2^*, z_3^*], [z_1, z_2, z_3]\Big]\Big] \\ &\quad - \Big[[x_1, x_2, x_3], [x_1^*, x_2^*, x_3^*], \Big[[y_1^*, y_2^*, y_3^*], [y_1, y_2, y_3], f([z_1, z_2, z_3])\Big]\Big]\Big\| \\ &\quad \leq \varphi(0, 0, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \end{aligned} \tag{14}$$

for all $\mu \in T$ and all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. If there exists a constant $m \in (0, 1)$ such that

$$\begin{aligned} &\varphi(2a, 2b, 2x_1, 2x_2, 2x_3, 2y_1, 2y_2, 2y_3, 2z_1, 2z_2, 2z_3) \\ &\quad \leq 4m\varphi(a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \end{aligned} \tag{15}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$, then there exists a unique ternary quadratic 3-derivation $D : A \to A$ satisfying

$$\|f(a) - D(a)\| \leq \frac{4m}{1-m}\psi(a) \tag{16}$$

for all $a \in A$, where $\psi(a) = \varphi(a, a, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Proof. Using (15), we obtain

$$\lim_{n\to\infty} \frac{\varphi(2^n a, 2^n b, 2^n x_1, 2^n x_2, 2^n x_3, 2^n y_1, 2^n y_2, 2^n y_3, 2^n z_1, 2^n z_2, 2^n z_3)}{4^n} = 0 \tag{17}$$

for all $a, b, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Putting $\mu = 1, a = b$ and replacing a by $2a$ in (13), we get

$$\left\| f(2a) - 4f(a) \right\| \leq \psi(a)$$

for all $a \in A$. By the last inequality, we have

$$\left\| \frac{1}{4} f(2a) - f(a) \right\| \leq \frac{1}{4}\psi(a) \tag{18}$$

for all $a \in A$. Similar to the proof of Theorem 2.3, we consider the set $\Omega := \{h : A \to A | h(0) = 0\}$ and introduce a generalized metric on Ω by

$$d(g, h) := inf\{C \in (0, \infty) : \|g(a) - h(a)\| \leq C\psi(a), \forall a \in A\},$$

if there exists such constant C, and $d(g,h)=\infty$, otherwise. Again, it is easy to check the fact that (Ω,d) is a complete metric space. We now define the linear mapping $T:\Omega\to\Omega$ by

$$T(h)(a)=\frac{1}{4}h(2a) \tag{19}$$

for all $a\in A$. For arbitrary elements $g,h\in\Omega$ and $C\in(0,\infty)$ with $d(g,h)\leq C$, we have

$$\|g(a)-h(a)\|\leq C\psi(a) \tag{20}$$

for all $a\in A$. Replacing a by 2a in the inequality (20) and using (15) and (19), we we have

$$\|(Tg)(a)-(Th)(a)\|=\frac{1}{4}\|G(2a)-h(2a)\|\leq\frac{1}{4}C\psi(2a)\leq Cm\psi(a)$$

for all $a\in A$. Thus $d(Tg,Th)\leq Cm$. Therefore, we conclude that $d(Tg,Th)\leq md(g,h)$ for all $g,h\in\Omega$. It follows from (18) that

$$d(Tf,f)\leq\frac{1}{4}. \tag{21}$$

Hence T is a strictly contractive mapping on Ω. Now, Theorem 2.2 shows that T has a unique fixed point $D:A\to A$ in the set $\Omega_1=\{h\in\Omega,d(f,h)<\infty\}$. On the other hand,

$$\lim_{n\to\infty}\frac{2^n a}{4^n}=D(a) \tag{22}$$

for all $a\in A$. By Theorem 2.2 and (21), we obtain

$$d(f,D)\leq\frac{d(Tf,f)}{1-m}\leq\frac{m}{4(1-m)},$$

i.e., the inequality (16) is true for all $a\in A$. Let us replace a and b in (13) by $2^n a$ and $2^n b$ respectively, and then divide both sides by 2^n. Passing to the limit as $n\to\infty$, we get

$$D(\mu a+\mu b)+D(\mu a-\mu b)=2\mu^2D(a)+2\mu^2D(b) \tag{23}$$

for all $a,b\in A$ and $\lambda\in T$. Putting $\mu=1$ in (23), we have

$$D(a+b)+D(a-b)=2D(a)+2D(b) \tag{24}$$

for all $a,b\in A$. Hence D is a quadratic mapping.

It follows from (14) that

$$\begin{aligned}
&\Big\|\frac{1}{4^{9n}}f\Big(\Big[[2^nx_1,2^nx_2,2^nx_3],[2^ny_1,2^ny_2,2^ny_3],[2^nz_1,2^nz_2,2^nz_3]\Big]\Big)\\
&\quad-\Big[\frac{1}{4^{3n}}f([2^nx_1,2^nx_2,2^nx_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[\frac{1}{4^{3n}}f([2^ny_1^*,2^ny_2^*,2^ny_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],\frac{1}{4^{3n}}f([2^nz_1,2^nz_2,2^nz_3])\Big]\Big]\Big\|\\
&\quad\leq\frac{1}{4^{9n}}\varphi(0,0,2^nx_1,2^nx_2,2^nx_3,2^ny_1,2^ny_2,2^ny_3,2^nz_1,2^nz_2,2^nz_3)
\end{aligned} \tag{25}$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$.

Taking the limit in the equality (25) and using (17), one obtains that

$$\begin{aligned}
&D\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&=\Big[D([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[D([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&+\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],D([z_1,z_2,z_3])\Big]\Big]
\end{aligned}$$

for all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Therefore, D is a ternary quadratic 3-derivation. This completes the proof. □

The following corollaries are some applications to show the stability and superstability of ternary quadratic 3-derivations under some conditions.

Corollary 2.7. *Let A be a ternary Banach algebra. Let p,θ be nonnegative real numbers such that $p<2$ and let f be a mapping on a C^*-ternary ring A with $f(0)=0$ and*

$$\Big\|f(\mu a+\mu b)+f(\mu a-\mu b)-2\mu^2(f(a)+f(b))\Big\|\le\theta(\|a\|^p+\|b\|\|^p),$$

$$\begin{aligned}
&\Big\|f\left(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\right)\\
&-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\
&\le\theta(\|x_1\|^p+\|x_2\|^p+\|x_3\|^p+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)
\end{aligned}$$

for all $\mu\in T$ and all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. Then there exists a unique ternary quadratic 3-derivation $D:A\to A$ satisfying

$$\|f(a)-D(a)\|\le\frac{2^p\theta}{4-2^p}\|a\|^p$$

for all $a\in A$.

Proof. The result follows from Theorem 2.6 by putting

$$\begin{aligned}
\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3):=\theta(&\|a\|^p+\|b\|\|^p+\|x_1\|^p+\|x_2\|^p+\|x_3\|^p\\
&+\|y_1\|^p+\|y_2\|^p+\|y_3\|^p+\|z_1\|^p+\|z_2\|^p+\|z_3\|^p)
\end{aligned}$$

for all $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. □

Corollary 2.8. *Let p,θ be nonnegative real numbers such that $11p<2$ and let f be a mapping on a C^*-ternary ring A with $f(0)=0$ and*

$$\Big\|f(\mu a+\mu b)+f(\mu a-\mu b)-2\mu^2(f(a)+f(b))\Big\|\le\theta(\|a\|^p\|b\|^p),\tag{26}$$

$$\begin{aligned}
&\Big\| f\Big(\Big[[x_1,x_2,x_3],[y_1,y_2,y_3],[z_1,z_2,z_3]\Big]\Big)\\
&\quad-\Big[f([x_1,x_2,x_3]),[y_1,y_2,y_3],\Big[[y_1^*,y_2^*,y_3^*],[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[f([y_1^*,y_2^*,y_3^*]),[z_1^*,z_2^*,z_3^*],[z_1,z_2,z_3]\Big]\Big]\\
&\quad-\Big[[x_1,x_2,x_3],[x_1^*,x_2^*,x_3^*],\Big[[y_1^*,y_2^*,y_3^*],[y_1,y_2,y_3],f([z_1,z_2,z_3])\Big]\Big]\Big\|\\
&\qquad\le\theta\cdot\|x_1\|^p\cdot\|x_2\|^p\cdot\|x_3\|^p\cdot\|y_1\|^p\cdot\|y_2\|^p\cdot\|y_3\|^p\cdot\|z_1\|^p\cdot\|z_2\|^p\cdot\|z_3\|^p
\end{aligned}$$

for all $\mu\in T$ *and all* $a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3\in A$. *Then* f *is a ternary quadratic* 3*-derivation on* A.

Proof. If we put $a=b=0$ in (26), then we have $f(0)=0$. Moreover, letting $b=0$, $\mu=1$ and replacing a by $2a$ in (26), we obtain $f(2a)=4f(a)$ for all $a\in A$. Similar to the proof of Corollary 2.5, we can show that f is a quadratic mapping. Putting

$$\begin{aligned}
&\varphi(a,b,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3)\\
&\qquad:=\theta\cdot\|a\|^p\cdot\|b\|^p\cdot\|x_1\|^p\cdot\|x_2\|^p\cdot\|x_3\|^p\cdot\|y_1\|^p\cdot\|y_2\|^p\cdot\|y_3\|^p\cdot\|z_1\|^p\cdot\|z_2\|^p\cdot\|z_3\|^p
\end{aligned}$$

in Theorem 2.6, we can obtain the desired result. □

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Hossein Piri, Shaghayegh Aslani

Department of Mathematics, University of Bonab, Bonab 5551761167, Iran

E-mail address: hossein.piri1979@yahoo.com; aslani.shaghayegh@gmail.com

Vahid Keshavarz

Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran

E-mail address: v.keshavarz68@yahoo.com

Themistocles M. Rassias

Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780, Athens, Greece

E-mail address: trassias@math.ntua.gr

Choonkil Park

Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea

E-mail address: baak@hanyang.ac.kr

Young Sun Park

Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea

E-mail address: pppppys@hanyang.ac.kr

Existence results for a coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals

Suthep Suantai [a], S.K. Ntouyas [b,c] and Jessada Tariboon [d,*]

[a] Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200 Thailand
e-mail: suthep.s@cmu.ac.th

[b] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

[c] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: sntouyas@uoi.gr

[d] Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
e-mail: jessada.t@sci.kmutnb.ac.th

Abstract

In this paper, we introduce a new class of coupled systems of boundary value problems for fractional differential equations which contains multiple orders of fractional derivatives and integrals, and discuss the existence and uniqueness of solutions. We apply Leray-Schauder's alternative and Banach's contraction mapping principle to obtain the desired results. Illustrative examples is also included.

Key words and phrases: Fractional differential systems; nonlocal boundary conditions; integral boundary conditions; fixed point theorem.
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1 Introduction

Differential equations of fractional order have played a significant role in engineering, science, and pure and applied mathematics in recent years. Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [1]-[4]. Fractional-order boundary value problems involving a variety of classical, nonlocal and integral boundary conditions have been addressed by many authors, for instance, see [5]-[13] and the references cited therein.

Coupled systems of fractional-order differential equations also constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [14], disease models [15]-[18], ecological models [19], synchronization of chaotic systems [20]-[22], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [23]-[28].

Recently in [29] a new class of fractional boundary valued problems was introduced, which contains four orders of Riemann-Liouville fractional derivatives, two in fractional differential equation and two

*Corresponding author

in boundary conditions of the form

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right) x(t) = f(t, x(t)), \quad t \in (0,T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \end{cases} \tag{1}$$

where D^{ϕ} is the Riemann-Liouville fractional derivative of order $\phi \in \{\alpha, \beta, \gamma_1, \gamma_2\}$ such that $1 < \alpha, \beta \leq 2$ and $0 < \gamma_1, \gamma_2 < \alpha - \beta$, $\gamma_3 \in \mathbb{R}$, the given constants $0 < \lambda \leq 1$, $0 \leq \mu \leq 1$ and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ is a continuous function. Existence and uniqueness results were obtained by means of Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative.

In this paper, we study a coupled system of fractional differential equations

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right) x(t) = f(t, x(t), y(t)), \quad t \in (0,T), \quad 1 < \alpha, \beta \leq 2 \\ \left(\lambda_1 D^{\alpha_1} + (1-\lambda_1)D^{\beta_1}\right) y(t) = g(t, x(t), y(t)), \quad t \in (0,T), \quad 1 < \alpha_1, \beta_1 \leq 2, \end{cases} \tag{2}$$

subject to the following type of boundary conditions

$$\begin{cases} x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \\ y(0) = 0, \quad \mu_1 I^{\delta_1} y(T) + (1-\mu_1) I^{\delta_2} y(T) = \delta_3, \end{cases} \tag{3}$$

where D^{ϕ} denotes the Caputo fractional derivatives of order $\phi \in \{\alpha, \beta, \alpha_1, \beta_1, \gamma_1, \gamma_2\}$, I^{χ} denotes the Riemann-Liouville fractional integral of order $\chi \in \{\delta_1, \delta_2\}$, $\gamma_3, \delta_3 \in \mathbb{R}$, $0 < \lambda, \lambda_1 \leq 1$, $0 \leq \mu, \mu_1 \leq 1$ and $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are appropriately chosen functions.

The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present two auxiliary lemmas. The main results are presented in Section 3. We give two results: the first one derives the existence of solutions via Leray-Schauder's alternative, whereas the second one concerning existence and uniqueness of solutions is established by Banach's contraction principle. We also discuss two examples for illustration of the existence-uniqueness results.

2 Preliminaries

Before presenting two auxiliary lemmas, we recall some basic definitions of fractional calculus [1, 2].

Definition 2.1 *For $(n-1)-$times absolutely continuous function $y : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^{c}D^{q} y(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} y^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^{q} y(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3 *The boundary value problem*

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)D^{\beta}\right) x(t) = \omega(t), \quad t \in (0,T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \end{cases} \tag{4}$$

is equivalent to the following integral equation

$$x(t) = \frac{\lambda - 1}{\lambda \Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds$$

$$
\begin{aligned}
&+\frac{t}{\Lambda_1}\Bigg(\gamma_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\
&-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds\\
&-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\\
&-\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}\omega(s)ds\Bigg),\quad t\in J:=[0,T],
\end{aligned}
\tag{5}
$$

where the non zero constant Λ_1 *is defined by*

$$
\Lambda_1=\frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}+\frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}. \tag{6}
$$

Proof. The first equation of (4) can be rewritten as

$$
D^\alpha x(t)=\frac{\lambda-1}{\lambda}D^\beta x(t)+\frac{1}{\lambda}\omega(t),\quad t\in J. \tag{7}
$$

Applying the Riemann-Liouville fractional integral of order α to both sides of (7), we obtain

$$
x(t) = \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\omega(s)ds+\ C_1+C_2t,
$$

where constants C_1, $C_2\in\mathbb{R}$. The first boundary condition of (4) implies that $C_1=0$. Hence

$$
x(t)=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\omega(s)ds+C_2t. \tag{8}
$$

Taking the Caputo fractional derivative of order $\psi\in\{\gamma_1,\gamma_2\}$ such that $0<\psi<\alpha-\beta$ to (8), we deduce that

$$
\begin{aligned}
D^\psi x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta-\psi)}\int_0^t(t-s)^{\alpha-\beta-\psi-1}x(s)ds\\
&\quad+\frac{1}{\lambda\Gamma(\alpha-\psi)}\int_0^t(t-s)^{\alpha-\psi-1}\omega(s)ds+C_2\frac{1}{\Gamma(2-\psi)}t^{1-\psi}.
\end{aligned}
$$

Substituting the values $\psi=\gamma_1$ and $\psi=\gamma_2$ to the above relation and using the second condition of (4), we obtain a constant γ_3 as

$$
\begin{aligned}
\gamma_3 &= \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\
&\quad+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds+\frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}C_2\\
&\quad+\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\\
&\quad+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}\omega(s)ds+\frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}C_2,
\end{aligned}
$$

which yields

$$
C_2 = \frac{1}{\Lambda_1}\Bigg[\gamma_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds
$$

$$-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds-\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}\omega(s)ds\Bigg].$$

Substituting the value of the constant C_2 into (8), we deduce the integral equation (5). The converse follows by direct computation. This completes the proof. □

In the same way, we obtain the following result with the Riemann-Liouville fractional integal boundary conditions.

Lemma 2.4 *The boundary value problem*

$$\begin{cases} \left(\lambda_1 D^{\alpha_1}+(1-\lambda_1)D^{\beta_1}\right)y(t)=\omega_1(t), \quad t\in(0,T), \\ y(0)=0, \quad \mu_1 I^{\delta_1}y(T)+(1-\mu_1)I^{\delta_2}y(T)=\delta_3, \end{cases} \tag{9}$$

is equivalent to the following integral equation

$$\begin{aligned} y(t) &= \frac{\lambda_1-1}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^t(t-s)^{\alpha_1-\beta_1-1}y(s)ds+\frac{1}{\lambda_1\Gamma(\alpha_1)}\int_0^t(t-s)^{\alpha_1-1}\omega_1(s)ds \\ &\quad+\frac{t}{\Lambda_2}\Bigg(\delta_3-\frac{\mu_1(\lambda_1-1)}{\lambda_1\Gamma(\delta_1+\alpha_1-\beta_1)}\int_0^T(T-s)^{\delta_1+\alpha_1-\beta_1-1}y(s)ds \\ &\quad-\frac{\mu_1}{\lambda_1\Gamma(\delta_1+\alpha_1)}\int_0^T(T-s)^{\delta_1+\alpha_1-1}\omega_1(s)ds \\ &\quad-\frac{(1-\mu_1)(\lambda_1-1)}{\lambda_1\Gamma(\delta_2+\alpha_1-\beta_1)}\int_0^T(T-s)^{\delta_2+\alpha_1-\beta_1-1}y(s)ds \\ &\quad-\frac{1-\mu_1}{\lambda_1\Gamma(\delta_2+\alpha_1)}\int_0^T(T-s)^{\delta_2+\alpha_1-1}\omega_1(s)ds\Bigg), \quad t\in J, \end{aligned} \tag{10}$$

where the non zero constant Λ_2 is defined by

$$\Lambda_2=\frac{\mu_1T^{1+\delta_1}}{\Gamma(2+\delta_1)}+\frac{(1-\mu_1)T^{1+\delta_2}}{\Gamma(2+\delta_2)}. \tag{11}$$

3 Main Results

Let us introduce the space $X=\{u(t)\,|\,u(t)\in C(J,\mathbb{R})\}$ endowed with the norm $\|u\|=\sup\{|u(t)|,t\in J\}$. Obviously $(X,\|\cdot\|)$ is a Banach space. Also $Y=\{v(t)|v(t)\in C(J,\mathbb{R})\}$ endowed with the norm $\|v\|=\sup\{|v(t)|,t\in J\}$ is a Banach space. Then the product space $(X\times Y,\|(u,v)\|)$ is also a Banach space equipped with norm $\|(u,v)\|=\|u\|+\|v\|$.

In view of Lemmas 2.3 and 2.4, we define the operator $\mathcal{T}:X\times Y\to X\times Y$ by

$$\mathcal{T}(u,v)(t)=\begin{pmatrix}\mathcal{T}_1(u,v)(t)\\ \mathcal{T}_2(u,v)(t)\end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}_1(u,v)(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}u(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s),v(s))ds \\ &\quad+\frac{t}{\Lambda_1}\Bigg(\gamma_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}u(s)ds \\ &\quad-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}f(s,u(s),v(s))ds \end{aligned}$$

$$- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}u(s)ds$$
$$- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}f(s,u(s),v(s))ds\Bigg),$$

and

$$\begin{aligned}\mathcal{T}_2(u,v)(t) &= \frac{\lambda_1-1}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^t (t-s)^{\alpha_1-\beta_1-1}v(s)ds + \frac{1}{\lambda_1\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}g(s,u(s),v(s))ds\\
&+ \frac{t}{\Lambda_2}\Bigg(\delta_3 - \frac{\mu_1(\lambda_1-1)}{\lambda_1\Gamma(\delta_1+\alpha_1-\beta_1)}\int_0^T (T-s)^{\delta_1+\alpha_1-\beta_1-1}v(s)ds\\
&- \frac{\mu_1}{\lambda_1\Gamma(\delta_1+\alpha_1)}\int_0^T (T-s)^{\delta_1+\alpha_1-1}g(s,u(s),v(s))ds\\
&- \frac{(1-\mu_1)(\lambda_1-1)}{\lambda_1\Gamma(\delta_2+\alpha_1-\beta_1)}\int_0^T (T-s)^{\delta_2+\alpha_1-\beta_1-1}v(s)ds\\
&- \frac{1-\mu_1}{\lambda_1\Gamma(\delta_2+\alpha_1)}\int_0^T (T-s)^{\delta_2+\alpha_1-1}g(s,u(s),v(s))ds\Bigg).\end{aligned}$$

Let us introduce the following hypotheses which are used hereafter.

(H$_1$) Assume that there exist real constants $k_i,\ \nu_i \geq 0$ $(i=1,2)$ and $k_0>0, \nu_0>0$ such that $\forall x_i \in \mathbb{R}$, $(i=1,2)$ we have

$$|f(t,x_1,x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|,$$
$$|g(t,x_1,x_2)| \leq \nu_0 + \nu_1|x_1| + \nu_2|x_2|.$$

(H$_2$) Assume that $f,h: J\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants m_i, n_i, $i=1,2$ such that for all $t\in J$ and $u_i, v_i \in \mathbb{R}$, $i=1,2$,

$$|f(t,u_1,u_2)-f(t,v_1,v_2)| \leq m_1|u_1-v_1| + m_2|u_2-v_2|$$

and

$$|g(t,u_1,u_2)-g(t,v_1,v_2)| \leq n_1|u_1-v_1| + n_2|u_2-v_2|.$$

For the sake of convenience, we set constants

$$M_1 = \frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}, \tag{12}$$

$$N_1 = \frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}, \tag{13}$$

$$M_2 = \frac{T^{\alpha_1}}{\lambda_1\Gamma(\alpha_1+1)} + \frac{T^{\delta_1+\alpha_1+1}\mu_1}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1+1)} + \frac{T^{\delta_2+\alpha_1+1}(1-\mu_1)}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1+1)}, \tag{14}$$

$$N_2 = \frac{T^{\alpha_1-\beta_1}|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{T^{\delta_1+\alpha_1-\beta_1+1}\mu_1|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1-\beta_1+1)} + \frac{T^{\delta_2+\alpha_1-\beta_1+1}(1-\mu_1)|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1-\beta_1+1)} \tag{15}$$

and

$$M_0 = \min\{1-(M_1k_1+N_1+M_2\nu_1),\ 1-(M_1k_2+M_2\nu_2+N_2)\},\quad k_i,\ \nu_i \geq 0\ (i=1,2). \tag{16}$$

The first result is based on Leray-Schauder alternative.

Lemma 3.1 *(Leray-Schauder alternative) ([30] p. 4.) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 *Assume that (H_1) holds. In addition, it is assumed that*

$$M_1k_1 + N_1 + M_2\nu_1 < 1 \quad \text{and} \quad M_1k_2 + M_2\nu_2 + N_2 < 1,$$

where M_1 and M_2 are given by (13) and (15) respectively. Then the system (2)-(3) has at least one solution.

Proof. First we show that the operator $\mathcal{T} : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g, the operator $\mathcal{T}$ is continuous.

Let $\Omega = \{(u, v) \in X \times Y : \|(u, v)\| \leq r\} \subset X \times Y$ be a bounded set. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \leq L_1, \quad |g(t, u(t), v(t))| \leq L_2, \quad \forall (u, v) \in \Omega.$$

Then for any $(u, v) \in \Omega$, we have

$$\begin{aligned}
|\mathcal{T}_1(u,v)(t)| &\leq \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|u(s)|ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s),v(s))|ds \\
&\quad + \frac{t}{\Lambda_1}\Bigg(|\gamma_3| + \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}|u(s)|ds \\
&\quad + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}|f(s,u(s),v(s))|ds \\
&\quad + \frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}|u(s)|ds \\
&\quad + \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}|f(s,u(s),v(s))|ds\Bigg) \\
&\leq L_1\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right] \\
&\quad + \|u\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right. \\
&\quad \left. + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right] + \frac{|\gamma_3|T}{\Lambda_1} \\
&= L_1M_1 + N_1r + |\gamma_3|T/\Lambda_1
\end{aligned}$$

and consequently,

$$\|\mathcal{T}_1(u, v)\| \leq L_1M_1 + N_1r + |\gamma_3|T/\Lambda_1.$$

Similarly, we get

$$\|\mathcal{T}_2(u, v)\| \leq L_2M_2 + N_2r + |\delta_3|T/\Lambda_2.$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$|\mathcal{T}_1(u(t_2), v(t_2)) - \mathcal{T}_1(u(t_1), v(t_1))|$$

$$
\begin{aligned}
&\leq \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\left[\int_0^{t_2}(t_2-s)^{\alpha-\beta-1}u(s)ds-\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}u(s)ds\right]\\
&\quad+\frac{1}{\lambda\Gamma(\alpha)}\left[\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,u(s),v(s))ds-\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u(s),v(s))ds\right]\\
&\quad+\frac{|t_2-t_1|}{\Lambda_1}\left(|\gamma_3|+\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}u(s)ds\right.\\
&\quad+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}f(s,u(s),v(s))ds\\
&\quad+\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}u(s)ds\\
&\quad\left.+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}f(s,u(s),v(s))ds\right)\\
&\leq r\left[\frac{[2(t_2-t_1)^{\alpha-\beta}+|t_2^{\alpha-\beta}-t_1^{\alpha-\beta}]|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{|t_2-t_1|}{\Lambda_1}\left(\frac{|\lambda-1|\mu T^{\alpha-\beta-\gamma_1}}{\lambda\Gamma(\alpha-\beta-\gamma_1+1)}\right.\right.\\
&\quad\left.\left.+\frac{|\lambda-1|(1-\mu)T^{\alpha-\beta-\gamma_2}}{\lambda\Gamma(\alpha-\beta-\gamma_2+1)}\right)\right]+\frac{|t_2-t_1||\gamma_3|}{\Lambda_1}\\
&\quad+L_1\left[\frac{2(t_2-t_1)^{\alpha-1}+|t_2^{\alpha-1}-t_1^{\alpha-1}|}{\lambda\Gamma(\alpha+1)}+\frac{|t_2-t_1|}{\Lambda_1}\left(\frac{\mu T^{\alpha-\gamma_1}}{\lambda\Gamma(\alpha-\gamma_1+1)}+\frac{(1-\mu)T^{\alpha-\gamma_2}}{\lambda\Gamma(\alpha-\gamma_2+1)}\right)\right].
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
&|\mathcal{T}_2(u(t_2),v(t_2))-\mathcal{T}_2(u(t_1),v(t_1))|\\
&\leq r\left[\frac{[2(t_2-t_1)^{\alpha_1-\beta_1}+|t_2^{\alpha_1-\beta_1}-t_1^{\alpha_1-\beta_1}]|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}+\frac{|t_2-t_1|}{\Lambda_2}\left(\frac{|\lambda_1-1|\mu_1T^{\delta_1+\alpha_1-\beta_1}}{\lambda_1\Gamma(\delta_1+\alpha_1-\beta_1+1)}\right.\right.\\
&\quad\left.\left.+\frac{|\lambda_1-1|(1-\mu)T^{\delta_2+\alpha_1-\beta_1}}{\lambda_1\Gamma(\delta_2+\alpha_1-\beta_1+1)}\right)\right]+\frac{|t_2-t_1||\delta_3|}{\Lambda_2}\\
&\quad+L_2\left[\frac{2(t_2-t_1)^{\alpha_1-1}+|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}|}{\lambda_1\Gamma(\alpha_1+1)}+\frac{|t_2-t_1|}{\Lambda_2}\left(\frac{\mu_1T^{\delta_1+\alpha_1}}{\lambda_1\Gamma(\delta_1+\alpha_1+1)}+\frac{(1-\mu_1)T^{\delta_2+\alpha_1}}{\lambda_1\Gamma(\delta_2+\alpha_1+1)}\right)\right].
\end{aligned}
$$

As $t_2-t_1\to 0$, the right-hand sides of the above inequalities tends to zero independently of $(u,v)\in\Omega$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{T}(u,v)$ is equicontinuous, and thus the operator $\mathcal{T}(u,v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(u,v)\in X\times Y|(u,v)=\theta\mathcal{T}(u,v),0\leq\theta\leq 1\}$ is bounded. Let $(u,v)\in\mathcal{E}$, with $(u,v)=\theta T(u,v)$. For any $t\in[0,T]$, we have

$$u(t)=\theta\mathcal{T}_1(u,v)(t),\quad v(t)=\theta\mathcal{T}_2(u,v)(t).$$

Then

$$
\begin{aligned}
|u(t)| &\leq (k_0+k_1\|u\|+k_2\|v\|)\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)}+\frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right]\\
&\quad+\|u\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right.\\
&\quad\left.+\frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right]+\frac{|\gamma_3|T}{\Lambda_1}
\end{aligned}
$$

and

$$\begin{aligned}|v(t)| &\leq (\nu_0+\nu_1\|u\|+\nu_2\|v\|)\bigg[\frac{T^{\alpha_1}}{\lambda_1\Gamma(\alpha_1+1)}+\frac{T^{\delta_1+\alpha_1+1}\mu_1}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1+1)}\\ &+\frac{T^{\delta_2+\alpha_1+1}(1-\mu_1)}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1+1)}\bigg]+\|v\|\bigg[\frac{T^{\alpha_1-\beta_1}|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}.\\ &+\frac{T^{\delta_1+\alpha_1-\beta_1+1}\mu_1|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1-\beta_1+1)}+\frac{T^{\delta_2+\alpha_1-\beta_1+1}(1-\mu_1)|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1-\beta_1+1)}\bigg]+\frac{|\delta_3|T}{\Lambda_2}.\end{aligned}$$

Hence we have

$$\|u\|\leq M_1(k_0+k_1\|u\|+k_2\|v\|)+N_1\|u\|+|\gamma_3|T/\Lambda_1$$

and

$$\|v\|\leq M_2(\nu_0+\nu_1\|u\|+\nu_2\|v\|)+N_2\|v\|+|\delta_3|T/\Lambda_2,$$

which imply that

$$\begin{aligned}\|u\|+\|v\| &= \Big(M_1k_0+M_2\nu_0+|\gamma_3|T/\Lambda_1+|\delta_3|T/\Lambda_2\Big)\\ &\quad+(M_1k_1+N_1+M_2\nu_1)\|u\|+(M_1k_2+M_2\nu_2+N_2)\|v\|.\end{aligned}$$

Consequently,

$$\|(u,v)\|\leq\frac{M_1k_0+M_2\nu_0+|\gamma_3|T/\Lambda_1+|\delta_3|T/\Lambda_2}{M_0},$$

for any $t\in[0,T]$, where M_0 is defined by (16), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Hence the boundary value problem (2)-(3) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (2)-(3) via Banach's contraction principle.

Theorem 3.3 *Assume that (H_2) holds. In addition, assume that*

$$M_1(m_1+m_2)+N_1+M_2(n_1+n_2)+N_2<1,$$

where M_1, N_1, M_2 and N_2 are given by (12) and (15), respectively. Then the system (2)-(3) has a unique solution on J.

Proof. Define $\sup_{t\in J}f(t,0,0)=F_0<\infty$ and $\sup_{t\in J}g(t,0,0)=G_0<\infty$ such that

$$r\geq\frac{N_1M_1+N_2M_2+|\gamma_3|T/\Lambda_1+|\delta_3|T/\Lambda_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)-(N_1+N_2)}.$$

We show that $\mathcal{T}B_r\subset B_r$, where $B_r=\{(u,v)\in X\times Y:\|(u,v)\|\leq r\}$.

For $(u,v)\in B_r$, we have

$$\begin{aligned}|\mathcal{T}_1(u,v)(t)| &\leq \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^T(T-s)^{\alpha-\beta-1}|u(s)|ds\\ &+\frac{1}{\lambda\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}[|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|]ds\\ &+\frac{T}{\Lambda_1}\bigg(|\gamma_3|+\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}|u(s)|ds\\ &+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}[|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|]ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}|u(s)|ds\\
&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}[|f(s,u(s),v(s))-f(s,0)|+|f(s,0,0)|]ds\Bigg)\\
\leq\ & (m_1\|u\|+m_2\|v\|+F_0)\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)}+\frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right]\\
&+\|x\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right]+\frac{|\gamma_3|T^{\alpha-1}}{\Lambda_1}\\
\leq\ & M_1[(m_1+m_2)r+F_0]+N_1r+|\gamma_3|T/\Lambda_1.
\end{aligned}
$$

Hence

$$\|\mathcal{T}_1(u,v)\|\leq M_1[(m_1+m_2)r+F_0]+N_1r+|\gamma_3|T/\Lambda_1.$$

In the same way, we can obtain that

$$\|\mathcal{T}_2(u,v)\|\leq M_2[(n_1+n_2)r+G_0]+N_2r+|\delta_3|T/\Lambda_2.$$

Consequently, $\|T(u,v)\|\leq r$.

Now for $(u_2,v_2),(u_1,v_1)\in X\times Y$, and for any $t\in[0,e]$, we get

$$
\begin{aligned}
&|\mathcal{T}_1(u_2,v_2)(t)-\mathcal{T}_1(u_1,v_1)(t)|\\
\leq\ & \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^T(T-s)^{\alpha-\beta-1}|u_2(s)-u_1(s)|ds\\
&+\frac{1}{\lambda\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&+\frac{T^{\alpha-1}}{\Lambda_1}\Bigg(\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}|u_2(s)-u_1(s)|ds\\
&+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&+\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T(T-s)^{\alpha-\beta-\gamma_2-1}|u_2(s)-u_1(s)|ds\\
&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T(T-s)^{\alpha-\gamma_2-1}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\Bigg)\\
\leq\ & (m_1\|u_2-u_1\|+m_2\|v_2-v_1\|)\left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)}+\frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\right]+\|u_2-u_1\|\left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)}\right.\\
&\left.+\frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\right]\\
\leq\ & M_1[(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|)+N_1\|u_2-u_1\|\\
\leq\ & M_1(m_1+m_2)[\|u_2-u_1\|+\|v_2-v_1\|)]+N_1[\|u_2-u_1\|+\|v_2-v_1\|]\\
\leq\ & [M_1(m_1+m_2)+N_1][\|u_2-u_1\|+\|v_2-v_1\|],
\end{aligned}
$$

and consequently we obtain

$$\|\mathcal{T}_1(u_2,v_2)-\mathcal{T}_1(u_1,v_1)\|\leq[M_1(m_1+m_2)+N_1][\|u_2-u_1\|+\|v_2-v_1\|]. \tag{17}$$

Similarly,

$$\|\mathcal{T}_2(u_2,v_2)-\mathcal{T}_2(u_1,v_1)\| \leq [M_2(n_1+n_2)+N_2][\|u_2-u_1\|+\|v_2-v_1\|]. \tag{18}$$

It follows from (17) and (18) that

$$\|\mathcal{T}(u_2,v_2)-\mathcal{T}(u_1,v_1)\| \leq [M_1(m_1+m_2)+N_1+M_2(n_1+n_2)+N_2](\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $M_1(m_1+m_2)+N_1+M_2(n_1+n_2)+N_2<1$, therefore, $\mathcal{T}$ is a contraction operator. So, By Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (2)-(3). This completes the proof. □

Example 3.4 *Consider the following coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals boundary conditions of the form*

$$\begin{cases} \left(\frac{33}{38}D^{29/15}+\frac{5}{38}D^{16/15}\right)x(t)=\frac{t}{t+1}+\frac{1}{3}\sin\left(\frac{|x(t)|}{4}\right)+\frac{y^2(t)}{10(1+|y(t)|)}, \ t\in[0,3/2], \\ \left(\frac{24}{27}D^{19/13}+\frac{3}{27}D^{15/13}\right)y(t)=\sqrt{t+3}+\frac{x(t)}{10}e^{-|x(t)|}+\frac{t}{3}\tan^{-1}\left(\frac{|y(t)|}{6}\right), \quad t\in[0,3/2], \\ x(0)=0, \qquad \frac{9}{16}D^{8/15}x\left(\frac{3}{2}\right)+\frac{7}{16}D^{11/15}x\left(\frac{3}{2}\right)=\frac{1}{3}, \\ y(0)=0, \qquad \frac{2}{5}I^{1/2}y\left(\frac{3}{2}\right)+\frac{3}{5}I^{3/2}y\left(\frac{3}{2}\right)=\frac{2}{7}. \end{cases} \tag{19}$$

Here $\lambda=33/38$, $\alpha=29/15$, $\beta=16/15$, $T=3/2$, $\lambda_1=24/27$, $\alpha_1=19/13$, $\beta_1=15/13$, $\mu=9/16$, $\gamma_1=8/15$, $\gamma_2=11/15$, $\gamma_3=1/3$, $\mu_1=2/5$, $\delta_1=1/2$, $\delta_2=3/2$, $\delta_3=2/7$. From all constants, we can compute that $\Lambda_1=1.307202573$, $\Lambda_2=1.050302214$. $M_1=3.248792650$, $N_1=0.4373542422$, $M_2=2.869543745$ and $N_2=0.3962406719$. Clearly,

$$\begin{aligned} |f(t,x,y)| &= \left|\frac{t}{t+1}+\frac{1}{3}\sin\left(\frac{|x|}{4}\right)+\frac{y^2}{10(1+|y|)}\right| \\ &\leq \frac{3}{5}+\frac{1}{12}|x|+\frac{1}{10}|y|, \end{aligned}$$

and

$$\begin{aligned} |g(t,x,y)| &= \left|\sqrt{t+3}+\frac{x}{10}e^{-|x|}+\frac{t}{3}\tan^{-1}\left(\frac{|y|}{6}\right)\right| \\ &\leq \frac{3}{\sqrt{2}}+\frac{1}{10}|x|+\frac{1}{12}|y|. \end{aligned}$$

Setting $k_0=3/5$, $k_1=1/12$, $k_2=1/10$, $\nu_0=3/\sqrt{2}$, $\nu_1=1/10$ and $\nu_2=1/12$, we have

$$M_1k_1+M_2\nu_1+N_1=0.9950413375<1 \quad \text{and} \quad M_1k_2+M_2\nu_2+N_2=0.9602485823<1.$$

Therefore, by applying Theorem 3.2, the boundary value problem (19) has at least one solution on $[0,3/2]$.

Example 3.5 *Consider the following coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals boundary conditions of the form*

$$\begin{cases} \left(\frac{49}{53}D^{17/9}+\frac{4}{53}D^{10/9}\right)x(t)=\frac{t+1}{2}+\frac{|x(t)|e^{-t^2}}{2(1+|x(t)|)}+\frac{1}{3}\sin|y(t)|\cos 2\pi t, \ t\in[0,1/2], \\ \left(\frac{41}{46}D^{13/7}+\frac{5}{46}D^{8/7}\right)y(t)=\frac{t}{4}+\tan^{-1}\left(\frac{|x(t)|}{3}\right)+\frac{1}{8}\left(\frac{y^2(t)+2|y(t)|}{1+|y(t)|}\right), \quad t\in[0,1/2], \\ x(0)=0, \qquad \frac{13}{31}D^{5/9}x\left(\frac{1}{2}\right)+\frac{18}{31}D^{4/9}x\left(\frac{1}{2}\right)=\frac{3}{4}, \\ y(0)=0, \qquad \frac{6}{11}I^{5/2}y\left(\frac{1}{2}\right)+\frac{5}{11}I^{7/2}y\left(\frac{1}{2}\right)=\frac{2}{3}. \end{cases} \tag{20}$$

Here $\lambda = 49/53$, $\alpha = 17/9$, $\beta = 10/9$, $T = 1/2$, $\lambda_1 = 41/46$, $\alpha_1 = 13/7$, $\beta_1 = 8/7$, $\mu = 13/31$, $\gamma_1 = 5/9$, $\gamma_2 = 4/9$, $\gamma_3 = 3/4$, $\mu_1 = 6/11$, $\delta_1 = 5/2$, $\delta_2 = 7/2$, $\delta_3 = 2/3$, $f(t,x,y) = ((t+1)/2) + ((|x|e^{-t^2})/(2(1+|x|))) + ((\sin|y|\cos 2\pi t)/(3))$ and $g(t,x,y) = (t/4) + \tan^{-1}(|x|/3) + ((y^2+2|y|)/(8(1+|y|)))$. From above information, we can calculate that $\Lambda_1 = 0.7921804090$, $\Lambda_2 = 0.004528637717$. $M_1 = 0.3706636539$, $N_1 = 0.09832444532$, $M_2 = 0.4209829845$ and $N_2 = 0.1927580748$. It is easy to see that

$$|f(t,x,y) - f(t,u,v)| \leq \frac{1}{2}|x-u| + \frac{1}{3}|y-v|,$$

and

$$|g(t,x,y) - g(t,u,v)| \leq \frac{1}{3}|x-u| + \frac{1}{4}|y-v|.$$

Putting $m_1 = 1/2$, $m_2 = 1/3$, $n_1 = 1/3$ and $n_2 = 1/4$, we deduce that

$$M_1(m_1+m_2) + N_1 + M_2(n_1+n_2) + N_2 = 0.8455423059 < 1.$$

Hence, by using Theorem 3.3, the boundary value problem (20) has a unique solution on $[0, 1/2]$.

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On solvability of a coupled system of fractional differential equations supplemented with a new kind of flux type integral boundary conditions

Bashir Ahmad[1], Sotiris K. Ntouyas[2,1] and Ahmed Alsaedi[1]

[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com (B. Ahmad), aalsaedi@hotmail.com (A. Alsaedi)

[2]Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr

Abstract

In this paper, we introduce a new kind of nonlocal nonlinear flux type integral boundary conditions and discuss the existence and uniqueness of solutions for a coupled system of fractional differential equations supplemented with these conditions. We apply Leray-Schauder's alternative and Banach's contraction mapping principle to obtain the desired results. An illustrative example is also included. Our results are new and enrich the existing material on coupled systems of fractional differential equations equipped with integral boundary conditions.

Key words and phrases: Fractional differential systems; nonlocal boundary conditions; integral boundary conditions; fixed point theorem
AMS (MOS) Subject Classifications: 34A08, 34B15.

1 Introduction

Fractional differential equations appear in the mathematical modeling of several systems and processes occurring in many branches of applied sciences such as blood flow phenomena, control theory, signal and image processing, reaction-diffusion models, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. [1]-[4]. Fractional order differential equations are also found to be of great support in describing the hereditary properties of various materials and processes. With this advantage, fractional-order models have become more realistic and practical than the corresponding classical integer-order models. Fractional-order boundary value problems involving a variety of classical, nonlocal and integral boundary conditions have been addressed by many authors, for instance, see [5]-[10] and the references cited therein.

Coupled systems of fractional-order differential equations also constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [11], disease models [12]-[15], ecological models [16], synchronization of chaotic systems [17]-[19], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [20]-[24].

The integral boundary conditions provide a descent approach to relax the limitation of circular cross-section of blood vessels with an arbitrary shaped cross-section of such vessels in the study of blood flow problems [25] and model the problem of bacterial self-organization [26]. Recently, in [27, 28], the authors investigated fractional-order differential inclusions and equations with nonlocal nonlinear flux type integral boundary conditions.

In this paper, we consider a more generalized version of flux type integral boundary conditions and develop the existence criteria for a coupled system of Caputo type fractional differential equations

equipped with these new conditions. Precisely, we investigate the following coupled system of Caputo type fractional differential equations

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), y(t)), & t \in [0,1], \quad 1 < q \leq 2, \\ {}^cD^p y(t) = h(t, x(t), y(t)), & t \in [0,1], \quad 1 < p \leq 2, \end{cases} \tag{1}$$

supplemented with the nonlocal nonlinear flux type integral boundary conditions:

$$\begin{cases} x'(0) = \alpha \displaystyle\int_0^\xi x'(s)ds, \quad x(1) = \beta \int_0^1 g(x'(s))ds, & 0 \leq \xi \leq 1, \\ y'(0) = \alpha_1 \displaystyle\int_0^\theta y'(s)ds, \quad y(1) = \beta_1 \int_0^1 g(y'(s))ds, & 0 \leq \theta \leq 1, \end{cases} \tag{2}$$

where ${}^cD^q$,${}^c D^p$ denote the Caputo fractional derivatives of order q and p respectively, $f, h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are appropriately chosen functions, and $\alpha, \beta, \alpha_1, \beta_1$ are real constants.

The objective of the present paper is to enhance the theoretical treatment of coupled systems further by considering a new boundary value problem of coupled fractional-order differential equations supplemented with nonlocal nonlinear flux type integral boundary conditions. The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma. The main results are presented in Section 3. We give two results: the first one derives the existence of solutions via Leray-Schauder's alternative, whereas the second one concerning existence and uniqueness of solutions is established by Banach's contraction principle. We also discuss an example for illustration of the existence-uniqueness result.

2 Preliminaries

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus [3, 2].

Definition 2.1 *For $(n-1)-$times absolutely continuous function $y : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q y(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} y^{(n)}(s)ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

To define the solution for the problem (1)-(2), we use the following lemma.

Lemma 2.3 *Let $\alpha\xi \neq 1$. For $\phi \in C([0,1], \mathbb{R})$, the linear problem consisting by the equation*

$${}^cD^q x(t) = \phi(t), \quad t \in [0,1], \quad 1 < q \leq 2, \tag{3}$$

supplemented with the boundary conditions

$$x'(0) = \alpha \int_0^\xi x'(s)ds, \quad x(1) = \beta \int_0^1 g(x'(s))ds, \quad 0 \leq \xi \leq 1, \tag{4}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) \quad &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \phi(s)ds + \frac{\alpha(t-1)}{1-\alpha\xi} \int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \phi(\tau)d\tau ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \phi(s)ds \\ &+\beta \int_0^1 g\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \phi(\tau)d\tau + \frac{\alpha}{1-\alpha\xi} \int_0^\xi \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \phi(\tau)d\tau ds \right) ds. \end{aligned} \tag{5}$$

Proof. It is well known that the general solution of the fractional differential equation (3) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\phi(s)ds, \tag{6}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions (4) in (6), we find that

$$\begin{aligned} c_0 &= \beta\int_0^1 g\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau + \frac{\alpha}{1-\alpha\xi}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau ds\right)ds \\ &\quad -\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\phi(s)ds - \frac{\alpha}{1-\alpha\xi}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau ds \end{aligned}$$

and

$$c_1 = \frac{\alpha}{1-\alpha\xi}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\phi(\tau)d\tau ds.$$

Substituting the values of c_0, c_1 in (6), we get (5). The converse follows by direct computation. This completes the proof. □

3 Main Results

Let us introduce the space $X = \{u(t)|u(t) \in C([0,1],\mathbb{R})\}$ endowed with the norm $\|u\| = \sup\{|u(t)|, t \in [0,1]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also $Y = \{v(t)|v(t) \in C([0,1],\mathbb{R})\}$ endowed with the norm $\|v\| = \sup\{|v(t)|, t \in [0,1]\}$ is a Banach space. Then the product space $(X \times Y, \|(u,v)\|)$ is also a Banach space equipped with norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemma 2.3, we define the operator $T : X \times Y \to X \times Y$ by $T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix}$, where

$$\begin{aligned} T_1(u,v)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,u(s),v(s))ds + \frac{\alpha(t-1)}{1-\alpha\xi}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,u(\tau),v(\tau))d\tau ds \\ &\quad -\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s,u(s),v(s))ds + \beta\int_0^1 g\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,u(\tau),v(\tau))d\tau \right. \\ &\quad \left. + \frac{\alpha}{1-\alpha\xi}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,u(\tau),v(\tau))d\tau ds\right)ds, \end{aligned}$$

and

$$\begin{aligned} T_2(u,v)(t) &= \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)}h(s,u(s),v(s))ds + \frac{\alpha_1(t-1)}{1-\alpha_1\theta}\int_0^\theta\int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}h(\tau,u(\tau),v(\tau))d\tau ds \\ &\quad -\int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)}h(s,u(s),v(s))ds + \beta_1\int_0^1 g\left(\int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}h(\tau,u(\tau),v(\tau))d\tau \right. \\ &\quad \left. + \frac{\alpha_1}{1-\alpha_1\theta}\int_0^\theta\int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}h(\tau,u(\tau),v(\tau))d\tau ds\right)ds,\ \alpha_1\theta \neq 1. \end{aligned}$$

Let us introduce the following hypotheses which are used hereafter.

(H$_1$) Assume that there exist real constants $k_i,\ \lambda_i \geq 0\ (i = 1,2)$ and $k_0 > 0, \lambda_0 > 0$ such that $\forall x_i \in \mathbb{R},\ (i = 1,2)$ we have

$$|f(t,x_1,x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|,\ |h(t,x_1,x_2)| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

(H$_2$) $|g(v)| \leq |v|,\ \ \forall v \in \mathbb{R}$.

(H$_3$) Assume that $f, h : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i, i = 1, 2$ such that for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le m_1|u_1 - v_1| + m_2|u_2 - v_2|$$

and

$$|h(t,u_1,u_2) - h(t,v_1,v_2)| \le n_1|u_1 - v_1| + n_2|u_2 - v_2|.$$

For the sake of convenience, we set

$$M_1 = \frac{1}{\Gamma(q+1)}\left(2 + \frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right) + \frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right), \tag{7}$$

$$M_2 = \frac{1}{\Gamma(p+1)}\left(2 + \frac{|\alpha_1|\theta^p}{|1-\alpha_1\theta|}\right) + \frac{|\beta_1|}{\Gamma(p+2)}\left(p+1+\frac{|\alpha_1|}{|1-\alpha_1\theta|}\right), \tag{8}$$

and

$$M_0 = \min\{1-(M_1k_1 + M_2\lambda_1),\ 1-(M_1k_2 + M_2\lambda_2)\},\quad k_i,\ \lambda_i \ge 0\ (i = 1, 2). \tag{9}$$

The first result is based on Leray-Schauder alternative.

Lemma 3.1 *(Leray-Schauder alternative) ([29] p. 4.) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 *Assume that (H$_1$), (H$_2$) hold. In addition it is assumed that*

$$M_1k_1 + M_2\lambda_1 < 1 \quad \text{and} \quad M_1k_2 + M_2\lambda_2 < 1,$$

where M_1 and M_2 are given by (7) and (8) respectively. Then the system (1)-(2) has at least one solution.

Proof. First we show that the operator $T : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f, h and g, the operator T is continuous.

Let $\Omega \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t,u(t),v(t)| \le L_1, \quad |h(t,u(t),v(t)| \le L_2, \quad \forall (u,v) \in \Omega.$$

Then for any $(u,v) \in \Omega$, we have

$$\begin{aligned}
|T_1(u,v)(t)| \le\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u(s),v(s))|ds + \frac{|\alpha(t-1)|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau ds \\
& + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,u(s),v(s))|ds + \beta\int_0^1 g\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau\right. \\
& \left. + \frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau ds\right)ds \\
\le\ & L_1\left\{\frac{1}{\Gamma(q+1)}\left(2 + \frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right) + \frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right)\right\},
\end{aligned}$$

which implies that

$$\|T_1(u,v)\| \le L_1\left\{\frac{1}{\Gamma(q+1)}\left(2 + \frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right) + \frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right)\right\} = L_1M_1.$$

Similarly, we get

$$\|T_2(u,v)\| \leq L_2\left\{\frac{1}{\Gamma(p+1)}\left(2+\frac{|\alpha_1|\theta^p}{|1-\alpha_1\theta|}\right)+\frac{|\beta_1|}{\Gamma(p+2)}\left(p+1+\frac{|\alpha_1|}{|1-\alpha_1\theta|}\right)\right\} = L_2M_2.$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|T_1(u(t_2),v(t_2)) - T_1(u(t_1),v(t_1))| \\
\leq\ & L_1\left\{\left|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}ds - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}vds\right|\right. \\
&\left.+\frac{|\alpha||t_2-t_1|}{|1-\alpha\xi|}\int_0^{\xi}\int_0^{s}\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}d\tau ds\right\} \\
\leq\ & L_1\left\{\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds + \frac{|\alpha||t_2-t_1|}{|1-\alpha\xi|}\frac{\xi^q}{\Gamma(q+1)}\right\} \\
\leq\ & \frac{L_1}{\Gamma(q+1)}[2(t_2-t_1)^q + |t_2^q - t_1^q|] + L_1\frac{|\alpha||t_2-t_1|}{|1-\alpha\xi|}\frac{\xi^q}{\Gamma(q+1)}.
\end{aligned}$$

Analogously, we can obtain

$$|T_2(u(t_2),v(t_2)) - T_2(u(t_1),v(t_1))| \leq \frac{L_2}{\Gamma(p+1)}[2(t_2-t_1)^p + |t_2^p - t_1^p|] + L_2\frac{|\alpha_1||t_2-t_1|}{|1-\alpha_1\theta|}\frac{\theta^p}{\Gamma(p+1)}.$$

Therefore, the operator $T(u,v)$ is equicontinuous, and thus the operator $T(u,v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(u,v) \in X \times Y | (u,v) = \lambda T(u,v), 0 \leq \lambda \leq 1\}$ is bounded. Let $(u,v) \in \mathcal{E}$, with $(u,v) = \lambda T(u,v)$. For any $t \in [0,1]$, we have

$$u(t) = \lambda T_1(u,v)(t), \quad v(t) = \lambda T_2(u,v)(t).$$

Then

$$|u(t)| \leq \left\{\frac{1}{\Gamma(q+1)}\left(2+\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right)+\frac{|\beta|}{\Gamma(q+2)}\left(q+1+\frac{|\alpha|}{|1-\alpha\xi|}\right)\right\}(k_0+k_1\|u\|+k_2\|v\|),$$

and

$$|v(t)| \leq \left\{\frac{1}{\Gamma(p+1)}\left(2+\frac{|\alpha_1|\theta^p}{|1-\alpha_1\theta|}\right)+\frac{|\beta_1|}{\Gamma(p+2)}\left(p+1+\frac{|\alpha_1|}{|1-\alpha_1\theta|}\right)\right\}(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|).$$

Hence we have

$$\|u\| \leq M_1(k_0+k_1\|u\|+k_2\|v\|),\ \|v\| \leq M_2(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|),$$

which imply that

$$\|u\|+\|v\| = (M_1k_0+M_2\lambda_0)+(M_1k_1+M_2\lambda_1)\|u\|+(M_1k_2+M_2\lambda_2)\|v\|.$$

Consequently,

$$\|(u,v)\| \leq \frac{M_1k_0+M_2\lambda_0}{M_0},$$

for any $t \in [0,1]$, where M_0 is defined by (9), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator T has at least one fixed point. Hence the boundary value problem (1)-(2) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (1)-(2) via Banach's contraction principle.

Theorem 3.3 *Assume that (H_2), (H_3) hold. In addition, assume that*

$$M_1(m_1+m_2)+M_2(n_1+n_2)<1,$$

where M_1 and M_2 are given by (7) and (8) respectively. Then the system (1)-(2) has a unique solution.

Proof. Define $\sup_{t\in[0,1]} f(t,0,0)=N_1<\infty$ and $\sup_{t\in[0,1]} g(t,0,0)=N_2<\infty$ such that

$$r\geq \frac{N_1M_1+N_2M_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)}.$$

We show that $TB_r\subset B_r$, where $B_r=\{(u,v)\in X\times Y:\|(u,v)\|\leq r\}$. For $(u,v)\in B_r$, we have

$$\begin{aligned}
|T_1(u,v)(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(|f(s,u(s),v(s))-f(t,0,0)|+|f(t,0,0)|)ds\\
&\quad+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}(|f(\tau,u(\tau),v(\tau))-f(t,0,0)|+|f(t,0,0)|)d\tau ds\\
&\quad+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}(|f(s,u(s),v(s))-f(t,0,0)|+|f(t,0,0)|)ds\\
&\quad+|\beta|\int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}(|f(\tau,u(\tau),v(\tau))-f(t,0,0)|+|f(t,0,0)|)d\tau\\
&\quad+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}(|f(\tau,u(\tau),v(\tau))-f(t,0,0)|+|f(t,0,0)|)d\tau ds\bigg)ds\\
&\leq \bigg\{\frac{1}{\Gamma(q+1)}+\frac{|\alpha|\xi^q}{|1-\alpha\xi|\Gamma(q+1)}+\frac{1}{\Gamma(q+1)}\\
&\quad+|\beta|\bigg(\frac{1}{\Gamma(q+1)}+\frac{|\alpha|}{|1-\alpha\xi|\Gamma(q+2)}\bigg)\bigg\}(m_1\|u\|+m_2\|v\|+N_1)\\
&= M_1[(m_1+m_2)r+N_1].
\end{aligned}$$

Hence

$$\|T_1(u,v)(t)\|\leq M_1[(m_1+m_2)r+N_1].$$

In the same way, we can obtain that

$$\|T_2(u,v)(t)\|\leq M_2[(n_1+n_2)r+N_2].$$

Consequently, $\|T(u,v)(t)\|\leq r$.

Now for $(u_2,v_2),(u_1,v_1)\in X\times Y$, and for any $t\in[0,e]$, we get

$$\begin{aligned}
&|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)|\\
\leq\ &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u_2(\tau),v_2(\tau))-|f(\tau,u_1(\tau),v_1(\tau))|d\tau ds\\
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|ds\\
&+|\beta|\int_0^1 g\bigg(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|d\tau\\
&+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|d\tau ds\bigg)ds
\end{aligned}$$

$$\begin{aligned}
\leq & \left\{\frac{1}{\Gamma(q+1)}+\frac{|\alpha|\xi^q}{|1-\alpha\xi|\Gamma(q+1)}+\frac{1}{\Gamma(q+1)}\right. \\
& \left.+|\beta|\left(\frac{1}{\Gamma(q+1)}+\frac{|\alpha|}{|1-\alpha\xi|\Gamma(q+2)}\right)\right\}(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|) \\
= & M_1(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|) \\
\leq & M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|),
\end{aligned}$$

and consequently we obtain

$$\|T_1(u_2,v_2)(t)-T_1(u_1,v_1)\| \leq M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{10}$$

Similarly,

$$\|T_2(u_2,v_2)(t)-T_2(u_1,v_1)\| \leq M_2(n_1+n_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{11}$$

It follows from (10) and (11) that

$$\|T(u_2,v_2)(t)-T(u_1,v_1)(t)\| \leq [M_1(m_1+m_2)+M_2(n_1+n_2)](\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $M_1(m_1+m_2)+M_2(n_1+n_2)<1$, therefore, T is a contraction operator. So, By Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1)-(2). This completes the proof. □

Example. Consider the following system of fractional boundary value problem

$$\begin{cases}
{}^cD^{3/2}x(t)=\dfrac{1}{4(t+2)^2}\dfrac{|x(t)|}{1+|x(t)|}+1+\dfrac{1}{32}\sin^2 y(t), & t\in[0,1], \\
{}^cD^{3/2}y(t)=\dfrac{1}{32\pi}\sin(2\pi x(t))+\dfrac{|y(t)|}{16(1+|y(t)|)}+\dfrac{1}{2}, & t\in[0,1], \\
x'(0)=\dfrac{1}{2}\displaystyle\int_0^{1/3}x'(s)ds, \quad x(1)=\dfrac{1}{3}\int_0^1 g(x'(s))ds, \\
y'(0)=\dfrac{4}{5}\displaystyle\int_0^{1/4}y'(s)ds, \quad y(1)=\dfrac{3}{4}\int_0^1 g(y'(s))ds.
\end{cases} \tag{12}$$

Here $q=p=3/2,\ \alpha=1/2,\ \alpha_1=4/5,\ \xi=1/3,\ \theta=1/4,\ \beta=1/3,\ \beta_1=3/4$, $g(v)=\begin{cases}\sqrt{v}, & |v|\geq 1,\\ v^2, & |v|<1.\end{cases}$ $f(t,u,v)=\dfrac{1}{4(t+2)^2}\dfrac{|u|}{1+|u|}+1+\dfrac{1}{32}\sin^2 v$, and $h(t,u,v)=\dfrac{1}{32\pi}\sin(2\pi u)+\dfrac{|v|}{16(1+|v|)}+\dfrac{1}{2}$. With the given data, we find that $M_1\approx 1.9027815,\quad M_2\approx 1.6365646$. Note that $|f(t,u_1,u_2)-f(t,v_1,v_2)|\leq \dfrac{1}{16}|u_1-u_2|+\dfrac{1}{16}|v_1-v_2|$, $|g(t,u_1,u_2)-g(t,v_1,v_2)|\leq \dfrac{1}{16}|u_1-u_2|+\dfrac{1}{16}|v_1-v_2|$, and $M_1(m_1+m_2)+M_2(n_1+n_2)\approx 0.4424181<1$. Thus all the conditions of Theorem 3.3 are satisfied and consequently, its conclusion applies to the problem (12).

4 Conclusions

We have obtained the existence criteria for the solutions of a coupled system of nonlinear Caputo type fractional differential equations equipped with a new kind of nonlocal nonlinear flux type integral boundary conditions. Our results are new in the sense of introduced integral boundary conditions (2) and contribute to the theory of coupled systems of fractional differential equations.

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Strong Convergence Theorems of Non-convex Hybrid Algorithm for Quasi-Lipschitz Mappings

Waqas Nazeer[1], Mobeen Munir[2], Shin Min Kang[3,4,*] and Samina Kausar[5]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[3]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[4]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[5]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: sminasaddique@gmail.com

Abstract

The aim of this paper is to introduce a new non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings. We establish strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space.

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1 Introduction

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects [13]. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed-point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute dose-deposition coefficient matrix, see [12]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly

*Corresponding author

and an immense literature is present currently. The construction of fixed point theorems (for example, Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some mapping f that is contracting with respect to some (complete) metric will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions, see [4]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence (see [2, 3, 5–9, 11] and references therein).

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi [9] in 2003. They introduced this modification for only one nonexpansive mapping in a Hilbert space, where Kim and Xu [5] introduced a modification for asymptotically nonexpansive mappings in the Hilbert space in 2006. In the same year Martinez-Yanes ad Xu [7] introduced a modification of the Ishikawa iteration process for a nonexpansive mapping for a Hilbert space. They also gave modification of the Halpern iteration method in a Hilbert space. Su and Qin [11] gave a monotone hybrid iteration process for nonexpansive mappings in a Hilbert space. Liu et al. [6] gave a novel iteration method for a finite family of quasi-asymptotically pseudo-contractive mappings in a Hilbert space.

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let $P_c(\cdot)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be *nonexpensive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T. It is well known that $F(T)$ is closed and convex. A mapping $T : C \to C$ is said to be *quasi-Lipschitz* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq L\|x - p\|$ for all $x \in C,\ p \in F(T)$, where $1 \leq L < \infty$ is a constant. If $L = 1$, then T is known as *quasi-nonexpansive*. It is well-known that T is said to be *closed* if $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ as for $n \to \infty$ implies $Tx = x$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings from C into itself with a nonempty common fixed points set F. Then $\{T_n\}$ is said to be *uniformly closed* if for any convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

In 1953 Mann [4] proposed an iterative scheme given as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

Guan et al. [3] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0. \end{cases}$$

They also established non-convex hybrid iteration algorithms and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotically

family of countable quasi-Lipschitz mappings in a Hilbert space. They applied their results for the finite case to obtain fixed points. In this article we established a kind of non-convex hybrid iteration algorithm concerning SP-iterative process [10] and proves strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space. We also present an application of our algorithm.

2 Main results

In this section we formulate our main results.

Definition 2.1. Let C be a closed convex subset of a Hilbert space H, and Let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself. $\{T_n\}$ is said to be *asymptotically* if $\lim_{n\to\infty} L_n = 1$.

The following lemmas is well known.

Proposition 2.2. *Let C be a closed convex subset of a Hilbert space H. For $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Proposition 2.3. *Let C be a closed convex subset of a Hilbert space H. For any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$ for all $z \in C$.*

Proposition 2.4. ([3]) *Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Then the common fixed point set F is closed and convex.*

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\} : C \to C$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Assume that $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T_n z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T_n t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ \qquad -2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n \\ \qquad +\beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n \\ \qquad +\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We give our proof in following steps.

STEP 1. We know that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)z_n + \alpha_n T_n z_n - p\| \\
&= \|(1-\alpha_n)((1-\beta_n)t_n + \beta_n T_n t_n) + \alpha_n T_n((1-\beta_n)t_n + \beta_n T_n t_n) - p\| \\
&= \|(1-\alpha_n)((1-\beta_n)[(1-\gamma_n)x_n + \gamma_n T_n x_n] + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n]) \\
&\quad + \alpha_n T_n((1-\beta_n)[(1-\gamma_n)x_n + \gamma_n T_n x_n] + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n]) - p\| \\
&= \|(1-\alpha_n-\beta_n-\gamma_n+\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n)(x_n-p) \\
&\quad + (\alpha_n+\beta_n+\gamma_n-2\alpha_n\beta_n-2\alpha_n\gamma_n-2\beta_n\gamma_n+3\alpha_n\beta_n\gamma_n)(T_n x_n - p) \\
&\quad + (\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-3\alpha_n\beta_n\gamma_n)(T_n^2 x_n - p) + \alpha_n\beta_n\gamma_n(T_n^3 x_n - p)\| \\
&\leq (1-\alpha_n-\beta_n-\gamma_n+\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n)\|x_n - p\| \\
&\quad + (\alpha_n+\beta_n+\gamma_n-2\alpha_n\beta_n-2\alpha_n\gamma_n-2\beta_n\gamma_n+3\alpha_n\beta_n\gamma_n)L_n\|x_n - p\| \\
&\quad + (\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-3\alpha_n\beta_n\gamma_n)L_n^2\|x_n - p\| + \alpha_n\beta_n\gamma_n L_n^3\|x_n - p\| \\
&= [1 + L_n(\alpha_n+\beta_n+\gamma_n-2\alpha_n\beta_n-2\alpha_n\gamma_n-2\beta_n\gamma_n+3\alpha_n\beta_n\gamma_n) \\
&\quad + L_n^2(\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 \\
&\quad - \alpha_n-\beta_n-\gamma_n+\alpha_n\beta_n+\alpha_n\gamma_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n]\|x_n - p\|
\end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

STEP 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$ for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

STEP3. We prove $\{x_n\}$ is bounded. Since F is a nonempty closed and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a Cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.3), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq 0$ and hence

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\to 0, \quad n \to \infty\end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP 5. We show that $y_n \to q$ as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}.$$

From the definition of D_n, we have

$$\begin{aligned}D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + (L_n^3 - 1)(L_n^3 + 1)\} \\ &= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\} \\ &= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}\end{aligned}$$

This shows that D_n is convex and closed, $n \in Z^+ \cup \{0\}$. Next, we want to prove that $C_n \subset D_n$, $n \geq 0$.

In fact, for any $z \in C_n$, we have

$$\begin{aligned}\|y_n - z\|^2 &\leq [1 + L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) \\ &\quad + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n \\ &\quad + \alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]^2\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + [2(L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) \\ &\quad + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n \\ &\quad + \alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n)\alpha_n + (L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ &\quad - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) \\ &\quad + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n + \alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n)^2]\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + [2(L_n^3 - 1) + (L_n^3 - 1)^2]\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\|x_n - z\|^2.\end{aligned}$$

From

$$\begin{aligned}C_n = \{z \in C : \|y_n - z\| &\leq [1 + L_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n - 2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) \\ &+ L_n^2(\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n L_n^3 - \alpha_n - \beta_n - \gamma_n + \alpha_n\beta_n \\ &+ \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0,\end{aligned}$$

We have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}\|y_n - z\| &\leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1).\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 - 1)(L_n^3 + 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

STEP 6. We show that $q \in F$. From the definition of y_n, we have

$$\begin{aligned}
&(\alpha_n + \beta_n + \gamma_n - \alpha_n\gamma_n - \beta_n\gamma_n - \alpha_n\beta_n + \alpha_n\beta_n\gamma_n) \\
&+ (\alpha_n\gamma_n + \beta_n\gamma_n + \alpha_n\beta_n - 2\alpha_n\beta_n\gamma_n)T_n + \alpha_n\beta_n\gamma_n T_n^2\|T_n x_n - x_n\| \\
&= \|y_n - x_n\| \to 0
\end{aligned}$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_n \to \infty \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

STEP 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\begin{aligned}
\|z_0 - x_n\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\
&= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.
\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. □

In [3], we show an example of C_n which does not involve a convex subset.

Corollary 2.6. *Let C be a closed convex subset of a Hilbert space H, and let T be a closed quasi-nonexpansive mapping from C into itself. Assume that $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases}
x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\
y_n = (1 - \alpha_n)z_n + \alpha_n T z_n, \quad n \geq 0, \\
z_n = (1 - \beta_n)t_n + \beta_n T t_n, \quad n \geq 0, \\
t_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is convex and closed and , for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.6. □

Corollary 2.7. *Let C be a closed convex subset of a Hilbert space H, and let T be a nonexpansive mapping from C into itself. Assume that $\alpha_n \in (0,1]$ and $\beta_n, \gamma_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

3 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C,\, p \in F,$$

where F is the common fixed point set of $\{T_n\}_{n=0}^{N-1}$ and $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is said to be *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C$$

for all $i \in \{0, 1, 2, ..., N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a uniformly L-Lipschitz finite family of asymptotically quasi-nonexpansive mappings with a nonempty common fixed point set F. Assume that $\alpha_n \in (0,1]$ and $\beta_n, \gamma_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T_{i(n)}^{j(n)} z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + k_{i(n),j(n)}(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ \qquad -2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + k_{i(n),j(n)}^2(\alpha_n\beta_n + \alpha_n\gamma_n \\ \qquad +\beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n k_{i(n),j(n)}^3 - \alpha_n - \beta_n - \gamma_n \\ \qquad +\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$, $n = (j(n)-1)N + i(n)$ *for all* $n \geq 0$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We can drive the prove from the following two conclusions.
Conclusion 1. $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.
Conclusion 2. $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings T_n. □

Corollary 3.2. *Let* C *be a closed convex subset of a Hilbert space* H, *and let* $T : C \to C$ *be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set* F. *Assume that* $\alpha_n \in (0, 1]$ *and* $\beta_n, \gamma_n \in [0, 1]$ *for all* $n \in N$. *Then* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n)z_n + \alpha_n T^n z_n, \quad n \geq 0, \\ z_n = (1-\beta_n)t_n + \beta_n T^n t_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + k_n(\alpha_n + \beta_n + \gamma_n - 2\alpha_n\beta_n \\ \qquad -2\alpha_n\gamma_n - 2\beta_n\gamma_n + 3\alpha_n\beta_n\gamma_n) + k_n^2(\alpha_n\beta_n + \alpha_n\gamma_n \\ \qquad +\beta_n\gamma_n - 3\alpha_n\beta_n\gamma_n) + \alpha_n\beta_n\gamma_n k_n^3 - \alpha_n - \beta_n - \gamma_n \\ \qquad +\alpha_n\beta_n + \alpha_n\gamma_n + \beta_n\gamma_n - \alpha_n\beta_n\gamma_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in Theorem 3.1, we get the desired result. □

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An Intermixed Algorithm for Three Strict Pseudo-contractions in Hilbert Spaces

Waqas Nazeer[1], Mobeen Munir[2] and Shin Min Kang[3,4,*]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[3]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[4]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

Abstract

We generalize an intermixed algorithm to three and m-strict pseudo-contractions in Hilbert spaces and show that this algorithm converges strongly to the fixed points of three and m-strict pseudo-contractions in Hilbert spaces, independently. Consequently, we can find the common fixed points of these mappings.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with its inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We use $Fix(T)$ to denote the set of fixed points of T. A mapping $T : C \to C$ is said to be *strictly pseudo-contractive* if there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

It is well known that every strictly pseudo-contractive mapping is also nonexpansive mapping but a nonexpansive mapping may not be pseudo-contractive mapping. For the

*Corresponding author

rest of this article, we reserve C to be a nonempty closed convex subset of a Hilbert space H.

Iterative construction of fixed points is a celebrated idea in these days in the realm of nonlinear mappings. $T : C \to C$ be a nonlinear mapping and $\{\alpha_n\}$ be a real number sequence in $(0, 1)$. For fixed $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ by the following manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

which is the Mann's iteration scheme ([11]). If T is a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $\{\alpha_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T ([14]). Now, it is a common fact that, in infinite-dimensional Hilbert spaces, Mann's algorithm fails to converge strongly . An active area of research today is to develop Iterative methods for nonexpansive mappings; see [1–4, 7–10, 14–19] . But for strict pseudo-contraction mappings, iterative methods are far less developed though Browder and Petryshyn [1] started this work in 1967. Because of some powerful applications, (see Scherzer [15]), we desired to create algorithms for computation of the fixed points of strict pseudo-contraction mappings. As Mann's algorithm is too strong enough to approximate fixed points of pseudo-contractions, we need to find other type of iterative algorithms, see [6, 12, 21]. The first attempt was made by Ishikawa [9] with the following Ishikawa algorithm which can be viewed as a double-step Mann's algorithm.

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$, T is a (nonlinear) self-mapping of C, where $x_0 \in C$ arbitrarily. Ishikawa proved that his algorithm converges in norm to a fixed point of a Lipschitz pseudo-contraction T if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy certain conditions and if T is compact.

In 2000, Noor [13] gave following three step Noor iterative scheme

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tz_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0, \end{cases}$$

In [20], the following algorithm for two strict pseudo-contraction mappings S and T is given which converges strongly.

Algorithm 1.1. *For given $x_0 \in C$, $y_0 \in C$ arbitrarily, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], \quad n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

The quest for the answer of the question, can we develop an iterative algorithm which strongly converges to fixed points of finite many strict pseudo-contractions? However the answer of this problem is still not known. In this paper, Our main purpose is to give a redundant intermixed algorithms for three and m-strict pseudo-contractions. It is shown that the above said algorithm converges strongly to the fixed points of three and m-strict pseudo-contractions, independently. As applications, we can find these common fixed points in the settings of Hilbert spaces.

2 Preliminaries

The metric projection from H onto C is defined as: for each point $x \in H$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C,$$

where P_C is given by

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x \rangle \leq 0, \quad y \in C.$$

Consequently, P_C is nonexpansive. Following well-known lemmas will be important for our results.

Lemma 2.1. ([12]) *Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at 0, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.2. ([10]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3. ([16]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n$, $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in R such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudo-contraction. Let $f : C \to H$ be a ρ_1-contraction, $g : C \to H$ be a ρ_2-contraction and $h : C \to H$ be a ρ_3-contraction. Let $k \in (0, 1 - \lambda)$ be a constant.

Now we give the following redundant intermixed algorithm for three strict pseudo-contractions T_1, T_2 and T_3.

Algorithm 3.1. *For given $x_0 \in C$, $y_0 \in C$ and $z_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$be generated iteratively by*

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1 x_n], \quad n \geq 0, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(z_n) + (1-k-\alpha_n)y_n + kT_2 y_n], \quad n \geq 0, \\ z_{n+1} = (1-\beta_n)z_n + \beta_n P_C[\alpha_n h(x_n) + (1-k-\alpha_n)z_n + kT_3 z_n], \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Remark 3.2. Note that this algorithm is said to be the redundant intermixed algorithm as $\{x_n\}$ in $\{z_n\}$ and $\{z_n\}$ is in $\{y_n\}$ and $\{y_n\}$ is in $\{x_n\}$. So we can use this algorithm to find the fixed points of T_1, T_2 and T_3, independently.

Theorem 3.3. *Suppose that $Fix(T_1) \neq \emptyset$, $Fix(T_2) \neq \emptyset$ and $Fix(T_3) \neq \emptyset$. Assume the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(C2)$ $\beta_n \in [\xi_1, \xi_2] \subset (0,1)$ for all $n \geq 0$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (3.1) converge strongly to the fixed points $P_{Fix(T_1)}f(y^*)$, $P_{Fix(T_2)}g(x^*)$ and $P_{Fix(T_3)}h(x^*)$ of T_1, T_2 and T_3, respectively, where $x^* \in Fix(T_1)$, $y^* \in Fix(T_2)$ and $z^* \in Fix(T_3)$.

Note that, $P_C[\alpha f + (1-k-\alpha)I + kT]$ is contractive for small enough α, see [20].

First, we give the following propositions.

Proposition 3.4. *The sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$are bounded.*

Proof. Since $Fix(T_1) \neq \emptyset$, $Fix(T_2) \neq \emptyset$ and $Fix(T_3) \neq \emptyset$, we can choose $x^* \in Fix(T_1)$, $y^* \in Fix(T_2)$ and $z^* \in Fix(T_3)$. From (3.1), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1 x_n] - x^*\| \\ &\leq \beta_n \|P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1 x_n] - x^*\| \\ &\quad + (1-\beta_n)\|x_n - x^*\| \\ &\leq \beta_n \alpha_n \|f(y_n) - x^*\| + \beta_n \|(1-k-\alpha_n)(x_n - x^*) + k(T_1 x_n - T_1 x^*)\| \\ &\quad + (1-\beta_n)\|x_n - x^*\| \\ &\leq \beta_n \alpha_n \|f(y_n) - f(y^*)\| + \beta_n \alpha_n \|f(y^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &\quad + \beta_n(1-\alpha_n)\|x_n - x^*\| \\ &\leq \rho_1 \beta_n \alpha_n \|y_n - y^*\| + \beta_n \alpha_n \|f(y^*) - x^*\| + (1-\alpha_n\beta_n)\|x_n - x^*\| \\ &\leq \rho \beta_n \alpha_n \|y_n - y^*\| + \beta_n \alpha_n \|f(y^*) - x^*\| + (1-\alpha_n\beta_n)\|x_n - x^*\|, \end{aligned} \tag{3.2}$$

where $\rho = \max\{\rho_1, \rho_2, \rho_3\}$.

Similarly, we have

$$\begin{aligned} \|y_{n+1} - y^*\| &\leq \rho_2 \beta_n \alpha_n \|z_n - z^*\| + \beta_n \alpha_n \|g(z^*) - y^*\| + (1-\alpha_n\beta_n)\|y_n - y^*\| \\ &\leq \rho \beta_n \alpha_n \|z_n - z^*\| + \beta_n \alpha_n \|g(z^*) - y^*\| + (1-\alpha_n\beta_n)\|y_n - y^*\| \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|z_{n+1} - z^*\| &\leq \rho_3 \beta_n \alpha_n \|x_n - x^*\| + \beta_n \alpha_n \|h(x^*) - z^*\| + (1-\alpha_n\beta_n)\|z_n - z^*\| \\ &\leq \rho \beta_n \alpha_n \|x_n - x^*\| + \beta_n \alpha_n \|h(x^*) - z^*\| + (1-\alpha_n\beta_n)\|z_n - z^*\|. \end{aligned} \tag{3.4}$$

By adding (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\|+\|z_{n+1}-z^*\| \\
&\leq [1-(1-\rho)\alpha_n\beta_n](\|x_n-x^*\|+\|y_n-y^*\|+\|z_n-z^*\|)+\alpha_n\beta_n(\|f(y^*)-x^*\| \\
&\quad +\|g(x^*)-y^*\|+\|h(z^*)-z^*\|) \\
&\leq \max\Bigg\{\|x_n-x^*\|+\|y_n-y^*\|+\|z_n-z^*\|, \\
&\qquad \frac{\|f(y^*)-x^*\|+\|g(x^*)-y^*\|+\|g(x^*)-z^*\|}{1-\rho}\Bigg\}.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&\|x_n-x^*\|+\|y_n-y^*\|+\|z_n-z^*\| \\
&\leq \max\Bigg\{\|x_0-x^*\|+\|y_0-y^*\|+\|z_0-z^*\|, \\
&\qquad \frac{\|f(y^*)-x^*\|+\|g(z^*)-y^*\|+\|h(x^*)-z^*\|}{1-\alpha}\Bigg\}.
\end{aligned}$$

So, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. This completes the proof. □

Proposition 3.5. *$\|x_n-T_1x_n\|\to 0$, $\|y_n-T_2y_n\|\to 0$ and $\|z_n-T_3z_n\|\to 0$.*

Proof. We will prove it for $\{x_n\}$ and $\{z_n\}$, for $\{y_n\}$ it is similar. We first estimate $\|x_{n+1}-x_n\|$. Set $u_n=P_C[\alpha_nf(y_n)+(1-k-\alpha_n)x_n+kT_1x_n], n\geq 0$. It follows that

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leq \|\alpha_{n+1}f(y_{n+1})+(1-k-\alpha_{n+1})x_{n+1}+kT_1x_{n+1} \\
&\quad -\alpha_nf(y_n)-(1-k-\alpha_n)x_n+kT_1x_n\| \\
&\leq \|(1-k-\alpha_{n+1})(x_{n+1}-x_n)+k(T_1x_{n+1}-T_1x_n)\| \\
&\quad +\alpha_{n+1}(\|f(y_{n+1})\|+\|x_n\|)+\alpha_n(\|f(y_n)\|+\|x_n\|) \\
&\leq (1-\alpha_{n+1})\|x_{n+1}-x_n\|+\alpha_{n+1}(\|f(y_{n+1})\|+\|x_n\|) \\
&\quad +\alpha_n(\|f(y_n)\|+\|x_n\|).
\end{aligned}$$

Since $\alpha_n\to 0$, we deduce that

$$\limsup_{n\to\infty}(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

From Lemma 2.2, we get

$$\lim_{n\to\infty}\|u_n-x_n\|=0 \text{ and } \lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

From (3.1), we derive

$$\begin{aligned}
\|x_{n+1}-T_1x_n\| &\leq (1-\beta_n)\|x_n-T_1x_n\|+\beta_n\alpha_n\|f(y_n)-T_1x_n\| \\
&\quad +\beta_n(1-k-\alpha_n)\|x_n-T_1x_n\| \\
&= [1-(k+\alpha_n)\beta_n]\|x_n-T_1x_n\|+\beta_n\alpha_n\|f(y_n)-T_1x_n\|.
\end{aligned}$$

Thus

$$\begin{aligned}\|x_n - T_1x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1x_n\| \\ &\leq [1-(k+\alpha_n)\beta_n]\|x_n - T_1x_n\| + \beta_n\alpha_n\|f(y_n) - T_1x_n\| \\ &\quad + \|x_n - x_{n+1}\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - T_1x_n\| &\leq \frac{1}{(k+\alpha_n)\beta_n}(\|x_n - x_{n+1}\| + \beta_n\alpha_n\|f(y_n) - T_1x_n\|) \\ &\to 0.\end{aligned}$$

Similarly, we can obtain

$$\lim_{n\to\infty}\|y_n - T_2y_n\| = 0.$$

Now, we will prove

$$\lim_{n\to\infty}\|z_n - T_3z_n\| = 0.$$

Set $w_n = P_C[\alpha_n h(z_n) + (1-k-\alpha_n)z_n + kT_3z_n]$, $n \geq 0$. It follows that

$$\begin{aligned}\|w_{n+1} - w_n\| &\leq \|\alpha_{n+1}h(x_{n+1}) + (1-k-\alpha_{n+1})z_{n+1} + kT_3z_{n+1} \\ &\quad - \alpha_n h(x_n) - (1-k-\alpha_n)z_n + kT_3z_n\| \\ &\leq \|(1-k-\alpha_{n+1})(z_{n+1} - z_n) + k(T_3z_{n+1} - T_3z_n)\| \\ &\quad + \alpha_{n+1}(\|h(x_{n+1})\| + \|z_n\|) + \alpha_n(\|h(x_n)\| + \|z_n\|) \\ &\leq (1-\alpha_{n+1})\|z_{n+1} - z_n\| + \alpha_{n+1}(\|h(x_{n+1})\| + \|z_n\|) \\ &\quad + \alpha_n(\|h(x_n)\| + \|z_n\|).\end{aligned}$$

Since $\alpha_n \to 0$, we deduce that

$$\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0.$$

From Lemma 2.2, we get

$$\lim_{n\to\infty}\|w_n - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|z_{n+1} - z_n\| = 0.$$

From (3.1), we derive

$$\begin{aligned}\|z_{n+1} - T_3z_n\| &\leq (1-\beta_n)\|z_n - T_3z_n\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\| \\ &\quad + \beta_n(1-k-\alpha_n)\|z_n - T_3z_n\| \\ &= [1-(k+\alpha_n)\beta_n]\|z_n - T_3z_n\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\|.\end{aligned}$$

Thus

$$\begin{aligned}\|z_n - T_3z_n\| &\leq \|z_n - z_{n+1}\| + \|z_{n+1} - T_3z_n\| \\ &\leq [1-(k+\alpha_n)\beta_n]\|z_n - T_3z_n\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\| \\ &\quad + \|z_n - z_{n+1}\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - T_3z_n\| &\leq \frac{1}{(k+\alpha_n)\beta_n}(\|z_n - z_{n+1}\| + \beta_n\alpha_n\|h(x_n) - T_3z_n\|) \\ &\to 0.\end{aligned}$$

This completes the proof. □

Note that the mapping $P_C[\alpha f + (1-k-\alpha)I + kT_1]$ is contractive for small enough α. Thus, the equation $x = P_C[tf(x) + (1-k-t)x + kT_1x]$ has a unique fixed point, denoted by x_t, that is,

$$x_t = P_C[tf(x_t) + (1-k-t)x_t + kT_1x_t] \tag{3.5}$$

for small enough t.

In order to prove Theorem 3.3, we need the following lemma.

Lemma 3.6. *Suppose $Fix(T_i) \neq \emptyset$, $i = 1, 2, 3$. Then as $t \to 0$, the net $\{x_t\}$ defined by (3.5) converges strongly to a fixed point of T_i.*

Proof. Let $z^* \in Fix(T_3)$. From (3.5), we have

$$\begin{aligned}\|z_t - z^*\| &= \|P_C[th(z_t) + (1-k-t)z_t + kT_3z_t] - z^*\| \\ &\leq t\|h(z_t) - z^*\| + \|(1-k-t)(z_t - z^*) + k(T_3z_t - z^*)\| \\ &\leq t\rho_1\|z_t - z^*\| + t\|h(z^*) - z^*\| + (1-t)\|z_t - z^*\|,\end{aligned}$$

hence

$$\|z_t - z^*\| \leq \frac{1}{1-\rho_1}\|h(z^*) - z^*\|.$$

Thus, $\{z_t\}$ is bounded. Again, from (3.5), we get

$$\|z_t - T_3z_t\| \leq t\|h(z_t) - T_3z_t\| + (1-k-t)\|z_t - T_3z_t\|.$$

It follows that

$$\|z_t - T_3z_t\| \leq \frac{t}{k+t}\|h(z_t) - T_3z_t\| \to 0.$$

Let $\{t_n\} \subset (0,1)$. Assume that $t_n \to 0$ as $n \to \infty$. Put $z_n := z_{t_n}$. We have $\lim_{n\to\infty}\|z_n - T_3z_n\| = 0$. Set $m_t = th(z_t) + (1-k-t)z_t + kT_3z_t$, for all t. Then, we have $z_t = P_Cm_t$, and for any $z^* \in Fix(T_3)$,

$$\begin{aligned}z_t - z^* &= z_t - m_t + m_t - z^* \\ &= z_t - m_t + t(h(z_t) - z^*) + (1-k-t)(z_t - z^*) + k(T_3z_t - z^*).\end{aligned}$$

From the property of the metric projection, we deduce

$$\langle z_t - m_t, z_t - z^*\rangle \leq 0.$$

So

$$\begin{aligned}\|z_t - z^*\|^2 &= \langle z_t - m_t, z_t - z^*\rangle + \langle (1-k-t)(z_t - z^*) + k(T_3z_t - z^*), z_t - z^*\rangle \\ &\quad + t\langle h(z_t) - z^*, z_t - z^*\rangle \\ &\leq \|(1-k-t)(z_t - z^*) + k(T_3z_t - z^*)\|\|z_t - z^*\| \\ &\quad + t\langle h(z_t) - h(z^*), z_t - z^*\rangle + t\langle h(z^*) - z^*, z_t - z^*\rangle \\ &\leq [1 - (1-\rho_1)t]\|z_t - z^*\|^2 + t\langle h(z^*) - z^*, z_t - z^*\rangle.\end{aligned}$$

Hence

$$\|z_t - z^*\|^2 \leq \frac{1}{(1-\rho_1)}\langle h(z^*) - z^*, z_t - z^*\rangle, \quad \forall z^* \in Fix(T).$$

By the similar arguments as that in [12], we can obtain that the net $\{z_t\}$ converges strongly to $z^* \in Fix(T_3)$. This completes the proof. □

From Lemma 3.6, we know that the net $\{z_t\}$ defined by $z_t = P_C[tu+(1-k-t)z_t+kT_3z_t]$, where $u \in H$ converges to $P_{Fix(T_3)}u$. Let $z^* \in Fix(T_3)$ and $y^* \in Fix(T_2)$ and $x^* \in Fix(T_1)$. If we take $u = h(z^*)$, then the net $\{z_t\}$ defined by $z_t = P_C[th(z^*) + (1-k-t)z_t + kT_3z_t]$ converges to $P_{Fix(T_3)}h(y^*)$.

Finally, we prove Theorem 3.3.

Proof. Now, we prove that $x_n \to P_{Fix(T_1)}f(y^*)$, $y_n \to P_{Fix(T_2)}g(z^*)$ and $z_n \to P_{Fix(T_3)}h(x^*)$, where $x^* \in Fix(T_1)$, $y^* \in Fix(T_2)$ and $z^* \in Fix(T_3)$. First we observe that, if the sequence $\{w_n\}$ is bounded and $\|w_n - Tw_n\| \to 0$, we easily deduce that

$$\limsup_{n\to\infty}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), w_n - P_{Fix(T_1)}f(y^*)\rangle \le 0,$$

$$\limsup_{n\to\infty}\langle g(P_{Fix(T_3)}h(x^*)) - P_{Fix(T_2)}g(z^*), w_n - P_{Fix(T_2)}g(z^*)\rangle \le 0$$

and

$$\limsup_{n\to\infty}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), w_n - P_{Fix(T_3)}h(x^*)\rangle \le 0.$$

We set

$$\begin{cases} u_n = P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n], \quad n \ge 0, \\ v_n = P_C[\alpha_n g(z_n) + (1-k-\alpha_n)y_n + kT_2y_n], \quad n \ge 0, \\ m_n = P_C[\alpha_n h(x_n) + (1-k-\alpha_n)z_n + kT_3z_n], \quad n \ge 0. \end{cases}$$

Thus, we deduce that the sequences $\{u_n\}$, $\{v_n\}$ and $\{m_n\}$ are bounded; and $\|u_n - T_1u_n\| \to 0$, $\|v_n - T_2v_n\| \to 0$ and $\|m_n - T_3m_n\| \to 0$. Therefore,

$$\limsup_{n\to\infty}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), u_n - P_{Fix(T_1)}f(y^*)\rangle \le 0,$$

$$\limsup_{n\to\infty}\langle g(P_{Fix(T_3)}h(x^*)) - P_{Fix(T_2)}g(z^*), v_n - P_{Fix(T_2)}g(z^*)\rangle \le 0$$

and

$$\limsup_{n\to\infty}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), m_n - P_{Fix(T_3)}h(x^*)\rangle \le 0.$$

Next, we estimate $\|u_n - P_{Fix(T_1)}f(y^*)\|$. Set $\tilde{u}_n = \alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n$, $\tilde{v}_n = \alpha_n g(z_n) + (1-k-\alpha_n)y_n + kT_2y_n$ and $\tilde{m}_n = \alpha_n h(x_n) + (1-k-\alpha_n)z_n + kT_3z_n$ for all n.

$$\begin{aligned}
&\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&= \|P_C[\tilde{U}_n] - P_{Fix(T_1)}f(y^*)\|^2 \\
&\le \langle \tilde{U}_n - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&= \langle \alpha_n f(y_n) + (1-k-\alpha_n)x_n + kT_1x_n - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\le \alpha_n\langle f(y_n) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\quad + (1-\alpha_n)\|x_n - P_{Fix(T_1)}f(y^*)\|\|U_n - P_{Fix(T_1)}f(y^*)\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1-\alpha_n}{2}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{1}{2}\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \alpha_n\langle f(y_n) - f(P_{Fix(T_2)}g(z^*)), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\quad + \alpha_n\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\leq \frac{1-\alpha_n}{2}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{1}{2}\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \alpha_n\rho\|y_n - P_{Fix(T_2)}g(z^*)\|\|U_n - P_{Fix(T_1)}f(y^*)\| \\
&\quad + \alpha_n\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\leq \frac{1-\alpha_n}{2}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{1}{2}\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \frac{\alpha_n\rho}{2}(\|y_n - P_{Fix(T_2)}g(z^*)\|^2 + \|U_n - P_{Fix(T_1)}f(y^*)\|^2) \\
&\quad + \alpha_n\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle,
\end{aligned}$$

so, we have

$$\begin{aligned}
&\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\leq \frac{1-\alpha_n}{1-\alpha_n\rho}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{\alpha_n\rho}{1-\alpha_n\rho}\|y_n - P_{Fix(T_2)}g(z^*)\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\rho}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|x_{n+1} - P_{Fix(T_1)}f(y^*)\|^2 \\
&\leq (1-\beta_n)\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \beta_n\|U_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\leq (1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n)\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|y_n - P_{Fix(T_2)}g(z^*)\|^2 \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|y_{n+1} - P_{Fix(T_2)}g(z^*)\|^2 \\
&\leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\|y_n - P_{Fix(T_2)}g(z^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|z_n - P_{Fix(T_3)}h(y^*)\|^2 \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle g(P_{Fix(T_3)}h(y^*)) - P_{Fix(T_2)}g(z^*), V_n - P_{Fix(T_2)}g(z^*)\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\|z_{n+1} - P_{Fix(T_3)}h(x^*)\|^2 \\
&\leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\|z_n - P_{Fix(T_3)}h(x^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|x_n - P_{Fix(T_1)}f(y^*)\|^2 \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), M_n - P_{Fix(T_3)}h(x^*)\rangle.
\end{aligned}$$

Combing all above, we have

$$\begin{aligned}
&\|x_{n+1} - P_{Fix(T_1)}f(y^*)\|^2 + \|y_{n+1} - P_{Fix(T_2)}g(z^*)\|^2 + \|z_{n+1} - P_{Fix(T_3)}h(x^*)\|^2 \\
&\leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)(\|x_n - P_{Fix(T_1)}f(y^*)\|^2 + \|y_n - P_{Fix(T_2)}g(z^*)\|^2) \\
&\quad + \|z_n - P_{Fix(T_3)}h(x^*)\|^2) \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle f(P_{Fix(T_2)}g(z^*)) - P_{Fix(T_1)}f(y^*), U_n - P_{Fix(T_1)}f(y^*)\rangle \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle g(P_{Fix(T_3)}h(x^*)) - P_{Fix(T_2)}g(z^*), V_n - P_{Fix(T_2)}g(z^*)\rangle \\
&\quad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle h(P_{Fix(T_1)}f(y^*)) - P_{Fix(T_3)}h(x^*), M_n - P_{Fix(T_3)}h(x^*)\rangle.
\end{aligned}$$

Therefore, $x_n \to P_{Fix(T_1)}f(y^*)$, $y_n \to P_{Fix(T_2)}g(z^*)$ and $z_n \to P_{Fix(T_3)}h(x^*)$. This completes the proof. □

4 An redundant intermixed algorithm for m-strict pseudo-contractions

Let $T_i : C \to C$ be λ-strict pseudo-contractions, $f_i : C \to H$ be ρ_i-contractions for $i = 1, 2, 3, ..., m$ and $k \in (0, 1-\lambda)$ be a constant.

We propose the following redundant intermixed algorithm for m-strict pseudo-contraction mappings T_i for $i = 1, 2, 3, ..., m$.

Algorithm 4.1.

$$\begin{cases}
x_{n+1}^1 = (1-\beta_n)x_n^1 + \beta_n P_C[\alpha_n f_1(x_n^2) + (1-k-\alpha_n)x_n^1 + kT_1x_n^1], & n \geq 0, \\
x_{n+1}^2 = (1-\beta_n)x_n^2 + \beta_n P_C[\alpha_n f_2(x_n^3) + (1-k-\alpha_n)x_n^2 + kT_2x_n^2], & n \geq 0, \\
x_{n+1}^3 = (1-\beta_n)x_n^3 + \beta_n P_C[\alpha_n f_3(x_n^4) + (1-k-\alpha_n)x_n^3 + kT_3x_n^3], & n \geq 0, \\
\quad\vdots \\
x_{n+1}^m = (1-\beta_n)x_n^m + \beta_n P_C[\alpha_n f_4(x_n^1) + (1-k-\alpha_n)x_n^m + kT_mx_n^m], & n \geq 0,
\end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two real number sequences in* $(0, 1)$.

Theorem 4.2. *Suppose that* $Fix(T_i) \neq \emptyset$. *Assume the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

$(C2)$ $\beta_n \in [\xi_1, \xi_2] \subset (0, 1)$ *for all* $n \geq 0$.

Then the sequences $\{x_n^i\}$ *generated by* (4.1) *converge strongly to the fixed points* $P_{Fix(T_i)}f_i(x^*)$ *of* T_i, *where* $x^{i*} \in Fix(T_i)$ *for all* $i = 1, 2, 3, ..., m$.

5 Conclusions

In this article, we presented an intermixed algorithm for three and m-strict pseudo-contractions in Hilbert spaces which are extensions of the results in [20]. We also proved

that, the above algorithm converges strongly to the fixed points for three and m-strict pseudo-contractions in Hilbert spaces, independently. Consequently, we can find the common fixed points of three and m-strict pseudo-contractions in Hilbert spaces.

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ON FIXED POINT THEOREMS IN DUALISTIC PARTIAL METRIC SPACES

MUHAMMAD NAZAM[1], MUHAMMAD ARSHAD[2], CHOONKIL PARK[3*] AND DONG YUN SHIN[4*]

Abstract. In this paper, we introduce dualistic contractive mappings and use such mappings to prove some fixed point theorems. The results extend various comparable results existing in the literature. Moreover, we give examples that show the superiority and effectiveness of our results among corresponding fixed point theorems in partial metric spaces.

Keywords: Fixed point, dualistic partial metric, monotone mapping.
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1. Introduction and preliminaries

In [6], Matthews introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. O'Neill [7] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. In [10], Oltra and Valero presented a Banach fixed point theorem on complete dualistic partial metric spaces. They also showed that the contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later, Valero [10] generalized the main theorem of [9] using nonlinear contractive condition instead of Banach contractive condition.

Alghamdi et. al.[1], presented the following theorems in partial metric spaces, which are stated below:

Theorem 1. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a weakly contractive mapping. Then T has a unique fixed point $x^* \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$. Moreover, $p(x^*, x^*) = 0$.*

Theorem 2. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a Kannan mapping. Then T has a unique fixed point $x \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$. Moreover, $p(x^*, x^*) = 0$.*

*Corresponding authors.
[1,2]Department of mathematics, International Islamic University Islamabad, Pakistan
[3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
[4]Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
Email: nazim.phdma47@iiu.edu.pk, marshadzia@iiu.edu.pk, baak@hanyang.ac.kr, dyshin@uos.ac.kr.

We shall prove new fixed point theorems that generalize fixed point theorems provided by Alghamdi, Shahzad and Valero in [1]. We will show, with the help of examples, that the new results allow us to find fixed points of mappings in some cases in which the results in partial metric spaces cannot be applied. The key feature in these fixed point theorems is that the contractivity condition on the nonlinear map is only assumed to hold on elements that are comparable in the partial order. However, the map is assumed to be monotone.

Throughout, in this paper, the letters $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, real numbers and positive integers, respectively.
Let us recall some mathematical basics of dualistic partial metric space to make this paper self-sufficient.

Definition 1. [7] A dualistic partial metric on a nonempty set X is a function $D : X \times X \longrightarrow \mathbb{R}$ satisfying the following properties, for all $x, y, z, \in X$:

(D_1) $x = y \Leftrightarrow D(x, x) = D(y, y) = D(x, y)$.
(D_2) $D(x, x) \leq D(x, y)$.
(D_3) $D(x, y) = D(y, x)$.
(D_4) $D(x, z) \leq D(x, y) + D(y, z) - D(y, y)$.

And the pair (X, D) represents a dualistic partial metric space.

If (X, D) is a dualistic partial metric space, then the function $d_D : X \times X \to \mathbb{R}^+$ defined by

$$d_D(x, y) = D(x, y) - D(x, x)$$

is a quasi metric on X such that $\tau(D) = \tau(d_D)$ for all $x, y \in X$.

Remark 1. It is obvious that every partial metric is a dualistic partial metric but the converse is not true. To support this comment, define $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$D_\vee(x, y) = x \vee y = \sup\{x, y\}$$

for all $x, y \in \mathbb{R}$. It is clear that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, since $D_\vee(-1, -2) = -1 \notin \mathbb{R}^+$. However, the restriction of $D_\vee$ to $\mathbb{R}^+$, $D_\vee|_{\mathbb{R}^+}$, is a partial metric.

Example 1. If (X, d) is a metric space and $c \in \mathbb{R}$ is arbitrary constant, then

$$D(x, y) = d(x, y) + c.$$

defines a dualistic partial metric on X.

Example 2. Let $X = \mathbb{R}$ and define the function $D : X \times X \to \mathbb{R}$ by

$$D(x, y) = x + y - xy$$

for all $x \leq y \wedge 1$. Then (X, D) is a dualistic partial metric space.

Following [7], each dualistic partial metric D on X generates a T_0 topology $\tau(D)$ on X which has, as a base, the family of D-balls $\{B_D(x, \epsilon) : x \in X, \epsilon > 0\}$ and $B_D(x, \epsilon) = \{y \in X : D(x, y) < \epsilon + D(x, x)\}$.

Definition 2. [7] Let (X, D) be a dualistic partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, D) converges to a point $x \in X$ if and only if $D(x, x) = \lim_{n\to\infty} D(x, x_n)$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(x_n, x_m)$ exists and is finite.
(3) A dualistic partial metric space (X, D) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(D)$, to a point $x \in X$ such that $D(x, x) = \lim_{n,m\to\infty} D(x_n, x_m)$.

Following lemma will be helpful in the sequel.

Lemma 1. [7, 10]

(1) A dualistic partial metric (X, D) is complete if and only if the metric space (X, d^s_D) is complete.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$, with respect to $\tau(d^s_D)$ if and only if $\lim_{n\to\infty} D(x, x_n) = D(x, x) = \lim_{n,m\to\infty} D(x_n, x_m)$.
(3) If $\lim_{n\to\infty} x_n = \upsilon$ such that $D(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} D(x_n, y) = D(\upsilon, y)$ for every $y \in X$.

Later on, Oltra and Valero [9] established a Banach fixed point theorem for dualistic partial metric spaces in such a way that the Matthews fixed point theorem is obtained as a particular case. The aforesaid result can be stated as follows:

Theorem 3. *Let (X, D) be a complete dualistic partial metric space and let $T : X \to X$ be a mapping such that there exists $\alpha \in [0, 1[$ satisfying*

$$|D(T(x), T(y))| \leq \alpha |D(x, y)|,$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, $D(x^*, x^*) = 0$ and the Picard iterative sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d^s_D)$, for every $x \in X$.

2. Main results

In this section, we shall prove the dualistic partial metric versions of Theorems 1 and 2.

Definition 3. *Let $(X, \preceq, D)$ be an ordered dualistic partial metric space. A self map T defined on X is said to be a Kannan type dualistic contractive mapping if there exists $k \in [0, 1[$ such that*

$$|D(T(x), T(y))| \leq \frac{k}{2}[|D(x, T(x))| + |D(y, T(y))|] \tag{2.1}$$

for all comparable $x, y \in X$.

Our first main result is given below.

Theorem 4. *Let $(X, \preceq)$ be a partially ordered set and (X, D) be a complete dualistic partial metric space. Let $T : X \to X$ be a nondecreasing mapping. If T satisfies following conditions;*

(1) *T is a Kannan type dualistic contractive mapping.*

(2) *there exists* $x_0 \in X$ *such that* $x_0 \preceq T(x_0)$.

(3) *if* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $\{x_n\} \to x \in X$, *then* $x_n \preceq x$.

Then T *has a fixed point* x^* *such that* $D(x^*, x^*) = 0$.

Proof. Let us consider the Picard iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ with initial point $x_0 \in X$ (i.e., $x_n = T(x_{n-1})$ for all $n \in \mathbb{N}$). Of course, if there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = T(x_n)$, then x_n is a fixed point of T. On the other hand, if $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n \preceq x_{n+1}$. Indeed by $x_0 \preceq T(x_0)$, we obtain $x_0 \preceq x_1$. Since T is nondecreasing, $x_0 \preceq x_1$ implies $T(x_0) \preceq T(x_1)$ and so $x_1 \preceq x_2$. Continuing in this way, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, using contractive condition (2.1), we have

$$\begin{aligned} |D(x_1, x_2)| &= |D(T(x_0), T(x_1))| \\ &\leq \frac{k}{2}[|D(x_0, T(x_0))| + |D(x_1, T(x_1))|] \\ &= \frac{k}{2}[|D(x_0, x_1)| + |D(x_1, x_2)|], \end{aligned}$$

which implies

$$(1 - \frac{k}{2})|D(x_1, x_2)| \leq \frac{k}{2}|D(x_0, x_1)|$$

and so

$$|D(x_1, x_2)| \leq \lambda |D(x_0, x_1)|,$$

where $\lambda = \frac{k}{2-k}$ and $0 < \lambda < 1$.

Similarly,

$$\begin{aligned} |D(x_2, x_3)| &= |D(T(x_1), T(x_2))| \\ &\leq \frac{k}{2}[|D(x_1, T(x_1))| + |D(x_2, T(x_2))|]. \end{aligned}$$

Thus,

$$|D(x_2, x_3)| \leq \lambda |D(x_1, T(x_1))| \leq \lambda^2 |D(x_0, x_1)|.$$

Continuing in this way, we have

$$|D(x_n, x_{n+1})| \leq \lambda^n |D(x_0, x_1)|. \tag{2.2}$$

Since $x_n \preceq x_n$, from the contractive condition (2.1), we get

$$|D(x_n, x_n)| \leq k\lambda^{n-1} |D(x_0, x_1)|. \tag{2.3}$$

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, D), we shall prove that $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). Clearly,

$$\begin{aligned} D(x_n, x_{n+1}) - D(x_n, x_n) &\leq |D(x_n, x_{n+1})| + |D(x_n, x_n)| \\ &\leq \lambda^n |D(x_0, x_1)| + k\lambda^{n-1}|D(x_0, x_1)| \\ &\leq \lambda^n (3-k)|D(x_0, x_1)| \end{aligned}$$

for all $n \in \mathbb{N}$. Thus for a fixed $p \in \mathbb{N}$,

$$D(x_{n+p-1}, x_{n+p}) - D(x_{n+p-1}, x_{n+p-1}) \leq \lambda^{n+p-1}(3-k)|D(x_0, x_1)| \tag{2.4}$$

for all $n \in \mathbb{N}$.

Now using (D_4) and (2.4), we have

$$\begin{aligned} D(x_n, x_{n+p}) - D(x_n, x_n) &\leq D(x_n, x_{n+1}) + D(x_{n+1}x_{n+2}) + \dots \\ &+ D(x_{n+p-1}, x_{n+p}) - \sum_{i=0}^{\eta-1} D(x_{n+i}, x_{n+i}) \\ &\leq (\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+p-1})(3-k)|D(x_0, x_1)| \\ &\leq \frac{\lambda^n}{1-\lambda}(3-k)|D(x_0, x_1)|. \end{aligned}$$

Similarly,

$$D(x_{n+p}, x_n) - D(x_{n+p}, x_{n+p}) \leq \frac{\lambda^n}{1-\lambda}(1+k)|D(x_0, x_1)|.$$

Consequently,

$$d_D^s(x_n, x_m) \leq 4\frac{\lambda^n}{1-\lambda}|D(x_0, x_1)|$$

for all $n + p = m > n \in \mathbb{N}$

This leads to $\lim_{n,m\to\infty} d_D^s(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). Since (X, D) is a complete dualistic partial metric space, by Lemma 1, (X, d_D^s) is also complete and there exists $x^* \in (X, d_D^s)$ such that $x_n \to x^*$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} d_D^s(x_n, x^*) = 0.$$

By Lemma 1, we have

$$\lim_{n\to\infty} D(x^*, x_n) = D(x^*, x^*) = \lim_{n,m\to\infty} D(x_n, x_m). \tag{2.5}$$

Since $\lim_{n,m\to\infty} d_D(x_n, x_m) = 0$, the inequality (2.3) implies that $\lim_{n,m\to\infty} D(x_n, x_m) = 0$, which shows that $\{x_n\}$ is a Cauchy sequence in (X, D). From (2.5), we get

$$D(x^*, x^*) = \lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.6}$$

Now, it follows from the hypotheses (3), (2.1) and (D_4) that

$$\begin{aligned} D(x^*, T(x^*)) &\leq D(x^*, x_n) + D(x_n, T(x^*)) - D(x_n, x_n), \\ &\leq D(x^*, x_n) + |D(x_n, T(x^*))| + |D(x_n, x_n)|, \\ &\leq D(x^*, x_n) + \frac{k}{2}[|D(x_{n-1}, x_n)| + D(x^*, T(x^*))] + |D(x_n, x_n)|. \end{aligned}$$

Hence we obtain

$$(1 - \frac{k}{2})D(x^*, T(x^*)) \leq D(x^*, x_n) + \frac{k}{2}|D(x_{n-1}, x_n)| + |D(x_n, x_n)|.$$

Letting $n \to \infty$ and using (2.3) and (2.2), we obtain

$$(1 - \frac{k}{2})D(x^*, T(x^*)) \leq 0.$$

and so $D(x^*, T(x^*)) \leq 0$, but also $0 = D(x^*, x^*) \leq D(x^*, T(x^*))$. We deduce that

$$D(x^*, Tx^*) = D(x^*, x^*) = D(T(x^*), T(x^*)) = 0.$$

This implies that $x^* = T(x^*)$. Hence x^* is a fixed point of T with $D(x^*, x^*) = 0$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_D^s)$ for any $x \in X$. □

Remark 2. *In case when $D(x, y) \in \mathbb{R}^+$ for all $x, y \in X$, Theorem 4 reduces to Theorem 2.*

A natural question that can be raised is whether the contractive condition in the statement of Theorem 4 can be replaced by the contractive condition in the statement of Theorem 2. The following easy example provides a negative answer to this question.

Example 3. *Consider the complete ordered dualistic partial metric $(\mathbb{R}, \leq, D_\vee)$. Define the self-mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by*

$$T_0(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}.$$

It is easy to check that T_0 is nondecreasing with respect to usual order on $\mathbb{R}$ and for all comparable $x, y \in \mathbb{R}$, contractive condition

$$D_\vee(T_0(x), T_0(y)) \leq \frac{1}{2}[D_\vee(x, T_0(x)) + D_\vee(y, T_0(y))]$$

holds. However, T_0 does not have a fixed point. Observe that T_0 does not satisfy the contractive condition in the statement of Theorem 4. Indeed, note that for all $k \in [0, 1[$, we have

$$\begin{aligned} 1 = |D_\vee(-1, -1)| = |D_\vee(T_0(0), T_0(0))| &> \frac{k}{2}\left[|D_\vee(0, T_0(0))| + |D_\vee(0, T_0(0))|\right] \\ &= k|(0 \vee (-1)| = 0. \end{aligned}$$

For next result, we begin with following definition.

Definition 4. *Let $(X, \preceq, D)$ be an ordered dualistic partial metric space. A mapping $T : X \to X$ is said to be a weakly dualistic contractive if there exists $\alpha : X \times X \to [0, 1[$ such that for all $0 \leq a \leq b$*

$$\theta(a, b) = \sup\{\alpha(x, y) : a \leq |D(x, y)| \leq b\} < 1,$$

and for all comparable $x, y \in X$

$$|D(T(x), T(y))| \leq \alpha(x, y)|D(x, y)|. \tag{2.7}$$

Example 4. *Consider $([-1, 1], \leq, D_\vee)$ an ordered dualistic partial metric space. Define the mapping $T_3 : X \to X$ by*

$$T_3(x) = \frac{x^3}{x^2 + 1}$$

for all $x \in X$. We define $\alpha : [-1, 1] \times [-1, 1] \to [0, 1]$ by

$$\alpha(x, y) = \begin{cases} \dfrac{D_\vee(T_3x, T_3y)}{D_\vee(x, y)} & \text{if } D_\vee(x, y) \neq 0 \\ \\ 0 & \text{if } D_\vee(x, y) = 0 \end{cases}.$$

Observe that $\dfrac{D_\vee(T_3x, T_3y)}{D_\vee(x, y)} > 0$ provided that $D_\vee(x, y) \neq 0$. It is easy to check that $\alpha(x, y) \leq \frac{1}{2}$ for all comparable $x, y \in [-1, 1]$ and that $\theta(a, b) < 1$ for all $a, b \in \mathbb{R}$ with $0 \leq a \leq b$. Moreover,

$$|D_\vee(T_3x, T_3y)| \leq \alpha(x, y)|D_\vee(x, y)|$$

for all comparable $x, y \in [-1, 1]$.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and (X, D) be a complete dualistic partial metric space. Let $T : X \to X$ be a nondecreasing mapping. Assume that T satisfies following conditions;*

1. *T is a weakly dualistic contractive mapping.*
2. *there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$.*
3. *either T is continuous or if $\{x_n\}$ is a nondecreasing sequence in X such that $\{x_n\} \to x \in X$, then $x_n \preceq x$.*

Then T has a fixed point x^* with $D(x^*, x^*) = 0$.

Proof. Consider the Picard iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ with an initial point $x_0 \in X$ (i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$). It is clear that if there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_n is a fixed point of T. On the other hand, if $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$, then $x_n \preceq x_{n+1}$. Indeed by $x_0 \preceq T(x_0)$, we obtain $x_0 \preceq x_1$. Since T is nondecreasing, $x_0 \preceq x_1$ implies $T(x_0) \preceq T(x_1)$, and so $x_1 \preceq x_2$. Continuing in this way, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, using contractive condition (2.7), we have

$$\begin{aligned} |D(x_n, x_{n+1})| &= |D(T(x_{n-1}), T(x_n))| \\ &\leq \alpha(x_{n-1}, x_n)|D(x_{n-1}, x_n)| \\ &\leq |D(x_{n-1}, x_n)|. \end{aligned}$$

This implies that the sequence $\{|D(x_n, x_{n+1})|\}_{n\in\mathbb{N}}$ is decreasing and bounded below. So it converges to $r \in \mathbb{R}$ with

$$r = \inf_{n\in\mathbb{N}} |D(x_{n-1}, x_n)| \geq 0.$$

We claim that $r = 0$. For the purpose of contradiction, assume $r > 0$.

$$0 < r \leq |D(x_n, x_{n+1})| \leq |D(x_{n-1}, x_n)| \leq \cdots \leq |D(x_0, x_1)|.$$

It implies $0 < r \leq |D(x_0, x_1)|$ and so we deduce that

$$\theta = \theta(r, |D(x_0, x_1)|) = \sup\{\alpha(x, y) : \quad r \leq |D(x, y)| \leq |D(x_0, x_1)|\} < 1.$$

Now from contractive conition (2.7), we get

$$\begin{aligned} r &\leq |D(x_n, x_{n+1})| \\ &\leq \alpha(x_{n-1}, x_n)|D(x_{n-1}, x_n)| \\ &\leq \theta(r, |D(x_0, x_1)|)|D(x_{n-1}, x_n)| \\ &\leq \theta^2(r, |D(x_0, x_1)|)|D(x_{n-2}, x_{n-1})| \leq \dots \\ &\leq \theta^n(r, |D(x_0, x_1)|)|D(x_0, x_1)|. \end{aligned}$$

Therefore,

$$r \leq \lim_{n\to\infty} \theta^n(r, |D(x_0, x_1)|)|D(x_0, x_1)|.$$

This implies that $r \leq 0$, which is a contradiction. Consequently, $r = 0$ and hence

$$\lim_{n\to\infty} |D(x_n, x_{n+1})| = 0 = \lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{2.8}$$

Now since $x_n \preceq x_n$, by arguing like above, we can show that $\lim_{n\to\infty} |D(x_n, x_n)| = \lim_{n\to\infty} D(x_n, x_n) = 0$, since

$$|D(x_n, x_n)| \leq \alpha(x_{n-1}, x_{n-1})|D(x_{n-1}, x_{n-1})|$$

for all $n \in \mathbb{N}$ and thus the sequence $\{|D(x_n, x_n)|\}_{n\in\mathbb{N}}$ is decreasing and bounded below.

Next we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_D^s). It is clear that

$$\begin{aligned} D(x_n, x_{n+1}) - D(x_n, x_n) &\leq \theta^n(0, |D(x_0, x_1)|)|D(x_0, x_1)| + \theta^n(0, |D(x_0, x_0)|)|D(x_0, x_0)| \\ &\leq \theta^n \left[|D(x_0, x_1)| + |D(x_0, x_0)|\right] \end{aligned}$$

for all $n \in \mathbb{N}$, where $\theta^n = (\theta^n(0, |D(x_0, x_1)|) \vee \theta^n(0, |D(x_0, x_0)|))$ for all $n \in \mathbb{N}$. This implies that, for a fixed $p \in \mathbb{N}$, we have

$$\begin{aligned} D(x_n, x_{n+p}) - D(x_n, x_n) &\leq D(x_n, x_{n+1}) + D(x_{n+1}x_{n+2}) + \dots \\ &+ D(x_{n+p-1}, x_{n+p}) - \sum_{i=0}^{p-1} D(x_{n+i}, x_{n+i}) \\ &\leq (\theta^n + \theta^{n+1} + \dots + \theta^{n+p-1})[|D(x_0, x_1)| + |D(x_0, x_0)|] \\ &\leq \frac{\theta^n}{1-\theta}[|D(x_0, x_1)| + |D(x_0, x_0)|] \end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, we can calculate that

$$D(x_{n+p}, x_n) - D(x_{n+p}, x_{n+p}) \leq \frac{\theta^n}{1-\theta}[|D(x_0, x_1)| + |D(x_0, x_0)|],$$

which implies that $\lim_{n\to\infty} d_D^s(x_n, x_{n+p}) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). Since (X, D) is a complete dualistic partial metric space, by Lemma 1, (X, d_D^s) is also complete and there exists $x^* \in (X, d_D^s)$ such that $x_n \to x^*$ as $n \to \infty$, i.e., $\lim_{n\to\infty} d_D^s(x_n, x^*) = 0$. Now again from Lemma 1, we get

$$D(x^*, x^*) = \lim_{n\to\infty} D(x_n, x^*) = \lim_{n,m\to\infty} D(x_n, x_m); \qquad m = n + p. \tag{2.9}$$

Now since $\lim_{n,m\to\infty} d_D(x_n, x_m) = 0$, $\lim_{n,m\to\infty}[D(x_n, x_m) - D(x_n, x_n)] = 0$ and

$$\lim_{n,m\to\infty} D(x_n, x_m) = \lim_{n\to\infty} D(x_n, x_n)$$

but (2.8) implies that

$$\lim_{n\to\infty} D(x_n, x_n) = 0.$$

It follows directly that

$$D(x^*, x^*) = \lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.10}$$

Now if T is continuous, then

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(x^*).$$

Now if T is discontinuous, then by the hypotheses (3), we have

$$\begin{aligned} D(x^*, T(x^*)) &\leq D(x^*, x_n) + D(x_n, T(x^*)) - D(x_n, x_n) \\ &\leq D(x^*, x_n) + |D(x_n, T(x^*))| + |D(x_n, x_n)| \\ &\leq D(x^*, x_n) + \alpha(x_{n-1}, x^*)|D(x_{n-1}, x^*)| + |D(x_n, x_n)| \\ &\leq D(x^*, x_n) + |D(x_{n-1}, x^*)| + |D(x_n, x_n)|. \end{aligned}$$

Since $\lim_{n\to\infty} D(x_n, x_n) = \lim_{n\to\infty} D(x_n, x^*) = 0$, $D(x^*, T(x^*)) \leq 0$, but also

$$0 = D(x^*, x^*) \leq D(x^*, T(x^*)).$$

We deduce that

$$D(x^*, Tx^*) = D(x^*, x^*) = D(T(x^*), T(x^*)) = 0.$$

This implies that $x^* = T(x^*)$. Hence x^* is a fixed point of T with $D(x^*, x^*)$. □

Remark 3. Since every dualistic partial metric is an extension of partial metric, Theorem 5 is an extension of Theorem 1.

There arises the following natural question:

Whether the contractive condition in the statement of Theorem 5 can be replaced by the contractive condition in Theorem 1?

The following example provides a negative answer to the above question.

Example 5. *Consider the complete ordered dualistic partial metric* $(\mathbb{R}, \leq, D_\vee)$ *and the self-mapping* T_0 *defined as in Example 3. Then, for fixed* $k \in [0,1[$*, it is easy to verify that for all comparable* $x, y \in \mathbb{R}$*, the contractive condition*

$$D_\vee(T_0(x), T_0(y)) \leq \alpha(x,y) D_\vee(x,y)$$

holds with $\alpha(x,y) = k$*. However,* T_0 *does not have a fixed point. Observe that* T_0 *does not satisfy the contractive condition of Theorem 5. Indeed, there is no mapping* $\alpha : \mathbb{R}\times\mathbb{R} \to [0,1[$ *such that*

$$1 = |D_\vee(-1,-1)| = |D_\vee(T_0(0), T_0(0))| > \alpha(0,0)|D_\vee(0,0)| = 0.$$

Remark 4. Significance of the above results lies in the fact that these results are true for all real numbers whereas such results proved in partial metric spaces are only true for positive real numbers.

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Approximation of a kind of new Stancu-Bézier type operators

Mei-Ying Ren[1*], Xiao-Ming Zeng[2*]
[1]School of Mathematics and Computer Science, Wuyi University,
Wuyishan 354300, China
[2]School of Mathematical Sciences, Xiamen University, Xiamen 361005, Chnia
E-mail: npmeiyingr@163.com, xmzeng@xmu.edu.cn

Abstract. In this paper, a kind of new Stancu-Bézier type operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is also obtained.

Keywords: Stancu-Bézier type operators; Korovich type approximation theorem; Rate of convergence; Modulus of continuity; Modulus of smoothness

Mathematical subject classification: 41A10, 41A25, 41A36

1. Introduction

In 2012, Ren [6] introduced Bernstein type operators as follows:

$$L_n(f;x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x)B_{n,k}(f) + f(1)P_{n,n}(x), \tag{1}$$

where $f \in C[0,1]$, $x \in [0,1]$, $P_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$ $(k = 0,1,...,n)$, and $B_{n,k}(f) = \frac{1}{B(nk,n(n-k))} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(t)dt$ $(k = 1,...,n-1)$, $B(.,.)$ is the beta function.

The moments of the operators $L_n(f;x)$ were obtained as follows (see [6]):

Remark 1. For $L_n(t^m;x)$, $m = 0,1,2$, we have

(i) $L_n(1;x) = 1$;
(ii) $L_n(t;x) = x$;
(iii) $L_n(t^2;x) = \frac{n(n-1)}{n^2+1}x^2 + \frac{n+1}{n^2+1}x$.

In 2015, Inspired by [1], Ren and Zeng [7] introduced new type Bézier operators, which is the Bézier variant of the Bernstein type operators $Ln(f;x)$, as follows:

$$L_{n,\alpha}(f;x) = f(0)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)B_{n,k}(f) + f(1)Q_{n,n}^{(\alpha)}(x), \tag{2}$$

*Corresponding authors: Mei-Ying Ren and Xiao-Ming Zeng.

where $f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,n+1}(x) = 0$, $J_{n,k}(x) = \sum_{i=k}^{n} P_{n,i}(x)$, $P_{n,k}(x)$ $(k = 0,1,...n)$, $B_{n,k}(f)$ $(k = 1,...,n-1)$ and $B(.,.)$ are as stated in (1).

In the present paper, we will study the Stancu variant of the new type Bézier operators $L_{n,\alpha}(f;x)$, which have been given by (2). We introduce new Stancu-Bézier type operators as follows:

$$L_{n,\alpha}^{(\beta,\gamma)}(f;x) = f(\frac{\beta}{n+\gamma})Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}(f) + f(\frac{n+\beta}{n+\gamma})Q_{n,n}^{(\alpha)}(x), \quad (3)$$

where f, x, α, $Q_{n,k}^{(\alpha)}(x)(k = 0,1,...,n)$ are as stated in (2), β, γ are two given real parameters satisfying the condition $0 \le \beta \le \gamma$, $B(.,.)$ is the beta function, and $B_{n,k}^{(\beta,\gamma)}(f) = \frac{1}{B(nk,n(n-k))} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\frac{nt+\beta}{n+\gamma})dt$, $k = 1,...,n-1$, .

It is clear that $L_{n,\alpha}^{(\beta,\gamma)}(f;x)$ are bounded and positive on C[0,1]. When $\beta = \gamma = 0$, $L_{n,\alpha}^{(\beta,\gamma)}(f;x)$ become the operators $L_{n,\alpha}(f;x)$.

The goal of this paper is to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rates of convergence of these operators by using a modulus of continuity. Then, we obtain the direct theorem concerned with an approximation for these operators by means of the Ditzian-Totik modulus of smoothness.

In the paper, for $f \in C[0,1]$, we denote $\|f\| = max\{|f(x)| : x \in [0,1]\}$. $\omega(f,\delta)$ $(\delta > 0)$ denotes the usual modulus of continuity of $f \in C[0,1]$.

2. Auxiliary results

Now, we give some lemmas, which are necessary to prove our results.

Lemma 1. Let $\alpha > 0, x \in [0,1]$, $0 \le \beta \le \gamma$. We have

(i) $L_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$;

(ii) $\lim_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t;x) = x$ *uniformly on* $[0,1]$;

(iii) $\lim_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = x^2$ *uniformly on* $[0,1]$.

Proof (i) Since $\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = 1$, so, by (3), we get $L_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$.

(ii) by (2) and (3), we have

$$L_{n,\alpha}^{(\beta,\gamma)}(t;x) = \frac{n}{n+\gamma} L_{n,\alpha}(t;x) + \frac{\beta}{n+\gamma},$$

thus, by Lemma 2 (ii) in [7], we have $\lim_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t;x) = x$ uniformly on [0,1].

(iii) by (2) and (3), we have

$$L_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = (\frac{n}{n+\gamma})^2 L_{n,\alpha}(t^2;x) + \frac{2n\beta}{(n+\gamma)^2} L_{n,\alpha}(t;x) + (\frac{\beta}{n+\gamma})^2,$$

thus, by Lemma 2 (iii) in [7], we have $\lim_{n\to\infty} L_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = x^2$ uniformly on [0,1].

Lemma 2.(see [4]) For $x \in [0,1]$, $k = 0,1,...,n$, we have

$$0 \le Q_{n,k}^{(\alpha)}(x) \le \begin{cases} \alpha P_{n,k}(x), & \alpha \ge 1; \\ P_{n,k}^{\alpha}(x), & 0 < \alpha < 1. \end{cases}$$

Lemma 3.(see [5]) For $0 < \alpha < 1$, $\nu > 0$, we have

$$\sum_{k=0}^{n} |k - nx|^{\nu} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha} (A_{\frac{\nu}{\alpha}})^{\alpha} n^{\frac{\nu}{2}},$$

where the constant A_s only depends on s.

Lemma 4. Let $\alpha > 0$, $0 \le \beta \le \gamma$, We have

(i) $B_{n,k}^{(\beta,\gamma)}(1) = 1$;

(ii) $B_{n,k}^{(\beta,\gamma)}(t) = \dfrac{k+\beta}{n+\gamma}$;

(iii) $B_{n,k}^{(\beta,\gamma)}(t^2) = \dfrac{n^2}{(n+\gamma)^2(n^2+1)}(k^2+\dfrac{k}{n}) + \dfrac{2k\beta}{(n+\gamma)^2} + \dfrac{\beta^2}{(n+\gamma)^2}$.

Proof By [7], we have $B_{n,k}(1) = 1,\ B_{n,k}(t) = \frac{k}{n},\ B_{n,k}(t^2) = \frac{1}{n^2+1}(k^2+\frac{k}{n})$, so, by simple calculation, we obtain

$$
\begin{aligned}
&\text{(i)} B_{n,k}^{(\beta,\gamma)}(1) = 1;\\
&\text{(ii)} B_{n,k}^{(\beta,\gamma)}(t) = \frac{n}{n+\gamma}B_{n,k}(t) + \frac{\beta}{n+\gamma}B_{n,k}(1) = \frac{k+\beta}{n+\gamma};\\
&\text{(iii)} B_{n,k}^{(\beta,\gamma)}(t^2) = \frac{n^2}{(n+\gamma)^2}B_{n,k}(t^2) + \frac{2n\beta}{(n+\gamma)^2}B_{n,k}(t) + \frac{\beta^2}{(n+\gamma)^2}B_{n,k}(1)\\
&\qquad\qquad = \frac{n^2}{(n+\gamma)^2(n^2+1)}(k^2+\frac{k}{n}) + \frac{2k\beta}{(n+\gamma)^2} + \frac{\beta^2}{(n+\gamma)^2}.
\end{aligned}
$$

Lemma 5. For $\alpha \ge 1, x \in [0,1]$, $0 \le \beta \le \gamma$, we have

$$
\begin{aligned}
&\text{(i)} L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x) \le \frac{2\alpha(1+\gamma)}{n+\gamma};\\
&\text{(ii)} L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x) \le \sqrt{\frac{2\alpha(1+\gamma)}{n+\gamma}}.
\end{aligned}
$$

Proof (i) For $\alpha \ge 1, x \in [0,1]$, $0 \le \beta \le \gamma$, by (3), Lemma 2, (1) and Remark 1, we obtain

$$
\begin{aligned}
&L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\\
&= (\frac{\beta}{n+\gamma}-x)^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}((t-x)^2) + (\frac{n+\beta}{n+\gamma}-x)^2 Q_{n,n}^{(\alpha)}(x)\\
&\le \alpha[(\frac{\beta}{n+\gamma}-x)^2 P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) B_{n,k}^{(\beta,\gamma)}((t-x)^2) + (\frac{n+\beta}{n+\gamma}-x)^2 P_{n,n}(x)]\\
&= \alpha[\frac{n^2}{(n+\gamma)^2}L_n(t^2;x) + \frac{2n\beta}{(n+\gamma)^2}L_n(t;x) + \frac{\beta^2}{(n+\gamma)^2}L_n(1;x)]\\
&\quad -2\alpha x(\frac{n}{n+\gamma}L_n(t;x) + \frac{\beta}{n+\gamma}L_n(1;x)) + \alpha x^2 L_n(1;x)\\
&= \alpha\{\frac{n^2}{(n+\gamma)^2}[\frac{n(n-1)}{n^2+1}x^2 + \frac{n+1}{n^2+1}x] + \frac{2n\beta x}{(n+\gamma)^2} + \frac{\beta^2}{(n+\gamma)^2}\} - 2\alpha x(\frac{nx+\beta}{n+\gamma})\\
&\quad +\alpha x^2\\
&= \alpha[\frac{n^3+n^2-2\beta\gamma n^2-2\beta\gamma}{(n+\gamma)^2(n^2+1)}x(1-x) + \frac{-2\beta\gamma n^2-2\beta\gamma+\gamma^2 n^2+\gamma^2}{(n+\gamma)^2(n^2+1)}x^2 + \frac{\beta^2}{(n+\gamma)^2}]\\
&\le \alpha[\frac{n+1}{(n+\gamma)^2}x(1-x) + \frac{\beta^2+\gamma^2}{(n+\gamma)^2}]\\
&\le \frac{\alpha(n+1+\beta^2+\gamma^2)}{(n+\gamma)^2}\\
&\le \frac{2\alpha(1+\gamma)}{n+\gamma}.
\end{aligned}
$$

(ii) In view of $L_{n,\alpha}^{(\beta,\gamma)}(1;x)=1$, by the Cauchy-Schwarz inequality, we have

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)},$$

thus, we get

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{\frac{2\alpha(1+\gamma)}{n+\gamma}}.$$

Lemma 6. For $0<\alpha<1, x\in[0,1]$, $0\leq\beta\leq\gamma$, we have

$$\text{(i) } L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\leq M_\alpha^{(\beta,\gamma)}n^{-\alpha};$$
$$\text{(ii) } L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{M_\alpha^{(\beta,\gamma)}}\cdot n^{-\frac{\alpha}{2}}.$$

where the constant $M_\alpha^{(\beta,\gamma)}$ only depends on α,β,γ.

Proof (i) For $0<\alpha<1, x\in[0,1]$, $0\leq\beta\leq\gamma$, by (3), Lemma 2 and Lemma 4, we obtain

$$\begin{aligned}
&L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\\
&=(\frac{\beta}{n+\gamma}-x)^2Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}((t-x)^2)+(\frac{n+\beta}{n+\gamma}-x)^2Q_{n,n}^{(\alpha)}(x)\\
&\leq(\frac{\beta}{n+\gamma}-x)^2P_{n,0}^{\alpha}(x)+\sum_{k=1}^{n-1}P_{n,k}^{\alpha}(x)B_{n,k}^{(\beta,\gamma)}((t-x)^2)+(\frac{n+\beta}{n+\gamma}-x)^2P_{n,n}^{\alpha}(x)\\
&\leq\sum_{k=0}^{n}P_{n,k}^{\alpha}(x)[\frac{1}{(n+\gamma)^2}(k^2+\frac{k}{n})+\frac{2k\beta}{(n+\gamma)^2}+\frac{\beta^2}{(n+\gamma)^2}-2x\frac{k+\beta}{n+\gamma}+x^2]\\
&=\frac{1}{(n+\gamma)^2}\sum_{k=0}^{n}(k-nx)^2P_{n,k}^{\alpha}(x)+\frac{2(\beta-\gamma x)}{(n+\gamma)^2}\sum_{k=0}^{n}(k-nx)P_{n,k}^{\alpha}(x)\\
&\quad+\frac{1}{(n+\gamma)^2}\sum_{k=0}^{n}(\frac{k}{n}+\beta^2-2\beta\gamma x+\gamma^2x^2)P_{n,k}^{\alpha}(x)\\
&=:I_1+I_2+I_3.
\end{aligned}$$

By Lemma 3, we get $I_1\leq\frac{n(n+1)}{(n+\gamma)^2}(n+1)^{-\alpha}(A_{\frac{2}{\alpha}})^\alpha\leq2(A_{\frac{2}{\alpha}})^\alpha n^{-\alpha}$, $I_2\leq\frac{2(\beta+\gamma)}{(n+\gamma)^2}\sum\limits_{k=0}^{n}|k-nx|P_{n,k}^{\alpha}(x)\leq\frac{2(\beta+\gamma)\sqrt{n}(n+1)}{(n+\gamma)^2}(n+1)^{-\alpha}(A_{\frac{1}{\alpha}})^\alpha\leq4(\beta+\gamma)(A_{\frac{1}{\alpha}})^\alpha n^{-\alpha}$, here the constant $A_{\frac{i}{\alpha}}(i=1,2)$ only depends on α.

Using the Hölder inequality, we have $\sum\limits_{k=0}^{n}P_{n,k}^{\alpha}(x)\leq(n+1)^{1-\alpha}[\sum\limits_{k=0}^{n}P_{n,k}(x)]^\alpha$, and $|\frac{k}{n}+\beta^2-2\beta\gamma x+\gamma^2x^2|\leq1+(\beta+\gamma)^2$, so, we have

$$I_3\leq\frac{1+(\beta+\gamma)^2}{(n+\gamma)^2}(n+1)^{1-\alpha}[\sum_{k=0}^{n}P_{n,k}(x)]^\alpha\leq2[1+(\beta+\gamma)^2]n^{-\alpha}.$$

Denote $M_\alpha^{(\beta,\gamma)}=2(A_{\frac{2}{\alpha}})^\alpha+4(\beta+\gamma)(A_{\frac{1}{\alpha}})^\alpha+2[1+(\beta+\gamma)^2]$, then we can get

$$L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\leq M_\alpha^{(\beta,\gamma)}n^{-\alpha}.$$

(ii) Since

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\leq\sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)},$$

thus, we get

$$L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x) \le \sqrt{M_\alpha^{(\beta,\gamma)}} \cdot n^{-\frac{\alpha}{2}}.$$

Lemma 7. *For $f \in C[0,1]$, $x \in [0,1]$, $\alpha \ge 0$, and $0 \le \beta \le \gamma$, we have*

$$| L_{n,\alpha}^{(\beta,\gamma)}(f;x) | \le \| f \| .$$

Proof By (3) and Lemma 1 (i), we have

$$| L_{n,\alpha}^{(\beta,\gamma)}(f;x) | \le \|f\| L_{n,\alpha}^{(\beta,\gamma)}(1;x) = \|f\|.$$

3. Main results

First of all we give the following convergence theorem for the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}(f;x)\}$.

Theorem 1. Let $\alpha > 0, x \in [0,1]$, $0 \le \beta \le \gamma$. *Then the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}(f;x)\}$ converges to f uniformly on $[0,1]$ for any $f \in C[0,1]$.*

Proof Since $L_{n,\alpha}^{(\beta,\gamma)}(f;x)$ is bounded and positive on $C[0,1]$, and by Lemma 1, we have $\lim\limits_{n\to\infty} \|L_{n,\alpha}^{(\beta,\gamma)}(e_m;\cdot) - e_m\| = 0$ for $e_m(t) = t^m$, $m = 0,1,2$. So, according to the well-known Bohman-korovkin theorem ([2, P.40, Theorem 1.9]), we see that the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}(f;x)\}$ converges to f uniformly on [0,1] for any $f \in C[0,1]$.

Next we estimate the rates of convergence of the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}\}$ by means of modulus of continuity.

Theorem 2. *Let $f \in C[0,1]$, $x \in [0,1]$, $0 \le \beta \le \gamma$.* Then

(i) when $\alpha \ge 1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;\cdot) - f\| \le [1 + \sqrt{2\alpha(1+\gamma)}]\omega(f, \frac{1}{\sqrt{n+\gamma}});$$

(ii) when $0 < \alpha < 1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \le (1 + \sqrt{M_\alpha^{(\beta,\gamma)}})\omega(f, n^{-\frac{\alpha}{2}}).$$

Here the constant $M_\alpha^{(\beta,\gamma)}$ only depends on α, β, γ.

Proof (i) When $\alpha \ge 1$, by Lemma 1 (i), we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \\
&\le |f(\frac{\beta}{n+\gamma}) - f(x)|Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}(|f(t) - f(x)|) \\
&\quad + |f(\frac{n+\beta}{n+\gamma}) - f(x)|Q_{n,n}^{(\alpha)}(x) \\
&\le \omega(f, |\frac{\beta}{n+\gamma} - x|)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}^{(\beta,\gamma)}(\omega(f, |t-x|)) \\
&\quad + \omega(f, |\frac{n+\beta}{n+\gamma} - x|)Q_{n,n}^{(\alpha)}(x) \\
&\le (1 + \sqrt{n+\gamma}|\frac{\beta}{n+\gamma} - x|)\omega(f, \frac{1}{\sqrt{n+\gamma}})Q_{n,0}^{(\alpha)}(x)
\end{aligned}$$

$$+\sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}((1+\sqrt{n+\gamma}|t-x|)\omega(f,\frac{1}{\sqrt{n+\gamma}}))$$
$$+(1+\sqrt{n+\gamma}|\frac{n+\beta}{n+\gamma}-x|)\omega(f,\frac{1}{\sqrt{n+\gamma}})Q_{n,n}^{(\alpha)}(x)$$
$$\leq \omega(f,\frac{1}{\sqrt{n+\gamma}})+\sqrt{n+\gamma}\omega(f,\frac{1}{\sqrt{n+\gamma}})L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x),$$

so, by Lemma 5 (ii), we obtain

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\leq[1+\sqrt{2\alpha(1+\gamma)}]\omega(f,\frac{1}{\sqrt{n+\gamma}}).$$

The desired result follows immediately.

(ii) When $0<\alpha<1$, by Lemma 1 (i), we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\\
&\leq \omega(f,|\frac{\beta}{n+\gamma}-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}(\omega(f,|t-x|))\\
&\quad+\omega(f,|\frac{n+\beta}{n+\gamma}-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq (1+n^{\frac{\alpha}{2}}|\frac{\beta}{n+\gamma}-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,0}^{(\alpha)}(x)\\
&\quad+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}^{(\beta,\gamma)}((1+n^{\frac{\alpha}{2}}|t-x|)\omega(f,n^{-\frac{\alpha}{2}}))\\
&\quad+(1+n^{\frac{\alpha}{2}}|\frac{n+\beta}{n+\gamma}-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,n}^{(\alpha)}(x)\\
&\leq \omega(f,n^{-\frac{\alpha}{2}})+n^{\frac{\alpha}{2}}\omega(f,n^{-\frac{\alpha}{2}})L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x),
\end{aligned}$$

so, by Lemma 6 (ii), we obtain $|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\leq(1+\sqrt{M_\alpha^{(\beta,\gamma)}})\omega(f,n^{-\frac{\alpha}{2}})$. The desired result follows immediately.

Theorem 3. *Let $f\in C^1[0,1]$, $x\in[0,1]$, $0\leq\beta\leq\gamma$. Then*

(i) when $\alpha\geq 1$, we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\\
&\leq[\|f'\|+\omega(f',\frac{1}{\sqrt{n+\gamma}})(1+\sqrt{2\alpha(1+\gamma)})]\sqrt{\frac{2\alpha(1+\gamma)}{n+\gamma}};
\end{aligned}$$

(ii) when $0<\alpha<1$, we have

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\leq[\|f'\|+\omega(f',n^{-\frac{\alpha}{2}})(1+\sqrt{M_\alpha^{(\beta,\gamma)}})]\sqrt{M_\alpha^{(\beta,\gamma)}n^{-\alpha}}.$$

Here the constant $M_\alpha^{(\beta,\gamma)}$ only depends on α,β,γ.

Proof Let $f\in C^1[0,1]$. For any $t,x\in[0,1]$, $\delta>0$, we have

$$\begin{aligned}
|f(t)-f(x)-f'(x)(t-x)| &\leq |\int_x^t|f'(u)-f'(x)|du|\\
&\leq \omega(f',|t-x|)|t-x|\\
&\leq \omega(f',\delta)(|t-x|+\delta^{-1}(t-x)^2),
\end{aligned}$$

hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f(t)-f(x)-f'(x)(t-x);x)|\\
&\leq \omega(f',\delta)\left(L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)+\delta^{-1}L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\right)\\
&\leq \omega(f',\delta)(\sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}+\delta^{-1}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x))}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)}.
\end{aligned}$$

So, we get

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)|\\
&\leq \|f'\|L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\\
&\quad+\omega(f',\delta)(1+\delta^{-1}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x))}\sqrt{L_{n,\alpha}((t-x)^2;x)}. \qquad (4)
\end{aligned}$$

(i) When $\alpha \geq 1$, taking $\delta = \frac{1}{\sqrt{n+\gamma}}$ in (4), by Lemma 5 and inequality (4), we obtain the desired result.

(ii)When $0 < \alpha < 1$, taking $\delta = n^{-\frac{\alpha}{2}}$ in (4), by Lemma 6 and inequality (4), we obtain the desired result.

Finally we study the direct theorem concerned with an approximation for the sequence $\{L_{n,\alpha}^{(\beta,\gamma)}\}$ by means of the Ditzian-Totik modulus of smoothness. For the next theorem we shall use some notations.

For $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 \leq \lambda \leq 1$, $x \in [0,1]$, let

$$\omega_{\varphi^\lambda}(f,t) = \sup_{0<h\leq t} \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]} |f(x+\frac{h\varphi^\lambda(x)}{2}) - f(x-\frac{h\varphi^\lambda(x)}{2})|$$

be the Ditzian-Totik modulus of first order, and let

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda}\{\| f-g \| + t\|\varphi^\lambda g'\| + t^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\} \qquad (5)$$

be the corresponding K-functional, where $W_\lambda = \{f | f \in AC_{loc}[0,1], \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\}$.

It is well known that (see [3])

$$K_{\varphi^\lambda}(f,t) \leq C\omega_{\varphi^\lambda}(f,t), \qquad (6)$$

for some absolute constant $C > 0$.

Now we state our next main result.

Theorem 4. Let $f \in C[0,1]$, $\alpha \geq 1$, $x \in [0,1]$, $0 \leq \beta \leq \gamma$, $\varphi(x) = \sqrt{x(1-x)}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n+\gamma}}$, $0 \leq \lambda \leq 1$. Then there exists an absolute constant $C > 0$ such that

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \leq C\omega_{\varphi^\lambda}(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}).$$

Proof Let $g \in W_\lambda$, by Lemma 1 (i) and Lemma 7, we have

$$\begin{aligned}
&|L_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)|\\
&\leq |L_{n,\alpha}^{(\beta,\gamma)}(f-g;x)| + |f(x)-g(x)| + |L_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x)|\\
&\leq 2\|f-g\| + |L_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x)|. \qquad (7)
\end{aligned}$$

Since $g(t)=\int_x^t g'(u)du+g(x)$, $L_{n,\alpha}^{(\beta,\gamma)}(1;x)=1$, so, we have

$$\begin{aligned}|L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)| &\leq |L_{n,\alpha}^{(\beta,\gamma)}(\int_x^t |g'(u)|du;x)| \\ &\leq \|\delta_n^{\lambda}g'\|L_{n,\alpha}^{(\beta,\gamma)}(|\int_x^t \delta_n^{-\lambda}(u)du|;x). \end{aligned} \tag{8}$$

By the Hölder inequality, we get

$$|\int_x^t \delta_n^{-\lambda}(u)du| \leq |\int_x^t \delta_n^{-1}(u)du|^{\lambda}|t-x|^{1-\lambda}. \tag{9}$$

Since

$$\delta_n^{-1}(x)\sim min(\varphi^{-1}(x),\sqrt{n+\gamma}), \tag{10}$$

here $a\sim b$ means that there exists some constant $C>0$, such that $C^{-1}b\leq a\leq Cb$.

Also, by (11) in [7] , we have

$$|\int_x^t \varphi^{-1}(u)du|\leq 4|t-x|\varphi^{-1}(x), \tag{11}$$

thus, by (9), (10) and (11), we obtain

$$|\int_x^t \delta_n^{-\lambda}(u)du|\leq C\delta_n^{-\lambda}(x)|t-x|, \tag{12}$$

also, by (8) and (12), we have

$$\begin{aligned}|L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)| &\leq C\|\delta_n^{\lambda}g'\|L_{n,\alpha}^{(\beta,\gamma)}(\delta_n^{-\lambda}(x)|t-x|;x) \\ &= C\|\delta_n^{\lambda}g'\|\delta_n^{-\lambda}(x)L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x). \end{aligned} \tag{13}$$

In view of the proof of Lemma 5 (i), we have

$$L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\leq \alpha[\frac{n+1}{(n+\gamma)^2}x(1-x)+\frac{\beta^2+\gamma^2}{(n+\gamma)^2}],$$

so, by the Cauchy-Schwarz inequality and Lemma 1 (i), we have

$$\begin{aligned}L_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x) &\leq \sqrt{L_{n,\alpha}^{(\beta,\gamma)}(1;x)}\sqrt{L_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)} \\ &\leq \sqrt{\alpha[\frac{n+1}{(n+\gamma)^2}x(1-x)+\frac{\beta^2+\gamma^2}{(n+\gamma)^2}]} \\ &\leq C\frac{\delta_n(x)}{\sqrt{n+\gamma}}, \end{aligned} \tag{14}$$

so, by (13) and (14), we obtain

$$|L_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)|\leq C\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\delta_n^{\lambda}g'\|, \tag{15}$$

thus, by (7), (15) and $\delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n+\gamma}}$, we have

$$\begin{aligned}&|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \\ &\leq C[\|f-g\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\delta_n^{\lambda}g'\|] \\ &\leq C[\|f-g\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\varphi^{\lambda}g'\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}(\frac{1}{\sqrt{n+\gamma}})^{\lambda}\|g'\|] \\ &\leq C[\|f-g\|+\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}\|\varphi^{\lambda}g'\|+(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}})^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|]. \end{aligned} \tag{16}$$

Then, in view of (16), (5) and (6), we obtain

$$|L_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \le CK_{\varphi^\lambda}(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}) \le C\omega_{\varphi^\lambda}(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n+\gamma}}),$$

where C is a positive constant, in different places, the value of C may be different.

Acknowledgements

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Dynamics of a Higher Order Difference Equations

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}}$$

M. M. El-Dessoky[1,2] and Aatef Hobiny[1,3]
[1]King Abdulaziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Mansoura University, Faculty of Science,
Department of Mathematics, Mansoura 35516, Egypt.
[3]Nonlinear Analysis and Applied Mathematics (NAAM) -Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
E-mail: dessokym@mans.edu.eg; ahobany@kau.edu.sa

ABSTRACT

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l} + bx_{n-k}}{cx_{n-l} + dx_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c,\ d \in (0,\ \infty)$ and the initial conditions $x_{-s},\ x_{-s+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $s = max\{l,\ k\}$. Examples to illustrate the importance of our results.

Keywords: difference equations, stability, global stability, boundedness, periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations have always played an important role in the construction and analysis of mathematical models of economic process, biology, ecology, physics and so forth. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

In [1] Papaschinopoulos et al. studied the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$$

Kalabušić et al. [2] investigated the global character of the solution of the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_{n-l}+\delta x_{n-k}}{Bx_{n-l}+Dx_{n-k}}.$$

Elsayed et al. [3] studied the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l}+cx_{n-k}}{dx_{n-l}+ex_{n-k}}.$$

Zayed et al. [4] investigated the behavior of the following rational recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n-dx_{n-k}}.$$

El-Moneam et al. [5] obtained the boundedness, the periodicity and the global stability of the positive solution of the difference equation,

$$x_{n+1} = ax_n + \frac{bx_{n-1}+cx_{n-2}+fx_{n-3}+rx_{n-4}}{dx_{n-1}+ex_{n-2}+gx_{n-3}+sx_{n-4}}.$$

El-Dessoky [6] studied the global stability, the boundedness and the periodicity of the nonlinear difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s}-\alpha x_{n-t}}, \qquad n = 0,\ 1,\$$

For other related results, see [1 - 30].

Our aim in this paper is to obtain some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}}, \quad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c,\ d\ \in (0,\ \infty)$ and the initial conditions $x_{-s},\ x_{-s+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $s = max\{l,\ k\}$.

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT

In this section, we study the local stability character of the equilibrium point of Eq. (1).

Eq. (1) has equilibrium point and is given by

$$x^* = \alpha x^* + \beta x^* + \gamma x^* + \frac{ax^*+bx^*}{cx^*+dx^*},$$

If $\alpha+\beta+\gamma < 1$, then the only positive equilibrium point x^* of Eq. (1) is given by $x^* = \frac{a+b}{[1-\alpha-\beta-\gamma](c+d)}$.

THEOREM 2.1. *(i) Let $ad > cb,\ \alpha+\beta+\gamma < 1$ and $\gamma > \frac{(ad-bc)(1-\alpha-\beta-\gamma))}{(c+d)(a+b)}$,then equilibrium x^* of Eq. (1) is locally asymptotically stable.*

(ii) Let $cb > ad,\ \alpha+\beta+\gamma < 1$ and $\beta > \frac{(ad-cb)(1-\alpha-\beta-\gamma))}{(c+d)(a+b)}$,then equilibrium x^* of Eq. (1) is locally asymptotically stable.

Proof: Suppose that $G : (0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$G(u_0,\ u_1,\ u_2) = \alpha u_0 + \beta u_1 + \gamma u_2 + \frac{au_1+bu_2}{cu_1+du_2}. \tag{2}$$

Therefore, it follows that

$$\frac{\partial G(u_0,\ u_1,\ u_2)}{\partial u_0} = \alpha,\ \frac{\partial G(u_0,\ u_1,\ u_2)}{\partial u_1} = \beta + \frac{(ad-cb)u_2}{(cu_1+du_2)^2},\ \frac{\partial G(u_0,\ u_1,\ u_2)}{\partial u_2} = \gamma - \frac{(ad-bc)u_0}{(cu_1+du_2)^2}.$$

Then, we see that

$$\frac{\partial G(x^*,\ x^*,\ x^*)}{\partial u_0} = \alpha,\ \frac{\partial G(x^*,\ x^*,\ x^*)}{\partial u_1} = \beta + \frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)},\ \frac{\partial G(x^*,\ x^*,\ x^*)}{\partial u_2} = \gamma - \frac{(ad-bc)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}.$$

Under the conditions of part (i), we get

$$\begin{aligned} |\alpha| + \left|\beta + \frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| + \left|\gamma - \frac{(ad-bc)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| &< 1, \\ \alpha+\beta+\frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} + \gamma - \frac{(ad-bc)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} &< 1, \end{aligned}$$

and so

$$\alpha+\beta+\gamma < 1.$$

Then the equilibrium x^* of Eq. (1) is locally asymptotically stable, the proof of part (i) is complete.

Under the conditions of part (ii), we get

$$\begin{aligned} |\alpha| + \left|\beta - \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| + \left|\gamma + \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}\right| &< 1, \\ \alpha + \beta - \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} + \gamma + \tfrac{(bc-ad)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)} &< 1, \end{aligned}$$

and so

$$\alpha + \beta + \gamma < 1.$$

Then the equilibrium x^* of Eq. (1) is locally asymptotically stable, the proof of part (ii) is complete.

Example 1. See Figure (1) when we take the Eq. (1) with $l = 4$, $k = 3, \alpha = 0.2$, $\beta = 0.1$, $\gamma = 0.5$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 0.6$, $x_{-3} = 7$, $x_{-2} = 2$, $x_{-1} = 3$ and $x_0 = 5$.

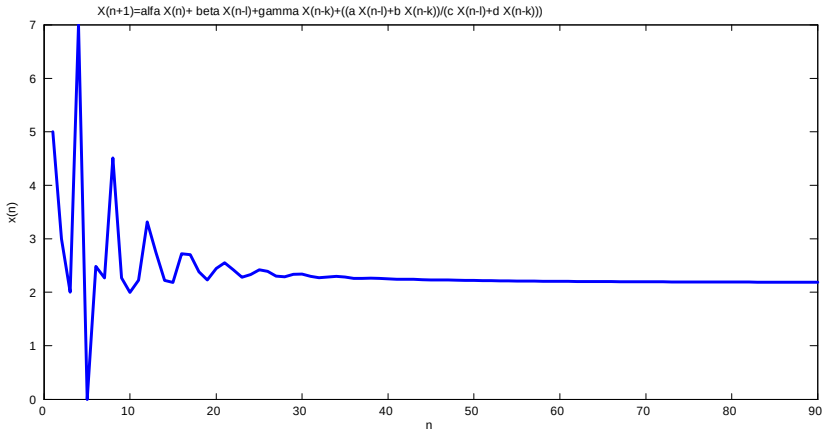


Fig. 1. sketch the behavior of the solution of Eq. (1).

Example 2. The solution of Eq. (1) is local stability if $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.1$, $\gamma = 0.2$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$ (See Fig. 2).

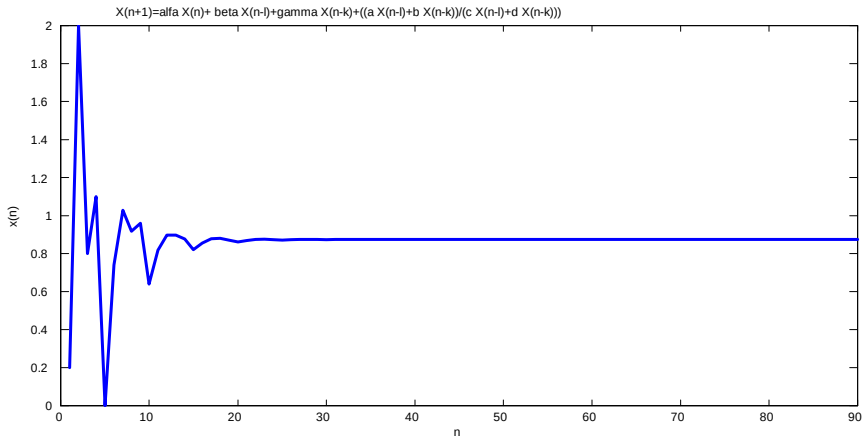


Fig. 2. Plot the behavior of the solution of equation (1) under the conditions (i).

Example 3. Figure (3) shows that if $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.3$, $\gamma = 0.2$, $a = 0.4$, $b = 2$, $c = 1.6$ and $d = 1$, then the solution of Eq. (1) is local stability with the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$.

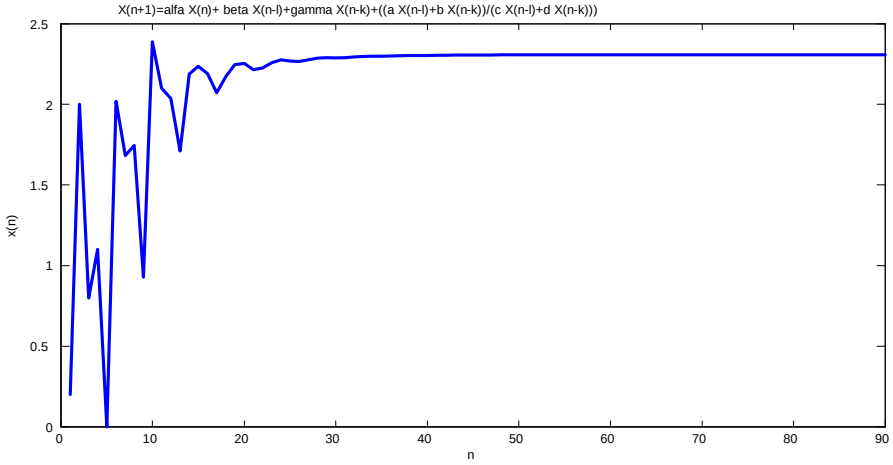


Fig. 3. Plot the behavior of the solution of equation (1) under the conditions (ii).

Example 4. See Figure (4) when we take Eq. (1) with $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.28$, $\gamma = 0.82$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$.

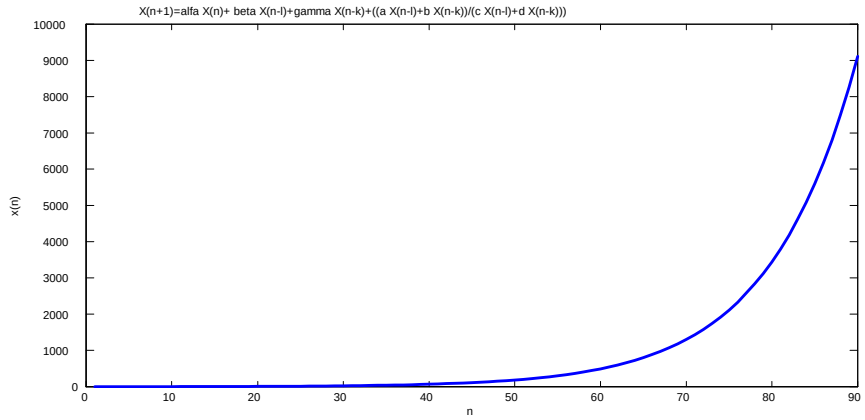


Fig. 4. Draw the behavior of the solution of Eq. (1).

3. GLOBAL STABILITY OF THE EQUILIBRIUM POINT

THEOREM 3.1. *The equilibrium point x^* is a global attractor of Eq. (1) if one of the following conditions holds:*

$$\begin{aligned} (i)\ ad - cb &\geqslant 0,\ b \geqslant a. \\ (ii)\ cb - ad &\geqslant 0,\ a \geqslant b. \end{aligned}$$

Proof. Let $r,\ s$ be nonnegative real numbers and assume that $H : [r, s]^3 \to [r, s]$ be a function defined by

$$H(u_0,\ u_1,\ u_1) = \alpha u_0 + \beta u_1 + \gamma u_2 + \tfrac{au_1+bu_2}{cu_1+du_2}.$$

Then
$$\frac{\partial H(u_0,\ u_1,\ u_1)}{\partial u_0} = a,\ \frac{\partial H(u_0,\ u_1,\ u_1)}{\partial u_1} = \beta + \frac{(ad-cb)u_1}{(cu_0+du_1)^2} \text{ and } \frac{\partial H(u_0,\ u_1,\ u_1)}{\partial u_2} = \gamma - \frac{(ad-bc)u_0}{(cu_0+du_1)^2}.$$

We consider two cases:

Case1: Assume that $ad-cb>0,\ \alpha+\beta+\gamma<1$ and $\gamma > \frac{(ad-bc)(1-\alpha-\beta-\gamma))}{(c+d)(a+b)}$ is true, then we can easily see that the function $H(u_0,\ u_1,\ u_2)$ is increasing in $u_0,\ u_1$ and decreasing in u_2. Suppose that (m,M) is a solution of the system
$$M = H(M,\ M,\ m) \quad \text{and} \quad m = H(m,\ m,\ M).$$
Then from Eq. (1), we see that
$$M = \alpha M + \beta M + \gamma m + \frac{aM+bm}{cM+dm} \quad \text{and} \quad m = \alpha m + \beta m + \gamma M + \frac{am+bM}{cm+dM},$$
then
$$\begin{aligned} c(1-\alpha-\beta)M^2 + d(1-\alpha-\beta)mM - c\gamma mM - d\gamma m^2 &= aM+bm, \\ c(1-\alpha-\beta)m^2 + d(1-\alpha-\beta)mM - c\gamma mM - d\gamma M^2 &= am+bM, \end{aligned}$$
Subtracting this two equations, we obtain
$$(M-m)\left\{(c(1-\alpha-\beta)+d\gamma)\left(M+m\right)\right] + (b-a)\} = 0,$$
under the condition $\alpha+\beta<1,\ b \geqslant a$, we see that $M=m$. Then x^* is a global attractor of Eq. (1).

Case 2: Assume that $cb>ad,\ \beta+\gamma<1$ and $\beta > \frac{(ad-cb)(1-\alpha-\beta-\gamma)}{(c+d)(a+b)}$ is true, then we can easily see that the function $H(u_0,\ u_1,\ u_2)$ is decreasing in $u_0,\ u_1$ and increasing in u_2. Suppose that (m,M) is a solution of the system
$$M = H(m,\ m,\ M) \quad \textit{and} \quad m = H(M,\ M,\ m).$$
Then from Eq. (1), we see that
$$M = \alpha m + \beta m + \gamma M + \frac{am+bM}{cm+dM},, \quad \text{and} \quad m = \alpha M + \beta M + \gamma m + \frac{aM+bm}{cM+dm},$$
then
$$\begin{aligned} d(1-\gamma)M^2 + c(1-\gamma)mM - c\left(\alpha+\beta\right)m^2 - d\left(\alpha+\beta\right)mM &= am+bM, \\ d(1-\gamma)m^2 + c(1-\gamma)mM - c\left(\alpha+\beta\right)M^2 - d\left(\alpha+\beta\right)mM &= aM+bm, \end{aligned}$$
Subtracting this two equations, we obtain
$$(M-m)\left\{(d(1-\gamma)+c\left(\alpha+\beta\right))\left(M+m\right)\right] + (a-b)\} = 0,$$
under the condition $\gamma \neq 1,\ a \neq b$, we see that $M=m$. Then x^* is a global attractor of Eq. (1).

Example 5. The solution of Eq. (1) is global stability if $l=4,\ k=3,\ \alpha=0.02,\ \beta=0.01,\ \gamma=0.03,\ a=0.4,\ b=1,\ c=0.2$ and $d=1$ and the initial conditions $x_{-4}=6,\ x_{-3}=1.1,\ x_{-2}=0.8,\ x_{-1}=2$ and $x_0=0.2$ (See Fig. 5).

Example 6. Figure (6) shows the global stability of the solution of Eq. (1) when $l=4,\ k=3,\ \alpha=0.02,\ \beta=0.2,\ \gamma=0.1,\ a=1.1,\ b=0.3,\ c=1$ and $d=0.3$ and the initial conditions $x_{-4}=6,\ x_{-3}=1.1,\ x_{-2}=0.8,\ x_{-1}=2$ and $x_0=0.2$.

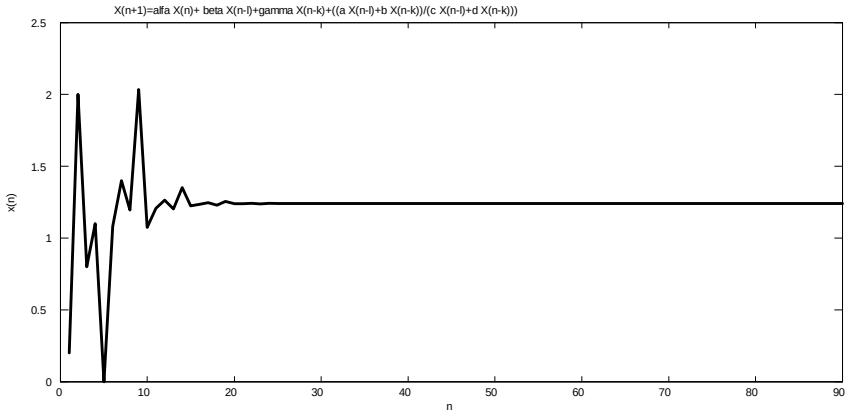


Fig. 5. sketch the behavior of the solution of Eq. (1) when $ad \geqslant cb$ and $b \geqslant a$.

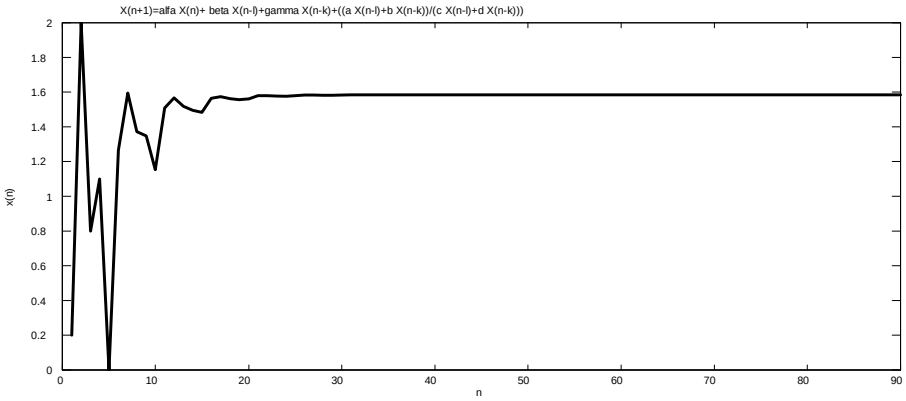


Fig. 6. Shows the behavior of the solution of Eq. (1) when $cb \geqslant ad,\ a \geqslant b$.

4. BOUNDEDNESS OF THE SOLUTIONS

THEOREM 4.1. *Every solution of Eq. (1) is bounded if* $\beta + \gamma < 1$.

Proof. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_{n-l} + b x_{n-k}}{c x_{n-l} + d x_{n-k}} \\ &= \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_{n-l}}{c x_{n-l} + d x_{n-k}} + \frac{b x_{n-k}}{c x_{n-l} + d x_{n-k}}. \end{aligned}$$

Then

$$x_{n+1} \leqslant \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_{n-l}}{c x_{n-l}} + \frac{b x_{n-k}}{d x_{n-k}} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a}{c} + \frac{b}{d} \text{ for all } n \geq 0.$$

By using a comparison, we can right hand side as follows

$$y_{n+1} = \alpha y_n + \beta y_{n-l} + \gamma y_{n-k} + \frac{a}{c} + \frac{b}{d}.$$

and this equation is locally asymptotically stable if $\alpha + \beta + \gamma < 1$, and converges to the equilibrium point $y^* = \frac{ad+bc}{cd(1-\alpha-\beta-\gamma)}$. Therefore

$$\lim_{n\to\infty} \sup y_n \leqslant \frac{ad+bc}{cd(1-\alpha-\beta-\gamma)}.$$

Thus the solution is bounded.

THEOREM 4.2. *Every solution of Eq. (1) is unbounded if* $\alpha > 1$ *or* $\beta > 1 or \gamma > 1$.

Proof. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1).Then from Eq. (1) we see that

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_{n-l} + b x_{n-k}}{c x_{n-l} + d x_{n-k}} > \alpha x_n \quad \text{for all} \quad n \geq 0.$$

We see that the right hand side can be written as follows $y_{n+1} = \alpha y_n$. Then

$$y_{n+1} = \alpha^n y_n + const.,$$

and this equation is unstable because $\alpha > 1$, and $\lim_{n\to\infty} y_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded. Using the same technique, we can prove the other cases.

Example 7. When $l = 4$, $k = 3$, $\alpha = 1.3$, $\beta = 0.5$, $\gamma = 0.2$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$, the solution of Eq. (1) with initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$, the solution of the difference equation is unbounded (See Fig. 7).

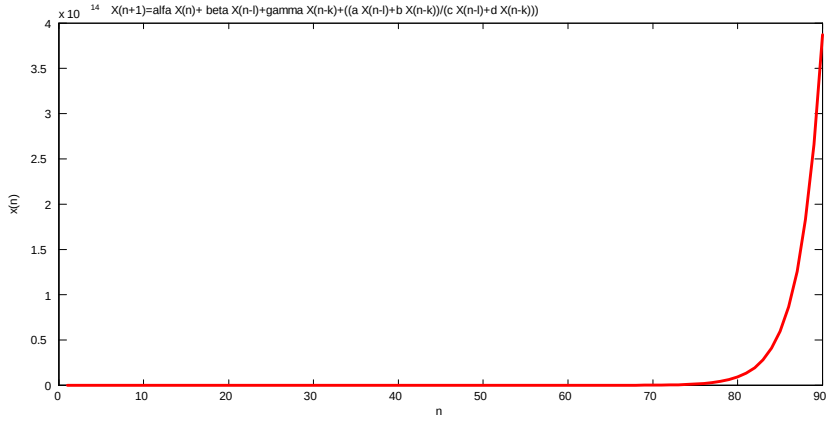


Fig. 7. Plot the behavior of the solution of equation (1) when $\alpha > 1$.

Example 8. Figure (8) shows that $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 1.5$, $\gamma = 0.5$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$, the solution of Eq. (1) with initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$ is unbounded.

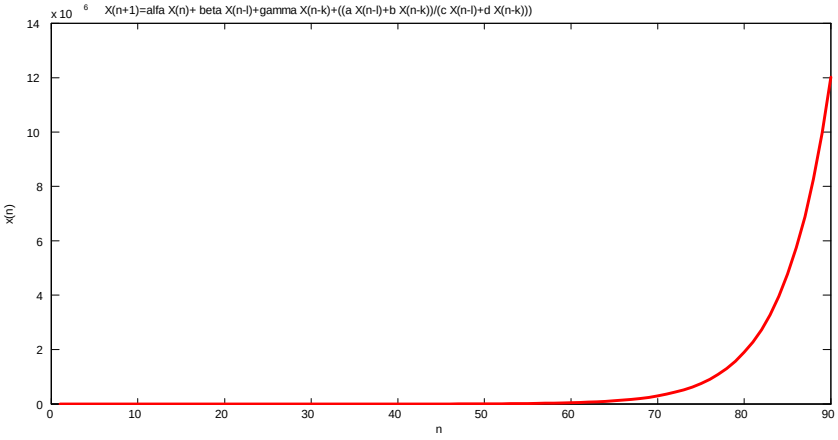


Fig. 8. Draw the behavior of the solution of equation (1) when $\beta > 1$.

Example 9. Figure (9) shows the solution of Eq. (1) is unbounded if $l = 4$, $k = 3$, $\alpha = 0.2$, $\beta = 0.4$, $\gamma = 1.2$, $a = 0.4$, $b = 0.3$, $c = 0.6$ and $d = 1$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 1.1$, $x_{-2} = 0.8$, $x_{-1} = 2$ and $x_0 = 0.2$.

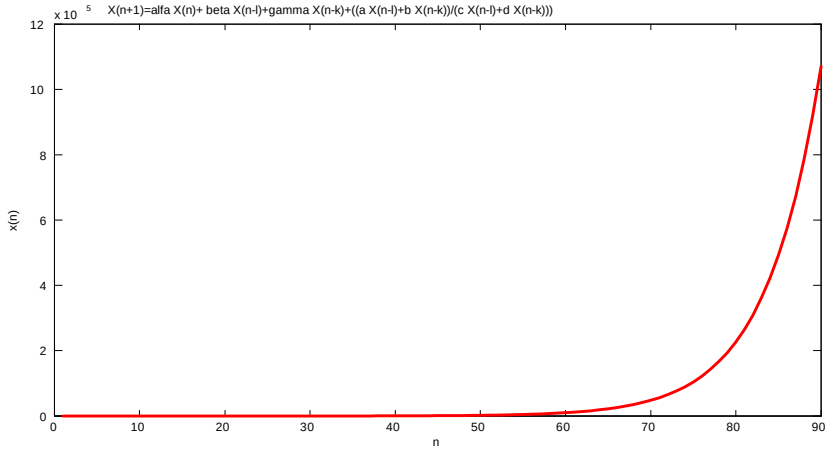


Fig. 9. Shows the behavior of the solution of equation (1) when $\gamma > 1$.

5. EXISTENCE OF PERIODIC SOLUTIONS

THEOREM 5.1. *If l is an even and k is an odd, then Eq. (1) has a prime period two solutions if*

$$(b-a)(c-d)(1+\alpha+\beta-\gamma) > 4(bc(\alpha+\beta)+ad(1-\gamma)), \tag{3}$$

where $a < b$, $c < d$, $\gamma < 1$ and $\gamma < 1+\alpha+\beta$.

Proof. Suppose that there exists a prime period two solution $...P, Q, P, Q, ...$, of Eq. (1). We see from Eq. (1) when l is even, and k is odd that

$$P = \alpha Q + \beta Q + \gamma P + \frac{aQ+bP}{cQ+dP} \quad \text{and} \quad Q = \alpha P + \beta P + \gamma Q + \frac{aP+bQ}{cP+dQ}.$$

Then

$$c(1-\gamma)PQ + d(1-\gamma)P^2 = c(\alpha+\beta)Q^2 + d(\alpha+\beta)PQ + aQ + bP, \tag{4}$$

$$c(1-\gamma)cPQ + d(1-\gamma)Q^2 = c(\alpha+\beta)P^2 + d(\alpha+\beta)PQ + aP + bQ. \tag{5}$$

Subtracting (4) from (5) gives

$$d(1-\gamma)(P^2-Q^2) = c(\alpha+\beta)(Q^2-P^2) - a(P-Q) + b(P-Q),$$

$$(d(1-\gamma)+c(\alpha+\beta))(P-Q)(P+Q) = (b-a)(P-Q),$$

Since $P \neq Q$, it follows that

$$(d(1-\gamma)+c(\alpha+\beta))(P+Q) = b-a,$$

$$P+Q = \frac{b-a}{d(1-\gamma)+c(\alpha+\beta)}. \tag{6}$$

Again, adding (4) and (5) yields

$$2PQ(c(1-\gamma)-d(\alpha+\beta)) = (c(\alpha+\beta)-d(1-\gamma))(P^2+Q^2) + (a+b)(P+Q), \tag{7}$$

It follows by (6), (7) and the relation

$$P^2+Q^2=(P+Q)^2-2PQ \quad for\ all \quad P,Q\in R,$$

that

$$2PQ\left(c\left(1-\gamma\right)-d\left(\alpha+\beta\right)\right)=\left(c\left(\alpha+\beta\right)-d\left(1-\gamma\right)\right)\left((P+Q)^2-2PQ\right)+(a+b)(P+Q),$$

$$\begin{aligned}
2\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)PQ &= \left(P+Q\right)\left[\left(c\left(\alpha+\beta\right)-d\left(1-\gamma\right)\right)\left(P+Q\right)+\left(a+b\right)\right],\\
2\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)PQ &= \left(\tfrac{b-a}{d(1-\gamma)+c(\alpha+\beta)}\right)\left(\tfrac{(c(\alpha+\beta)-d(1-\gamma))(b-a)+(a+b)(d(1-\gamma)+c(\alpha+\beta))}{d(1-\gamma)+c(\alpha+\beta)}\right),\\
2\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)PQ &= 2\left(\tfrac{b-a}{d(1-\gamma)+c(\alpha+\beta)}\right)\left(\tfrac{bc(\alpha+\beta)+ad(1-\gamma)}{d(1-\gamma)+c(\alpha+\beta)}\right)
\end{aligned}$$

$$PQ=\tfrac{(b-a)(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))^2}. \tag{8}$$

Now it is clear from equations (6) and (8) that P and Q are the two distinct roots of the quadratic equation

$$\left(d\left(1-\gamma\right)+c\left(\alpha+\beta\right)\right)t^2-\left(b-a\right)t+\tfrac{(b-a)(da-\gamma ad+\beta cb)}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))}=0. \tag{9}$$

and so

$$\begin{aligned}
\left(\tfrac{b-a}{d(1-\gamma)+c(\alpha+\beta)}\right)^2 &> \tfrac{4(b-a)(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))^2},\\
\left(b-a\right) &> \tfrac{4(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)(d(1-\gamma)+c(\alpha+\beta))^2}.
\end{aligned}$$

For $b>a,\ c>d$ and $\gamma<1,\ \gamma<1+\alpha+\beta$, then

$$\left(b-a\right)\left(c-d\right)\left(1+\alpha+\beta-\gamma\right)>4\left(bc\left(\alpha+\beta\right)+ad\left(1-\gamma\right)\right).$$

Therefore Inequality (3) holds and the proof is complete.

Example 10. Figure (10) shows the Eq. (1) has a prime period two solution when $l=4,\ k=3,\ \alpha=0.001,\ \beta=0.03,\ \gamma=0.06,\ a=0.1,\ b=0.9,\ c=0.8$ and $d=0.06$ and the initial conditions $x_{-4}=Q,\ x_{-3}=P,\ x_{-2}=Q,\ x_{-1}=P$ and $x_0=Q$ such that $P=\frac{b-a+\xi}{2(d(1-\gamma)+c(\alpha+\beta))}$ and $Q=\frac{b-a-\xi}{2(d(1-\gamma)+c(\alpha+\beta))}$ where $\xi=\sqrt{(b-a)^2-\frac{4(b-a)(bc(\alpha+\beta)+ad(1-\gamma))}{(c-d)(1+\alpha+\beta-\gamma)}}$.

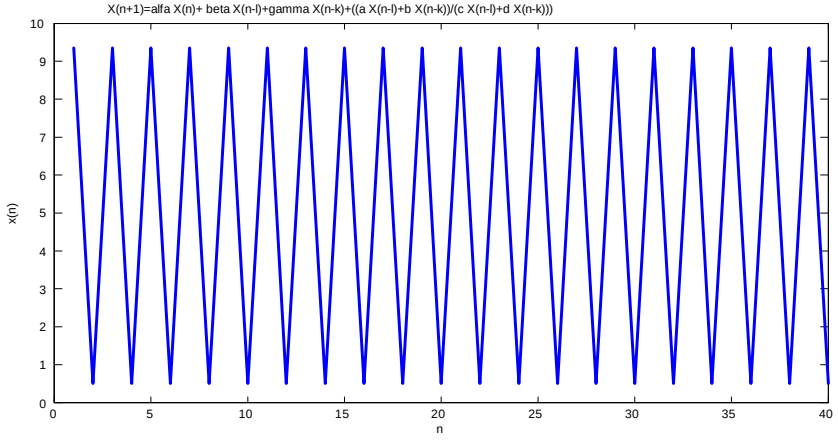


Fig. 9. Plot the solution of Eq. (1) has a periodic solution.

THEOREM 5.2. *If l is an odd and k is an even, then Eq. (1) has a prime period two solutions if*

$$(a-b)(d-c)(d-c)(1+\alpha+\gamma-\beta) > 4(ad(\alpha+\gamma)+cb(1-\beta)), \tag{10}$$

where $b<a$, $c<d$ and $\beta<\alpha+\gamma+1$.

Proof. Suppose that there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...$, of Eq. (1). We see from Eq. (1) when l is odd, and k is even that

$$P=\alpha Q+\beta P+\gamma Q+\tfrac{aP+bQ}{cP+dQ} \quad \text{and} \quad Q=\alpha P+\beta Q+\gamma P+\tfrac{aQ+bP}{cQ+dP}.$$

Then

$$\begin{aligned} c(1-\beta)P^2+d(1-\beta)PQ &= c(\alpha+\gamma)PQ+d(\alpha+\gamma)Q^2+aP+bQ, &(11)\\ c(1-\beta)Q^2+d(1-\beta)PQ &= c(\alpha+\gamma)PQ+d(\alpha+\gamma)P^2+aQ+bP. &(12)\end{aligned}$$

Subtracting (11) from (12) gives

$$c(1-\beta)(P^2-Q^2)=-d(\alpha+\gamma)(P^2-Q^2)+a(P-Q)-b(P-Q),$$

$$(c(1-\beta)+d(\alpha+\gamma))(P-Q)(P+Q)=(a-b)(P-Q),$$

Since $P\neq Q$, it follows that

$$(c(1-\beta)+d(\alpha+\gamma))(P+Q)=a-b,$$

$$P+Q=\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}. \tag{13}$$

Again, adding (11) and (12) yields

$$2d(1-\beta)PQ+c(1-\beta)(P^2+Q^2)=2c(\alpha+\gamma)PQ+d(\alpha+\gamma)(P^2+Q^2)+(a+b)(P+Q),$$

$$2PQ(d(1-\beta)-c(\alpha+\gamma))=(d(\alpha+\gamma)-c(1-\beta))(P^2+Q^2)+(a+b)(P+Q), \tag{14}$$

It follows by (13), (14) and the relation

$$P^2+Q^2=(P+Q)^2-2PQ \quad for\ all \quad P,Q\in R,$$

that

$$2PQ(d(1-\beta)-c(\alpha+\gamma))=(d(\alpha+\gamma)-c(1-\beta))\left((P+Q)^2-2PQ\right)+(a+b)(P+Q),$$

$$\begin{aligned} 2(d-c)(1+\alpha+\gamma-\beta)PQ &= (P+Q)[(d(\alpha+\gamma)-c(1-\beta))(P+Q)+(a+b)],\\ &= \left(\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}\right)\left[\tfrac{(d(\alpha+\gamma)-c(1-\beta))(a-b)+(c(1-\beta)+d(\alpha+\gamma))(a+b)}{c(1-\beta)+d(\alpha+\gamma)}\right]\\ &= 2\left(\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}\right)\left(\tfrac{ad(\alpha+\gamma)+cb(1-\beta)}{c(1-\beta)+d(\alpha+\gamma)}\right)\end{aligned}$$

$$PQ=\tfrac{(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)(c(1-\beta)+d(\alpha+\gamma))^2}. \tag{15}$$

Now it is clear from equations (13) and (15) that P and Q are the two distinct roots of the quadratic equation

$$(c(1-\beta)+d(\alpha+\gamma))t^2-(a-b)t+\tfrac{(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)(c(1-\beta)+d(\alpha+\gamma))}=0. \tag{16}$$

and so

$$\begin{aligned} \left(\tfrac{a-b}{c(1-\beta)+d(\alpha+\gamma)}\right)^2 &> \tfrac{4(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)(c(1-\beta)+d(\alpha+\gamma))^2},\\ (a-b) &> \tfrac{4(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)}.\end{aligned}$$

For $b < a,\ c < d$ and $\beta < 1 + \alpha + \gamma$, then

$$(a-b)(d-c)(d-c)(1+\alpha+\gamma-\beta) > 4\left(ad(\alpha+\gamma)+cb(1-\beta)\right).$$

Therefore Inequality (10) holds and the proof is complete.

Example 11. Figure (11) shows the Eq. (1) has a prime two solution when $l = 1$, $k = 4$, $\alpha = 0.001$, $\beta = 0.6$, $\gamma = 0.02$, $a = 0.9$, $b = 0.2$, $c = 0.05$ and $d = 0.55$ and the initial conditions $x_{-4} = Q$, $x_{-3} = P$, $x_{-2} = Q$, $x_{-1} = P$ and $x_0 = Q$, such that $P = \frac{a-b+\xi}{2(c(1-\beta)+d(\alpha+\gamma))}$ and $Q = \frac{a-b-\xi}{2(c(1-\beta)+d(\alpha+\gamma))}$ where $\xi = \sqrt{(a-b)^2 - \frac{4(a-b)(ad(\alpha+\gamma)+cb(1-\beta))}{(d-c)(1+\alpha+\gamma-\beta)}}$

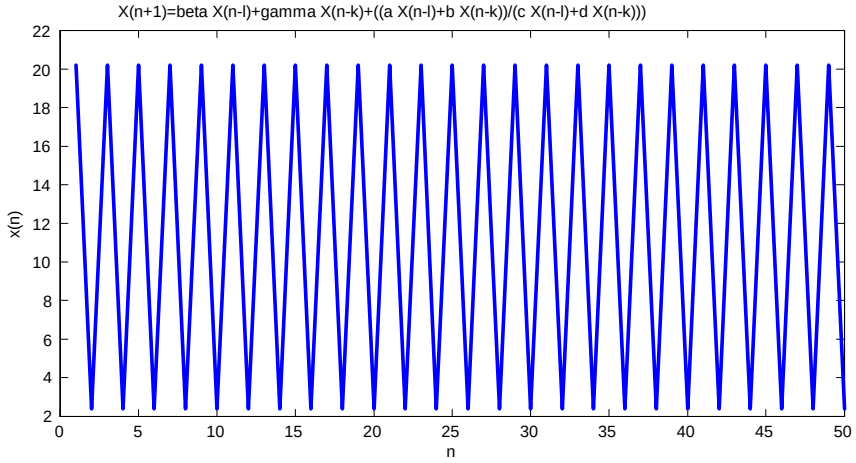


Fig. 11. sketch the solution of Eq. (1) has a periodic solution.

THEOREM 5.3. *Equation (1) has no prime period two solutions if l and k are even and $\beta + \gamma + 1 \neq 0$.*

Proof. Suppose that there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...$, of Equation (1). We see from Equation (1) when l and k are even that

$$P = \alpha Q + \beta Q + \gamma Q + \frac{aQ+bQ}{cQ+dQ}, \tag{17}$$

$$Q = \alpha P + \beta P + \gamma P + \frac{aP+bP}{cP+dP}. \tag{18}$$

Subtracting (17) from (18) gives

$$(\alpha+\beta+\gamma+1)(P-Q) = 0,$$

Since $\alpha + \beta + \gamma + 1 \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is completed.

Example 12. Figure (12) shows the Eq. (1) has no period two solution when $l = 4$, $k = 4$, $\alpha = 0.2$, $\beta = 0.7$, $\gamma = 0.4$, $a = 0.8$, $b = 0.3$, $c = 0.6$ and $d = 0.9$ and the initial conditions $x_{-4} = 6$, $x_{-3} = 7$, $x_{-2} = 2$, $x_{-1} = 3$ and $x_0 = 5$.

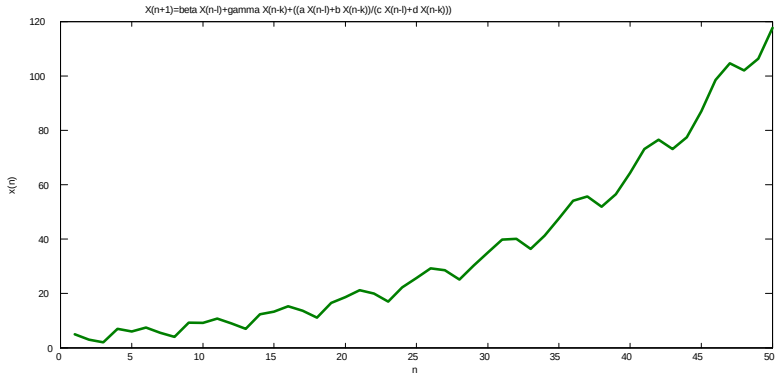


Fig. 12. Draw the solution of Eq. (1) has no periodic when l and k are even.

THEOREM 5.4. *Equation (1) has no prime period two solutions if l and k are odd and $1-\beta-\gamma \neq 0$.*

Proof. Suppose that there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...$, of Eq. (1). We see from Eq. (1) when l and k are odd that

$$P = \alpha Q + \beta P + \gamma P + \frac{aP+bP}{cP+dP}, \tag{10}$$

$$Q = \alpha P + \beta Q + \gamma Q + \frac{aQ+bQ}{cQ+dQ}. \tag{20}$$

Subtracting (19) from (20) gives

$$(1-\alpha-\beta-\gamma)(P-Q)=0,$$

Since $1-\alpha-\beta-\gamma \neq 0$, then $P=Q$. This is a contradiction. Thus, the proof is completed.

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OSCILLATION OF SOLUTIONS OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS

YONG LIU AND XIAOGUANG QI

ABSTRACT. In this article, we mainly investigate the growth of solutions of certain higher order linear differential equations. The results we obtain generalize some previous results of P. C. Wu and J. Zhu.

1 INTRODUCTION

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory(e.g.see [8, 14, 15]). In addition, we will use the notations $\sigma(f), \mu(f), \lambda(f), \lambda(\frac{1}{f})$ to denote the order, the lower order, the exponents of the convergence of the zero-sequence and the exponents of convergence of pole-sequence of a meromorphic function $f(z)$, respectively.

For a set $E \subset R^+$, let $m(H)$, respectively $m_l(H)$, denote the linear measure, respectively the logarithmic measure, of H. By $\chi H(t)$, we denote the characteristic function of H. Moreover, the upper logarithmic density and the lower logarithmic density of H are defined by

$$\overline{\log dens} H = \limsup_{r\to\infty}(\int_1^r (\chi H(t)/t)dt)/\log r$$

$$\underline{\log dens} H = \liminf_{r\to\infty}(\int_1^r (\chi H(t)/t)dt)/\log r,$$

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Key words:deficient functions, complex differential equations, value distribution, infinite order.

where $\chi H(t)$ is the characteristic function of the set H.

For the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

many authors have investigated the growth of solutions of (1.1), where $A(z)$ and $B(z)$ are entire functions. It is well known that if $B(z)$ is transcendental and f_1, f_2 are two linearly independent solutions of equation (1.1), then at least one of f_1, f_2 must have infinite order. On the other hand, there exist some equations of the form (1.1) that possess a solution $f \not\equiv 0$ of finite order; for example, $f(z) = e^{2z}$ satisfies $f'' + e^{-2z}f' - (2e^{-2z} + 4)f = 0$. Thus a natural question is: what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors have focused on this subject, such that ([1-3, 9-13])

Recently, P. C. Wu and J. Zhu [12] proved the following result:

Theorem A. [12] Let $A(z)$ be a meromorphic function with finite order having a finite deficient value. Suppose that $B(z)$ is a meromorphic function satisfying the following condition:

$$\lambda(\frac{1}{B}) < \mu(B) < \frac{1}{2}.$$

Then every solution $f \not\equiv 0$ of equation (1.1) is of infinite order.

Thus a natural question arises: whether does the conclusion hold when $\mu(B) = \frac{1}{2}$? We give an affirmative answer, and get the following interesting result:

Theorem 1.1. Let $A_0(z)$ be a meromorphic function with $\lambda(\frac{1}{A_0}) < \mu(A_0) \leq \frac{1}{2}$. And let $A_j(z)(j = 1, 2, \cdots, k-1)$ be meromorphic functions with finite order having a finite deficient value. Then every solution $f \not\equiv 0$ of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f'(z) + A_0(z)f = 0 \tag{1.2}$$

satisfies $\sigma(f) = \infty$.

2 SOME LEMMAS

Lemma 2.1. [7] *Let $f(z)$ be a transcendental meromorphic function of finite-order σ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin H \cup [0, 1]$ and for all $k, j, 0 \leq j < k$, one has*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\sigma-1+\varepsilon)} \tag{2.1}$$

Similarly, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0, 2\pi) \setminus E$, and for all $k, j, 0 \leq j < k$, the inequality (2.1) holds.

Lemma 2.2. [5] *Let $f(z)$ be a meromorphic function of finite order σ. Given $\zeta > 0$ and $l, 0 < l < \frac{1}{2}$, there exist a constant $K(\sigma, \zeta)$ and a set $E_\zeta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that for all $r \in E_\zeta$ and for every interval J of length l*

$$r \int_J \Big| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \Big| d\theta < K(\sigma, \zeta) \Big(l \log \frac{1}{l} \Big) T(r, f).$$

By the remarks following Theorem 8.1 in [4] and ref [9], we can obtain that

Lemma 2.3. *Suppose that $g(z)$ is an entire function of $\mu(f) = \frac{1}{2}$, and satisfies*

$$\log L(r, g) = o(\log M(r, g))$$

where $L(r, g) = \min_{|z|=r} |g(z)|, M(r, g) = \max_{|z|=r} |g(z)|$. There exists a set G of logarithmic density 1, a set H of density 0, a real-valued function $\varphi(r)$, and a positive function $\Psi(r)$ varying slowing in the sense that

$$\lim_{r \to \infty} \frac{\Psi(\sigma(r))}{\Psi(r)} = 1, r \in G \tag{2.1}$$

for all $\sigma > 0$, such that for $r \in G - H$

$$\log |g(re^{i(\varphi + \varphi(r))})| = (\cos(\frac{\varphi}{2}) + o(1)) r^{\frac{1}{2}} \Psi(r), r \to \infty, \tag{2.2}$$

uniformly for $\varphi \in [-\pi, \pi]$.

Lemma 2.4. [6] *Suppose that $g(z)$ is transcendental and meromorphic in the plane, of lower order $\mu < \alpha < 1$, and define $L(r, g) = \min\{|g(z)| : |z| = r\}$ and*

$$Y_1 = \{r > 1 : \log L(r, g) > \gamma(\cos \pi\alpha + \delta(\infty, g) - 1) T(r, g)\}, .$$

where $\gamma = \frac{\pi\alpha}{\sin \pi\alpha}$. Then Y_1 has upper logarithmic density at least $1 - \frac{\mu}{\alpha}$.

Lemma 2.5. *Suppose $f(z)$ is meromorphic and $\lambda\Big(\frac{1}{f}\Big) < \mu(f) \leq \frac{1}{2}$. Then either, for every $\delta < \mu(f)$, there exists $r_m \to \infty$ such that*

$$\log |f(z)| > r_m^\delta \tag{2.3}$$

for all z satisfying $|z| = r_m$. Or, for every $\delta < \mu(f)$, if

$$k_r = \{\theta \in [0, 2\pi) : \log |f(re^{i\theta})| < r^\delta\}$$

there exists a set $E_1 \subset [1, \infty)$ of upper logarithmic density 1 such that for $r \in E_1$,

$$m(K_r) \to 0, \ r \to \infty.$$

Proof. Let $f(z) = \frac{g(z)}{l(z)}$, where $l(z)$ is canonical products(or polynomial) formed by the poles of $f(z)$, and $g(z)$ is entire. From $\lambda(\frac{1}{f}) < \mu(f)$, we have

$$\lambda\Big(\frac{1}{f}\Big) = \lambda(l) = \sigma(l) < \mu(f), \ \mu(f) = \mu(g).$$

We divide our proof into two cases:

Case 1: $\mu(f) < \frac{1}{2}$. Since $\lambda(\frac{1}{f}) < \mu(f)$, we have $\delta(\infty, f) = 1$. Let $\lambda\left(\frac{1}{f}\right) < \delta < \alpha_1 < \mu(f) < \alpha < \frac{1}{2}$. By Lemma 2.4, then there exists a set $E_1\ of\ (1, \infty)$, having lower logarithmic density $1 - \frac{\mu(f)}{\alpha}$, such that for all $r \in E_1$ we have

$$\log L(r, f) > \gamma \cos \pi\alpha T(r, f) \geq r^{\alpha_1} > r^{\delta},$$

where $\gamma = \frac{\pi\alpha}{\sin \pi\alpha}$. Hence, for every $\delta < \mu(f)$, there exists $r_m \to \infty$ such that

$$\log |f(z)| > r_m^{\delta}$$

for all z satisfying $|z| = r_m$. So (i) holds.

Case 2: $\mu(f) = \frac{1}{2}$. There exist the following two subcases:

Subcase 2.1. If there exists $r_n \to \infty$ with

$$\log L(r_n, g) > \alpha \log M(r_n, g) \ \ as \ \ r_n \to \infty \tag{2.4}$$

for some $\alpha > 0$. Hence for given $0 < \varepsilon < \min\{\frac{\delta - \sigma(l)}{2}, \frac{\mu(g) - \delta}{2}\}$, by (2.4) we have

$$\begin{aligned} & \log L(r_n, f) \geq \log L(r_n, g) - \log M(r_n, l) \\ \geq \ & \alpha r_n^{\delta+\varepsilon} - r_n^{\sigma(l)+\varepsilon} > r_n^{\delta}. \end{aligned} \tag{2.5}$$

So (i) also hold.

Subcase 2.2. Otherwise

$$\log L(r, g) = o(\log M(r, g)). \tag{2.6}$$

We choose $\max\{\delta, \lambda(\frac{1}{f})\} < \xi < \alpha < \mu(g)$. We note that $E^* = G - H$ has logarithmic density 1, where G, H are defined as in Lemma 2.3. By Lemma 2.3, (2.1), (2.2), (2.6) and the fact that E^* has logarithmic density 1, we obtain

$$\Psi(r) r^{\frac{1}{2} - \xi} \to \infty, \tag{2.7}$$

as $r \to \infty$. Defining

$$K*_r = \{\theta \in [0, 2\pi] : \log |g(re^{i\theta})| < r^{\xi}\}.$$

By (2.2) and (2.7), for all $r \in E^*$ we have that

$$m(K*_r) \to 0.$$

Set $F = \{z | f(z) = \infty\}$, since $\lambda(\frac{1}{f}) < \frac{1}{2}$, we have $m_l(F) < \infty$. Obviously, $E^{**} = E^* - F$ has logarithmic density 1. If $\theta \in K_r$ for all $r \in E^{**}$, we have

$$\log |g(re^{i\theta})| < \log |f(z)| + \log M(r, d) < r^{\delta} + r^{\xi} < r^{\alpha}.$$

So, $K_r \subset K_r^*$. Thus Lemma 2.5 holds. □

Lemma 2.6. *Suppose $f(z)$ is a nonconstant meromorphic function of order $\sigma < \sigma_1 < \infty$. For a positive number α, there exists a set $E(\alpha) \subset [1, \infty)$ with finite linear measure such that*

$$m(E(\alpha) \cap [\frac{r}{e}, er]) < \exp(-r^{\alpha}), r > r_0(f),$$

and that, for $|z| = r \notin E(\alpha)$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| < \exp(r^{3\alpha}),\ r > r_0(f),\ j = 1, 2, \cdots, k.$$

Proof. Let $\Delta(a, \delta) = \{z : |z - a| < \delta\}$ and let $\{l_\mu\}, \{m_\nu\}$ denote all the zeros and poles of $f(z)$, respectively. Let $A = A_1 \cup A_2$, where $A_1 = \cup_\mu \Delta(l_\mu, \frac{1}{k}\exp(-3|l_\mu|^{2\alpha}))$, $A_2 = \cup_\nu \Delta(m_\nu, \frac{1}{k}\exp(-3|m_\nu|^{2\alpha}))$.
Suppose $E_1 = \{t \geq 1 : A \cap \{|z| = t\} \neq \emptyset\}$. Obviously

$$\begin{aligned} & m(E_1(\alpha) \cap [\frac{r}{e}, er]) \\ < & \frac{1}{k}\{n(3r, f) + n(3r, \frac{1}{f})\}\exp(-r^{2\alpha}) \\ < & \frac{2}{k}(3r)^{\sigma_1}\exp(-r^{2\alpha}) < \frac{1}{k}\exp(-r^{\alpha}), r > r_1(\alpha) \end{aligned}$$

for $|z| = r \notin E_1(\alpha)$. We consider the differentiated Poisson-Jensen formula, for $|z| = r$ and $R = 3r$, we have

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(3re^{i\theta})| \frac{2z(3re^{i\theta})}{(3re^{i\theta} - z)^2} d\theta \\ &+ \sum_{|l_\mu|<3r} (\frac{z}{z - l_\mu} + \frac{\overline{l_\mu} z}{(3r)^2 - \overline{l_\mu} z}) \\ &+ \sum_{|m_\nu|<3r} (\frac{z}{z - m_\nu} + \frac{\overline{m_\nu} z}{(3r)^2 - \overline{m_\nu} z}). \end{aligned}$$

We use the method of ref [9], we also obtain

$$\begin{aligned} \Big|\frac{zf'(z)}{f(z)}\Big| &\leq \frac{3}{2}\{m(3r, f) + m(3r, \frac{1}{f})\} \\ &+4r(n(3r, f) + n(3r, \frac{1}{f}))e^{(9r)^{2\alpha}} + O(1) \\ &\leq (8r + 3)r^{\sigma_1}e^{(9r)^{2\alpha}}. \end{aligned}$$

Hence

$$\Big|\frac{f'(z)}{f(z)}\Big| < (8r + 3)r^{\sigma_1 - 1}e^{(9r)^{2\alpha}} < e^{r^{3\alpha}}.$$

We use the same method to each of the functions $f', \cdots, f^{(k)}$, we get there exists a set $E(\alpha)$ such that

$$m(E(\alpha) \cap [\frac{r}{e}, er]) < \exp(-r^{\alpha}),$$

and if $|z| = r \notin E(\alpha)$, we obtain

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| < e^{r^{3\alpha}}, 1 \leq j \leq k$$

Evidently

$$m_l(E(\alpha) \cap [\frac{r}{e}, er]) \leq \frac{em(E \cap [\frac{r}{e}, er])}{r} = o(1).$$

Since $m_l([\frac{r}{e}, er]) = 2$, we obtain that the logarithmic density of E is 0. □

By the remarks following Theorem 8.1 in [4] and ref [9], we can easily get that

Lemma 2.7. *Suppose $f(z)$ is meromorphic of $\lambda(\frac{1}{f}) < \mu(f) < 1$ and $0 < \varepsilon < \min(\frac{\mu(f)-\lambda(\frac{1}{f})}{2}, 1-\mu(f))$. Suppose there exists an unbounded set of r-valued such that*

$$\log|f(re^{i\theta})| > r^{\mu(f)-\varepsilon}$$

for all $\theta \in [0, 2\pi]$. Suppose also that $E_3 \subset [1, \infty)$ satisfies

$$m(E_3 \cap [\frac{r}{e}, er]) < \exp(-r^{6\varepsilon}), r > R_0.$$

Then there is an unbounded set of s-values with $s \notin E_3$ such that

$$\log|f(se^{i\theta})| > s^{\mu(f)-2\varepsilon}$$

for all $\theta \in [0, 2\pi]$.

3 Proof of Theorem 1.1

Suppose that $A_j(z)$ $(j = 1, 2, \cdots, k-1)$ has a finite deficiency $\delta(a_j, f) = 2\alpha_j > 0$ at $a_j \in C$. By the definition of deficiency, for all sufficiently r, we get

$$m(r, \frac{1}{A_j - a_j}) \geq \alpha_j T(r, A_j).$$

Hence, for all sufficiently r, there exists a point z_r satisfying $|z_r| = r$ and

$$\log|A_j(z_r) - a_j| \leq -\alpha_j T(r, A_j). \tag{3.1}$$

By Lemma 2.2, we choose $l > 0$ so small that

$$K(\rho, \varphi)(l, \log\frac{1}{l}) < \frac{\alpha_j}{2}.$$

Then for all $r \in E_\varphi$ and for every interval J, we obtain that

$$r\int_J \Big|\frac{A_j'(re^{i\theta})}{A_j(re^{i\theta})}\Big| d\theta < \frac{\alpha_j}{2} T(r, A_j),$$

where E_φ is a set with lower logarithmic density greater than $1-\varphi$. Suppose $z_r = re^{i\theta_r}$ and $\varphi > 0$ be a sufficiently small number, we choose a number $\theta_0 > 0, |\theta_r - \theta_0| \le l$, and a set $E_\varphi \subset [0,\infty)$ with lower logarithmic density greater than $1-\varphi$. For all given $r \in E_\varphi$ and for all $\theta \in [\theta_r - \beta, \theta_r + \beta]$, we have

$$\begin{aligned}
&\log|A_j(re^{i\theta}) - a_j| \\
&= \log|A_j(re^{i\theta_r}) - a_j| + \int_{\theta_r}^{\theta} \frac{d}{dt}\log|A_j(e^{it}) - a_j|dt \\
&\le -\alpha T(r, A_j) + r\int_{\theta_r}^{\theta} \Big|\frac{A_j'(re^{it})}{A_j(re^{it})}\Big||dt| \\
&\le -\frac{\alpha}{2}T(r, A_j) \le 0.
\end{aligned}$$

Thus for $|z_r| = r \in E(\varphi)\setminus[0, r_1]$ and $\theta \in [\theta_r - \theta_0, \theta_r + \theta_0]$, we obtain

$$|A_j(re^{i\theta})| \le |a_j| + 1. \tag{3.2}$$

Let transcendental $f \not\equiv 0$ be a finite order solution of (1.2), and suppose $\lambda(\frac{1}{A_0}) < \mu(A_0)$. We divide the proof into two cases depending on the growth property of $A_0(z)$ by Lemma 2.5.

Case 1. For given $\varepsilon, 0 < \varepsilon < \min\{\frac{\mu(f)-\lambda(\frac{1}{f})}{2}, 1-\mu(f), \frac{\mu(A_0)}{20}\}$. there exists a sequence $r_m \to \infty$ such that

$$\log|A_0(z)| > r_m^{\mu(A_0)-\varepsilon}. \tag{3.3}$$

From (1.2), we get

$$|A_0(z)| \le \Big|\frac{f^{(k)}(z)}{f(z)}\Big| + |A_{k-1}|\Big|\frac{f^{(k-1)}(z)}{f(z)}\Big| + \cdots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big|. \tag{3.4}$$

By Lemma 2.6, set $\alpha = 7\varepsilon$, there exists a set $E_\alpha \subset [1,\infty)$ with finite linear measure satisfying

$$m(E_\alpha \cap [\frac{r}{e}, er]) < e^{-r^{6\varepsilon}},\ r > R_0 \tag{3.5}$$

and if $|z| = r \notin E_\alpha$, we get

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| < e^{r^{15\varepsilon}},\ 1 \le j \le k,\ r > R_0 \tag{3.6}$$

By Lemma 2.7, there exists a sequence $s_m \to \infty, s_m \notin E_\alpha$ such that for all $\theta \in [0, 2\pi]$,

$$\log|A_0(s_m e^{i\theta})| > s_m^{\mu(A_0)-2\varepsilon}. \tag{3.7}$$

With (3.1), (3.4), (3.6), (3.7), as $s_m \to \infty$, we have

$$\exp(s_m^{\mu(A_0)-2\varepsilon}) \le (|a_1| + \cdots + |a_{k-1}| + k)\exp(s_m^{16\varepsilon}). \tag{3.8}$$

Thus, (3.8) is impossible.

Case 2. For given $\varepsilon, 0 < \varepsilon < \frac{\mu(A_0)}{2}$, If

$$K_r = \{\theta \in [0, 2\pi) : \log|g(r^{i\theta})| < r^{\mu(A_0)-\varepsilon}\},$$

there exists a set $E_2 \subset [0, \infty)$ having logarithmic density 1 such that $m(K_r) \to 0$, as $r \to \infty$ in E_2.

By Lemma 2.1, there exists a set $E_3 \subset [0, \infty)$ with linear measure zero such that for all $|z| = r \notin E_3$, we get

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leq |z|^{k\sigma(f)-1+\varepsilon}. \tag{3.9}$$

Note that $E_4 = E(\varphi) \cap E_2 \backslash E_3$ has a positive lower logarithmic density, and for all sufficiently large $r \in E_4$, we have $[\theta_r - \phi, \theta_r + \phi] - K_r \neq \emptyset$. Hence, there exist unbounded points $z = re^{i\theta}$ such that (3.2), (3.9) and $\log|A_0(re^{i\theta})| \geq r^{\mu(A_0)-\varepsilon}$, we obtain that

$$\exp\{r^{\mu(A_0)-\varepsilon}\} \leq (k + |a|)r^{k\sigma(f)+1}. \tag{3.10}$$

Obviously, (3.10) is impossible. By case 1, and case 2, we obtain that $\sigma(f) = \infty$.

If rational function $f \not\equiv 0$ be a solution of (1.2), using the above similar method, we can get a contradiction. Hence, Theorem 1.1 hold.

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YONG LIU
Department of Mathematics, Shaoxing College of Arts and Sciences, Shaoxing, Zhejiang 312000, China
E-mail address: liuyongsdu@aliyun.com

XIAOGUANG QI

University of Jinan, School of Mathematics, Jinan, Shandong, 250022, P. R. China
E-mail address: xiaogqi@gmail.com or xiaogqi@mail.sdu.edu.cn

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Francesco Altomare
Dipartimento di Matematica
Universita' di Bari
Via E.Orabona, 4
70125 Bari, ITALY
Tel+39-080-5442690 office
+39-080-3944046 home
+39-080-5963612 Fax
altomare@dm.uniba.it
Approximation Theory, Functional Analysis, Semigroups and Partial Differential Equations, Positive Operators.

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
tel: 361-593-2600
Agarwal@tamuk.edu
Differential Equations, Difference Equations, Inequalities

George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152,U.S.A
Tel.901-678-3144
e-mail: ganastss@memphis.edu
Approximation Theory, Real Analysis,
Wavelets, Neural Networks,
Probability, Inequalities.

J. Marshall Ash
Department of Mathematics
De Paul University
2219 North Kenmore Ave.
Chicago, IL 60614-3504
773-325-4216
e-mail: mash@math.depaul.edu
Real and Harmonic Analysis

Dumitru Baleanu
Department of Mathematics and Computer Sciences,
Cankaya University, Faculty of Art and Sciences,
06530 Balgat, Ankara,
Turkey, dumitru@cankaya.edu.tr
Fractional Differential Equations
Nonlinear Analysis, Fractional Dynamics

Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site:
http://www.unipg.it/~bardaro/
Functional Analysis and Approximation Theory, Signal Analysis, Measure Theory, Real Analysis.

Martin Bohner
Department of Mathematics and Statistics, Missouri S&T
Rolla, MO 65409-0020, USA
bohner@mst.edu
web.mst.edu/~bohner
Difference equations, differential equations, dynamic equations on time scale, applications in economics, finance, biology.

Jerry L. Bona
Department of Mathematics
The University of Illinois at Chicago
851 S. Morgan St. CS 249
Chicago, IL 60601
e-mail:bona@math.uic.edu
Partial Differential Equations, Fluid Dynamics

Luis A. Caffarelli
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082
512-471-3160
e-mail: caffarel@math.utexas.edu
Partial Differential Equations

George Cybenko
Thayer School of Engineering

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces, Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School
and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
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OSCILLATION CRITERIA FOR FORCED AND DAMPED NABLA FRACTIONAL DIFFERENCE EQUATIONS

JEHAD ALZABUT, THABET ABDELJAWAD, HUSSAM ALRABAIAH

Abstract. Based on the properties of Riemann–Liouville difference and sum operators, sufficient conditions are established to guarantee the oscillation of solutions for forced and damped nabla fractional difference equations. Numerical examples are presented to show the applicability of the proposed results. We finish the paper by a concluding remark.

1. Introduction and preliminaries

The study of oscillation of solutions for various type of equations including differential, difference and dynamic equations on time scales has been the object of many researchers in the last five decades. Indeed, great efforts have been put in the direction of establishing new oscillation criteria for these type of equations; see the monographs [1, 2, 3, 4].

Due to their widespread applications in science and technology, the fractional differential equations have started to attract more attention among physicists and mathematicians. Indeed, it was found that various interdisciplinary applications can be elegantly modeled by the help of these equations. For instance, the nonlinear oscillation of earthquake and heart beats and the stability of brain tumor growth can be modeled by using fractional derivatives; the reader can consult the paper [5, 6] for more details. In parallel to the recent developments in research, the investigation of oscillation property for fractional differential equations has continued and thus many significant results have lately appeared [7, 8, 9, 10, 11, 12, 13]. On the other hand, difference equations have shown tremendous applications in numerical and computational simulations [14, 15]. However, the oscillation of damped or forced difference equations have comparably gained less attention among researchers [16, 17, 18, 19]. For fractional difference equations, fairly few papers have recently published on the oscillation of their solutions [20, 21, 22].

Following this trend in this paper, we consider the following forced and damped nabla fractional difference equation

$$\begin{aligned} \big(1-p(n)\big)\nabla\nabla_0^{\alpha}y(n)+p(n)\nabla_0^{\alpha}y(n)+q(n)f(y(n))=g(n), \quad n\in\mathbb{N}_1,\\ \nabla_0^{-(1-\alpha)}y(1)=y(1)=c, \end{aligned} \tag{1.1}$$

where $\nabla_0^{\alpha}y$ and $\nabla_0^{-\alpha}y$ are the Riemann–Liouville fractional difference and sum operators of y of order α, respectively, $\alpha\in(0,1)$ is a real number, c is a constant, $\mathbb{N}_1=\{1,2,\ldots\}$ and

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(H) p, g are real sequences from $\mathbb{N}_1 \to \mathbb{R}$, $p(n) < 1$, q is a positive real sequence from $\mathbb{N}_1 \to \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ such that $\frac{f(s)}{s} > 0$ for all $s \neq 0$.

To the best of their observations, the authors claim that there is no paper in the literature concerning with the oscillation of solutions of nabla fractional difference equation involving forcing and damping terms. Therefore, the equation under consideration and the obtained results are essentially new and have their own merits.

Definition 1. *A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is called non–oscillatory.*

Throughout this paper, we will make use of the following notations, definitions and some known results of ∇–fractional operators [23]. For any $\alpha, t \in \mathbb{R}$, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \ t \in \mathbb{R}\backslash\{\ldots,-2,-1,0\}, \tag{1.2}$$

with the convention that $0^{\overline{\alpha}} = 0$. Γ is the well known Gamma function satisfying the property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Definition 2. [23] *For $\alpha \in (0,1)$ and $\rho(s) = s-1$, we have*

I. *The nabla operator is defined by $\nabla y(n) = y(n) - y(n-1), \ \ n \in \mathbb{N}_1$.*
II. *The Riemann–Liouville fractional sum $\nabla_0^{-\alpha}$ of order α is defined by*

$$\nabla_0^{-\alpha} y(n) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{n} (n-\rho(s))^{\overline{\alpha-1}} y(s), \ \ n \in \mathbb{N}_1. \tag{1.3}$$

III. *The Riemann–Liouville fractional difference ∇_0^{α} of order α is defined by*

$$\nabla_0^{\alpha} y(n) = \nabla \nabla_0^{-(1-\alpha)} y(n), \ \ n \in \mathbb{N}_1 \tag{1.4}$$

and hence

$$\nabla_0^{\alpha} y(n) = \frac{\nabla}{\Gamma(1-\alpha)} \sum_{s=1}^{n} (n-\rho(s))^{-\overline{\alpha}} y(s), \ \ n \in \mathbb{N}_1. \tag{1.5}$$

Lemma 1. [23] *For $\alpha \in (0,1)$ and $n \in \mathbb{N}_0$, we have*

1. *The relation*

$$\nabla_1^{-\alpha} \nabla_0^{\alpha} y(n) = y(n) - \frac{n^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(1). \tag{1.6}$$

2. *The power rule is defined by*

$$\nabla_0^{-\alpha} n^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} n^{\overline{\alpha+\mu}}, \ \mu > -1. \tag{1.7}$$

Lemma 2. [24] *For $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \frac{\Gamma(n) n^{\varepsilon}}{\Gamma(n+\varepsilon)} = 1. \tag{1.8}$$

2. The main results

In this section, we establish the main results of this paper. Indeed, sufficient conditions are established for the oscillation of the solutions of equation (1.1).

Theorem 1. *Let the assumption (H) and the following conditions hold*

$$\liminf_{n\to\infty}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big]<0, \tag{2.1}$$

and

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big]>0, \tag{2.2}$$

where M is a constant and $P(n)=\prod_{s=n_0}^{n}\Big(\frac{1}{1-p(s)}\Big)$ for $n_0\in\mathbb{N}_1$. Then every solution of (1.1) is oscillatory.

Proof. For the sake of contradiction, assume that $y(n)$ is a non–oscillatory solution of (1.1). Then, the solution y is either $y(n)>0$ or $y(n)<0$ for $n\in\mathbb{N}_1$. Let $y(n)>0,\ \ n\in\mathbb{N}_1$. From equation (1.1), we obtain

$$\big(1-p(n)\big)\nabla\nabla_0^{\alpha}y(n)+p(n)(\nabla_0^{\alpha}y)(n)=-q(n)f(y(n))+g(n)<g(n)$$

Multiplying both sides of the above inequality by $P(n)$, we get

$$\nabla\nabla_0^{\alpha}y(n)P(n-1)+\nabla_0^{\alpha}y(n)\nabla P(n)<g(n)P(n), \tag{2.3}$$

where $P(n-1)=\big(1-p(n)\big)P(n)$ and $\nabla P(n)=p(n)P(n)$ have been used. However, the left side of (2.3) can be written in the form

$$\nabla\Big(\nabla_0^{\alpha}y(n)P(n)\Big)<g(n)P(n). \tag{2.4}$$

Taking the sum of both sides from n_0+1 to n, we have

$$\nabla_0^{\alpha}y(n)P(n)<\nabla_0^{\alpha}y(n_0)P(n_0)+\sum_{s=n_0+1}^{n}g(s)P(s),$$

or

$$\nabla_0^{\alpha}y(n)<\frac{M}{P(n)}+\frac{1}{P(n)}\sum_{s=n_0+1}^{n}g(s)P(s), \tag{2.5}$$

where $M=\nabla_0^{\alpha}y(n_0)P(n_0)$. Applying the operator $\nabla_1^{-\alpha}$ on both sides of the above inequality, we get

$$\nabla_1^{-\alpha}\nabla_0^{\alpha}y(n)<\nabla_1^{-\alpha}\Big[\frac{M}{P(n)}+\frac{1}{P(n)}\sum_{s=n_0+1}^{n}g(s)P(s)\Big]. \tag{2.6}$$

In view of (1.3) and (1.6), we observe that

$$\nabla_1^{-\alpha}\nabla_0^{\alpha}y(n)=y(n)-\frac{n^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(1). \tag{2.7}$$

and

$$\nabla^{-\alpha}\Big[\frac{M}{P(n)}+\frac{1}{P(n)}\sum_{s=n_0+1}^{n}g(s)P(s)\Big]$$

$$=\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{n}(n-\rho(s))^{\overline{\alpha-1}}\Big[\frac{M}{P(s)}+\frac{1}{P(s)}\sum_{r=n_0+1}^{s}g(r)P(r)\Big]. \tag{2.8}$$

Combining (2.7) and (2.8), the inequality (2.6) becomes

$$y(n)<\frac{n^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(1)+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big].$$

Multiplying the above inequality by $n^{1-\alpha}$, we get

$$n^{1-\alpha}y(n)<n^{1-\alpha}n^{\overline{\alpha-1}}\frac{y(1)}{\Gamma(\alpha)}+\frac{n^{1-\alpha}}{\Gamma(\alpha)}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big].$$

However, we note that $n^{1-\alpha}n^{\overline{\alpha-1}}=\frac{n\Gamma(n+\alpha-1)}{n^{\alpha}\Gamma(n)}$. By the help of Stirling formula (1.8), we observe that

$$\begin{aligned}\lim_{n\to\infty}n^{1-\alpha}n^{\overline{\alpha-1}} &= \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{n^{\alpha}\Gamma(n)}\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha)}\\ &= \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)}=\lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{\Gamma(n+\alpha-1+1)}\\ &= \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{(n+\alpha-1)\Gamma(n+\alpha-1)}=1. \qquad (2.9)\end{aligned}$$

By virtue of (2.9) and condition (2.1), we conclude that

$$\liminf_{n\to\infty}n^{1-\alpha}y(n)\leq-\infty,$$

which contradicts the assumption that $y(n)>0$.

In case $y(n)<0,\ n\in\mathbb{N}_1$, nevertheless, we get

$$\nabla\Big(\nabla_0^{\alpha}y(n)P(n)\Big)>g(n)P(n)$$

or

$$\nabla_0^{\alpha}y(n)>\frac{M}{P(n)}+\frac{1}{P(n)}\sum_{s=n_0+1}^{n}g(s)P(s),$$

where $M=\nabla_0^{\alpha}y(n_0)P(n_0)$. Following similar steps, we end up with

$$n^{1-\alpha}y(n)>n^{1-\alpha}n^{\overline{\alpha-1}}\frac{y(1)}{\Gamma(\alpha)}+\frac{n^{1-\alpha}}{\Gamma(\alpha)}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big].$$

The analysis in (2.9) and condition (2.2) implies that $\limsup_{n\to\infty}n^{1-\alpha}y(n)\geq\infty$, which contradicts the assumption that $y(n)<0$. The proof is finished. □

Let $z(n)=\sum_{s=1}^{n}(n-\rho(s))^{-\overline{\alpha}}y(s)$ where $y(t)$ is a solution of equation (1.1). Then, by relations (1.3) and (1.4) we get

$$\nabla z(n)=\Gamma(1-\alpha)\nabla_0^{\alpha}y(n). \tag{2.10}$$

It is clear that z and y have the same dynamical character. We will use relation (2.10) to prove the following theorem.

Theorem 2. *Let the assumption (H) and the following conditions hold*

$$\liminf_{n\to\infty}\sum_{s=1}^{n}\frac{1}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big]=-\infty, \tag{2.11}$$

and

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\frac{1}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big]=\infty, \tag{2.12}$$

where M is a constant and $P(n)=\prod_{s=n_0}^{n}\left(\frac{1}{1-p(s)}\right)$ for $n_0\in\mathbb{N}_0$. Then every solution of (1.1) is oscillatory.

Proof. For the sake of contradiction, assume that $y(n)$ is a non–oscillatory solution of (1.1). Then, the solution y is either $y(n)>0$ or $y(n)<0$ for $n\in\mathbb{N}_1$. Let $y(n)>0,\ \ n\in\mathbb{N}_1$. Following similar steps as in the proof of Theorem 1, we reach to

$$\nabla_0^{\alpha}y(n)<\frac{M}{P(n)}+\frac{1}{P(n)}\sum_{s=n_0+1}^{n}g(s)P(s),$$

It follows from (2.10) that

$$\nabla z(n)<\frac{\Gamma(1-\alpha)}{P(n)}\Big[M+\sum_{s=n_0+1}^{n}g(s)P(s)\Big].$$

Taking the sum of both sides from n_0+1 to n, we have

$$z(n)<z(n_0)+\Gamma(1-\alpha)\sum_{s=n_0+1}^{n}\frac{1}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big].$$

In view of condition (2.11), one can easily see that the right hand side of the above inequality tends to $-\infty$ as $n\to\infty$ which contradicts with the fact that $z(n)>0$.

In case $y(n)<0,\ n\in\mathbb{N}_1$, then in the same manner we obtain

$$z(n)>z(n_0)+\Gamma(1-\alpha)\sum_{s=n_0+1}^{n}\frac{1}{P(s)}\Big[M+\sum_{r=n_0+1}^{s}g(r)P(r)\Big].$$

In view of condition (2.12), the right hand side of the above inequality tends to ∞ as $n\to\infty$ which contradicts the fact that $z(n)<0$. □

3. Examples

In this section, we present two examples to demonstrate the validity of the results in Theorem 1 and Theorem 2. By finding a non–oscillatory solution, the first example shows that the assumptions of Theorem 1 cannot be ignored whereas in the second example we verify that the assumptions of Theorem 2 are satisfied thus we conclude that all solutions are oscillatory.

Example 1. *Consider the following forced and damped nabla fractional difference equation*

$$\begin{aligned} &\frac{3}{2}\nabla\nabla_0^{\frac{3}{4}}y(n)-\frac{1}{2}\nabla_0^{\frac{3}{4}}y(n)+\frac{3\Gamma(n)}{4\Gamma(n+\frac{3}{4})}y(n)=\frac{6-3\Gamma(\frac{3}{4})}{8},\quad n\in\mathbb{N}_1,\\ &\nabla_0^{-\frac{1}{4}}y(1)=y(1)=\frac{3}{4}\Gamma\Big(\frac{3}{4}\Big), \end{aligned} \tag{3.1}$$

where $\alpha=\frac{3}{4}$, $p(n)=-\frac{1}{2}$, $q(n)=\frac{3\Gamma(n)}{4\Gamma(n+\frac{3}{4})}$, $f(y)=y$ *and* $g(n)=\frac{6-3\Gamma(\frac{3}{4})}{8}$. *Therefore,* $P(n)=\prod_{s=1}^{n}\left(\frac{2}{3}\right)=(\frac{2}{3})^n$. *By simple calculations, we get*

$$\sum_{s=1}^{n}(n-s+1)^{-\frac{1}{4}}\Big(\frac{2}{3}\Big)^s\Big[M+\sum_{r=2}^{s}\frac{6-3\Gamma(\frac{3}{4})}{8}\Big(\frac{3}{2}\Big)^r\Big]>0. \tag{3.2}$$

Inequality (3.2) implies that condition (2.1) of Theorem 1 is not satisfied. On the other hand, one can easily figure out that $y(n)=n^{\frac{3}{4}}$ *is a non–oscillatory solution for equation (3.1).*

Example 2. *Consider the following forced and damped nabla fractional difference equation*

$$\begin{aligned} &\frac{2}{3}\nabla\nabla_0^{\frac{1}{4}}y(n)-\frac{1}{3}\nabla_0^{\frac{1}{4}}y(n)+q(n)f(y(n))=n^2\Gamma(n),\quad n\in\mathbb{N}_1,\\ &\nabla_0^{-\frac{3}{4}}y(1)=y(1)=c, \end{aligned} \tag{3.3}$$

where $\alpha=\frac{1}{4}$, $p(n)=-\frac{1}{3}$ *and* $g(n)=n^2\Gamma(n)$. *Therefore,* $P(n)=\prod_{s=1}^{n}\left(\frac{3}{4}\right)=\left(\frac{3}{4}\right)^n$. *Furthermore, we get*

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\Big(\frac{4}{3}\Big)^s\Big[M+\sum_{r=2}^{s}r^2\Gamma(r)\Big(\frac{3}{4}\Big)^r\Big]=\infty,$$

which implies that condition (2.11) of Theorem 2 holds. Therefore, every solution of (3.3) is oscillatory.

4. A concluding remarks

The oscillation of solutions of fractional differential equations is considered to be one of the hottest topics among researchers nowdays. For fractional difference equations, however, the oscillation of solutions is still at its first stages of progress. We consider here this problem for a general form of equations involving forcing and damping terms which make the equation under consideration adequately represents real phenomena. Two main results are established to guarantee the oscillation of solutions. For numerical treatment, we examined our results by presenting two examples which show that the proposed assumptions are sufficient.

There are some points that feature the results of this paper and make them innovative. Equation (1.1) is considered in general form which covers many particular cases. No restriction is imposed on the forced term g in equation (1.1). The results of this paper can be extended to equation involving delay argument

$$\big(1-p(n)\big)\nabla\nabla_0^{\alpha}y(n)+p(n)(\nabla_0^{\alpha}y)(n)+q(n)f(y(n-\tau))=g(n),\ \ n\in\mathbb{N}_1,$$

where $\tau>0$. It is worthy to mention that every solution of the above equation is also oscillatory under assumption (H). That is, the delay argument has no influence on the oscillatory behavior. The results of this paper can be also carried out upon

employing the right fractional difference operator, we recommend the reader to consult the paper [23] for more details. The above equation with delay argument is then converted to an equation with advanced argument. Unlike most of the papers appeared in the literature which studied oscillation property for difference equations involving delta operator, we consider here an equation involving nabla operator which is more appropriate upon calculations.

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J. Alzabut: Department of Mathematics and Physical Sciences, Prince Sultan University, P. O. Box 66833, 11586 Riyadh, Saudi Arabia
E-mail address: jalzabut@psu.edu.sa

T. Abdeljawad: Department of Mathematics and Physical Sciences, Prince Sultan University, P. O. Box 66833, 11586 Riyadh, Saudi Arabia
E-mail address: tabdeljawad@psu.edu.sa

H. Alrabaiah: Department of Mathematics, Tafila Technical University, Tafila, Jordan
E-mail address: hussam89@yahoo.com

A type of C^2 smooth surfaces generated by bivariate rational spline interpolation

Jianxun Pan[1], Qinghua Sun[2]*, Ling Zhang[2], Fangxun Bao[2]
1 Department of Basic, Shandong Woman's University, Jinan, 250300, China
2 School of Mathematics, Shandong University, Jinan, 250100, China

Abstract

In this paper, we present a novel C^2 surface modeling method only based on the values of the original function. There are two schemes to generate a family of the interpolation surfaces. Identified uniquely by the values of the tension parameters $\alpha_{i,j}$ and $\beta_{i,j}$, each interpolant of the family is C^2 continuous in the whole interpolating region, and which can be represented by using basis functions clearly. More important, since there are free positive parameters in the interpolants, the shape of the interpolating surfaces can be modified by selecting suitable parameters for the unchanged interpolating data. Also, the interpolants are stable for any positive parameters, and the error estimate formula of the interpolators are derived. Numerical examples show that the interpolators give a good approximation to the interpolated function and the shape of the interpolating surfaces can be modified.

Keywords: C^2-spline; bivariate rational interpolation; surface modeling; shape control

1 Introduction

In various applications such as industrial design and manufacture, atmospheric analysis, geology and medical imaging, etc., it is often necessary to generate a smooth surface that interpolates a prescribed set of data. For most applications, C^1 smoothness is generally sufficient. However, curvature continuity sometimes is needed and this leads to the need for C^2 smoothness.

For generating a C^1 smooth surface, there are many ways to tackle this problem [3, 9, 10, 16, 22]. But, generating a C^2 bivariate interpolation is a more difficult task. In [9], a bicubic spline interpolation scheme was proposed as a extension of the theory of cubic splines to two dimension, this type of interpolation scheme has become the standard scheme for rectangular regions, and which has studied in many literatures [4, 7, 17]. In recent years, some of the literatures have contributed to the C^2 bivariate interpolation also. For example, in [5], Brou and Méhauté proposed a construction of C^r bivariate rational splines over a triangulation, via a finite element approach; In [6], a novel surface modeling scheme was presented based on an envelope template, and G^2 or C^2 composite surfaces can be obtained utilizing the envelope template sweeping over the data points; In [8], a rationally corrected quintic Bézier triangular patch scheme of degree 9 over degree 4 and controlled by 27 Bézier points was used to define a smooth surface through scattered data, and a convex combination technique was employed to enable C^2 continuity conditions on the boundaries of the

*Corresponding author: sunqh@sdu.edu.cn

triangle to be satisfied; In [11], a refinable function vector of C^2-quartic splines was introduced for generating approximation quadrilateral subdivisions, and that of C^2-quintic splines was constructed for generating a second order Hermite interpolatory quadrilateral subdivision; In [12], two families of solutions provided by two Hermite subdivision schemes HD^2 and HR^2 were investigate, and a C^2 interpolant on any semiregular rectangular mesh was generated with Hermite data of degree 2; In [18], two C^2 shape-preserving bivariate interpolant on rectangular grids were developed by using polynomial splines; In [20], C^1- and C^2-continuous spline-interpolation surfaces were constructed in a regular triangular net with the help of polynomial basic functions; In [21], author proved that there exists a C^3 piecewise polynomial of degree 7 on the twice CT type split of a triangle, which interpolate arbitrarily given values and derivatives of orders up to 3 at the vertices and on the edges of the triangle.

Most of above C^2 bivariate spline interpolations are in fact polynomial interpolations. However, one of the disadvantages of the polynomial spline method is that the local shape can not be modified for the interpolating surfaces while interpolating data is unchanged. Further, generating a C^2 bivariate interpolation usually requires up to second-order partial derivative values of the interpolated function, or to solve a system of consistency equations for second-order partial derivatives at the data sites. Unfortunately, in many practical problems, the partial derivatives are difficult to get. Thus, in order to generate the C^2 surfaces required for CAGD, the following conditions must be satisfied: (a) the surfaces are constructed based only on function values; (b) the parameters of constructed surfaces can be modified without changing the given data.

In fact, in recent years, motivated by the univariate rational spline interpolation, the C^1 bivariate rational spline, which has a simple and explicit mathematical representation with parameters, has been studied [1, 13, 14, 15]. Since the parameters in the interpolation function are selective according to the control constrains, the constrained control of the shape becomes possible.

In this paper, motivated by C^2 rational spline curve [2], a C^2 piecewise bivariate rational spline interpolation which can be modified by using new parameters, will be concerned based only on function values. To solve the problem, a new approach is proposed by using a constructed interpolation function comprising a simple and explicit mathematical representation with the new parameters $\alpha_{i,j}$ and $\beta_{i,j}$. This paper is arranged as follows. In Section 2, a piecewise bivariate rational spline interpolation with parameters is constructed over rectangular mesh. Section 3 discusses the C^2 continuity of the interpolant. In Section 4, the basis of this interpolator is derived, and the bounded property is obtained. Sections 5 deals with the error estimates of the interpolator. Some numerical examples are given in Section 6, which show that this interpolator gives a good approximation to the interpolated function and the shape of the interpolating surfaces can be modified by selecting suitable parameters.

2 Construction of bivariate rational spline interpolant

Let $\Omega : [a, b; c, d]$ be the plane region, and $\{(x_i, y_i, f_{i,j}), i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_m = d$ are the knot spacings, $f_{i,j} = f(x_i, y_j)$. $d_{i,j}$ and $e_{i,j}$ are chosen partial derivative values $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ at the knots (x_i, y_j), respectively. Let $h_i = x_{i+1} - x_i$, and $l_j = y_{j+1} - y_j$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the xy-plane, let $\theta = (x - x_i)/h_i$ and $\eta = (y - y_j)/l_j$. Denoting

$$\Delta_{i,j}^{(x)} = \frac{f_{i+1,j} - f_{i,j}}{h_i}, \ \Delta_{i,j}^{(y)} = \frac{f_{i,j+1} - f_{i,j}}{l_j}.$$

First, for each $y = y_j, j = 1, 2, \cdots, m$, construct the x-direction interpolating curve, this is given

by

$$P^*_{i,j}(x) = \frac{p^*_{i,j}(x)}{q^*_{i,j}(x)}, \quad i = 1, 2, \cdots, n-1, \tag{1}$$

where

$$\begin{aligned} p^*_{i,j}(x) &= (1-\theta)^3 f_{i,j} + \theta(1-\theta)^2 V^*_{i,j}(x) + \theta^2(1-\theta) W^*_{i,j}(x) + \theta^3 f_{i+1,j}, \\ q^*_{i,j}(x) &= (1-\theta)^2 + \theta(1-\theta)\alpha_{i,j} + \theta^2, \end{aligned}$$

with

$$\begin{aligned} V^*_{i,j}(x) &= (\alpha_{i,j}+1) f_{i,j} + h_i d_{i,j} + \theta(1-(1-\theta)\alpha_{i,j})(f_{i+1,j} - f_{i,j} - h_i d_{i,j}), \\ W^*_{i,j}(x) &= (\alpha_{i,j}+1) f_{i+1,j} - h_i d_{i+1,j} - (1-\theta)(1-\theta\alpha_{i,j})(f_{i+1,j} - f_{i,j} - h_i d_{i+1,j}), \end{aligned}$$

and $\alpha_{i,j} > 0$. This interpolation $P^*_{i,j}(x)$ defined by (1) is called the rational quintic interpolator which satisfies

$$P^*_{i,j}(x_i) = f_{i,j},\ P^*_{i,j}(x_{i+1}) = f_{i+1,j},\ {P^*_{i,j}}'(x_i) = d_{i,j},\ {P^*_{i,j}}'(x_{i+1}) = d_{i+1,j}.$$

Further, when $\alpha_{i,j} \to +\infty$, the interpolant is the well-known standard cubic Hermite interpolation. That is to say, in this special case, the interpolant $P^*_{i,j}(x)$ defined by (1) will give approximately the Hermite interpolation.

If the partial derivative values $d_{i,j}$ at the data sites are estimated using the arithmetic mean method:

$$d_{i,j} = \frac{h_{i-1}\Delta^{(x)}_{i,j} + h_i \Delta^{(x)}_{i-1,j}}{h_{i-1} + h_i},\ i = 2, 3, \cdots, n-1, \tag{2}$$

then the interpolation function $P^*_{i,j}(x)$ defined by (1) is C^2 continuous in $[a, b]$, and which satisfies

$$P''(x_i) = \frac{2}{h_{i-1} + h_i}(\Delta^{(x)}_{i,j} - \Delta^{(x)}_{i-1,j}),\ i = 2, 3, \cdots, n-1.$$

Remark 1. At the end knots x_1, x_n, the derivative values are given as

$$\begin{aligned} d_{1,j} &= \Delta^{(x)}_{1,j} - \frac{h_1}{h_1 + h_2}(\Delta^{(x)}_{2,j} - \Delta^{(x)}_{1,j}), \\ d_{n,j} &= \Delta^{(x)}_{n-1,j} + \frac{h_{n-1}}{h_{n-1} + h_{n-2}}(\Delta^{(x)}_{n-1,j} - \Delta^{(x)}_{n-2,j}), \end{aligned} \tag{3}$$

For each pair of $(i, j), i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, using the x-direction interpolation $P^*_{i,j}(x)$, define the interpolation function $P_{i,j}(x, y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \quad i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m-1, \tag{4}$$

where

$$\begin{aligned} p_{i,j}(x, y) &= (1-\eta)^3 P^*_{i,j}(x) + \eta(1-\eta)^2 V_{i,j} + \eta^2(1-\eta) W_{i,j} + \eta^3 P^*_{i,j+1}(x), \\ q_{i,j}(y) &= (1-\eta)^2 + \eta(1-\eta)\beta_{i,j} + \eta^2, \end{aligned}$$

with

$$V_{i,j} = (\beta_{i,j}+1)P^*_{i,j}(x) + l_j\phi_{i,j}(x) + \varphi_{i,j}(x,y),$$
$$W_{i,j} = (\beta_{i,j}+1)P^*_{i,j+1}(x) - l_j\phi_{i,j+1}(x) + \psi_{i,j}(x,y),$$

and

$$\phi_{i,s}(x) = (1-\theta)^3(1+4\theta+9\theta^2)e_{i,s} + \theta^3(6-8\theta+3\theta^2)e_{i+1,s},\ s = j, j+1,$$
$$\varphi_{i,j}(x,y) = (\eta - \eta(1-\eta)(\beta_{i,j}+1))(P^*_{i,j+1}(x) - P^*_{i,j}(x) - l_j\phi_{i,j}(x)),$$
$$\psi_{i,j}(x,y) = (1-\eta-\eta(1-\eta)(\beta_{i,j}+1))(P^*_{i,j}(x) - P^*_{i,j+1}(x) + l_j\phi_{i,j+1}(x)),$$

and $\beta_{i,j} > 0$. The interpolation function $P_{i,j}(x,y)$ defined by (4) is called a bivariate piecewise rational interpolator, which satisfies

$$P_{i,j}(x_r,y_s) = f(x_r,y_s),\ \frac{\partial P_{i,j}(x_r,y_s)}{\partial x} = d_{r,s},\ \frac{\partial P_{i,j}(x_r,y_s)}{\partial y} = e_{r,s},\quad r = i, i+1,\ s = j, j+1.$$

The interpolating scheme above begins in x-direction first. Now, let the interpolation begins with y-direction first. For each $x = x_i, i = 1,2,\cdots,n$, denote the y-direction interpolation in $[y_j, y_{j+1}]$ by

$$Q^*_{i,j}(y) = \frac{(1-\eta)^3 f_{i,j} + \eta(1-\eta)^2\overline{V}^*_{i,j}(y) + \eta^2(1-\eta)\overline{W}^*_{i,j}(y) + \eta^3 f_{i,j+1}}{(1-\eta)^2 + \eta(1-\eta)\beta_{i,j} + \eta^2},\ j = 1,2,\cdots,m-1, \quad (5)$$

where

$$\overline{V}^*_{i,j}(y) = (\beta_{i,j}+1)f_{i,j} + l_j e_{i,j} + \eta(1-(1-\eta)\beta_{i,j})(f_{i,j+1} - f_{i,j} - l_j e_{i,j}),$$
$$\overline{W}^*_{i,j}(y) = (\beta_{i,j}+1)f_{i,j+1} - l_j d_{i,j+1} - (1-\eta)(1-\eta\beta_{i,j})(f_{i,j+1} - f_{i,j} - l_j e_{i,j+1}),$$

with $\beta_{i,j} > 0$. This interpolant $Q^*_{i,j}(y)$ defined by (5) is C^1-continuous in $[c,d]$, and which satisfies

$$Q^*_{i,j}(y_j) = f_{i,j},\ Q^*_{i,j}(y_{j+1}) = f_{i,j+1},\ {Q^*_{i,j}}'(y_j) = e_{i,j},\ {Q^*_{i,j}}'(y_{j+1}) = e_{i,j+1}.$$

If the partial derivative values $e_{i,j}$ at the data sites are estimated using the arithmetic mean method:

$$\begin{aligned} e_{i,j} &= \frac{l_{j-1}\Delta^{(y)}_{i,j} + l_j\Delta^{(y)}_{i,j-1}}{l_{j-1}+l_j}, j = 2,3,\cdots,m-1,\\ e_{i,1} &= \Delta^{(y)}_{i,1} - \frac{l_1}{l_1+l_2}(\Delta^{(y)}_{i,2} - \Delta^{(y)}_{i,1}),\\ e_{i,m} &= \Delta^{(y)}_{i,m-1} + \frac{l_{m-1}}{l_{m-1}+l_{m-2}}(\Delta^{(y)}_{i,m-1} - \Delta^{(y)}_{i,m-2}), \end{aligned} \quad (6)$$

then the interpolation function $Q^*_{i,j}(y)$ defined by (5) is C^2-continuous in $[c,d]$, and which satisfies

$$Q''(y_j) = \frac{2}{l_{j-1}+l_j}(\Delta^{(y)}_{i,j} - \Delta^{(y)}_{i,j-1}),\ j = 2,3,\cdots,m-1.$$

For each pair (i,j), $i = 1,2,\cdots,n-1$ and $j = 1,2,\cdots,m-1$, using the y-direction interpolation function $Q^*_{i,j}(y)$, define the bivariate rational Hermite interpolating function $Q_{i,j}(x,y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$Q_{i,j}(x,y) = \frac{(1-\theta)^3 Q^*_{i,j}(y) + \theta(1-\theta)^2\overline{V}_{i,j} + \theta^2(1-\theta)\overline{W}_{i,j} + \theta^3 Q^*_{i+1,j}(y)}{(1-\theta)^2 + \theta(1-\theta)\alpha_{i,j} + \theta^2}, \quad (7)$$

where

$$\overline{V}_{i,j} = (\alpha_{i,j}+1)Q^*_{i,j}(y) + h_i\overline{\phi}_{i,j}(y) + \overline{\varphi}_{i,j}(x,y),$$
$$\overline{W}_{i,j} = (\alpha_{i,j}+1)Q^*_{i+1,j}(y) - h_i\overline{\phi}_{i+1,j}(y) + \overline{\psi}_{i,j}(x,y),$$

with

$$\overline{\phi}_{i,s}(y) = (1-\eta)^3(1+4\eta+9\eta^2)d_{r,j} + \eta^3(6-8\eta+3\eta^2)d_{r,j+1},\ r = i, i+1,$$
$$\overline{\varphi}_{i,j}(x,y) = (\theta-\theta(1-\theta)(\alpha_{i,j}+1))(Q^*_{i+1,j}(y) - Q^*_{i,j}(y) - h_i\overline{\phi}_{i,j}(y)),$$
$$\overline{\psi}_{i,j}(x,y) = (1-\theta-\theta(1-\theta)(\alpha_{i,j}+1))(Q^*_{i,j}(y) - Q^*_{i+1,j}(y) + h_i\overline{\phi}_{i+1,j}(y)),$$

and $\alpha_{i,j} > 0$. The interpolation function $Q_{i,j}(x,y)$ defined by (7) satisfies

$$Q_{i,j}(x_r,y_s) = f(x_r,y_s),\ \frac{\partial Q_{i,j}(x_r,y_s)}{\partial x} = d_{r,s},\ \frac{\partial Q_{i,j}(x_r,y_s)}{\partial y} = e_{r,s},\quad r = i, i+1,\ s = j, j+1.$$

The interpolating functions $P_{i,j}(x,y)$ defined by (4) and $Q_{i,j}(x,y)$ defined by (7) satisfy the same interpolating data, but they are not the same interpolation functions. In the following, unless pointed out specifically, the bivariate rational interpolation based on function values means they are defined by (4).

3 C^2 continuity of the interpolant

For the C^2 continuity of the interpolation function $P_{i,j}(x,y)$ defined by (4), we have the following theorem.

Theorem 1. *If the knots are equally spaced for variable x, namely, $h_i = (b-a)/n$, a sufficient condition for the interpolation function $P_{i,j}(x,y), i = 1,2,\cdots,n-1; j = 1,2,\cdots,m-1$, to be C^2 in the whole interpolating region $[x_1,x_n;y_1,y_m]$ is that the parameters $\beta_{i,j}$ =constant for each $j \in \{1,2,\cdots,m-1\}$ and all $i = 1,2,\cdots,n-1$, no matter what the parameters $\alpha_{i,j}$ might be.*

Proof. When the conditions of the theorem are satisfied, we can easily obtain that the interpolation function $P_{i,j}(x,y)$ is C^1 continuous in the interpolating region $[x_1,x_n;y_1,y_m]$ (see [13]).

Furthermore, since the rational interpolation function $P^*_{i,j}(x)$ defined by (1) is C^2 continuous in $[x_1,x_n]$, it is easy to show that the bivariate interpolation function $P_{i,j}(x,y)$ has continuous second-order partial derivatives $\frac{\partial P^2_{i,j}(x,y)}{\partial x^2}$, $\frac{\partial P^2_{i,j}(x,y)}{\partial x\partial y}$ and $\frac{\partial P^2_{i,j}(x,y)}{\partial y^2}$ in the interpolating region $[x_1,x_n;y_1,y_m]$ except $\frac{\partial P^2_{i,j}(x,y)}{\partial x^2}$ for every $y \in [y_j,y_{j+1}], j = 1,2,\cdots,m-1$, at the points $(x_i,y), i = 2,3,\cdots,n-1$, and $\frac{\partial P^2_{i,j}(x,y)}{\partial y^2}$ for every $x \in [x_i,x_{i+1}], i = 1,2,\cdots,n-1$, at the points $(x,y_j), j = 2,3,\cdots,m-1$. Thus, it is sufficient for $P_{i,j}(x,y) \in C^2$ in the whole interpolating region $[x_1,x_n;y_1,y_m]$ if $\frac{\partial P^2_{i,j}(x_i^+,y)}{\partial x^2} = \frac{\partial P^2_{i,j}(x_i^-,y)}{\partial x^2}$ and $\frac{\partial P^2_{i,j}(x,y_j^+)}{\partial y^2} = \frac{\partial P^2_{i,j}(x,y_j^-)}{\partial y^2}$ hold.

From (4), it can be derived

$$\begin{aligned}\frac{\partial P^2_{i,j}(x,y)}{\partial x^2} = \frac{1}{q_{i,j}(y)}[&(1-\eta)^3(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})\frac{d^2P^*_{i,j}(x)}{dx^2}\\ &+\eta^3(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})\frac{d^2P^*_{i,j+1}(x)}{dx^2}\\ &+l_j\eta(1-\eta)^3(1+\eta+\eta\beta_{i,j})\frac{d^2\phi_{i,j}(x)}{dx^2}\\ &-l_j\eta^3(1-\eta)(2-\eta+(1-\eta)\beta_{i,j})\frac{d^2\phi_{i,j+1}(x)}{dx^2}].\end{aligned} \tag{8}$$

Since

$$\frac{d^2\phi_{i,s}(x_i^+)}{dx^2}=0,\ \frac{d^2\phi_{i,s}(x_{i+1}^-)}{dx^2}=0,\ s=j,j+1,$$

and the interpolation function $P_{i,j}^*(x)$ is C^2 continuous, it is easy to see from (8) that $\frac{\partial P_{i,j}^2(x,y)}{\partial x^2}$ is continuous at the points $(x_i,y), i=2,3,\cdots,n-1$ for every $y\in[y_j,y_{j+1}], j=1,2,\cdots,m-1$, when $\beta_{i-1,j}=\beta_{i,j}$ and $h_{i-1}=h_i$. The proof of the case which $\frac{\partial P_{i,j}^2(x,y)}{\partial y^2}$ is continuous at the points (x,y_j) is similar. This completes the proof. □

Similarly, for the interpolation function $Q_{i,j}(x,y)$ defined by (7), the following theorem can be derived.

Theorem 2. *If the knots are equally spaced for variable y, namely, $l_j=(b-a)/m$, a sufficient condition for the interpolating function $Q_{i,j}(x,y), i=1,2,\cdots,n-1; j=1,2,\cdots,m-1$, to be C^2 in the whole interpolating region $[x_1,x_n;y_1,y_m]$ is that the parameters $\alpha_{i,j}$ =constant for each $i\in\{1,2,\cdots,n-1\}$ and all $j=1,2,\cdots,m-1$, no matter what the parameters $\beta_{i,j}$ might be.*

4 Basis of the interpolant

From (1) and (4), the interpolation function $P_{i,j}(x,y)$ can be written as follows:

$$P_{i,j}(x,y)=\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}[a_{r,s}(\theta,\eta)f_{r,s}+b_{r,s}(\theta,\eta)h_i d_{r,s}+c_{r,s}(\theta,\eta)l_j e_{r,s}], \tag{9}$$

where

$$a_{i,j}(\theta,\eta)=\frac{(1-\theta)^2(1-\eta)^3(1+\theta(1+\theta-2\theta^2)\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$a_{i,j+1}(\theta,\eta)=\frac{(1-\theta)^2\eta^3(1+\theta(1+\theta-2\theta^2)\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$a_{i+1,j}(\theta,\eta)=\frac{\theta^2(1-\eta)^3(1+\theta(3-5\theta+2\theta^2)\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$a_{i+1,j+1}(\theta,\eta)=\frac{\theta^2\eta^3(1+\theta(3-5\theta+2\theta^2)\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i,j}(\theta,\eta)=\frac{\theta(1-\theta)^3(1-\eta)^3(1+\theta\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i,j+1}(\theta,\eta)=\frac{\theta(1-\theta)^3\eta^3(1+\theta\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i+1,j}(\theta,\eta)=-\frac{\theta^3(1-\theta)(1-\eta)^3(1+(1-\theta)\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i+1,j+1}(\theta,\eta)=-\frac{\theta^3(1-\theta)\eta^3(1+(1-\theta)\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$c_{i,j}(\theta,\eta)=\frac{(1-\theta)^3\eta(1-\eta)^3(1+4\theta+9\theta^2)(1+\eta+\eta\beta_{i,j})}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2},$$

$$c_{i,j+1}(\theta,\eta)=-\frac{(1-\theta)^3\eta^3(1-\eta)(1+4\theta+9\theta^2)(2-\eta+(1-\eta)\beta_{i,j})}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2},$$

$$c_{i+1,j}(\theta,\eta)=\frac{\theta^3\eta(1-\eta)^3(6-8\theta+3\theta^2)(1+\eta+\eta\beta_{i,j})}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2},$$
$$c_{i+1,j+1}(\theta,\eta)=-\frac{\theta^3\eta^3(1-\eta)(6-8\theta+3\theta^2)(2-\eta+(1-\eta)\beta_{i,j})}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2}.$$

The teams $a_{r,s}(\theta,\eta), b_{r,s}(\theta,\eta), c_{r,s}(\theta,\eta), r=i,i+1, s=j,j+1$ are called the basis of the interpolant defined by (4), which satisfy

$$\begin{aligned}
&a_{i,j}(\theta,\eta)+a_{i,j+1}(\theta,\eta)+a_{i+1,j}(\theta,\eta)+a_{i+1,j+1s}(\theta,\eta)=1,\\
&b_{i,j}(\theta,\eta)+b_{i,j+1}(\theta,\eta)-b_{i+1,j}(\theta,\eta)-b_{i+1,j+1s}(\theta,\eta)=\theta(1-\theta),\\
&c_{i,j}(\theta,\eta)-c_{i,j+1}(\theta,\eta)+c_{i+1,j}(\theta,\eta)-c_{i+1,j+1s}(\theta,\eta)\\
&\quad=\frac{\eta(1-\eta)(1-\eta+\eta^2+\eta(1-\eta)\beta_{i,j})(1+\theta-10\theta^3+15\theta^4-6\theta^5)}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2}.
\end{aligned}\tag{10}$$

Denote

$$\begin{aligned}
M&=\max\{|f_{r,s}|,r=i,i+1;s=j,j+1\},\\
Q_1&=\max\{h_i|d_{r,s}|,r=i,i+1;s=j,j+1\},\\
Q_2&=\max\{l_j|e_{r,s}|,r=i,i+1;s=j,j+1\}.
\end{aligned}$$

For the given data, the values of the piecewise bivariate interpolation function $P_{i,j}(x,y)$ defined by (4) are bounded in the interpolating interval as described by the following theorem.

Theorem 3. *Let $P_{i,j}(x,y)$ is the interpolation function over $[x_i,x_{i+1};y_j,y_{j+1}]$ defined by (4). No matter what positive number the parameters $\alpha_{i,s}$ $(s=j,j+1)$ and $\beta_{i,j}$ take, the values of $P_{i,j}(x,y)$ in $[x_i,x_{i+1};y_j,y_{j+1}]$ satisfy*

$$|P_{i,j}(x,y)|\le M+\frac{1}{4}Q_1+0.430029Q_2.$$

Proof. From (9) and (10), it is easy to derive that

$$\begin{aligned}
|P_{i,j}(x,y)|&\le M\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|a_{r,s}(\theta,\eta)|+Q_1\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|b_{r,s}(\theta,\eta)|+Q_2\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|c_{r,s}(\theta,\eta)|\\
&\le M+\theta(1-\theta)Q_1+Q_2\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|c_{r,s}(\theta,\eta)|\le M+\frac{1}{4}Q_1+Q_2\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|c_{r,s}(\theta,\eta)|.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|c_{r,s}(\theta,\eta)|&=\frac{\eta(1-\eta)(1-\eta+\eta^2+\eta(1-\eta)\beta_{i,j})(1+\theta-10\theta^3+15\theta^4-6\theta^5)}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2}\\
&\le(1+\theta-10\theta^3+15\theta^4-6\theta^5)\frac{\eta(1-\eta)(1-\eta+\eta^2)}{1-2\eta+2\eta^2},
\end{aligned}$$

and

$$\begin{aligned}
\max_{\theta\in[0,1]}(1+\theta-10\theta^3+15\theta^4-6\theta^5)&=1.14675,\\
\max_{\eta\in[0,1]}\frac{\eta(1-\eta)(1-\eta+\eta^2)}{1-2\eta+2\eta^2}&=\frac{3}{8},
\end{aligned}$$

thus, the proof is completed. □

5 Error estimates of the interpolant

Note that the interpolant defined by (4) is local, without loss of generality, it is only necessary to consider the interpolating region $[x_i, x_{i+1}; y_j, y_{j+1}]$ in order to process its error estimates. Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be the interpolation function defined by (4) over $[x_i, x_{i+1}; y_j, y_{j+1}]$.

Denoting

$$\|\frac{\partial f}{\partial y}\| = \max_{(x,y)\in D} |\frac{\partial f(x,y)}{\partial y}|, \quad \|\frac{\partial P}{\partial y}\| = \max_{(x,y)\in D} |\frac{\partial P_{i,j}(x,y)}{\partial y}|,$$

where $D = [x_i, x_{i+1}; y_j, y_{j+1}]$. By the Taylor expansion and the Peano-Kernel Theorem [19] gives the following:

$$\begin{aligned} |f(x,y) - P_{i,j}(x,y)| &\le |f(x,y) - f(x,y_j)| + |P_{i,j}(x,y_j) - P_{i,j}(x,y)| + |f(x,y_j) - P_{i,j}(x,y_j)| \\ &\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + |\int_{x_i}^{x_i+1} \frac{\partial^2 f(\tau, y_j)}{\partial x^2} R_x[(x-\tau)_+]d\tau| \\ &\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + \|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| \int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+]|d\tau, \end{aligned} \tag{11}$$

where $\|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| = \max_{x\in[x_i,x_{i+1}]} |\frac{\partial^2 f(x,y_j)}{\partial x^2}|$, and

$$\begin{aligned} R_x[(x-\tau)_+] &= \begin{cases} (x-\tau) - a_{i+1,j}(\theta,0)(x_{i+1}-\tau) - b_{i+1,j}(\theta,0)h_i, x_i < \tau < x; \\ -a_{i+1,j}(\theta,0)(x_{i+1}-\tau) - b_{i+1,j}(\theta,0)h_i, x < \tau < x_{i+1}, \end{cases} \\ &= \begin{cases} r(\tau), & x_i < \tau < x; \\ s(\tau), & x < \tau < x_{i+1}. \end{cases} \end{aligned}$$

Thus, by simple integral calculation, it can be derived that

$$\int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+]|d\tau == h_i^2 B(\theta, \alpha_{i,j}), \tag{12}$$

where

$$B(\theta, \alpha_{i,j}) = \frac{\theta^2(1-\theta)^2(1+2\theta(1-\theta)\alpha_{i,j})^2}{(1+\theta(3-5\theta+2\theta^2)\alpha_{i,j})(1+\theta(1+\theta-2\theta^2)\alpha_{i,j})}. \tag{13}$$

For the fixed $\alpha_{i,j}$, let

$$B_{i,j}^{(x)} = \max_{\theta\in[0,1]} B(\theta, \alpha_{i,j}). \tag{14}$$

This leads to the following theorem.

Theorem 4. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (4) in $[x_i, x_{i+1}; y_j, y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s}, \beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y) - P_{i,j}(x,y)| \le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + h_i^2 \|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| B_{i,j}^{(x)},$$

where $B_{i,j}^{(x)}$ defined by (14).

Similarly, denoting $\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\| = \max_{x\in[x_i,x_{i+1}]} |\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}|$, then the following theorem holds.

Theorem 5. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolation function defined by (4) in $[x_i, x_{i+1}; y_j, y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s}, \beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y) - P_{i,j}(x,y)| \le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + h_i^2 \|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\| B_{i,j+1}^{(x)},$$

where $B_{i,j+1}^{(x)} = \max_{\theta\in[0,1]} B(\theta, \alpha_{i,j+1})$, and $B(\theta, \alpha_{i,j})$ defined by (13).

Furthermore, for $B_{i,s}^{(x)}$, we can conclude the following theorem.

Theorem 6. *For any positive parameters $\alpha_{i,s}, s = j, j+1$, $B_{i,s}^{(x)}$ are bounded, and*

$$\frac{1}{16} \le B_{i,s}^{(x)} \le \frac{3}{16}.$$

6 Numerical examples

For the bivariate rational spline interpolant defined by (4) , since there are three shape parameters in the interpolation function, when the parameters vary, the interpolation function can be changed for the unchanged interpolating data. Thus, the shape of the interpolating surface can be modified by selecting suitable shape parameters according to the control need. Also, the interpolator can give a good approximation to the interpolated function. In this section, in order to show the effectiveness which the interpolator defined by (4) approximate a function, and to describe that the shape of the interpolating surface can be modified by free shape parameters, some examples will be given.

Example 1. Let the interpolated function be $f(x,y) = \sin(x^2+y)$, $(x,y) \in [0, 0.8; 0, 0.8]$, and let $h_i = l_j = 0.2$, then $x_i = 0.2(i-1)$, $y_j = 0.2(j-1)$, $i,j = 1,2,3,4,5$. Also let $\alpha_{i,j} = 0.3 + 0.2i + 0.1j$, $\beta_{i,j} = 0.6 + 0.1j$. The partial derivative values $d_{i,j}$ at the knots (x_i, y_j) $(i,j = 1,2,3,4,5)$ are conducted by using (2) and (3). The partial derivative values $e_{i,j}$ at the knots (x_i, y_j) $(i,j = 1,2,3,4,5)$ are given by (6).

Figure 1 shows the graph of the interpolated function $f(x,y)$. Figure 2 shows the graph of the interpolation function $P(x,y)$ defined by (4). Figure 3 shows the surface of the error $f(x,y) - P(x,y)$. From Figure 3, it is evident that the error of the interpolation is smaller than $\pm 8 \times 10^{-3}$, this means the interpolator defined by (4) gives a good approximation to the interpolated function.

Example 2. Let $\Omega : [0, 1.5; 0, 1.5]$ be the plane region, and the interpolation data are given in Table 1. The interpolation function $P_{i,j}(x,y)$ defined by (4) can be constructed in $[0, 1.5; 0, 1.5]$ for the given positive parameters $\alpha_{i,j}, \alpha_{i,j+1}$ and $\beta_{i,j}$. In order to show that the shape of the interpolating surface can be modified by selecting suitable parameters according to control need, we consider the value control of the interpolating surface. Assume $\alpha_{i,j} = \alpha_{i,j+1}$ and $\beta_{i,j}$ =constant for each $j \in \{1,2,3\}$ and all $i = 1,2,3$, then the interpolant $P_{i,j}(x,y)$ defined by (4) is C^2 in interpolating region $[0, 1.5; 0, 1.5]$. Without loss of generality, we only consider a subinterval $[0.5, 1; 0.5, 1]$.

Let $\alpha_{i,j} = 0.5, \beta_{i,j} = 0.8$. The partial derivative values $d_{i,j}$ and $e_{i,j}$ at the knots (x_i, y_j) are conducted by using (2) and (6), respectively. For the given interpolation data, denote the interpolation function by $P_1(x,y)$ which is defined over $[0.5, 1; 0.5, 1]$. Figure 4 shows the graph of the bivariate rational interpolating surface $P_1(x,y)$ with the parameters $\alpha_{i,j} = 0.5, \beta_{i,j} = 0.8$. It is

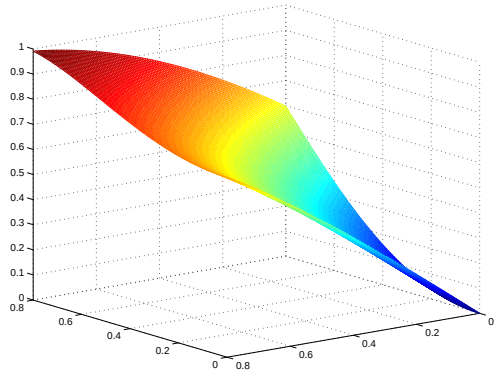


Figure 1: Graph of surface $f(x, y)$.

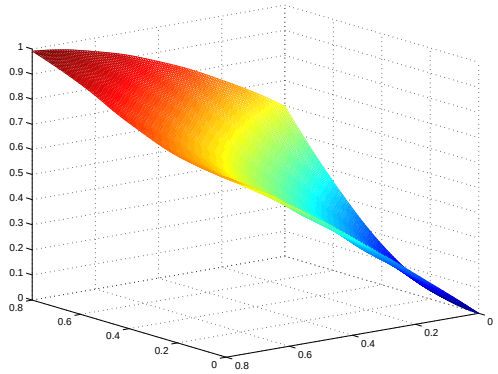


Figure 2: Graph of surface $P(x, y)$.

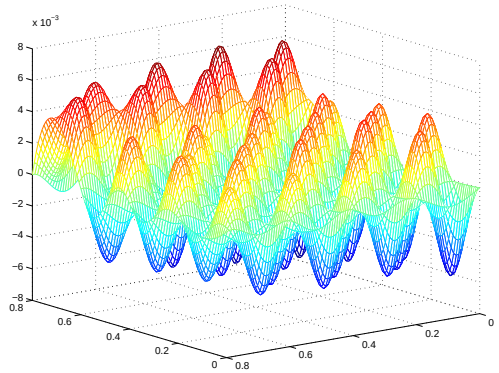


Figure 3: Graph of surface $f(x, y) - P(x, y)$.

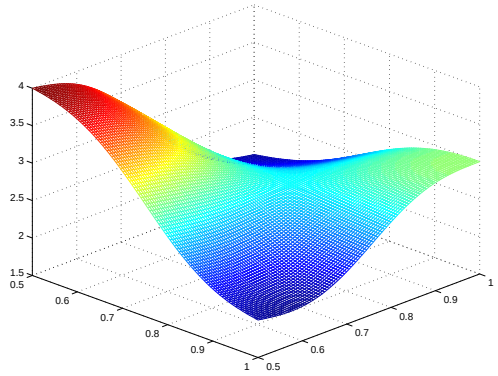


Figure 4: Graph of surface $P_1(x, y)$ with $\alpha_{i,j} = 0.5$, $\beta_{i,j} = 0.8$.

easy to compute that $P_1(0.875, 0.875) = 2.84818$. If the practical design requires $P(0.875, 0.875) = 2.8$, then $\alpha_{i,j} = 5$ and $\beta_{i,j} = 11.2921$ can be obtained. Denote the interpolation by $P_2(x, y)$. Figure 5 shows the graph of the surface $P_2(x, y)$ with the parameters $\alpha_{i,j} = 5, \beta_{i,j} = 11.2921$, and in this case $P_2(0.875, 0.875) = 2.8$.

Furthermore, if the practical design requires $P(0.875, 0.875) = 2.86$, then $\alpha_{i,j} = 0.112391$ and $\beta_{i,j} = 0.2$ can be derived. Denote the interpolation by $P_3(x, y)$. Figure 6 shows the graph of the surface $P_3(x, y)$ with the parameters $\alpha_{i,j} = 0.112391, \beta_{i,j} = 0.2$, and in this case $P_3(0.875, 0.875) = 2.86$.

Remark 2. Each interpolant of the family of the C^2 bivariate rational spline interpolation defined by (4) is identified uniquely by the values of the shape parameters $\alpha_{i,j}$ and $\beta_{i,j}$. For different shape

Table 1: Set of the interpolating data.

(x_i, y_j)	(0,0)	(0,0.5)	(0,1)	(0,1.5)	(0.5,0)	(0.5,0.5)	(0.5,1)	(0.5,1.5)
$f_{i,j}$	3	3	2	4	3	4	2	3
(x_i, y_j)	(1,0)	(1,0.5)	(1,1)	(1,1.5)	(1.5,0)	(1.5,0.5)	(1.5,1)	(1.5,1.5)
$f_{i,j}$	4	2	3	2	3	3	2	4

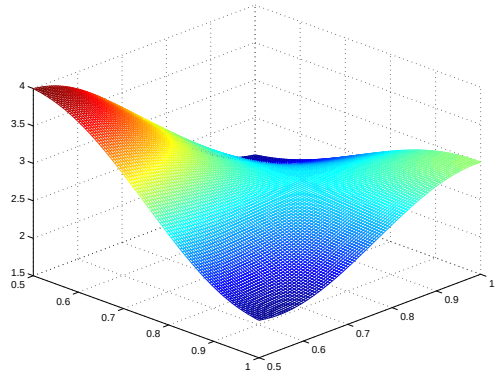


Figure 5: Graph of surface $P_2(x, y)$ with $\alpha_{i,j} = 5$, $\beta_{i,j} = 11.2921$.

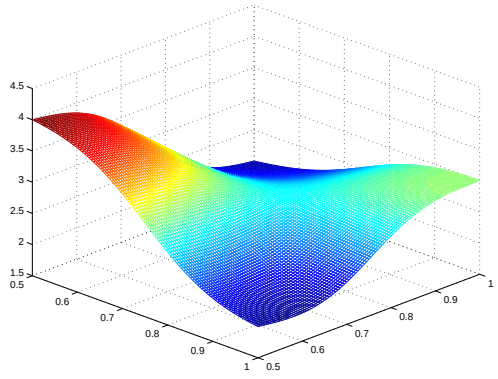


Figure 6: Graph of surface $P_3(x, y)$ with $\alpha_{i,j} = 0.112391$, $\beta_{i,j} = 0.2$.

parameters, from Figures 4 to 6, we can catch sight of some minor changes of the surfaces in shape. It means that the shape modification of interpolating surface can be achieved by selecting suitable shape parameters according to needs of practical design.

7 Concluding remarks

Generally speaking, generating a C^2 bivariate interpolation is a very difficult task, it requires up to second-order partial derivatives values of the interpolated function. Usual NURBS method is the most popular technology in modern surface modeling, however, preset weights are needed to generate a C^2 rational surface, the given points play the role of the control points.

This paper develops a new interpolating approach for construction of C^2 bivariate rational spline interpolants only based on the values of a function. There are two schemes to generate this type of interpolation function, one is interpolating from the x-direction first, another is from the y-direction first. In the interpolant beginning from the x-direction first, there are three positive shape parameters: $\alpha_{i,j}, \alpha_{i,j+1}$ and $\beta_{i,j}$; In the interpolant beginning from the y-direction first, there are also three positive shape parameters: $\beta_{i,j}, \beta_{i+1,j}$ and $\alpha_{i,j}$. Generally, when the interpolating data are given, because of the uniqueness of the interpolation function, the shape of interpolating surface is fixed. However, note that there are some free shape parameters in (4) and (7), the shape of the interpolating surfaces can be modified by selecting suitable shape parameters for the unchanged interpolating data according to the control need, and numerical examples illustrate this case. It means that the uniqueness of the interpolating surfaces for the given interpolating data becomes that of for the given interpolating data and the shape parameters.

For each pitch of the interpolating surface, the value of the interpolation function depends on the interpolating data. Theorem 3 shows that the values of the interpolant is bounded in whole interpolating region, it means that the interpolant is stable for the positive shape parameters. Also, numerical example shows that the interpolator can give a good approximation to the original function.

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Brunn-Minkowski type inequalities for width-integrals of index i

Yibin Feng[1] and Shanhe Wu[2,*]

1 School of Mathematics and Statistics, Hexi University, Zhangye, 734000, China.
E-mail: fengyibin001@163.com
2 Department of Mathematics, Longyan University, Longyan, 364012, China.
E-mail: shanhewu@163.com
*Corresponding author

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Abstract

In this paper, we investigate Brunn-Minkowski type inequalities for width-integrals of index i related to the Blaschke Minkowski homomorphism. Some inequalities similar to Lutwak's inequality are established.

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1 Introduction and Main Results

Let $\mathcal{K}^n$ be the set of convex bodies, which is a compact, convex subsets with nonempty interiors in Euclidean space $\mathbb{R}^n$. S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. We denote by $V(K)$ the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we denote its volume by $\omega_n = V(B)$.

The support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, of a convex body $K \in \mathcal{K}^n$ is defined by (see [5, 14])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The study of width-integral has a long history, and has received considerable attention. It was first considered by Blaschke [2], and a book by Hadwiger [6] detailed this problem. In 1975, width-integral was extended to width-integrals of index i by Lutwak in the reference [10]. For the more results associated with width-integrals of index i, we refer the interested reader to [8, 11-13].

Define by $\overline{B}_i(K)$, $i \in \mathbb{R}$, the width-integrals of index i of $K \in \mathcal{K}^n$ (see [10])

$$\overline{B}_i(K) = \frac{1}{n}\int_{S^{n-1}} \bar{b}(K,u)^{n-i} dS(u), \tag{1.1}$$

where $dS(u)$ and $\bar{b}(K,u)$ denote the $(n-1)$-dimensional volume element on S^{n-1} and the half width, $\bar{b}(K,u) = \frac{1}{2}(h(K,u) + h(K,-u))$, of K in the direction u, respectively. If we take $i = 0$ in (1.1), $\overline{B}_i(K)$ is just width-integral $\overline{B}(K)$.

The map, $\overline{B}_i : \mathcal{K}^n \to \mathbb{R}$, is continuous, positive, invariant under motion and homogeneous of degree $n - i$. If there exists a constant $\lambda > 0$ such that $\bar{b}(K,u) = \lambda \bar{b}(L,u)$ for all $u \in S^{n-1}$, then we call K and L with similar width.

Lutwak [10] showed that if $K, L \in \mathcal{K}^n$, then for $u \in S^{n-1}$

$$\bar{b}(K+L,u) = \bar{b}(K,u) + \bar{b}(L,u). \tag{1.2}$$

From the above formula (1.2), Lutwak [10] established the following Brunn-Minkowski inequality for width-integrals of index i.

Theorem 1.A *If $K, L \in \mathcal{K}^n$ and $i \le n-1$, then*

$$\overline{B}_i(K+L)^{\frac{1}{n-i}} \le \overline{B}_i(K)^{\frac{1}{n-i}} + \overline{B}_i(L)^{\frac{1}{n-i}},$$

with equality if and only if K and L have similar width.

The main results of the present paper are the following:

We first establish a Brunn-Minkowski type inequality for width-integrals of index i similar to Theorem 1.A.

Theorem 1.1 *If $K, L \in \mathcal{K}^n$, then for $i \le n-1$*

$$\overline{B}_i(\Phi(K\hat{+}L))^{\frac{1}{n-i}} \le \overline{B}_i(\Phi K)^{\frac{1}{n-i}} + \overline{B}_i(\Phi L)^{\frac{1}{n-i}}, \tag{1.3}$$

with equality if and only if ΦK and ΦL are homothetic.

Here $K\hat{+}L$ is the Blaschke sum of $K, L \in \mathcal{K}^n$ (see (2.7)) and Φ denotes Blaschke Minkowski homomorphism (see Section 2.2). In fact, the more general result than Theorem 1.1 will be obtained at the beginning of Section 3.

Moreover, we get the following Brunn-Minkowski type inequality of width-integrals of index i based on the Blaschke Minkowski homomorphism.

Theorem 1.2 *If $K_1, K_2, \cdots, K_{n-2}, L_1, L_2 \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then for $i \leq n-1$*

$$\overline{B}_i(\Phi(C, L_1 + L_2))^{\frac{1}{n-i}} \leq \overline{B}_i(\Phi(C, L_1))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C, L_2))^{\frac{1}{n-i}}; \tag{1.4}$$

for $i > n$

$$\overline{B}_i(\Phi(C, L_1 + L_2))^{\frac{1}{n-i}} \geq \overline{B}_i(\Phi(C, L_1))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C, L_2))^{\frac{1}{n-i}}, \tag{1.5}$$

with equality in every inequality if and only if $\Phi(C, L_1)$ and $\Phi(C, L_2)$ are homothetic.

Here, we should note that Φ denotes mixed Blaschke-Minkowski homomorphism (see Section 2.2 for precise definition), and $L_1 + L_2$ is the Minkowski sum of $L_1, L_2 \in \mathcal{K}^n$ (see (2.5)).

Finally, we show the following result which is the more general form of the Brunn-Minkowski type inequality of width-integrals of index i.

Theorem 1.3 *If $K_1, K_2, \cdots, K_{n-2}, K, L \in \mathcal{K}^n$, let $C = (K_1, K_2, \cdots, K_{n-2})$ and $i, j \in \mathbb{R}$, then for $i \leq n-1 \leq j \leq n$ and $i \neq j$*

$$\left(\frac{\overline{B}_i(\Phi(C, (K +_{-1} L)^*))}{\overline{B}_j(\Phi(C, (K +_{-1} L)^*))}\right)^{\frac{1}{j-i}} \leq \left(\frac{\overline{B}_i(\Phi(C, K^*))}{\overline{B}_j(\Phi(C, K^*))}\right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi(C, L^*))}{\overline{B}_j(\Phi(C, L^*))}\right)^{\frac{1}{j-i}}; \tag{1.6}$$

for $n-1 \leq i \leq n \leq j$ and $i \neq j$

$$\left(\frac{\overline{B}_i(\Phi(C, (K +_{-1} L)^*))}{\overline{B}_j(\Phi(C, (K +_{-1} L)^*))}\right)^{\frac{1}{j-i}} \geq \left(\frac{\overline{B}_i(\Phi(C, K^*))}{\overline{B}_j(\Phi(C, K^*))}\right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi(C, L^*))}{\overline{B}_j(\Phi(C, L^*))}\right)^{\frac{1}{j-i}}, \tag{1.7}$$

with equality in every inequality if and only if $\Phi(C, K^*)$ and $\Phi(C, L^*)$ are homothetic.

Here $K +_{-1} L$ denotes the harmonic radial sum of $K, L \in \mathcal{S}^n$ (see (2.4)) and K^* denotes the polar of K (see (2.2)).

2 Preliminaries

2.1 Radial function and polar of convex bodies

For a compact star-shaped set K in $\mathbb{R}^n$, let $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ denote the radial function of K (see[5, 14]), that is

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}. \tag{2.1}$$

If ρ_K is positive and continuous, then K is said to be a star body, and $\mathcal{S}^n$ denotes the set of star bodies in Euclidean space $\mathbb{R}^n$.

For $K \in \mathcal{K}^n$, its polar body is defined by (see[5, 14])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.2}$$

Obviously, it follows from (2.2) that $(K^*)^* = K$ and

$$h_{K^*} = \rho_K^{-1}, \quad \rho_{K^*} = h_K^{-1}. \tag{2.3}$$

For $\lambda, \mu \geq 0$ (not both zero), define by $\lambda \star K +_{-1} \mu \star L$ the harmonic radial combination of $K, L \in \mathcal{S}^n$ (see [4]). Namely,

$$\rho^{-1}(\lambda \star K +_{-1} \mu \star L, \cdot) = \lambda \rho^{-1}(K, \cdot) + \mu \rho^{-1}(L, \cdot). \tag{2.4}$$

For $\lambda, \mu \geq 0$ (not both zero), define by $\lambda \cdot K + \mu \cdot L$ the Minkowski linear combination of $K, L \in \mathcal{K}^n$ (see [5, 14]), Namely,

$$h(\lambda \cdot K + \mu \cdot L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot). \tag{2.5}$$

Combining (2.3), (2.4) with (2.5), we obtain that for $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$ (not both zero)

$$(\lambda \star K +_{-1} \mu \star L)^* = \lambda \cdot K^* + \mu \cdot L^*. \tag{2.6}$$

2.2 Blaschke Minkowski homomorphism

For $\lambda, \mu \geq 0$ (not both zero), define by $\lambda \circ K \hat{+} \mu \circ L$ the Blaschke addition of $K, L \in \mathcal{K}^n$ such that (see [5, 14])

$$S(\lambda \circ K \hat{+} \mu \circ L, \cdot) = \lambda S(K, \cdot) + \mu S(L, \cdot), \tag{2.7}$$

where $S(K, \cdot)$ denotes the surface area measure of K.

Schuster [15] introduced the definition of Blaschke Minkowski homomorphism as follows: A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called Blaschke Minkowski homomorphism if it satisfies the following conditions

(a) Φ is continuous.

(b) Φ is Blaschke Minkowski additive, i.e., for all $K, L \in \mathcal{K}^n$

$$\Phi(K \hat{+} L) = \Phi K + \Phi L. \tag{2.8}$$

(c) Φ intertwines rotation, i.e., for all $K \in \mathcal{K}^n$ and $\vartheta \in SO(n)$

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

Here $SO(n)$ is the group of rotation in n dimensions.

The following result is a direct extension for the Blaschke Minkowski homomorphism which is said to be the mixed Blaschke Minkowski homomorphism.

Theorem 2.A*([15]) There is a continuous operator*

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments such that for $K_1, \cdots, K_m \in \mathcal{K}^n$ *and* $\lambda_1, \cdots, \lambda_m \geq 0$,

$$\Phi(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \cdots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \cdots, K_{i_{n-1}}),$$

where the operator $\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ is called mixed Blaschke Minkowski homomorphism.

Further, the author of [15] established the following properties for the mixed Blaschke Minkowski homomorphism.

(i) $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \to \mathcal{K}^n$ is continuous and symmetric with respect to origin.

(ii) If $K, L, K_1, K_2, \cdots, K_{n-2} \in \mathcal{K}^n$, $\lambda, \mu \geq 0$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then

$$\Phi(C, \lambda \cdot K + \mu \cdot L) = \lambda \Phi(C, K) + \mu \Phi(C, L). \tag{2.9}$$

3 The Proofs of Main Results

Here, we first establish Theorem 3.1 which is the more general version of Theorem 1.1. Next, we will prove Theorems 1.2 and 1.3.

Theorem 3.1 *If* $K, L \in \mathcal{K}^n$ *and* $i, j \in \mathbb{R}$, *then for* $i \leq n-1 \leq j \leq n$ *and* $i \neq j$

$$\left(\frac{\overline{B}_i(\Phi(K \hat{+} L))}{\overline{B}_j(\Phi(K \hat{+} L))}\right)^{\frac{1}{j-i}} \leq \left(\frac{\overline{B}_i(\Phi K)}{\overline{B}_j(\Phi K)}\right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi L)}{\overline{B}_j(\Phi L)}\right)^{\frac{1}{j-i}}; \tag{3.1}$$

for $n-1 \leq i \leq n \leq j$ *and* $i \neq j$

$$\left(\frac{\overline{B}_i(\Phi(K \hat{+} L))}{\overline{B}_j(\Phi(K \hat{+} L))}\right)^{\frac{1}{j-i}} \geq \left(\frac{\overline{B}_i(\Phi K)}{\overline{B}_j(\Phi K)}\right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi L)}{\overline{B}_j(\Phi L)}\right)^{\frac{1}{j-i}}, \tag{3.2}$$

with equality in every inequality if and only if ΦK and ΦL are homothetic.

The proof of Theorem 3.1 requires the following lemmas.

Lemma 3.1*([10]) If K is a convex body in $\mathbb{R}^n$, then*

$$\overline{B}_{2n}(K) \leq V(K^*), \tag{3.3}$$

with equality if and only if K is origin-symmetric.

Beckenbach-Dresher inequality [3] is an extension of Beckenbach's inequality [1] which is proved by Dresher through the method of moment-space techniques.

Lemma 3.2*(The Beckenbach-Dresher inequality) If $p \geq 1 \geq r \geq 0$, $p \neq r$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.4}$$

with equality if and only if the functions f and g are positively proportional. Here $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

The inverse Beckenbach-Dresher inequality was established in [9].

Lemma 3.3*(The inverse Beckenbach-Dresher inequality) If $1 \geq p \geq 0 \geq r$, $p \neq r$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \geq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.5}$$

with equality if and only if the functions f and g are positively proportional.

Proof of Theorem 3.1. According to (1.1) and (2.8), we obtain that for $p \geq 1 \geq r \geq 0$

$$\begin{aligned}\overline{B}_{n-p}(\Phi(K\hat{+}L)) &= \frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(K\hat{+}L), u)^p dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi K, u) + \bar{b}(\Phi L, u))^p dS(u).\end{aligned} \tag{3.6}$$

Similarly,

$$\overline{B}_{n-r}(\Phi(K\hat{+}L)) = \frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi K, u) + \bar{b}(\Phi L, u))^r dS(u). \tag{3.7}$$

From Lemma 3.2, (3.6) and (3.7), this implies

$$\begin{aligned}
&\left(\frac{\overline{B}_{n-p}(\Phi(K\hat{+}L))}{\overline{B}_{n-r}(\Phi(K\hat{+}L))}\right)^{\frac{1}{p-r}}\\
&= \left(\frac{\int_{S^{n-1}}(\bar{b}(\Phi K,u)+\bar{b}(\Phi L,u))^p dS(u)}{\int_{S^{n-1}}(\bar{b}(\Phi K,u)+\bar{b}(\Phi L,u))^r dS(u)}\right)^{\frac{1}{p-r}}\\
&\le \left(\frac{\int_{S^{n-1}}\bar{b}(\Phi K,u)^p dS(u)}{\int_{S^{n-1}}\bar{b}(\Phi K,u)^r dS(u)}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{S^{n-1}}\bar{b}(\Phi L,u)^p dS(u)}{\int_{S^{n-1}}\bar{b}(\Phi L,u)^r dS(u)}\right)^{\frac{1}{p-r}}\\
&= \left(\frac{\overline{B}_{n-p}(\Phi K)}{\overline{B}_{n-r}(\Phi K)}\right)^{\frac{1}{p-r}} + \left(\frac{\overline{B}_{n-p}(\Phi L)}{\overline{B}_{n-r}(\Phi L)}\right)^{\frac{1}{p-r}}. \qquad (3.8)
\end{aligned}$$

Let $p = n-i$ and $r = n-j$. By $0 \le r \le 1 \le p$ and $p \ne r$, we obtain that $i \le n-1 \le j \le n$ and $i \ne j$. Suppose $p = n-i$ and $r = n-j$ in (3.8), this gives inequality (3.1). Similar to the above method, it follows from Lemma 3.3 that inequality (3.2).

The equality conditions of Lemmas 3.2 and 3.3 imply that equality holds in inequalities (3.1) and (3.2) if and only if $\bar{b}(\Phi K,u)$ and $\bar{b}(\Phi L,u)$ are positively proportional, namely, ΦK and ΦL have similar width. Since ΦK and ΦL are origin-symmetric, we have ΦK and ΦL are homothetic. Therefore, equality holds in every inequality if and only if ΦK and ΦL are homothetic.□

If $j = n$ in (3.1), then $\overline{B}_n(K) = \frac{1}{n}\int_{S^{n-1}} dS(u) = \omega_n$ is a constant. thus we get Theorem 1.1.

Let $i = 2n$ and $j = n$ in (3.2). Note that ΦK, ΦL and $\Phi(K\hat{+}L)$ are origin-symmetric. Thus Lemma 3.1 implies that inequality (3.2) has the following result.

Corollary 3.1 *If $K, L \in \mathcal{K}^n$, then*

$$V(\Phi^*(K\hat{+}L))^{-\frac{1}{n}} \ge V(\Phi^*K)^{-\frac{1}{n}} + V(\Phi^*L)^{-\frac{1}{n}},$$

with equality if and only if ΦK and ΦL are homothetic.

Proof of Theorem 1.2. We first prove inequality (1.4). From (1.1), (1.2), (2.9) and the Minkowski's integral inequality (see [7]), we obtain that for $i \le n-1$

$$\overline{B}_i(\Phi(C, L_1+L_2))^{\frac{1}{n-i}} = \left(\frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(C, L_1+L_2),u)^{n-i} dS(u)\right)^{\frac{1}{n-i}}$$

$$
\begin{aligned}
&= \left(\frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi(C,L_1),u)+\bar{b}(\Phi(C,L_2),u))^{n-i}dS(u)\right)^{\frac{1}{n-i}}\\
&\le \left(\frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(C,L_1),u)^{n-i}dS(u)\right)^{\frac{1}{n-i}}\\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(C,L_2),u)^{n-i}dS(u)\right)^{\frac{1}{n-i}}\\
&= \overline{B}_i(\Phi(C,L_1))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C,L_2))^{\frac{1}{n-i}}.
\end{aligned}
$$

This implies inequality (1.4). Similarly, it follows from the inverse Minkowski's integral inequality that inequality (1.5).

From the equality condition of Minkowski's integrals inequalities, it follows that equality holds in inequalities (1.4) and (1.5) if and only if $\Phi(C,L_1)$ and $\Phi(C,L_2)$ have similar width. Since $\Phi(C,L_1)$ and $\Phi(C,L_2)$ are origin-symmetric, we obtain that equality holds in every inequality if and only if $\Phi(C,L_1)$ and $\Phi(C,L_2)$ are homothetic.□

If $i=0$ in (1.4), then inequality (1.4) implies the following result.

Corollary 3.2 *If $K_1,K_2,\cdots,K_{n-2},L_1,L_2\in\mathcal{K}^n$, and let $C=(K_1,K_2,\cdots,K_{n-2})$, then*

$$\overline{B}(\Phi(C,L_1+L_2))^{\frac{1}{n}} \le \overline{B}(\Phi(C,L_1))^{\frac{1}{n}} + \overline{B}(\Phi(C,L_2))^{\frac{1}{n}},$$

with equality if and only if $\Phi(C,L_1)$ and $\Phi(C,L_2)$ are homothetic.

If we let $i=2n$ in (1.5), and note that $\Phi(C,L_1)$, $\Phi(C,L_2)$ and $\Phi(C,L_1+L_2)$ are origin-symmetric, then Lemma 3.1 implies that inequality (1.5) has the following result.

Corollary 3.3 *If $K_1,K_2,\cdots,K_{n-2},L_1,L_2\in\mathcal{K}^n$, and let $C=(K_1,K_2,\cdots,K_{n-2})$, then*

$$V(\Phi^*(C,L_1+L_2))^{-\frac{1}{n}} \ge V(\Phi^*(C,L_1))^{-\frac{1}{n}} + V(\Phi^*(C,L_2))^{-\frac{1}{n}},$$

with equality if and only if $\Phi(C,L_1)$ and $\Phi(C,L_2)$ are homothetic.

Proof of Theorem 1.3. Combining (1.1) with (2.6), we obtain that for $p\ge 1\ge r\ge 0$

$$
\begin{aligned}
&\overline{B}_{n-p}(\Phi(C,(K+_{-1}L)^*))\\
&= \frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(C,(K+_{-1}L)^*),u)^p dS(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(C,K^*+L^*),u)^p dS(u)
\end{aligned}
$$

$$= \frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi(C,K^*),u) + \bar{b}(\Phi(C,L^*),u))^p dS(u). \tag{3.9}$$

Similarly,

$$\overline{B}_{n-r}(\Phi(C,(K +_{-1} L)^*))$$
$$= \frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi(C,K^*),u) + \bar{b}(\Phi(C,L^*),u))^r dS(u). \tag{3.10}$$

By Lemma 3.2, (3.9) and (3.10), it follows that

$$\left(\frac{\overline{B}_{n-p}(\Phi(C,(K +_{-1} L)^*))}{\overline{B}_{n-r}(\Phi(C,(K +_{-1} L)^*))}\right)^{\frac{1}{p-r}}$$

$$= \left(\frac{\int_{S^{n-1}} (\bar{b}(\Phi(C,K^*),u) + \bar{b}(\Phi(C,L^*),u))^p dS(u)}{\int_{S^{n-1}} (\bar{b}(\Phi(C,K^*),u) + \bar{b}(\Phi(C,L^*),u))^r dS(u)}\right)^{\frac{1}{p-r}}$$

$$\leq \left(\frac{\int_{S^{n-1}} \bar{b}(\Phi(C,K^*),u)^p dS(u)}{\int_{S^{n-1}} \bar{b}(\Phi(C,K^*),u)^r dS(u)}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{S^{n-1}} \bar{b}(\Phi(C,L^*),u)^p dS(u)}{\int_{S^{n-1}} \bar{b}(\Phi(C,L^*),u)^r dS(u)}\right)^{\frac{1}{p-r}}$$

$$= \left(\frac{\overline{B}_{n-p}(\Phi(C,K^*))}{\overline{B}_{n-r}(\Phi(C,K^*))}\right)^{\frac{1}{p-r}} + \left(\frac{\overline{B}_{n-p}(\Phi(C,L^*))}{\overline{B}_{n-r}(\Phi(C,L^*))}\right)^{\frac{1}{p-r}}. \tag{3.11}$$

If $p = n-i$ and $r = n-j$ in (3.11), then $0 \leq r \leq 1 \leq p$ and $p \neq r \Rightarrow i \leq n-1 \leq j \leq n$ and $i \neq j$.

This implies that inequality (1.6) is given. Similar to the above method, Lemma 3.3 implies that inequality (1.7) holds.

The equality conditions of Lemmas 3.2 and 3.3 see that with equality in inequalities (1.6) and (1.7) if and only if $\bar{b}(\Phi(C,K^*),u)$ and $\bar{b}(\Phi(C,L^*),u)$ are positively proportional. Since $\Phi(C,K^*)$ and $\Phi(C,L^*)$ are origin-symmetric, $\Phi(C,K^*)$ and $\Phi(C,L^*)$ are homothetic. Therefore, equality holds in every inequality if and only if $\Phi(C,K^*)$ and $\Phi(C,L^*)$ are homothetic. □

Analogue to the proofs of Theorem 1.1 and Corollary 3.1, Theorem 1.3 has the following facts.

Corollary 3.4 *If $K_1, K_2, \cdots, K_{n-2}, K, L \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then for $i \leq n-1$*

$$\overline{B}_i(\Phi(C,(K +_{-1} L)^*))^{\frac{1}{n-i}} \leq \overline{B}_i(\Phi(C,K^*))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C,L^*))^{\frac{1}{n-i}},$$

with equality if and only if $\Phi(C,K^*)$ and $\Phi(C,L^*)$ are homothetic.

Corollary 3.5 *If $K_1, K_2, \cdots, K_{n-2}, K, L \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$,*

then

$$V(\Phi^*(C,(K+_{-1}L)^*))^{-\frac{1}{n}} \geq V(\Phi^*(C,K^*))^{-\frac{1}{n}} + V(\Phi^*(C,L^*))^{-\frac{1}{n}},$$

with equality if and only if $\Phi(C,K^*)$ *and* $\Phi(C,L^*)$ *are homothetic.*

4 Conclusions

We first establish some Brunn-Minkowski type inequalities of width-integrals of index i which are related to the Blaschke Minkowski homomorphism. Then together with the Blaschke Minkowski homomorphism, we use the Beckenbach-Dresher inequalities to give more general Brunn-Minkowski type inequalities of width-integrals of index i, in which some inequalities similar to Lutwak's inequality are established.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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On fuzzy mighty filters in BE-algebras

Sun Shin Ahn[1] and Jeong Soon Han[2,*]

[1]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

[2]*Department of Applied Mathematics, Hanyang University, Ahnsan, 15588, Korea*

Abstract. In this paper, we study several degrees in defining a fuzzy mighty filter, which is a generalization of a fuzzy filter in BE-algebras.

1. Introduction

In [6], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra. S. S. Ahn and K. S. So [3, 4] introduced the notion of ideals in BE-algebras. S. S. Ahn et al. [2] fuzzified the concept of BE-algebras, and investigated some of their properties. H. R. Lee and S. S. Ahn [7] defined the notions of a mighty filter and an n-fold mighty filter, and considered some related properties of them.

In this paper, we study several degrees in defining a fuzzy mighty filter, which is a generalization of a fuzzy filter in BE-algebras.

2. Prelimiaries

We recall some definitions and results discussed in [3, 6].

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$,
(BE2) $x * 1 = 1$ for all $x \in X$,
(BE3) $1 * x = x$ for all $x \in X$,
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*).

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

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* The corresponding author.

[0]E-mail: sunshine@dongguk.edu (S. S. Ahn); han@hanyang.ac.kr (J. S. Han)

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Let X be a BE-algebra and n denote a positive integer. For any elements $x, y \in X$, let $x^n * y$ denote $x * (\cdots (x * (x * y)) \cdots)$, in which x occurs n times, and $x^0 * y = y$.

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* ([6]) of X if

(F1) $1 \in F$,

(F2) $x * y \in F$ and $x \in F$ imply $y \in F$

for all $x, y \in X$. A non-empty subset F of a BE-algebra X is called a *mighty filter* ([7]) of X if it satisfies (F1) and

(F3) $x * (y * z) \in F$ and $x \in F$ imply $((z * y) * y) * z \in F$

for all $x, y, z \in X$. A non-empty subset F of a BE-algebra X is called an *n-fold mighty filter* ([7]) of X if it satisfies (F1) and

(F4) $x * (y * z) \in F$ and $x \in F$ imply $((z^n * y) * y) * z \in F$

for all $x, y, z \in X$.

Note that every mighty filter of a BE-algebra X is a filter of X.

Proposition 2.2. *Let $(X; *, 1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.3 *Let $(X; *, 1)$ be a self distributive BE-algebra. Then the following hold, for any $x, y, z \in X$:*

(i) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$,*

(ii) $y * z \leq (z * x) * (y * x)$,

(iii) $y * z \leq (x * y) * (x * z)$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.3 (iii). If a BE-algebra X is transitive, then $y \leq z$ imply $x * y \leq x * z$ and $z * x \leq y * x$ for all $x, y, z \in X$.

Definition 2.4. ([5]) A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X if it satisfies for all $x, y \in X$

(d1) $\mu(1) \geq \mu(x)$,

(d2) $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$.

Proposition 2.5. *Every fuzzy filter of a BE-algebra X satisfies the following assertions:*

(i) $(\forall x, y \in X)(y \leq x \Rightarrow \mu(y) \leq \mu(x))$,

(ii) $(\forall x, y, z \in X)(x \leq y * z \Rightarrow \mu(z) \geq min\{\mu(y), \mu(x)\})$.

Definition 2.6. ([5]) Let F be a non-empty subset of a BE-algebra X which is not necessary a filter of X. One says that a subset G of X is an *enlarged filter* of X related to F if it satisfies:

(1) F is a subset of G,

(2) $1 \in G$,
(3) $(\forall x, y \in X)(\forall x \in F)(x * y \in F \Rightarrow y \in G)$.

Definition 2.7. ([5]) A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X with degree (λ, κ) if it satisfies:

(e1) $(\forall x \in X)(\mu(1) \geq \lambda\mu(x))$,
(e2) $(\forall x, y \in X)(\mu(x) \geq \kappa\min\{\mu(y * x), \mu(y)\})$.

Proposition 2.8. ([5]) *Every fuzzy filter of a BE-algebra X with degree (λ, κ) satisfies the following assertions:*

(i) $(\forall x, y \in X)(\mu(x * y) \geq \lambda\kappa\mu(y))$,
(ii) $(\forall x, y \in X)(y \leq x \Rightarrow \mu(x) \geq \lambda\kappa\mu(y))$,
(iii) $(\forall x, y, z \in X)(x \leq y * z \Rightarrow \mu(z) \geq \min\{\kappa\mu(y), \lambda\kappa^2\mu(x)\}$.

3. Fuzzy mighty filters of BE-algebras

In what follows, let X denote a BE-algebra unless specified otherwise.

Definition 3.1. A fuzzy subset μ of a BE-algebra X is called a *fuzzy mighty filter* of X if it satisfies (d1) and

(d3) $\mu(((x * y) * y) * x) \geq \min\{\mu(z * (y * x)), \mu(z)\}$,

for all $x, y, z \in X$.

Example 3.2. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([7]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.7 & 0.4 & 0.7 & 0.7 & 0.4 & 0.4 \end{pmatrix}$$

Then μ is a fuzzy mighty filter of X.

Proposition 3.3. *Every fuzzy mighty filter of a BE-algebra X is a fuzzy filter of X.*

Proof. Let $y := 1$ in (d3). Then $\mu(x) = \mu(((x * 1) * 1) * x) \geq \min\{\mu(z * (1 * x)), \mu(z)\} = \min\{\mu(z * x), \mu(z)\}$. Hence (d2) holds. □

The converse of Proposition 3.3 may not be true in general (see Example 3.4).

Example 3.4. Let $X := \{1, a, b, c, d\}$ be a BE-algebra ([6]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c & d \\ 0.7 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

Then ν is a fuzzy filter of X, but not a fuzzy mighty filter of X, since $\mu(((a*c)*c)*a) = \mu(a) = 0.3 \not\geq 0.7 = \mu(1) = \min\{\mu(1*(c*a)), \mu(1)\}$.

Theorem 3.5. *A fuzzy filter μ of a BE-algebra X is mighty if and only if it satisfies the following inequality:*

(3.1) $\mu(((x*y)*y)*x) \geq \mu(y*x)$ *for all* $x, y \in X$.

Proof. Suppose that a fuzzy filter μ of a BE-algebra X is mighty. Putting $z := 1$ in (d3), we have $\mu(((x*y)*y)*x) \geq \min\{\mu(1*(y*x)), \mu(1)\} = \mu(y*x)$ for all $x, y \in X$.

Conversely, assume that μ is a fuzzy filter of X satisfying (3.1). It follows from (d2) and (3.1) that $\mu(((x*y)*y)*x) \geq \mu(y*x) \geq \min\{\mu(z*(y*x)), \mu(z)\}$. Hence μ is mighty. □

Theorem 3.6. *Let μ, ν be fuzzy filters of a transitive BE-algebra X such that $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If μ is mighty, then so is ν.*

Proof. Let $x, y \in X$. Since μ is a fuzzy mighty filter of a BE-algebra X, by (3.1) and $\mu \subseteq \nu$ we have $\mu(1) = \mu(y*((y*x)*x)) \leq \mu(((((y*x)*x)*y)*y)*((y*x)*x)) \leq \nu(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $\mu(1) = \nu(1)$, we get $\nu((y*x)*((((y*x)*x)*y)*y)*x)) = \nu(((((y*x)*x)*y)*y)*((y*x)*x)) = \nu(1)$. It follows from (d1) and (d2) that

$$\begin{aligned} \nu(y*x) &= \min\{\nu(1), \nu(y*x)\} \\ &= \min\{\nu((y*x)*(((((y*x)*x)*y)*y)*x)), \nu(y*x)\} \\ &\leq \nu(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{3.2}$$

Since X is transitive, we get

$$\begin{aligned}[[((((y*x)*x)*y)*y)*x]*[((x*y)*y)*x] & \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1.\end{aligned}$$

It follows from Proposition 2.5 that $\min\{\nu(((((y*x)*x)*y)*y)*x), \nu(1)\} = \nu(((((y*x)*x)*y)*y)*x) \leq \nu(((x*y)*y)*x)$. Using (3.2), we have $\nu(y*x) \leq \nu(((((y*x)*x)*y)*y)*x) \leq \nu(((x*y)*y)*x)$. Therefore $\nu(y*x) \leq \nu(((x*y)*y)*x)$. By Theorem 3.5, ν is a fuzzy mighty filter of X.

Proposition 3.7. *Let μ be a fuzzy mighty filter of a BE-algebra. Then $X_\mu := \{x \in X | \mu(x) = \mu(1)\}$ is a mighty filter.*

Proof. Clearly, $1 \in X_\mu$. Let $x*(y*z), x \in X_\mu$. Then $\mu(x*(y*z)) = \mu(1)$ and $\mu(x) = \mu(1)$. It follows from (d3) that $\mu(1) = \min\{\mu(x*(y*z)), \mu(x)\} \leq \mu((((z*y)*y)*z)$. By (d1), $\mu((((z*y)*y)*z) = \mu(1)$. Hence $((z*y)*y)*z \in X_\mu$, Therefore X_μ is a mighty filter of X. □

Definition 3.8. A fuzzy subset μ of a BE-algebra X is called a *fuzzy n-fold mighty filter* of X if it satisfies (d1) and

(d4) $\mu(((x^n*y)*y)*x) \geq \min\{\mu(z*(y*x)), \mu(z)\}$,

for all $x, y, z \in X$.

Putting $n := 1$ in (d4), every fuzzy 1-fold mighty filter of a BE-algebra X is a fuzzy mighty filter of X.

Theorem 3.9. *A fuzzy filter μ of a BE-algebra X is a fuzzy n-fold mighty if and only if it satisfies the following inequality*

(3.3) $\mu(((x^n*y)*y)*x) \geq \mu(y*x)$ *for all $x, y \in X$.*

Proof. Suppose that a fuzzy filter μ is a fuzzy n-fold mighty. Letting $z := 1$ in (d4), we have $\mu(((x^n*y)*y)*x) \geq \min\{\mu(1*(y*x)), \mu(1)\} = \mu(y*x)$ for all $x, y \in X$.

Conversely, assume that μ is a fuzzy filter of X satisfying (3.3). It follows from (d2) and (3.3) that $\mu(((x^n*y)*y)*x) \geq \mu(y*x) \geq \min\{\mu(z*(y*x)), \mu(z)\}$. Hence μ is a fuzzy n-fold mighty filter of X. □

Theorem 3.10. *Let μ and ν be fuzzy filters of a transitive BE-algebra X with $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If μ is a fuzzy n-fold mighty filter of X, then so is ν.*

Proof. Let $x, y \in X$. Setting $w := (y * x) * x$, we have $y * w = y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since μ is a fuzzy n-fold mighty filter of a BE-algebra X, by (3.3) and $\mu \subseteq \nu$ we have $\mu(1) = \mu(y * w)) \leq \mu(((w^n * y) * y) * w) \leq \nu(((w^n * y) * y) * w)$. Since $\mu(1) = \nu(1)$, we get $\nu(((w^n * y) * y) * w) = \nu(1)$. It follows from (d1) and (d2) that

$$\begin{aligned} \nu(y * x) &= \min\{\nu(1), \nu(y * x)\} \\ &= \min\{\nu(((w^n * y) * y) * w), \nu(y * x)\} \\ &= \min\{\nu(((w^n * y) * y) * ((y * x) * x)), \nu(y * x)\} \\ &= \min\{\nu((y * x) * (((w^n * y) * y) * *x)), \nu(y * x)\} \\ &\leq \nu(((w^n * y) * y) * x). \end{aligned} \tag{3.4}$$

Since $y \leq w$, we have $w^n * y \leq x^n * y$, and so $((w^n * y) * y) * x \leq ((x^n * y) * y) * x$. Using (3.4) and Proposition 2.5(i), we have $\nu(y * x) \leq \nu((w^n * y) * y) * x) \leq \nu(((x^n * y) * y) * x)$. Therefore $\nu(y * x) \leq \nu(((x^n * y) * y) * x)$. By Theorem 3.9, ν is a fuzzy n-fold mighty filter of X. □

4. **Fuzzy mighty filters of BE-algebras with degrees in the interval** $(0, 1]$

.

Definition 4.1. Let F be a non-empty subset of a BE-algebra X, which is not necessary a mighty filter of X. One says that a subset G of X is an *enlarged mighty filter* of X related to F if it satisfies:

(1) F is a subset of G,
(2) $1 \in G$,
(3) $(\forall x, y \in X)(\forall z \in F)(z * (y * x) \in F \Rightarrow ((x * y) * y) * x \in G)$.

Obviously, every mighty filter is an enlarged mighty filter of X related to itself. Note that there exists an enlarged mighty filter of X related to any non-empty subset F of X.

Example 4.2. Consider a BE-algebra $X = \{1, a, b, c, d\}$ as in Example 3.4. Then $F := \{1\}$ is not a mighty filter of X, since $1 * (c * a) = 1 \in F$ and $((a * c) * c) * a = a \notin F$. But $G := \{1, a, b, c\}$ is an enlarged mighty filter of X related to F and G is not a mighty filter of X, since $c * (1 * d) = b \in G, c \in G$ and $((d * 1) * 1) * d = d \notin G$.

Proposition 4.3. *Let F be a non-empty subset of a BE-algebra X. Every enlarged mighty filter of X related to F is an enlarged filter of X related to F.*

Proof. Let G be an enlarged mighty filter of X related to F. Putting $y := 1$ in Definition 4.1 (3), we have for all $x \in X$, all $z \in F$ $z * (1 * x) = z * x \in F$ imply $((x * 1) * 1) * x = x \in G$. Hence G is an enlarged filter of X related to F. □

The converse of Proposition 4.3 is not true in general as seen in the following example.

Example 4.4. Consider a BE-algebra $X = \{1, a, b, c, d\}$ as in Example 3.4. Let $H := \{1, b\}$. Then H is an enlarged filter of X related to $F := \{1\}$, but it is not an enlarged mighty filter of X related to F since $1 * (c * a) = 1 \in F$ and $((a * c) * c) * a = a \notin H$.

In what follows let λ and κ be members of $(0, 1]$, and let n and k denote a natural number and a real number, respectively, such that $k < n$ unless otherwise specified.

Definition 4.5. A fuzzy subset μ of a BE-algebra X is called a *fuzzy mighty filter* of X with degree (λ, κ) if it satisfies (e1) and

(e3) $(\forall x, y, z \in X)(\mu(((x * y) * y) * x) \geq \kappa\min\{\mu(z * (y * x)), \mu(z)\})$.

Example 4.6. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 3.2. Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.6 & 0.4 & 0.7 & 0.7 & 0.4 & 0.4 \end{pmatrix}$$

Then ν is a fuzzy filter of X with degree $(\frac{3}{5}, \frac{2}{5})$, but it not a fuzzy mighty filter of X, since $\mu(1) = 0.6 \not\geq \mu(b) = 0.7$.

Note that a fuzzy mighty filter with degree (λ, κ) is a fuzzy mighty filter if and only if $(\lambda, \kappa) = (1, 1)$.

Proposition 4.7. *If μ is a fuzzy mighty filter of a BE-algebra X with degree (λ, κ), then μ is a fuzzy filter of X with degree (λ, κ).*

Proof. Putting $y := 1$ in (e3), we have

$$\begin{aligned} \mu(x) = \mu(((x * 1) * 1) * x) \geq & \kappa\min\{\mu(z * (1 * x)), \mu(z)\} \\ = & \kappa\min\{\mu(z * x), \mu(z)\} \end{aligned}$$

for any $x, y \in X$. Thus μ is a fuzzy filter of X with degree (λ, κ). □

The converse of Proposition 4.7 is not true in general (see Example 4.8).

Example 4.8. Consider a BE-algebra $X = \{1, a, b, c, d\}$ which is given in Example 3.4. Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c & d \\ 0.6 & 0.4 & 0.7 & 0.4 & 0.4 \end{pmatrix}$$

Then ν is a fuzzy filter of X with degree $(\frac{4}{5}, \frac{4}{5})$, but it is not a fuzzy mighty filter of X with degree $(\frac{4}{5}, \frac{4}{5})$, since

$$\begin{aligned}\nu(((a*c)*c)*a) = \nu(a) = 0.4 \not\geq 0.48 = \frac{4}{5} \times 0.6 \\ = &\frac{4}{5} \times \nu(1) \\ = &\frac{4}{5} \times \min\{\nu(1*(c*a)) = \nu(1), \nu(1)\}.\end{aligned}$$

Proposition 4.9. *Let μ be a fuzzy mighty filter of a BE-algebra X with degree (λ, κ). Then the following holds:*

$$(\forall x, y \in X)(\mu(((x*y)*y)*x) \geq \kappa\lambda\mu(y*x)).$$

Proof. Assume that μ is a fuzzy mighty filter of a BE-algebra X with degree (λ, κ) and let $x, y \in X$. Let $z := 1$ in (e3). Then we have

$$\begin{aligned}\mu(((x*y)*y)*x) \geq &\kappa\min\{\mu(1*(y*x)), \mu(1)\} \\ \geq &\kappa\min\{\mu(y*x), \lambda\mu(y*x)\} \\ = &\kappa\lambda\mu(y*x).\end{aligned}$$

This completes the proof. □

Proposition 4.10. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ) satisfying*

$$(\forall x, y \in X)(\mu(((x*y)*y)*x) \geq \mu(y*x)).$$

Then μ is a fuzzy mighty filter of X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using (e2), we have

$$\begin{aligned}\mu(((x*y)*y)*x) \geq &\mu(y*x) \\ \geq &\kappa\min\{\mu(z*(y*x)), \mu(z)\}.\end{aligned}$$

Thus μ is a fuzzy mighty filter of a BE-algebra X with degree (λ, κ). □

Corollary 4.11. *Let μ be a fuzzy filter of X. Then μ is a fuzzy mighty filter of X if and only if*

$$(\forall x, y \in X)(\mu(((x*y)*y)*x) \geq \mu(y*x)).$$

Proof. It follows from Proposition 4.9 and Proposition 4.10. □

Definition 4.12. A fuzzy subset μ of a BE-algebra X is called a *fuzzy n-fold mighty filter* of X with degree (λ, κ) if it satisfies (e1) and

(e4) $(\forall x, y, z \in X)(\mu(((x^n*y)*y)*x) \geq \kappa\min\{\mu(z*(y*x)), \mu(z)\})$.

A fuzzy subset μ of a BE-algebra X is called a *fuzzy weak n-fold mighty filter* of X with degree (λ, κ) if it satisfies (e1) and

(e5) $(\forall x, y, z \in X)(\mu((x * y) * y) \geq \kappa \min\{\mu(z * ((y^n * x) * x)), \mu(z)\})$.

Putting $y := 1$ and $x := y$ in (e4), and using (BE1), (BE2) and (BE3), we know that every fuzzy n-fold mighty filter with degree (λ, κ) is a fuzzy mighty filter of X with degree (λ, κ). Setting $x := y$ in (e5) and using (BE1), (BE2) and (BE3), we know that every fuzzy weak n-fold mighty filter with degree (λ, κ) is a fuzzy mighty filter of X with degree (λ, κ). Hence every fuzzy (weak) n-fold mighty filter of X with degree (λ, κ) is a fuzzy filter of X with degree (λ, κ).

Example 4.13. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([6]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.6 & 0.7 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

Then μ is a fuzzy $n(\geq 2)$-fold mighty filter of X with degree $(\frac{5}{7}, \frac{4}{7})$, but it is not a fuzzy mighty filter of X with degree $(\frac{5}{7}, \frac{4}{7})$, since $\mu(((b * c) * c) * b) = \mu(b) = 0.3 \not\geq 0.399 = \frac{4}{7} \times 0.7 = \mu(a) = \min\{\mu(a * (c * b)), \mu(a)\}$.

Proposition 4.14. *Let μ be a fuzzy n-fold mighty filter of a BE-algebra X with degree (λ, κ). Then the following holds:*

$$(\forall x, y \in X)(\mu(((x^n * y) * y) * x) \geq \kappa\lambda\mu(y * x)).$$

Proof. Assume that μ is a fuzzy n-fold mighty filter of a BE-algebra X with degree (λ, κ) and let $x, y \in X$. Let $z := 1$ in (e4). Then we have

$$\begin{aligned} \mu(((x^n * y) * y) * x) &\geq \kappa \min\{\mu(1 * (y * x)), \mu(1)\} \\ &\geq \kappa \min\{\mu(y * x), \lambda\mu(y * x)\} \\ &= \kappa\lambda\mu(y * x). \end{aligned}$$

This completes the proof. □

Proposition 4.15. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ) satisfying*

$$(\forall x, y \in X)(\mu(((x^n * y) * y) * x) \geq \mu(y * x)).$$

Then μ is a fuzzy n-fold mighty filter of X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using (e2), we have

$$\begin{aligned}\mu(((x^n * y) * y) * x) \geq & \mu(y * x) \\ \geq & \kappa \min\{\mu(z * (y * x)), \mu(z)\}.\end{aligned}$$

Thus μ is a fuzzy n-fold mighty filter of a BE-algebra X with degree (λ, κ). □

Proposition 4.16. *If μ is a fuzzy weak n-fold filter of X, then the following holds:*

(4.1) $(\forall x, y \in X)(\mu((x * y) * y) \geq \kappa\lambda\mu((y^n * x) * x))$.

Proof. Let $z := 1$ in (e5). Then $\mu((x * y) * y) \geq \kappa \min\{\mu(1 * ((y^n * x) * x)), \mu(1)\} \geq \kappa \min\{\mu((y^n * x) * x), \lambda\mu((y^n * x) * x)\} = \kappa\lambda\mu((y^n * x) * x)$. This completes the proof. □

Proposition 4.17. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ) satisfying*

$$(\forall x, y \in X)(\mu((y^n * x) * x) \leq \mu((x * y) * y)).$$

Then μ is a fuzzy weak n-fold mighty filter of X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using (e2), we have

$$\begin{aligned}\mu((x * y) * y) \geq & \mu((y^n * x) * x) \\ \geq & \kappa \min\{\mu(z * ((y^n * x) * x)), \mu(z)\}.\end{aligned}$$

Thus μ is a fuzzy weak n-fold mighty filter of a BE-algebra X with degree (λ, κ). □

A BE-algebra X is said to be *n-fold mighty* if $((x^n * y) * y) * x = y * x$ for all $x, y \in X$.

Lemma 4.18. *Let X be an n-fold mighty BE-algebra. If μ is a fuzzy filter of X with degree (λ, κ), then the following holds:*

(4.2) $(\forall x, y \in X)(\mu((y * x) * x) \geq \lambda\kappa\mu((x^n * y) * y))$.

Proof. Since X is an n-fold mighty BE-algebra, $((x^n * y) * y) * ((y * x) * x) = (y * x) * (((x^n * y) * y) * x) = (y * x) * (y * x) = 1$, for all $x, y \in X$. Hence $(x^n * y) * y \leq ((y * x) * x)$. Using Proposition 2.8(ii), we have $\mu((y * x) * x) \geq \lambda\kappa\mu((x^n * y) * y))$. This completes the proof. □

Proposition 4.19. *Let X be an n-fold mighty transitive BE-algebra. If μ is a fuzzy filter of X with degree (λ, κ), then it is a fuzzy weak n-fold mighty filter of X with degree $(\lambda, \lambda\kappa^2)$.*

Proof. Let μ be a fuzzy filter of X with degree (λ, κ). Using Lemma 4.18 and (e2), we have $\mu((y * x) * x) \geq \lambda\kappa\mu((x^n * y) * y)) \geq \lambda\kappa^2 \min\{\mu(z * ((x^n * y) * y), \mu(z)\}$. Hence μ is a fuzzy weak n-fold mighty filter of X with degree $(\lambda, \lambda\kappa^2)$.

Proposition 4.20. *Let μ be a fuzzy filter with degree (λ, κ) of an n-fold mighty BE-algebra X. Then μ is a fuzzy n-fold mighty filter of X with degree (λ, κ).*

Proof. Let μ be a fuzzy filter with degree (λ, κ) of an n-fold mighty BE-algebra X. Since X is an n-fold mighty BE-algebra, we have $\mu(((x^n * y) * y) * x) = \mu(y * x) \geq \mu(y * x)$. By Proposition 4.15, μ is a fuzzy n-fold mighty filter of X with degree (λ, κ). □

Denote by $\mathcal{F}_M(X)$ the set of all mighty filters of a BE-algebra X. Note that a fuzzy subset μ of a BE-algebra X is a fuzzy mighty filter of X if and only if

$$(\forall t \in [0,1])(U(\mu; t) := \{x \in X | \mu(x) \geq t\} \in \mathcal{F}_M(X) \cup \{\emptyset\}).$$

But we know that for any fuzzy subset μ of a BE-algebra X there exist $\lambda, \kappa \in (0,1)$ and $t \in [0,1]$ such that

(1) μ is a fuzzy mighty filter of X with degree (λ, κ),
(2) $U(\mu; t) \notin \mathcal{F}_M(X) \cup \{\emptyset\}$.

Example 4.21. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given Example 3.2. Define a fuzzy subset $\mu : X \to [0,1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.8 & 0.3 & 0.9 & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

If $t \in (0.8, 1]$, then $U(\mu; t) = \{1, b\}$ is not a mighty filter of X, since $b * (a * c) = b \in U(\mu; t)$ and $((c * a) * a) * c = c \notin U(\mu; t)$. But μ is a fuzzy mighty filter of X with degree $(0.6, 0.3)$.

Theorem 4.22. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \max\{\lambda, \kappa\}$, if $U(\mu; t)$ is an enlarged mighty filter of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, then μ is a fuzzy mighty filter of X with degree (λ, κ).*

Proof. Assume that $\mu(1) < t \leq \lambda\mu(x)$ for some $x \in X$ and $t \in (0, \lambda]$. Then $\mu(x) \geq \frac{t}{\lambda} \geq \frac{t}{\max\{\lambda,\kappa\}}$ and so $x \in U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, i.e., $U(\mu; \frac{t}{\max\{\lambda,\kappa\}}) \neq \emptyset$. Since $U(\mu; t)$ is an enlarged filter of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, we have $1 \in U(\mu; t)$, i.e., $\mu(1) \geq t$. This is a contradiction, and thus $\mu(1) \geq \lambda\mu(x)$ for all $x \in X$.

Now suppose that there exist $a, b, c \in X$ such that $\mu(((a*b)*b)*a) < \kappa\min\{\mu(c*(b*a)), \mu(c)\}$. If we take $t := \kappa\min\{\mu(c * (b * a)), \mu(c)\}$, then $t \in (0, \kappa] \subseteq (0, \max\{\lambda, \kappa\}]$. Hence $c * (b * a) \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$ and $c \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$. It follows from an enlarged mighty filter that $((a * b) * b) * a \in U(\mu; t)$ so that $\mu(((a * b) * b) * a) \geq t$, which is impossible. Therefore

$$\mu(((x * y) * y) * x) \geq \kappa\min\{\mu(z * (y * x)), \mu(z)\}$$

for all $x, y, z \in X$. Thus μ is a fuzzy mighty filter of X with degree (λ, κ). □

Corollary 4.23. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \frac{k}{n}$, if $U(\mu;t)$ is an enlarged mighty filter of X related to $U(\mu;\frac{n}{k}t)$, then μ is a fuzzy mighty filter of X with degree $(\frac{k}{n},\frac{k}{n})$.*

Theorem 4.24. *Let $t \in [0,1]$ be such that $U(\mu;t)(\neq \emptyset)$ is not necessary a mighty filter of a BE-algebra X. If μ is a fuzzy mighty filter of X with degree (λ,κ), then $U(\mu;t\min\{\lambda,\kappa\})$ is an enlarged mighty filter of X related to $U(\mu;t)$.*

Proof. Since $t\min\{\lambda,\kappa\} \leq t$, we have $U(\mu;t) \subseteq U(\mu;t\min\{\lambda,\kappa\})$. Since $U(\mu;t) \neq \emptyset$, there exists $x \in U(\mu;t)$ and so $\mu(x) \geq t$. By (e1), we obtain $\mu(1) \geq \lambda\mu(x) \geq \lambda t \geq t\min\{\lambda,\kappa\}$. Therefore $1 \in U(\mu;t\min\{\lambda,\kappa\})$.

Let $x,y,z \in X$ be such that $x*(y*z) \in U(\mu;t)$ and $x \in U(\mu;t)$. Then $\mu(x*(y*z)) \geq t$ and $\mu(x) \geq t$. It follows from (e3) that

$$\begin{aligned}\mu(((z*y)*y)*z) \geq& \kappa\min\{\mu(x*(y*z)),\mu(x)\}\\ \geq& \kappa t \geq t\min\{\lambda,\kappa\}.\end{aligned}$$

so that $((z*y)*y)*z \in U(\mu;t\min\{\lambda,\kappa\})$. Thus $U(\mu;t\min\{\lambda,\kappa\})$ is an enlarged mighty filter of X related to $U(\mu;t)$. □

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SYMMETRIC IDENTITIES FOR (h,q)-EXTENSIONS OF THE GENERALIZED HIGHER ORDER MODIFIED q-EULER POLYNOMIALS

JONGKYUM KWON[1], GYOYONG SOHN[2], AND JIN-WOO PARK[3,*]

Abstract. In this paper, we introduce the (h,q)-extensions of the generalized modified q-Euler polynomials. The main objective of this paper is to consider symmetric identities of the (h,q)-extensions of the generalized modified q-Euler polynomials attached to χ which are derived from the p-adic fermionic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|q-1|_p < p^{-\frac{1}{p-1}}$. The *$q$-analogue of number x* is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the *p-adic fermionic q-integral on $\mathbb{Z}_p$* is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty}\frac{1+q}{1+q^{p^N}}\sum_{x=0}^{p^N-1} f(x)(-q)^x\\ &= \frac{[2]_q}{2}\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x q^x, \text{ (see [1, 3, 4, 6, 7, 9, 11, 12, 14])}.\end{aligned} \tag{1.1}$$

Thus, by (1.1), we get

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0), \tag{1.2}$$

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* corresponding author.

and

$$q^n \int_{\mathbb{Z}_p} f(x+n)d\mu_{-q}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} f(l)q^l(-1)^{n-1-l}, \quad (1.3)$$

where $n \in \mathbb{N}$ (see [5, 10-13]).

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{ (see [2, 8])}, \quad (1.4)$$

when $x = 0$, $E_n = E_n(0), (n \geq 0)$, are called Euler numbers.

It is known that the *q-Euler polynomials* are given by the generating function as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, (\text{see } [1, 3-7, 9-14]), \quad (1.5)$$

when $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called *q-Euler numbers.*

It is known that

$$\int_X f(x)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(x). \quad (1.6)$$

Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then, by (1.3), we get

$$\begin{aligned}\int_X \chi(y)e^{(x+y)t}d\mu_{-1}(y) &= \left(\frac{2}{e^{dt}+1}\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}\right)e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi}(x)\frac{t^n}{n!}, (\text{see } [4, 11])\end{aligned} \quad (1.7)$$

where $E_{n,\chi}(x)$ are called the generalized Euler polynomials attached to χ and $x = 0$, $E_{n,\chi} = E_{n,\chi}(0)$ is called the n-th generalized Euler number attached to χ.

From (1.7), we can derive the generalized higher order Euler polynomials attached to χ as follows:

$$\begin{aligned}&\int_X \cdots \int_X \left(\prod_{l=1}^{r}\chi(y_l)\right)e^{(y_1+y_2+\cdots+y_r+x)t}d\mu_{-1}(y_1)d\mu_{-1}(y_2)\cdots d\mu_{-1}(y_r) \\ &= \left(\frac{2}{e^{dt}+1}\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x)\frac{t^n}{n!}, \text{ (see [9, 11])}.\end{aligned} \quad (1.8)$$

From (1.8), we consider the (h,q)-extensions of higher order modified Euler polynomials attached to χ as follows:

$$\begin{aligned}&\int_X \cdots \int_X q^{\sum_{l=1}^r (h-l)y_l} \Big(\prod_{l=1}^r \chi(y_l)\Big) e^{[y_1+y_2+\cdots+y_r+x]_q t} d\mu_{-1}(y_1) d\mu_{-1}(y_2)\cdots d\mu_{-1}(y_r)\\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,r)}(x)\frac{t^n}{n!}, \text{ (see [9, 11])},\end{aligned} \tag{1.9}$$

where $r \in \mathbb{N}, h \in \mathbb{Z}$.

Recently, T. Kim and D.Kim studied (h,q)-extensions of the generalized higher order Euler polynomials (see [11]). So, we introduce (h,q)-extensions of the generalized higher order modified q-Euler polynomials. The purpose of this paper is to investigate symmetric identities of (h,q)-extensions of the generalized higher order modified q-Euler polynomials.

2. Symmetric identities for (h,q)-extensions of the generalized higher order modified q-Euler polynomials

We introduce the (h,q)-extensions of the generalized higher order modified q-Euler polynomials attached to χ as follows:

$$\begin{aligned}&\int_X \cdots \int_X q^{\sum_{l=1}^r (h-l)y_l} \Big(\prod_{l=1}^r \chi(y_l)\Big) e^{[y_1+y_2+\cdots+y_r+x]_q t} d\mu_{-q}(y_1) d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)\\ &= \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi,q}^{(h,r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

Now, we consider the symmetric identities of (h,q)-extensions of the generalized higher order modified q-Euler polynomials.
Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. Then, we consider the following identity.

$$\begin{aligned}&\int_X \cdots \int_X q^{\sum_{l=1}^r (h-l)y_l} \Big(\prod_{l=1}^r \chi(y_l)\Big)\\ &\times e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l]_q t} d\mu_{-q}(y_1) d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)\\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{i_1,i_2,\cdots,i_r=0}^{dw_2-1} \sum_{y_1,y_2,\cdots,y_r=0}^{p^N-1} q^{w_1\sum_{l=1}^r (h-l)y_l} \Big(\prod_{l=1}^r \chi(y_l)\Big)\\ &\times e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r (i_l + dw_2 y_l)]_q t} (-q)^{\sum_{l=1}^r (i_l + dw_2 y_l)}.\end{aligned} \tag{2.2}$$

From (1.9), we have

$$\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}(i_l+dw_2y_l)}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_1\sum_{l=1}^{r}(h-l)y_l}$$
$$\times(\prod_{l=1}^{r}\chi(y_l))e^{[w_1w_2x+w_2\sum_{l=1}^{r}j_l+w_1\sum_{l=1}^{r}y_l]_qt}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$
$$=\lim_{N\to\infty}\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}\sum_{i_1,i_2,\cdots,i_r=0}^{dw_2-1}\sum_{y_1,y_2,\cdots,y_r=0}^{p^N-1}(-q)^{\sum_{l=1}^{r}(i_l+j_l+y_l)}q^{\sum_{l=1}^{r}(h-l)(w_2j_l+w_1i_l+w_1w_2y_l)}$$
$$\times e^{[w_1w_2(x+d\sum_{l=1}^{r}y_l)+\sum_{l=1}^{r}(w_2j_l+w_1i_l)]_qt}(\prod_{l=1}^{r}\chi(i_lj_l)). \tag{2.3}$$

By the similar method as (2.3), we get

$$\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1}(-q)^{\sum_{l=1}^{r}(i_l+dw_2y_l)}q^{w_1\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_2\sum_{l=1}^{r}(h-l)y_l}$$
$$\times(\prod_{l=1}^{r}\chi(y_l))e^{[w_1w_2x+w_1\sum_{l=1}^{r}j_l+w_2\sum_{l=1}^{r}y_l]_qt}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$
$$=\lim_{N\to\infty}\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1}\sum_{i_1,i_2,\cdots,i_r=0}^{dw_1-1}\sum_{y_1,y_2,\cdots,y_r=0}^{p^N-1}(-q)^{\sum_{l=1}^{r}(i_l+j_l+y_l)}q^{\sum_{l=1}^{r}(h-l)(w_1j_l+w_2i_l+w_1w_2y_l)}$$
$$\times e^{[w_1w_2(x+d\sum_{l=1}^{r}y_l)+\sum_{l=1}^{r}(w_1j_l+w_2i_l)]_qt}(\prod_{l=1}^{r}\chi(i_lj_l)). \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1\ (mod\ 2)$, $w_2 \equiv 1\ (mod\ 2)$. Then, we have*

$$\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}(i_l+dw_2y_l)}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_1\sum_{l=1}^{r}(h-l)y_l}$$
$$\times(\prod_{l=1}^{r}\chi(y_l))e^{[w_1w_2x+w_2\sum_{l=1}^{r}j_l+w_1\sum_{l=1}^{r}y_l]_qt}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$
$$=\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1}(-q)^{\sum_{l=1}^{r}(i_l+dw_2y_l)}q^{w_1\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_2\sum_{l=1}^{r}(h-l)y_l}$$
$$\times(\prod_{l=1}^{r}\chi(y_l))e^{[w_1w_2x+w_1\sum_{l=1}^{r}j_l+w_2\sum_{l=1}^{r}y_l]_qt}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r). \tag{2.5}$$

Now, we consider that

$$\left[w_1w_2x+w_2\sum_{l=1}^{r}j_l+w_1\sum_{l=1}^{r}y_l\right]_q=[w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_q^{w_1}, \tag{2.6}$$

and

$$\left[w_1w_2x+w_1\sum_{l=1}^{r}j_l+w_2\sum_{l=1}^{r}y_l\right]_q=[w_2]_q\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_q^{w_2}, \quad (2.7)$$

Therefore, by Theorem 2.1 , (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *Let $w_1,w_2\in\mathbb{N}$ such that $w_1\equiv 1\ (mod\ 2)$, $w_2\equiv 1\ (mod\ 2)$ and $n\geq 0$. Then, we have*

$$\begin{aligned}
&[w_1]_q^n\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_1\sum_{l=1}^{r}(h-l)y_l}\\
&\times(\prod_{l=1}^{r}\chi(y_l))\left[w_2x+\frac{w_2}{w_1}(j_1+j_2+\cdots+j_r)\sum_{l=1}^{r}y_l\right]_{q^{w_1}}^{n}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)\\
&=[w_2]_q^n\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_1\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_2\sum_{l=1}^{r}(h-l)y_l}\\
&\times(\prod_{l=1}^{r}\chi(y_l))\left[w_1x+\frac{w_1}{w_2}(j_1+j_2+\cdots+j_r)\sum_{l=1}^{r}y_l\right]_{q^{w_2}}^{n}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r).
\end{aligned} \quad (2.8)$$

Thus, by (1.7) and Theorem 2.2, we obtain the following theorem.

Theorem 2.3. *Let $w_1,w_2\in\mathbb{N}$ such that $w_1\equiv 1\ (mod\ 2)$, $w_2\equiv 1\ (mod\ 2)$ and $n\geq 0$. Then, we have*

$$\begin{aligned}
&[w_1]_q^n\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))E_{n,\chi,q^{w_1}}^{(h,r)}(w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l)\\
&=[w_2]_q^n\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))E_{n,\chi,q^{w_2}}^{(h,r)}(w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}j_l).
\end{aligned} \quad (2.9)$$

Now, we consider that

$$
\begin{aligned}
&\int_X \cdots \int_X q^{w_1 \sum_{l=1}^{r}(h-l)y_l} \Big(\prod_{l=1}^{r} \chi(y_l)\Big) \\
&\times \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q_1^w}^{n} d\mu_{-q}(y_1) d\mu_{-q}(y_2) \cdots d\mu_{-q}(y_r) \\
&= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [\sum_{l=1}^{r} j_l]_{q^{w_2}}^{i} q^{w_2(n-i)\sum_{l=1}^{r} j_l} \int_X \cdots \int_X \left[w_2 x + \sum_{l=1}^{r} y_l \right]_{q_1^w}^{n-i} \\
&\times (q)^{w_1 \sum_{l=1}^{r}(h-l)y_l)} \Big(\prod_{l=1}^{r} \chi(y_l)\Big) d\mu_{-q}(y_1) d\mu_{-q}(y_2) \cdots d\mu_{-q}(y_r) \\
&= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i \left[\sum_{l=1}^{r} j_l \right]_{q^{w_2}}^{i} q^{w_2(n-i)\sum_{l=1}^{r} j_l} E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2 x) \\
&= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^{n-i} \left[\sum_{l=1}^{r} j_l \right]_{q^{w_2}}^{n-i} q^{w_2 i \sum_{l=1}^{r} j_l} E_{i,\chi,q^{w_1}}^{(h,r)}(w_2 x).
\end{aligned}
\tag{2.10}
$$

From (2.6), we have

$$
\begin{aligned}
&[w_1]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} \Big(\prod_{l=1}^{r} \chi(j_l)\Big) \int_X \cdots \int_X q^{w_1 \sum_{l=1}^{r}(h-l)y_l} \\
&\times \Big(\prod_{l=1}^{r} \chi(y_l)\Big) \left[w_2 x + \frac{w_2}{w_1}(j_1 + j_2 + \cdots + j_r) + \sum_{l=1}^{r} y_l \right]_{q^{w_1}}^{n} d\mu_{-q}(y_1) d\mu_{-q}(y_2) \cdots d\mu_{-q}(y_r) \\
&= \sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} \Big(\prod_{l=1}^{r} \chi(j_l)\Big) \\
&\times \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \left[\sum_{l=1}^{r} j_l \right]_{q^{w_2}}^{n-i} q^{w_2 \sum_{l=1}^{r} j_l} E_{i,\chi,q^{w_1}}^{(h,r)}(w_2 x) \\
&= \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{i,\chi,q^{w_1}}^{(h,r)}(w_2 x) \sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} \Big(\prod_{l=1}^{r} \chi(j_l)\Big) \\
&\times \left[\sum_{l=1}^{r} j_l \right]_{q^{w_2}}^{n-i} q^{w_2 \sum_{l=1}^{r}(h-l-i)j_l} \\
&= \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{i,\chi,q^{w_1}}^{(h,r)}(w_2 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_1-1} (-q)^{\sum_{l=1}^{r} j_l} q^{\sum_{l=1}^{r}(h-l)j_l} \Big(\prod_{l=1}^{r} \chi(j_l)\Big) \\
&\times \left[\sum_{l=1}^{r} j_l \right]_{q^{w_2}}^{n-i}.
\end{aligned}
\tag{2.11}
$$

By the similar method as (2.6), we get

$$\begin{aligned}
&[w_2]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1} (-q)^{\sum_{l=1}^r j_l} q^{w_1\sum_{l=1}^r (h-l)j_l} (\prod_{l=1}^r \chi(j_l)) \int_X \cdots \int_X q^{w_2\sum_{l=1}^r (h-l)y_l} \\
&\times (\prod_{l=1}^r \chi(y_l)) \left[w_1 x + \frac{w_1}{w_2}(j_1+j_2+\cdots+j_r) + \sum_{l=1}^r y_l \right]_{q^{w_2}}^n d\mu_{-q}(y_1) d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r) \\
&= \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{i,\chi,q^{w_2}}^{(h,r)}(w_1 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_2-1} (-q)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l)j_l} (\prod_{l=1}^r \chi(j_l)) \\
&\times \left[\sum_{l=1}^r j_l \right]_{q^{w_1}}^{n-i}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.4. *Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1 \ (mod\ 2)$, $w_2 \equiv 1 \ (mod\ 2)$ and $n \geq 0$. Then, we have*

$$\begin{aligned}
&\binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{i,\chi,q^{w_1}}^{(h,r)}(w_2 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_1-1} (-q)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l)j_l} \\
&\times (\prod_{l=1}^r \chi(j_l)) \left[\sum_{l=1}^r j_l \right]_{q^{w_2}}^{n-i} \\
&= \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{i,\chi,q^{w_2}}^{(h,r)}(w_1 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_2-1} (-q)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l)j_l} \\
&\times (\prod_{l=1}^r \chi(j_l)) \left[\sum_{l=1}^r j_l \right]_{q^{w_1}}^{n-i}.
\end{aligned} \tag{2.13}$$

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[1]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
E-mail address: mathkjk26@gnu.ac.kr

[2] Department of Mathematics Education, Daegu National University of Education, Daegu, 42411, Republic of Korea
E-mail address: gysohn@dnue.ac.kr

[3,*] Department of Mathematics Education, Daegu University, Gyeongsan-si, Kyungsang-bukdo, 38453, Republic of Korea
E-mail address: a0417001@knu.ac.kr

Some properties of (p, q)-tangent polynomials

R. P. Agarwal [1], J. Y. Kang[2*], C. S. Ryoo[3]

Abstract We introduce (p, q)-tangent polynomials and their basic properties including (p, q)-derivative and (p, q)-integral. By using Mathematica, we find roots of (p, q)-tangent polynomials. We also investigate relations of zeros between (p, q)-tangent polynomials and classical tangent polynomials.

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1. Introduction

In [1, 2, 3], R. Chakrabarti and R. Jagannathan, G. Brodimas, et al. and M. Arik, et al. introduced the (p, q)-number in order to unify various forms of q-oscillator algebras.

For any $n \in \mathbb{C}$, the (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is clear that (p, q)-number contains symmetric property, and this number reduces to q-number when $p = 1$. In particular, we can see that $\lim_{q\to 1}[n]_{p,q} = n$ with $p = 1$(see [9]). By using the above numbers, many researchers have studied (p, q)-calculus(see [4, 5, 8]).

Definition 1.1. We define the (p, q)-derivative operator

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,$$

and $D_{p,q}f(0) = f'(0)$. The following properties of (p, q)-derivative operator are immediate.

[1, 2] *Department of Mathematics, Texas A & M University, Kingsville, USA*
[3] *Department of Mathematics, Hanman University, Daejeon, KOREA*

Theorem 1.2. For the operator $D_{p,q}$ the following hold

(i) Derivative of a product
$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

(ii) Derivative of a ratio
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

In [4], R. B. Corcino found (p,q)-extension of binomials coefficients and used it to establish various properties related to horizontal function, the triangular function, and the vertical function.

Definition 1.3. The (p,q)-analogue of $(x+a)^n$ is defined by

(i) $$(x+a)_{p,q}^n = \begin{cases} 1 & \text{if } n = 0 \\ (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}) & \text{if } n \neq 0 \end{cases},$$

(ii) $$(x+a)_{p,q}^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ is (p,q)-Gauss-Binomial coefficients.

Definition 1.4. Let z be any complex numbers with $|z| < 1$. Two forms of (p,q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$
$$\mathcal{E}_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}.$$

These forms are connected by the following interesting relations

$$e_{p,q}(z)\mathcal{E}_{p,q}(-z) = 1, \qquad \mathcal{E}_{p,q}(z) = e_{p^{-1},q^{-1}}(z).$$

Bernoulli, Euler, and Genocchi polynomials have been studied extensively by many mathematicians(see [5-7, 10-13]). In 2013, C. S. Ryoo introduced tangent polynomials and he developed several properties of these polynomials(see [10]). The tangent numbers are closely related to Euler numbers.

Definition 1.5. Tangent numbers T_n and tangent polynomials $T_n(x)$ are defined by

means of the generating functions

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} = 2\sum_{m=0}^{\infty}(-1)^m e^{2mt},$$
$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{tx} 2\sum_{m=0}^{\infty}(-1)^m e^{(2m+x)t}.$$

Theorem 1.6. For any positive integer n, we have

$$T_n(x) = (-1)^n T_n(2-x).$$

Theorem 1.7. For any positive integer $m(= odd)$, we have

$$T_n(x) = m^n \sum_{i=0}^{m-1}(-1)^i T_n\left(\frac{2i+x}{m}\right), \quad n \in \mathbb{Z}_+.$$

Theorem 1.8. For $n \in \mathbb{Z}_+$, we have

$$T_n(x+y) = \sum_{k=0}^{n}\binom{n}{k} T_k(x) y^{n-k}.$$

The main aim of this paper is to extend tangent numbers and polynomials, and study some of their properties. Our paper is organised as follows: In Section 2, we define (p,q)-tangent polynomials and find some properties of these polynomials. In Section 3, we consider (p,q)-tangent polynomials in two parameters and establish some relations between (p,q)-tangent polynomials and (p,q)-Euler or Bernoulli polynomials. In Section 4, we observe roots distributions of (p,q)-tangent polynomials and demonstrate interesting phenomenon.

2. Some properties of the (p,q)-tangent polynomials

In this section we define the (p,q)-tangent numbers and polynomials and establish some of their basic properties. We also define (p,q)-derivative and (p,q)-integral of (p,q)-tangent polynomials.

Definition 2.1. For $x, p, q \in \mathbb{C}$, we define (p,q)-tangent polynomials as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx), \quad |t| < \frac{\pi}{2}.$$

From Definition 2.1, it follows that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(0)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1},$$

where $\mathcal{T}_{n,p,q}$ is (p,q)-tangent number. If $p=1$, $q \to 1$, then it reduces to the classical tangent polynomial(see [10]).

Theorem 2.2. Let $x, p, q \in \mathbb{C}$. Then, the following hold

$$\text{(i)} \qquad \mathcal{T}_{n,p,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q} = \begin{cases} [2]_{p,q} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases},$$

$$\text{(ii)} \qquad \mathcal{T}_{n,p,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) = [2]_{p,q} p^{\binom{n}{2}} x^n.$$

Proof. From the Definition 2.1, we have

$$\begin{aligned} [2]_{p,q} &= (1 + e_{p,q}(2t)) \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\mathcal{T}_{n,p,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q} \right) \frac{t^n}{n!}. \end{aligned}$$

Now comparing the coefficients of t^n we find (i). For (ii) we use the relation

$$\begin{aligned} [2]_{p,q} e_{p,q}(tx) &= (1 + e_{p,q}(2t)) \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left(\mathcal{T}_{n,p,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) \right) \frac{t^n}{n!}, \end{aligned}$$

and again compare the coefficients of t^n.

□

Theorem 2.3. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q} x^k.$$

Proof. From the definition of the (p, q)-exponential function, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(tx) &= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q}(x) x^k \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

The required relation now follows on comparing the coefficients of t^n on both sides.

□

Theorem 2.4. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) x^{n-k}.$$

Proof. From the property of (p,q)-exponential function, it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} &= \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) e_{p^{-1},q^1}(-tx) \\ &= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-1)^n x^n \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) x^{n-k} \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

The required relation now follows immediately.

□

In what follows, we consider (p,q)-derivative of $e_{p,q}(tx)$. Using the Mathematical Induction, we find

$$\text{(i)} \qquad k = 1: \qquad D_{p,q}^{(1)} e_{p,q}(tx) = \sum_{n=1}^{\infty} p^{\binom{n}{2}} x^{n-1} \frac{t^n}{[n-1]_{p,q}!}.$$

$$\text{(ii)} \qquad k = i: \qquad D_{p,q}^{(i)} e_{p,q}(tx) = \sum_{n=i}^{\infty} p^{\binom{n}{2}} x^{n-i} \frac{t^n}{[n-i]_{p,q}!}.$$

If (ii) is true, then it follows that

$$\begin{aligned} \text{(iii)} \qquad k = i+1: \qquad D_{p,q}^{(i+1)} e_{p,q}(tx) &= D_{p,q;x}^{(1)} \left(\sum_{n=i}^{\infty} p^{\binom{n}{2}} x^{n-i} \frac{t^n}{[n-i]_{p,q}!} \right) \\ &= \sum_{n=i+1}^{\infty} p^{\binom{n}{2}} x^{n-(i+1)} \frac{t^n}{[n-(i+1)]_{p,q}!} \\ &= t^{i+1} p^{\binom{i+1}{2}} e_{p,q}(p^{i+1} tx). \end{aligned}$$

We are now in the position to prove the following theorem.

Theorem 2.5. For $k \in \mathbb{N}$, the following holds

$$D_{p,q}^{(k)} \mathcal{T}_{n,p,q}(x) = \frac{[n]_{p,q}!}{[n-k]_{p,q}!} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q}(p^k x).$$

Proof. Considering (p,q)-derivative of $e_{p,q}(tx)$, we find

$$\begin{aligned}
D_{p,q}^{(i+1)}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty}D_{p,q}^{(i+1)}\mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\\
&= \frac{[2]_{p,q}}{e_{p,q}(2t)+1}D_{p,q}^{(i+1)}e_{p,q}(tx)\\
&= t^{i+1}p^{\binom{i+1}{2}}\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(p^{i+1}tx)\\
&= p^{\binom{i+1}{2}}\sum_{n=0}^{\infty}[n+(i+1)]_{p,q}\cdots[n+2]_{p,q}[n+1]_{p,q}\\
&\times \mathcal{T}_{n,p,q}(p^{i+1}x)\frac{t^{n+i+1}}{[n+(i+1)]_{p,q}!}\\
&= p^{\binom{i+1}{2}}\sum_{n=0}^{\infty}\frac{[n]_{p,q}}{[n+(i+1)]_{p,q}!}\mathcal{T}_{n-(i+1),p,q}(p^{i+1}x)\frac{t^n}{[n]_{p,q}!},
\end{aligned}$$

which immediately gives the required result.

□

Theorem 2.6. Let a,b be any real numbers. Then, we have

$$\int_a^b \mathcal{T}_{n,p,q}(x)d_{p,q}x = \sum_{k=0}^{n+1}\frac{p^k}{p^n[n+1]_{p,q}}\left(\mathcal{T}_{n+1,p,q}(b)-\mathcal{T}_{n+1,p,q}(a)\right).$$

Proof. From Theorem 2.3, we find

$$\begin{aligned}
\int_a^b \mathcal{T}_{n,p,q}(x)d_{p,q}x &= \int_a^b\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathcal{T}_{k,p,q}x^{n-k}d_{p,q}x\\
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathcal{T}_{k,p,q}\frac{1}{[n-k+1]_{p,q}}x^{n-k+1}\Big|_a^b\\
&= \sum_{k=0}^{n+1}\frac{p^k(\mathcal{T}_{n+1,p,q}(b)-\mathcal{T}_{n+1,p,q}(a))}{p^n[n+1]_{p,q}}.
\end{aligned}$$

□

3. Some properties of the (p,q)-tangent polynomial in two parameters

In this section, we shall study the (p,q)-tangent polynomials involving two parameters. We shall also find some important relations between these polynomials and other polynomials.

Definition 3.1. For $x, y \in \mathbb{C}$, we define (p,q)-tangent polynomial with two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx)e_{p,q}(ty), \quad |t| < \frac{\pi}{2}.$$

From the Definition 3.1, it is clear that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,0)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx),$$
$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(0,0)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1},$$

where $\mathcal{T}_{n,p,q}$ is (p,q)-tangent number. We also note that the original tangent number, $\mathcal{T}_n$,

$$\lim_{q\to 1} \sum_{n=0}^{\infty} \mathcal{T}_{n,1,q}\frac{t^n}{[n]_{1,q}!} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1},$$

where $p = 1$ and $q \to 1$.

Theorem 3.2. Let x, y be any complex numbers. Then, the following hold

(i) $\mathcal{T}_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q}(x) y^k,$

(ii) $\mathcal{T}_{n,p,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{T}_{n-l,p,q} \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} p^{\frac{(l+k)^2-l}{2}} x^{l-k} y^k.$

Proof. From the Definition 3.1, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx)e_{p,q}(ty)$$
$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} y^n \frac{t^n}{[n]_{p,q}!}.$$

Using Cauchy's product and the method of coefficient comparison in the above relation, we find (i). Next, we transform (p,q)-tangent polynomials in two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx)e_{p,q}(ty)$$
$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}\frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} y^n \frac{t^n}{[n]_{p,q}!}.$$

Now following same procedure as in (i), we obtain (ii).

□

Theorem 3.3. Setting $y = 2$ in (p, q)-tangent polynomials with two parameters, the following relation holds

$$[2]_{p,q} p^{\binom{n}{2}} x^n = \mathcal{T}_{n,p,q}(x, 2) + \mathcal{T}_{n,p,q}(x).$$

Proof. Using (p, q)-tangent polynomials and its polynomials with two parameters, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x, 2) \frac{t^n}{[n]_{p,q}!} + \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \frac{[2]_{p,q} e_{p,q}(2t)}{e_{p,q}(2t) + 1} e_{p,q}(tx) + \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(tx) \\ &= [2]_{p,q} e_{p,q}(tx)\end{aligned}$$

Now from the definition of (p, q)-exponential function, the required relation follows.

□

Theorem 3.3 is interesting as it leads to the relation

$$x^n = \frac{\mathcal{T}_{n,p,q}(x, 2) + \mathcal{T}_{n,p,q}(x)}{[2]_{p,q} p^{\binom{n}{2}}}.$$

Theorem 3.4. Let $\left|\frac{q}{p}\right| < 1$. Then, the following holds

$$\mathcal{T}_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k p^{\binom{k}{2}} \mathcal{T}_{k,1,\frac{p}{q}}(2) x^{n-k}.$$

Proof. To prove the relation, we note that

$$e_{1,\frac{p}{q}}(-2t) = \mathcal{E}_{p,q}(-2t),$$

where $\mathcal{E}_{p,q}(t) = e_{p^{-1},q^{-1}}(t)$. Using the above equation we can represent the (p, q)-tangent polynomials as

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(tx) \\ &= \frac{[2]_{p,q}}{1 + \mathcal{E}_{p,q}(-2t)} \mathcal{E}_{p,q}(-2t) e_{p,q}(tx) \\ &= \frac{[2]_{p,q}}{e_{1,\frac{p}{q}}(-2t) + 1} e_{1,\frac{p}{q}}(-2t) e_{p,q}(tx) \\ &= \sum_{n=0}^{\infty} \mathcal{T}_{n,1,\frac{p}{q}}(2) \frac{(-t)^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k p^{\binom{k}{2}} T_{k,1,\frac{p}{q}}(2) x^{n-k} \right\} \frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which leads to the required relation immediately.

□

Now we shall find relations between (p,q)-tangent polynomials and others polynomials. For this, first we introduce well known polynomials by using (p,q)-numbers.

Definition 3.5. We define (p,q)-Euler polynomials, $E_{n,p,q}(x)$, and (p,q)-Bernoulli polynomials, $B_{n,p,q}(x)$, as

$$\sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(t)+1}e_{p,q}(tx), \quad |t|<\pi,$$
$$\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{t}{e_{p,q}(t)-1}e_{p,q}(tx), \quad |t|<2\pi.$$

Theorem 3.6. For $x,y\in\mathbb{C}$, the following relation holds

$$\mathcal{T}_{n,p,q}(x,y)$$
$$= \frac{1}{[2]_{p,q}}\sum_{l=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(\frac{\mathcal{T}_{n-l,p,q}(x)}{m^l}+\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{p,q}\frac{p^{\binom{n-l-k}{2}}\mathcal{T}_{k,p,q}(x)}{m^{n-k}}\right)E_{l,p,q}(my).$$

Proof. Transforming (p,q)-tangent polynomials containing two parameters, we find

$$\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(tx)e_{p,q}(ty)$$
$$= \left(\frac{[2]_{p,q}}{e_{p,q}(\frac{t}{m})+1}e_{p,q}(ty)\right)\left(\frac{e_{p,q}(\frac{t}{m})+1}{[2]_{p,q}}\right)\left(\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(tx)\right).$$

Thus, for the relation between (p,q)-tangent polynomials of two parameters and (p,q)-Euler polynomials, we have

$$\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty}E_{n,p,q}(my)\frac{t^n}{m^n[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty}\frac{1}{[2]_{p,q}}p^{\binom{n}{2}}\frac{t^n}{m^n[n]_{p,q}!}+\frac{1}{[2]_{p,q}}\right)$$
$$= \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}E_{l,p,q}(my)\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{p,q}p^{\binom{n-l-k}{2}}\frac{\mathcal{T}_{k,p,q}(x)}{m^{n-k}}\frac{t^n}{[n]_{p,q}!}$$
$$+\frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}E_{l,p,q}(my)\frac{\mathcal{T}_{n-l,p,q}(x)}{m^l}\frac{t^n}{[n]_{p,q}!},$$

which on comparing the coefficients immediately gives the required relation.

□

Corollary 3.7. From Theorem 3.6, the following hold

$$\text{(i)}\quad \mathcal{T}_{n,q}(x,y)=\frac{1}{[2]_q}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\left(\frac{\mathcal{T}_{n-l,q}(x)}{m^l}+\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}\right)E_{l,q}(my).$$

$$\text{(ii)}\quad \mathcal{T}_{n}(x,y)=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\mathcal{T}_{n-l}(x)}{m^l}+\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\mathcal{T}_{k}(x)}{m^{n-k}}\right)E_{l}(my).$$

Theorem 3.8. For $x,y\in\mathbb{C}$, the following relation holds

$$\mathcal{T}_{n-1,p,q}(x,y)$$
$$=\frac{1}{[n]_{p,q}}\sum_{l=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{p,q}\frac{p^{\binom{n-l-k}{2}}\mathcal{T}_{k,p,q}(x)}{m^{n-k}}-\frac{\mathcal{T}_{n-l,p,q}(x)}{m^l}\right)B_{l,p,q}(my).$$

Proof. We note that

$$\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(tx)e_{p,q}(ty)$$
$$=\left(\frac{t}{e_{p,q}(\frac{t}{m})-1}e_{p,q}(ty)\right)\left(\frac{e_{p,q}(\frac{t}{m})-1}{t}\right)\left(\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(tx)\right).$$

Thus as in Theorem 3.6, we have

$$\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{t^{n-1}}{m^n[n]_{p,q}!}-\frac{1}{t}\right)\sum_{n=0}^{\infty}B_{n,p,q}(my)\frac{t^n}{m^n[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{p,q}p^{\binom{n-l-k}{2}}\frac{\mathcal{T}_{k,p,q}(x)}{m^{n-k}}B_{l,p,q}(my)\right)\frac{t^{n-1}}{[n]_{p,q}!}$$
$$-\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}\frac{\mathcal{T}_{n-l,p,q}(x)}{m^l}B_{l,p,q}(my)\right)\frac{t^{n-1}}{[n]_{p,q}!}.$$

The required relation now follows on comparing the coefficients.

□

Corollary 3.9. From the Theorem 3.8, the following relations hold

$$\text{(i)}\quad \mathcal{T}_{n-1,q}(x,y)=\frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}-\frac{\mathcal{T}_{n-l,q}(x)}{m^l}\right)B_{l,q}(my).$$

$$\text{(ii)}\quad \mathcal{T}_{n-1}(x,y)=\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\mathcal{T}_{k}(x)}{m^{n-k}}-\frac{\mathcal{T}_{n-l}(x)}{m^l}\right)B_{l}(my).$$

4. The observation of scattering zeros of the (p,q)-tangent polynomials

In this section, our aim is to find zeros of the (p,q)-tangent polynomials. From this work, we can investigate relations between (p,q)-tangent polynomials and classical tangent polynomials. In addition, we shall observe scattering of zeros of the (p,q)-tangent polynomials in three dimension. For this, we use Theorem 2.2 to calculate some elements of (p,q)-tangent numbers and polynomials. The first few (p,q)-tangent numbers are

$$\begin{aligned}
\mathcal{T}_{0,p,q} &= \frac{[2]_{p,q}}{3},\\
\mathcal{T}_{1,p,q} &= -\frac{2[2]_{p,q}}{9},\\
\mathcal{T}_{2,p,q} &= -\frac{4[2]_{p,q}}{9}p + \frac{4[2]_{p,q}^2}{27},\\
\mathcal{T}_{3,p,q} &= -\frac{72[2]_{p,q}}{81}p^3 + \left(\frac{24[2]_{p,q}^2}{81} - \frac{8[2]_{p,q}[3]_{p,q}}{81}\right)p - \frac{8[2]_{p,q}[3]_{p,q}}{81},\\
&\cdots .
\end{aligned}$$

To compute (p,q)-tangent polynomials we employ Mathematica. The first few (p,q)-tangent polynomials are

$$\begin{aligned}
\mathcal{T}_{0,p,q}(x) &= \frac{p+q}{2},\\
\mathcal{T}_{1,p,q}(x) &= \frac{1}{2}(p+q)(-1+x),\\
\mathcal{T}_{2,p,q}(x) &= \frac{1}{2}(-q^2(-1+x)+pq(-2+x)x+p^2(-1-x+x^2)),\\
\mathcal{T}_{3,p,q}(x) &= \frac{1}{2}(p+q)(q^3(-1+x)-p^2q(-2+x^2)-pq^2(-2+x^2)+p^3(-1-x-x^2+x^3)),\\
&\cdots .
\end{aligned}$$

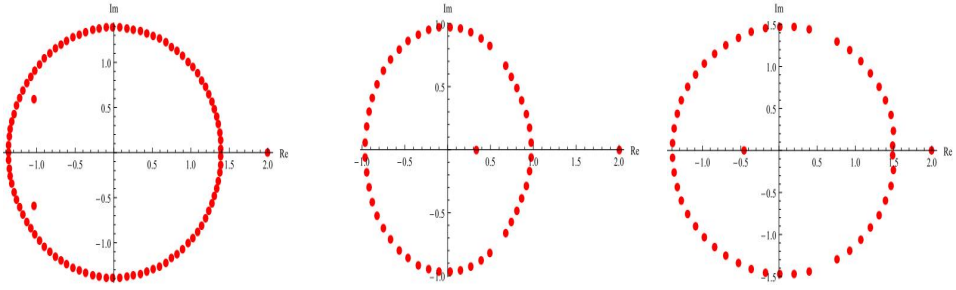


Figure 1: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $q = 0.19, 0.39, 0.59$

Using computer we can investigate the zeros of $\mathcal{T}_{n,p,q}(x)$. Here, our expectation is that a plot of $\mathcal{T}_{n,p,q}(x)$ will approach to a plot of $T_n(x)$ when $p = 1$ and $q \to 1$. We let $n = 100$. In Figure 1, for $q = 0.19, 0.39, 0.59$, we observe from left to the right that the middle shape is similar to a sphere, but it seems like an ellipse because one of real roots is exists near to 2.

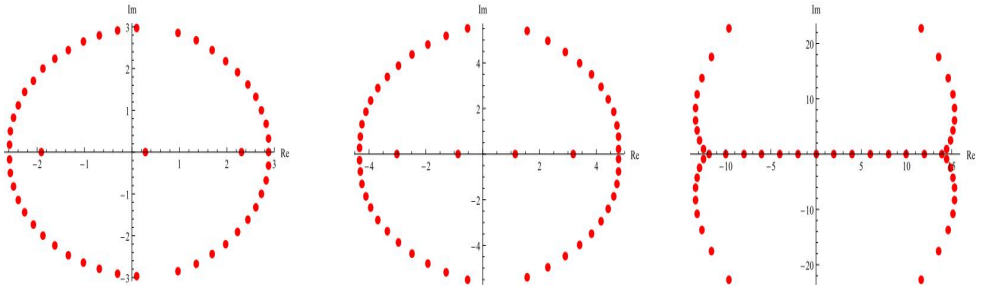


Figure 2: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $q = 0.79, 0.89, 0.99$

A similar pattern we see in Figure 2 for $q = 0.79, 0.89, 0.99$. From Figures 1 and 2, it is clear that zeros of $T_{n,p,q}(x)$ are very similar to zeros of $T_n(x)$ given in([11]) for $n = 100, p = 1, q = 0.99$. We can also expect that $Im(x) = 0$ of (p, q)-tangent polynomials have reflective and symmetric properties. An interesting point is the location of roots. We can see the empty spot in the middle of Figure 1 and observe spread of empty space in Figure 2.

For $n = 100, p = 1$, and $q = 0.19$, the approximate zeros of (p, q)-tangent polynomials are listed in the following table.

degree	Approximate roots of (p, q)-tangent polynomials for $p = 1, q = 0.19$
100	-1.36843, -1.36556-0.087566i, -1.36556+0.087566i, -1.35692-0.174899i, -1.35692+0.174899i, -1.34243-0.261765i, -1.34243+0.261765i, -1.32191-0.347931i, -1.32191+0.347931i, -1.29512-0.433163i, -1.29512+0.433163i, -1.26178-0.517337i, -1.26178+0.517337i, -1.2219-0.600756i, -1.2219+0.600756i, -1.17691-0.683948i, -1.17691+0.683948i, -1.12913-0.764992i, -1.12913+0.764992, -1.07803-0.84129i, -1.07803+0.84129i, -1.03576-0.587811i, -1.03576+0.587811i, -1.02268-0.912499i-1.02268+0.912499i, -0.963125-0.978877i, -0.963125+0.978877i, -0.899648-1.0407i, -0.899648+1.0407i, -0.832427-1.09814i, -0.832427+1.09814i, -0.761516-1.15116i,-0.761516+1.15116i, -0.68703-1.19947i, -0.68703+1.19947i, -0.609252-1.24274i, -0.609252+1.24274i, -0.528594-1.28064i, -0.528594+1.28064, -0.445526-1.31298i, -0.445526+1.31298i, -0.36053-1.33962i, -0.36053+1.33962i,

	-0.274073-1.36053i, -0.274073+1.36053i, -0.186589-1.37571i, -0.186589+1.37571i, -0.0984547-1.38522i-0.0984547+1.38522i, -0.0099887-1.38914i, -0.0099887+1.38914i, 0.0785479-1.38753i, 0.0785479+1.38753i, 0.166936-1.38039i, 0.166936+1.38039i, 0.254936-1.36763i, 0.254936+1.36763i, 0.342202-1.34913i, 0.342202+1.34913i, 0.428262-1.32482i, 0.428262+1.32482i, 0.512581-1.29476i, 0.512581+1.29476i, 0.594658-1.25916i, 0.594658+1.25916i, 0.674095-1.21831i, 0.674095+1.21831i, 0.750625-1.17257i, 0.750625+1.17257i, 0.8241-1.1222i, 0.8241+1.1222i, 0.894413-1.06734i, 0.894413+1.06734i, 0.961386-1.00806i, 0.961386+1.00806i, 1.02471-0.944367i, 1.02471+0.944367i, 1.08399-0.876392i, 1.08399+0.876392i, 1.13879-0.804367i, 1.13879+0.804367i, 1.18869-0.728642i, 1.18869+0.728642i, 1.23339-0.649655i, 1.23339+0.649655i, 1.27264-0.567879i, 1.27264+0.567879i, 1.30629-0.48378i, 1.30629+0.48378i, 1.33425-0.397798i, 1.33425+0.397798i, 1.35643-0.310367i, 1.35643+0.310367i, 1.37275-0.221972i, 1.37275+0.221972i, 1.38331-0.133155i, 1.38331+0.133155i, 1.3884-0.0443556i, 1.3884+0.0443556i, 2

The above table for $\mathcal{T}_{100,1,0.19}(x)$ shows that there are only two real roots, and the shape is similar to a sphere.

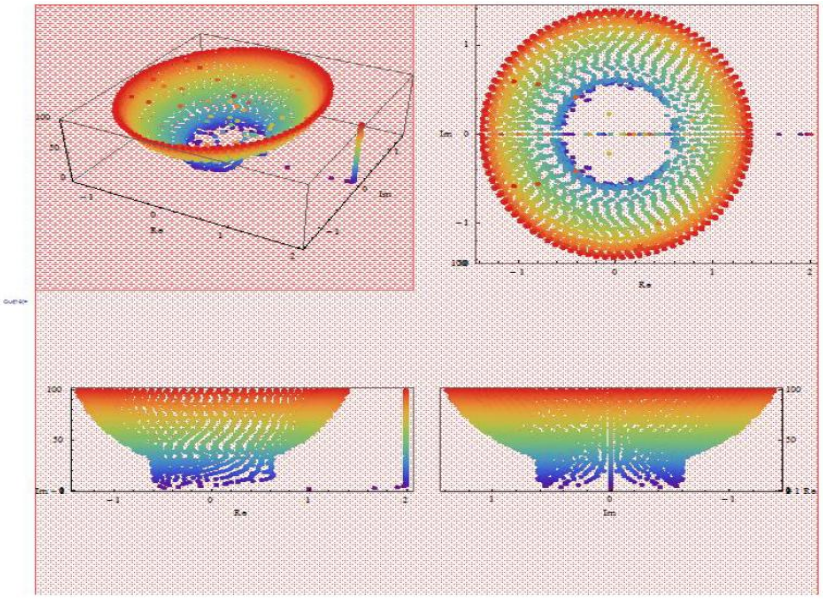


Figure 3: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $p = 1, q = 0.19, 1 < n < 100$

From Figure 3 and the following table, we observe that (p,q)-tangent polynomials for different values of n do not have a regular pattern of the number of zeros.

degree	Approximate roots of (p,q)-tangent polynomials for $p=1, q=0.19$
100	-1.36843, 2
99	2
98	-1.37033, 0.141925, 1.39158, 2
97	-1.37233, 1.38851, 2
96	1.37639, 2
95	-1.16958, 1.36103, 2
94	1.32543, 2
93	1.32543, 2
92	0.362552, 1.33302, 2

In Figure 4, we stack the zeros of (p,q)-tangent polynomials for $1 \leq n \leq 100$. We put $q = 0.19$(the top-left), 0.39(the top-middle), 0.59(the top-right),0.79(the bottom-left), 0.89(the bottom-middle), 0.99(the bottom-right) and $p = 1$. We observe scattering of zeros of (p,q)-tangent polynomials including a stick when $q \leq 0.59$. We expect that this stick exists for all values of q smaller than 0.59. From Figures 1-3, it appears that the shape of zeros of (p,q)-tangent polynomials is similar to a sphere for all large values of n and small values of q.

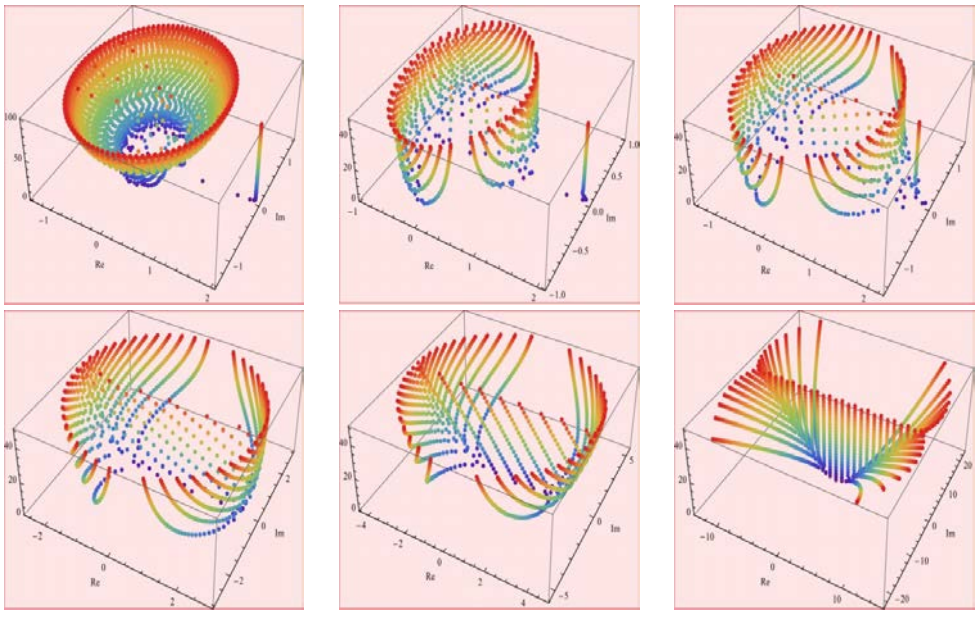


Figure 4: Stacks of zeros of $\mathcal{T}_{n,p,q}(x)$ for $q = 0.19, 0.39, 0.59, 0.79, 0.89, 0.99$

Figure 5 shows the front view of Figure 4.

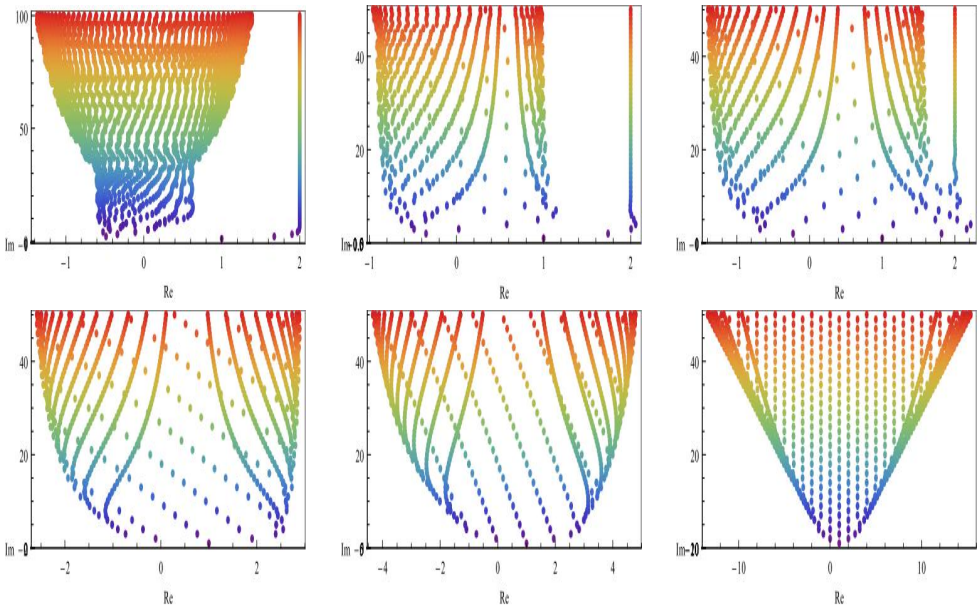


Figure 5: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $1 \leq n \leq 100$

Conclusion

Our first main contribution in the work is to show that the zeros of (p,q)-tangent polynomials approach to zeros of classical tangent polynomials as $q \to 1$. To support our claim we obtain some specific results. Next, we show that the shape of the roots in (p,q)-tangent polynomials is almost identical to a circle when q is small. We also observe that as q is large the zeros are separated into two parts maintaining the symmetry.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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R. P. Agarwal,
Department of Mathematics,
Texas A & M University,
Kingsville, Texas 78363-8202
E-mail : agarwal@tamuk.edu

Jung Yoog Kang,
Department of Mathematics,
Texas A & M University,
Kingsville, Texas 78363-8202
E-mail : KUJK2016@tamuk.edu

Cheon Seoung Ryoo,
Department of Mathematics,
Hannam University, Daejeon, Republic of Korea
E-mail : ryoocs@hnu.kr

Strong Convergence Theorems for a Non-convex Hybrid Method for Quasi-Lipschitz Mappings and Applications

Waqas Nazeer[1], Shin Min Kang[2,3,*] Mobeen Munir[4] and Samina Kausar[5]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[3]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[5]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: sminasaddique@gmail.com

Abstract

In this note, the strong convergence theorems of a non-convex hybrid iteration algorithm corresponding to Noor iterative scheme about common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space has been proved. Moreover some applications of developed algorithm is presented.

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1 Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest on its on and has a various applications in related fields like image recovery, signal processing and geometry of objects [13]. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration

*Corresponding author

scheme has been applied in intensity modulated radiation therapy optimization optimization to pre-compute dose-deposition coefficient matrix, see [12]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now. Constructive fixed point theorems (for example, Banach fixed point theorem) which not only claims the existence of a fixed point but yields an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some mapping f that is contracting with respect to some (complete) metric will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see [6]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and guarantee the strong convergence (see [2–5, 7–9, 11] and references therein).

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi [9] in 2003. They introduced this modification for only one nonexpansive mapping, where Kim and Xu [4] introduced a variant for asymptotically nonexpansive mappings in a Hilbert space in 2006. In the same year Martinez-Yanes and Xu [7] introduced a variation of the Ishikawa iteration process for a nonexpansive mapping in a Hilbert space. They also gave a variant of the Halpern method. Su and Qin [11] gave a hybrid iteration process for nonexpansive mappings which is monotone. Liu et al. [5] gave a novel method for a finite family of quasi-asymptotically pseudo-contractive mappings.

Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Let $P_c(\cdot)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be *nonexpensive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T. It is well known that $F(T)$ is closed and convex. A mapping $T : C \to C$ is said to be *quasi-Lipschitz* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq L\|x - p\|$ for all $x \in C,\ p \in F(T)$, where $1 \leq L < \infty$ is a constant. If $L = 1$, then T is known as *quasi-nonexpansive*. It is well-known that T is said to be *closed* if $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. It is obvious that a weak closed mapping should be closed, but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings from C into itself with a nonempty common fixed points set F. Then $\{T_n\}$ is said to be *uniformly closed* if for any convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

In 1953 Mann [6] proposed an iterative scheme given as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

Guan et al. [3] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

and proved some strong convergence results about common fixed points relating to a family of countable uniformly closed asymptotically quasi-Lipschitz mappings in H. They applied their results for the finite case to obtain fixed points.

The Noor iterative scheme [10] was defined in 2000 as

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_nT(y_n), \\ y_n = (1-b_n)x_n + b_nT(z_n), \\ z_n = (1-c_n)x_n + c_nT(x_n), \quad n \geq 0. \end{cases}$$

In this article, we established a non-convex hybrid algorithm corresponding to Noor iterative scheme. Then we also establish strong convergence theorems with proofs about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space. Some applications of presented algorithm were also given.

2 Main results

In this section we formulate our main results.

Definition 2.1. Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself. $\{T_n\}$ is said to be *asymptotically* if $\lim_{n\to\infty} L_n = 1$.

The following lemmas are well known and useful for our conclusions

Proposition 2.2. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if we have the relation $\langle x - z, z - y\rangle \geq 1$ for all $y \in C$.*

Proposition 2.3. ([3]) *Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Then the common fixed point set F is closed and convex.*

Proposition 2.4. *Let C be a closed convex subset of a Hilbert space H. For any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$ for all $z \in C$.*

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\} : C \to C$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{chosen arbitrarily}, \\ y_n = (1-a_n)x_n + a_nT_nz_n, \quad n \geq 0, \\ z_n = (1-b_n)x_n + b_nT_nt_n, \quad n \geq 0, \\ t_n = (1-c_n)x_n + c_nT_nx_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1-b_n) + L_n^2(1-c_n)b_n \\ \qquad +b_nc_nL_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We divide our proof in following seven steps.

Step 1. It is obvious that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-a_n)x_n + a_n T_n z_n - p\| \\
&= \|(1-a_n)x_n + a_n T_n[(a-b_n)x_n + b_n T_n t_n] - p\| \\
&= \|(1-a_n)x_n + a_n T_n[(a-b_n)x_n + b_n T_n((1-c_n)x_n + c_n T_n x_n)] - p\| \\
&= \|(1-a_n)(x_n - p) + (a_n - a_n b_n)(T_n x_n - p) \\
&\quad + (a_n b_n - a_n b_n c_n)(T_n^2 x_n - p) + a_n b_n c_n)(T_n^3 x_n - p)\| \\
&\leq (1-a_n)\|x_n - p\| + (a_n - a_n b_n)L_n\|x_n - p\| \\
&\quad + (a_n b_n - a_n b_n c_n)L_n^2\|x_n - p\| + a_n b_n c_n)L_n^3\|x_n - p\| \\
&= [1 + (L_n(1-b_n) + L_n^2(1-c_n)b_n + b_n c_n L_n^3 - 1)a_n]\|x_n - p\|,
\end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

Step 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

Step 3. We prove that $\{x_n\}$ is bounded. Using closeness and convexity properties of F, we can say that we have a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This shows that $\{x_n\}$ is bounded.

Step 4. We show that $\{x_n\}$ converges strongly to a point of C by showing that $\{x_n\}$ is a Cauchy sequenc. As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq$

0 and hence

$$\begin{aligned}\|x_{n+m}-x_n\|^2 &= \|(x_{n+m}-x_0)-(x_n-x_0)\|^2\\ &\le \|x_{n+m}-x_0\|^2-\|x_n-x_0\|^2-2\langle x_{n+m}-x_n,x_n-x_0\rangle\\ &\le \|x_{n+m}-x_0\|^2-\|x_n-x_0\|^2\to 0,\ n\to\infty\end{aligned}$$

for any $m\ge 1$. Therefore $\{x_n\}$ is a Cauchy sequence in C, then there exists a point $q\in C$ such that $\lim_{n\to\infty}x_n=q$.

Step 5. We show that $y_n\to q$, as $n\to\infty$. Let

$$D_n=\{z\in C:\|y_n-z\|^2\le\|x_n-z\|^2+(L_n^3-1)(L_n^3+1)\}.$$

From the definition of D_n, we have

$$\begin{aligned}D_n&=\{z\in C:\langle y_n-z,y_n-z\rangle\le\langle x_n-z,x_n-z\rangle+(L_n^3-1)(L_n^3+1)\}\\ &=\{z\in C:\|y_n\|^2-2\langle y_n,z\rangle+\|z\|^2\le\|x_n\|^2-2\langle x_n,z\rangle+\|z\|^2+(L_n^3-1)(L_n^3+1)\}\\ &=\{z\in C:2\langle x_n-y_n,z\rangle\le\|x_n\|^2-\|y_n\|^2+(L_n^3-1)(L_n^3+1)\}.\end{aligned}$$

This shows that D_n is convex and closed, $n\in\mathbb{Z}^+\cup\{0\}$. Next, we want to prove that $C_n\subset D_n$, $n\ge 0$.

In fact, for any $z\in C_n$, we have

$$\begin{aligned}\|y_n-z\|^2&\le[1+(L_n(1-b_n)+L_n^2(1-c_n)b_n+b_nc_nL_n^3-1)a_n]^2\|x_n-z\|^2\\ &=\|x_n-z\|^2+[2(L_n(1-b_n)+L_n^2(1-c_n)b_n+b_nc_nL_n^3-1)a_n\\ &\quad+(L_n(1-b_n)+L_n^2(1-c_n)b_n+b_nc_nL_n^3-1)^2a_n^2]\|x_n-z\|^2\\ &\le\|x_n-z\|^2+[2(L_n^3-1)+(L_n^3-1)^2]\|x_n-z\|^2\\ &=\|x_n-z\|^2+(L_n^3-1)(L_n^3+1)\|x_n-z\|^2.\end{aligned}$$

From

$$\begin{aligned}C_n=\{z\in C:\|y_n-z\|\le[1+(L_n(1-b_n)+L_n^2(1-c_n)b_n\\ +b_nc_nL_n^3-1)a_n]\|x_n-z\|\}\cap A,\quad n\ge 0,\end{aligned}$$

we have $C_n\subset A$, $n\ge 0$. Using convexity of A, we have $\overline{co}C_n\subset A$, $n\ge 0$. Consider $x_n\in\overline{co}C_{n-1}$, we know that

$$\begin{aligned}\|y_n-z\|&\le\|x_n-z\|^2+(L_n^3-1)(L_n^3+1)\|x_n-z\|^2\\ &\le\|x_n-z\|^2+(L_n^3-1)(L_n^3+1).\end{aligned}$$

This implies that $z\in D_n$ and hence $C_n\subset D_n$, $n\ge 0$. Since D_n is convex, we have $\overline{co}(C_n)\subset D_n$, $n\ge 0$. Therefore

$$\|y_n-x_{n+1}\|^2\le\|x_n-x_{n+1}\|^2+(L_n^3-1)(L_n^3+1)\to 0$$

as $n\to\infty$. That is, $y_n\to q$ as $n\to\infty$.

Step 6. To prove that $q\in F$, we use definition of y_n. So we have

$$(a_n+a_nb_nT_n+a_nb_nc_nT_n^2)\|T_nx_n-x_n\|=\|y_n-x_n\|\to 0$$

as $n \to \infty$. Since $a_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\begin{aligned}\|z_0 - x_n\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\ &= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$, which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, which is a contradiction. This completes the proof. □

In [3], we show an example of C_n which does not involve a convex subset.

Corollary 2.6. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a closed quasi-nonexpansive mapping from C into itself. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{chosen arbitrarily}, \\ y_n = (1 - a_n)x_n + a_n T z_n, \quad \geq 0, \\ z_n = (1 - b_n)x_n + b_n T t_n, \quad n \geq 0, \\ t_n = (1 - c_n)x_n + c_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n \\ \qquad +b_n c_n L_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is convex and closed for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.6. □

Corollary 2.7. *Let C be a closed convex subset of a Hilbert space H, and let T be a nonexpansive mapping from C into itself. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{chosen arbitrarily}, \\ y_n = (1 - a_n)x_n + a_n T z_n, \quad n \geq 0, \\ z_n = (1 - b_n)x_n + b_n T t_n, \quad n \geq 0, \\ t_n = (1 - c_n)x_n + c_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n \\ \qquad +b_n c_n L_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

3 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C,\, p \in F,$$

where F is the common fixed point set of $\{T_n\}_{n=0}^{N-1}$ and $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is said to be *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C$$

for all $i \in \{0, 1, 2, ..., N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a uniformly L-Lipschitz finite family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{chosen arbitrarily,} \\ y_n = (1-a_n)x_n + a_n T_{i(n)}^{j(n)} z_n, \quad n \geq 0, \\ z_n = (1-b_n)x_n + b_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ t_n = (1-c_n)x_n + c_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_{i(n),j(n)}(1-b_n) + k_{i(n),j(n)}^2(1-c_n)b_n \\ \qquad +b_n c_n k_{i(n),j(n)}^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$, $n = (j(n)-1)N + i(n)$ for all $n \geq 0$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We can acquire the prove from the conclusions:
Conclusion 1. $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is asymptotically family of uniformly closed countable quasi-L_n-Lipschitz mappings from C into itself.
Conclusion 2. $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ is the fixed point set of T_n. □

Corollary 3.2. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated*

by

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{chosen arbitrarily}, \\ y_n = (1-a_n)x_n + a_n T^n z_n, \quad n \geq 0, \\ z_n = (1-b_n)x_n + b_n T^n t_n, \quad n \geq 0, \\ t_n = (1-c_n)x_n + c_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_n(1-b_n) + k_n^2(1-c_n)b_n \\ \qquad +b_n c_n k_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in 3.1, we get the desired result. □

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Certain subclasses of k-uniformly starlike functions associated with symmetric q-derivative operator

Nanjundan Magesh[1], Şahsene Altınkaya[2,*], Sibel Yalçın[2]
[1]Post-Graduate and Research Department of Mathematics
Government Arts College for Men
Krishnagiri 635001, Tamilnadu, India.
e-mail: nmagi_2000@yahoo.co.in
[2]Department of Mathematics, Faculty of Arts and Science,
Uludag University, 16059, Bursa, Turkey.
*corresponding author. e-mail: sahsene@uludag.edu.tr
[2]syalcin@uludag.edu.tr

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Abstract

In this paper, we have established the inclusion relations for k-uniformly starlike functions under the $(\widetilde{D}_q f)(z)$ operator. We define two new subclass of k-uniformly starlike functions of order α. Moreover, for functions belonging to these function classes, we investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points.

Keywords: Analytic functions, k-uniformly starlike functions, symmetric q-derivative operator.

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1 Introduction, Definitions and Notations

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and S be the subclass of A consisting of the form (1) which are also univalent in U. A function $f \in A$ is called starlike of order α, $0 \leq \alpha < 1$, if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ $(z \in U)$. We denote by $S(\alpha)$ the subset of A consisting of all

starlike functions of order α. For $\alpha = 0$ we get the class S of functions f that maps U onto a starlike domain with respect to the origin.

Also represent by T the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{2}$$

A function $f \in A$ is said to be in $US(k, \alpha)$, the class of k-uniformly starlike functions of order α, $0 \leq \alpha < 1$, if f satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (k \geq 0),$$

(for details see [5]).

We remark here that the class of k-uniformly starlike functions is an extension of the relatively more familiar class of uniformly starlike functions investigated earlier by (for example) Goodman [9], Rønning [18], (see also the more recent contributions on this function class by Srivastava and Mishra [21] and others [11, 12, 19]).

In the field of Geometric Function Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [20]). In fact, the theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, [1, 2, 4, 14] and [16]). In particular, Purohit and Raina [17] investigated applications of fractional q-calculus operators to define several classes of functions which are analytic in the open unit disk $\mathbb{U}$. On the other hand, Mohammed and Darus [13] studied approximation and geometric properties of these q-operators in regard to some subclasses of analytic functions in a compact disk.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology as in [8]. We recall the definitions of fractional q-calculus operators of complex valued function $f(z)$.

Definition 1 *Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^{\lambda}}{1 - q} & (\lambda \in \mathbb{C}) \\ \\ \displaystyle\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N} = \{1, 2, \ldots\}). \end{cases} \tag{3}$$

Definition 2 *Let $q \in (0,1)$ and define the q-fractional $[n]_q!$ by*

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}.$$

Definition 3 *For $\alpha \in \mathbb{C}$, the q-shifted factorial is defined as a product of $n \in \mathbb{N}$ factors by*

$$(\alpha;q)_0 = 1, \quad (\alpha;q)_n = \prod_{i=0}^{n-1}(1-\alpha q^i), \quad (\alpha;q)_\infty = \prod_{i=0}^{\infty}(1-\alpha q^i).$$

Definition 4 *(see [10]) The q-derivative of a function f is defined on a subset of $\mathbb{C}$ is given by*

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad if \quad z \neq 0, \tag{4}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{df(z)}{dz}$$

if f is differentiable. From (4), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{5}$$

Definition 5 *(see [6]) The symmetric q-derivative $\widetilde{D}_q f$ of a function f given by (1) is defined as follows:*

$$(\widetilde{D}_q f)(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}, \qquad if \quad z \neq 0, \tag{6}$$

and $(\widetilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.

From (6), we deduce that

$$(\widetilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1},$$

where the symbol $\widetilde{[n]}_q$ denotes the number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

frequently occurring in the study of q-deformed quantum mechanical simple harmonic oscillator (see [7]).

The following properties hold

$$\widetilde{D}_q(f(z)+g(z)) = (\widetilde{D}_q f)(z) + (\widetilde{D}_q g)(z)$$

$$\begin{aligned}\widetilde{D}_q(f(z)g(z)) &= g(q^{-1}z)(\widetilde{D}_q f)(z) + f(qz)(\widetilde{D}_q g)(z) \\ &= g(qz)(\widetilde{D}_q f)(z) + f(q^{-1}z)(\widetilde{D}_q g)(z)\end{aligned}$$

$$\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}.$$

Finally, we have the following relation

$$\widetilde{D}_q f(z) = D_{q^2} f(q^{-1}z).$$

Definition 6 *Let $0 \leq k < \infty$ and $0 \leq \alpha < 1$. By $US(q;k,\lambda,\alpha)$ we denote the class of functions $f \in A$ satisfying the condition*

$$\Re\left(\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)} - \alpha\right) > k\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)} - 1\right|, \ (z \in U).$$

We also let $UTS(q;k,\lambda,\alpha) = US(q;k,\lambda,\alpha) \cap T$.

We note that

$$\lim_{q\to 1^-} US(q;k,\lambda,\alpha) = US(k,\lambda,\alpha)$$

where $US(k,\lambda,\alpha)$ is the class of defined by Murugusundaramoorthy and Magesh [15].

Murugusundaramoorthy and Magesh [15] and Srivastava and Mishra [21] defined the new subclasses of the families UCV and US making use of hypergeometric functions and fractional calculus respectively and obtained various interesting properties. In light of this in this paper, we study the classes $US(q;k,\lambda,\alpha)$ and $UTS(q;k,\lambda,\alpha)$ defining by symmetric q-derivative operator. We provide necessary and sufficent coefficient conditions, distortion bounds, extreme points for functions in $UTS(q;k,\lambda,\alpha)$.

2 Main Results

Theorem 7 *Let $f \in A$ be given by (1). If the inequality*

$$\sum_{n=2}^{\infty}\left[\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)\right]|a_n| \leq 1-\alpha \tag{7}$$

holds true for some k $(0 \leq k < \infty)$ and α $(0 \leq \alpha < 1)$, then $f \in US(q;k,\lambda,\alpha)$.

From Definition 6, it suffices to prove that

$$k\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right|-\Re\left(\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right)\leq 1-\alpha.$$

Observe that

$$\begin{aligned}
&k\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right|-\Re\left(\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right)\\
&\leq (k+1)\left|\tfrac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right|\\
&=(k+1)\left|\frac{\sum\limits_{n=2}^{\infty}\left(\widetilde{[n]}_q-\lambda\widetilde{[n]}_q-1+\lambda\right)a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\left(1-\lambda+\lambda\widetilde{[n]}_q\right)a_n z^{n-1}}\right|;\ (|z|<1)\\
&\leq (k+1)\frac{\sum\limits_{n=2}^{\infty}\left(\widetilde{[n]}_q-\lambda\widetilde{[n]}_q-1+\lambda\right)|a_n|}{1-\sum\limits_{n=2}^{\infty}\left(1-\lambda+\lambda\widetilde{[n]}_q\right)|a_n|}.
\end{aligned}$$

The last expression is bounded by $1-\alpha$ if the inequality (7) holds.

The next Corollary can be easily obtained from Theorem 7.

Corollary 8 *Let $f(z)=z+a_n z^n$. If*

$$|a_n|\leq\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)},\quad (n\geq 2)$$

then $f\in US(q;k,\lambda,\alpha)$.

Theorem 9 *A necessary and sufficient condition for $f(z)$ of the form (2) to be in the class $UTS(q;k,\lambda,\alpha)$, is that*

$$\begin{gathered}
\sum_{n=2}^{\infty}\left[\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)\right]a_n\leq 1-\alpha,\\
(0\leq k<\infty;\ 0\leq\alpha<1;\ z\in U).
\end{gathered}\tag{8}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z-\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}z^n.$$

Proof. In view of Theorem 7, we need only to prove the necessity. If $f\in UTS(q;k,\lambda,\alpha)$, using the fact that $|\Re(z)|\leq|z|$ for any z, then

$$\left|\frac{1-\sum\limits_{n=2}^{\infty}\widetilde{[n]}_q a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}(1-\lambda+\lambda\widetilde{[n]}_q)a_n z^{n-1}}-\alpha\right|\geq k\left|\frac{\sum\limits_{n=2}^{\infty}\left(\widetilde{[n]}_q-\lambda\widetilde{[n]}_q-1+\lambda\right)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}(1-\lambda+\lambda\widetilde{[n]}_q)a_n z^{n-1}}\right|.\tag{9}$$

Choose values of z on the real axis so that $\widetilde{D}_q f(z)$ is real. Upon clearing the dominator in (9) and letting $z \to 1^-$ through the real values, we obtain (8). ■

This completes the proof.

Letting $q \to 1^-$ we get desired Corollary.

Corollary 10 *As special cases of Theorem 9, for $k = 0$ and $\lambda = 0$, see [3].*

3 Distortion Theorems

Theorem 11 *Let the function f defined by (2) in the class $UTS(q; k, \lambda, \alpha)$. Then*

$$r - \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r^2 \leq |f(z)| \leq r + \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r^2 \tag{10}$$

$$(|z| = r < 1)$$

for $z \in U$.

Equality in (10) holds true for the function $f(z)$ given by

$$|f(z)| = |z| + \frac{q\,(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}\,|z|^2\,. \tag{11}$$

Proof. Since $f \in UTS(q; k, \lambda, \alpha)$, in view of Theorem 9, we have

$$\left[\widetilde{[2]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[2]}_q)\right]\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)\right]a_n$$

$$\leq 1-\alpha,$$

which gives

$$\sum_{n=2}^{\infty}a_n \leq \frac{1-\alpha}{\widetilde{[2]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[2]}_q)}. \tag{12}$$

Therefore

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty}a_n\,|z|^n$$

$$\leq r + \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r^2.$$

On the other hand,

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty}a_n\,|z|^n$$

$$\geq r - \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r^2.$$

■

Theorem 12 *Let the function f defined by (2) in the class $UTS(q;k,\lambda,\alpha)$. Then*

$$1-\frac{2q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r \quad \leq \quad |f'(z)| \leq 1+\frac{2q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r \tag{13}$$

$$|z| \quad = \quad r<1$$

for $z \in U$.

From (2),

$$|f'(z)| \leq 1+\sum_{n=2}^{\infty} na_n\,|z|^{n-1} \leq 1+r\sum_{n=2}^{\infty} na_n, \tag{14}$$

and

$$|f'(z)| \geq 1-\sum_{n=2}^{\infty} na_n\,|z|^{n-1} \geq 1-r\sum_{n=2}^{\infty} na_n. \tag{15}$$

The assertion (13) of Theorem 12 would now follow from (14) and (15) by means of a rather simple consequence of (12) given by

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\,(1-\alpha)}{\widetilde{[2]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[2]}_q)}.$$

This completes the proof of Theorem 12.

4 Extreme Points of the Function Class $UTS(q;k,\lambda,\alpha)$

Theorem 13 *Let*

$$f_1(z)=z$$

and

$$f_n(z)=z-\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}z^n \quad (n=2,3,\ldots).$$

Then $f \in UTS(q;k,\lambda,\alpha)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z),$$

where $\lambda_n>0$ and $\sum\limits_{n=1}^{\infty}\lambda_n=1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\ &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n \left[z - \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)} z^n \right] \\ &= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)} z^n \\ &= \left(\sum_{n=1}^{\infty} \lambda_n \right) z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)} z^n \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)} z^n \ . \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)} . \frac{\widetilde{[n]}_q(k+1) - (k+\alpha)}{1-\alpha} &= \sum_{n=2}^{\infty} \lambda_n \\ &= \sum_{n=2}^{\infty} \lambda_n - \lambda_1 \\ &= 1 - \lambda_1 \leq 1. \end{aligned}$$

Thus we have $f \in UTS(q;k,\lambda,\alpha)$. ■

Conversely, suppose that $f \in UTS(q;k,\lambda,\alpha)$. Since

$$|a_n| \leq \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)},$$

we may set

$$\lambda_n = \frac{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}{1-\alpha} |a_n| \quad \textit{and} \quad \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then

$$\begin{aligned} f(z) &= z+\sum_{n=2}^{\infty}a_nz^n = z+\sum_{n=2}^{\infty}\lambda_n\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}z^n \\ &= z+\sum_{n=2}^{\infty}\lambda_n(z+f_n(z)) = z+\sum_{n=2}^{\infty}\lambda_nz+\sum_{n=2}^{\infty}\lambda_nf_n(z) \\ &= \left(1-\sum_{n=2}^{\infty}\lambda_n\right)z+\sum_{n=2}^{\infty}\lambda_nf_n(z) = \lambda_1z+\sum_{n=2}^{\infty}\lambda_nf_n(z) \\ &= \lambda_1f_1(z)+\sum_{n=2}^{\infty}\lambda_nf_n(z) = \sum_{n=1}^{\infty}\lambda_nf_n(z) \end{aligned}$$

This completes the proof.

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ON SOLUTION OF SYSTEM OF INTEGRAL EQUATIONS VIA FIXED POINT METHOD

MUHAMMAD NAZAM, MUHAMMAD ARSHAD, CHOONKIL PARK*, AFTAB HUSSAIN, AND DONG YUN SHIN*

Abstract. In this article, following the approach of F-contractions, we establish a common fixed point theorem for a pair of self-mappings satisfying F-contraction of rational type in complete metric spaces. An example is constructed to illustrate this result. An application to system of integral equations is presented.

Keywords: metric space; fixed point; rational type F-contraction; integral equation.
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[1]

1. Introduction and preliminaries

Banach Contraction Principle has been extended and generalized in many directions (see [3, 7, 8, 9, 10]). One of the most interesting generalization of it was given by Wardowski [22]. Later on, Abbas et al.[1] further generalized the concept of F-contraction and proved certain fixed point results. Hussain and Salimi [13] introduced α-GF contraction with respect to a general family of functions G and established Wardowski type fixed point results in ordered metric spaces. Batra et al. [5, 6] extended the concept of F-contraction on graphs and altered distances. Batra also proved some fixed point and coincidence point results. Recently, Cosentino and Vetro [11] followed the approach of F-contraction and obtained some fixed point theorems for Hardy-Rogers-type self-mappings in complete metric spaces and complete ordered metric spaces.

In this article, following Cosentino and Vetro [11], we prove common fixed point theorems for a pair of self-mappings satisfying F-contraction of rational type in complete metric spaces. An example is constructed to illustrate this result. An application to system of Volterra type integral equations is presented.

Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. The following concepts and results will be required for the proofs of main results.

Definition 1. A mapping $T : X \to X$ is said to be an F-contraction if it satisfies the following condition:

$$d(T(x),T(y)) > 0 \Rightarrow\ t + F(d(T(x),T(y))) \leq F(d(x,y)) \tag{1.1}$$

for all $x, y \in X$ and some $t > 0$. Here $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following properties:

[1]*Corresponding authors.

(F_1) F is strictly increasing.
(F_2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
(F_3) There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

Wardowski [22] established the following result using F-contraction:

Theorem 1. [22] Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $\upsilon \in X$ and for every $X_0 \in X$ a sequence $\{T^n(X_0)\}$ is convergent to υ.

We denote by Δ_F the set of all functions satisfying the conditions $(F_1) - (F_3)$.

Example 1. [22] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies $(F_1) - (F_3)$ for any $\kappa \in (0,1)$. Each mapping $T : X \to X$ satisfying (1.1) is an F-contraction such that

$$d(T(x), T(y)) \leq e^{-\tau} d(x,y) \text{ for all } x, y \in X \text{ with } T(x) \neq T(y).$$

Obviously, for all $x, y \in X$ such that $T(x) = T(y)$, the inequality $d(T(x), T(y)) \leq e^{-\tau} d(x,y)$ holds, that is, T is a Banach contraction.

Remark 1. From (F_1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

2. Main result

We begin with the following definitions.

Definition 2. Let (X,d) be a metric space. A mapping $T : X \to X$ is called a rational type F-contraction if, for all $x, y \in X$, we have

$$\tau + F(d(T(x), T(y))) \leq F\left(N(x,y)\right), \tag{2.1}$$

where $F \in \Delta_F$ and $\tau > 0$, and

$$N(x,y) = \max\left\{ d(x,y), \frac{d(x,T(x))d(y,T(y))}{1+d(x,y)}, \frac{d(x,T(x))d(y,T(y))}{1+d(T(x),T(y))} \right\}.$$

Definition 3. Let (X,d) be a metric space. Mappings $S, T : X \to X$ are called a pair of rational type F-contractions if for all $x, y \in X$, we have

$$\tau + F(d(S(x), T(y))) \leq F\left(M(x,y)\right), \tag{2.2}$$

where $F \in \Delta_F$ and $\tau > 0$, and

$$M(x,y) = \max\left\{ d(x,y), \frac{d(x,S(x))d(y,T(y))}{1+d(x,y)}, \frac{d(x,S(x))d(y,T(y))}{1+d(S(x),T(y))} \right\}.$$

The following theorem is one of our main results.

Theorem 2. *Let (X,d) be a complete metric space and $S, T : X \to X$ be a pair of mappings such that*

(1) *(S, T) is a pair of continuous mappings,*

(2) (S, T) *is a pair of rational type* F*-contractions.*

Then there exists a common fixed point υ *of a pair* (S, T) *in* X*.*

Proof. We begin with the following observation:

$$M(x,y)=0 \text{ if and only if } x=y \text{ is a common fixed point of } (S,T).$$

Indeed, if $x = y$ is a common fixed point of (S, T), then $T(y) = T(x) = x = y = S(y) = S(x)$ and

$$M(x,y)=\max\left\{d(x,x),\frac{d(x,S(x))d(y,T(y))}{1+d(x,y)},\frac{d(x,S(x))d(y,T(y))}{1+d(S(x),T(y))}\right\}=0.$$

Conversely, if $M(x, y) = 0$, it is easy to check that $x = y$ is a fixed point of S and T. In order to find common fixed points of S and T for the situation when $M(x, y) > 0$ for all $x, y \in X$ with $x \neq y$, we construct an iterative sequence $\{x_n\}$ of points in X such a way that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$ where $i = 0, 1, 2, \ldots$. If $x_n \neq x_{n+1}$ for all $n \geq 0$, then from contractive condition (2.2), we get

$$F\left(d(x_{2i+1},x_{2i+2})\right)=F\left(d(S(x_{2i}),T(x_{2i+1}))\right)\leq F\left(M(x_{2i},x_{2i+1})\right)-\tau$$

for all $i \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
M(x_{2i},x_{2i+1}) &= \max\left\{\begin{array}{l} d(x_{2i},x_{2i+1}),\dfrac{d(x_{2i},S(x_{2i}))d(x_{2i+1},T(x_{2i+1}))}{1+d(x_{2i},x_{2i+1})},\\ \dfrac{d(x_{2i},S(x_{2i}))d(x_{2i+1},T(x_{2i+1}))}{1+d(S(x_{2i}),T(x_{2i+1}))}\end{array}\right\}\\
&= \max\left\{d(x_{2i},x_{2i+1}),\frac{d(x_{2i},x_{2i+1})d(x_{2i+1},x_{2i+2})}{1+d(x_{2i},x_{2i+1})},\frac{d(x_{2i},x_{2i+1})d(x_{2i+1},x_{2i+2})}{1+d(x_{2i+1},x_{2i+2})}\right\}\\
&\leq \max\left\{d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2})\right\}.
\end{aligned}$$

If $M(x_{2i}, x_{2i+1}) = d(x_{2i+1}, x_{2i+2})$, then

$$F\left(d(x_{2i+1},x_{2i+2})\right)\leq F\left(d(x_{2i+1},x_{2i+2})\right)-\tau,$$

which is a contradiction due to F_1. Therefore,

$$F\left(d(x_{2i+1},x_{2i+2})\right)\leq F\left(d(x_{2i},x_{2i+1})\right)-\tau,$$

for all $i \in \mathbb{N} \cup \{0\}$. Hence

$$F\left(d(x_{n+1},x_{n+2})\right)\leq F\left(d(x_n,x_{n+1})\right)-\tau, \tag{2.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. By (2.3), we obtain

$$F\left(d(x_n,x_{n+1})\right)\leq F\left(d(x_{n-2},x_{n-1})\right)-2\tau.$$

Repeating these steps, we get

$$F\left(d(x_n,x_{n+1})\right)\leq F\left(d(x_0,x_1)\right)-n\tau. \tag{2.4}$$

From (2.4), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{2.5}$$

From the property (F_3) of F-contraction, there exists $\kappa \in (0,1)$ such that

$$\lim_{n\to\infty} \left((d(x_n, x_{n+1}))^{\kappa} F\left(d(x_n, x_{n+1})\right)\right) = 0. \tag{2.6}$$

By (2.4), for all $n \in \mathbb{N}$, we obtain

$$(d(x_n, x_{n+1}))^{\kappa} \left(F\left(d(x_n, x_{n+1})\right) - F\left(d(x_0, x_1)\right)\right) \leq -\left(d(x_n, x_{n+1})\right)^{\kappa} n\tau \leq 0. \tag{2.7}$$

Considering (2.5), (2.6) and letting $n \to \infty$ in (2.7), we have

$$\lim_{n\to\infty} \left(n\left(d(x_n, x_{n+1})\right)^{\kappa}\right) = 0. \tag{2.8}$$

Since (2.8) holds, there exists $n_1 \in \mathbb{N}$, such that $n\left(d(x_n, x_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$ or,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{2.9}$$

Using (2.9), we get for $m > n \geq n_1$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m); \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists $\upsilon \in X$ such that $x_n \to \upsilon$ as $n \to \infty$, moreover, $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$.

Now the continuity of T implies

$$\upsilon = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T(x_{2n+1}) = T(\lim_{n\to\infty} x_{2n+1}) = T(\upsilon).$$

Analogously, $\upsilon = S(\upsilon)$. Thus we have $S(\upsilon) = T(\upsilon) = \upsilon$. Hence (S, T) has a common fixed point. Now we show that υ is the unique common fixed point of S and T. Assume the contrary, that is, there exists $\omega \in X$ such that $\omega = T(\omega)$. From the contractive condition (2.2), we have

$$\tau + F(d(S(\upsilon), T(\omega))) \leq F\left(M(\upsilon, \omega)\right), \tag{2.10}$$

where

$$M(\upsilon, \omega) = \max\left\{d(\upsilon, \omega), \frac{d(\upsilon, S(\upsilon))d(\omega, T(\omega))}{1 + d(\upsilon, y)}, \frac{d(\upsilon, S(\upsilon))d(\omega, T(\omega))}{1 + d(S(\upsilon), T(\omega))}\right\}.$$

From (2.10), we have

$$\tau + F(d(\upsilon, \omega)) \leq F\left(d(\upsilon, \omega)\right), , \tag{2.11}$$

which implies

$$d(\upsilon, \omega) < d(\upsilon, \omega),$$

which is a contradiction. Hence $\upsilon = \omega$ and υ is a unique common fixed point of a pair (S, T). □

Let us consider an example to illustrate Theorem 2.

Example 2. Let $X = [1, \infty]$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define the mappings $S, T : X \to X$ as follows:

$$S(x) = x^2 \text{ and } T(x) = x + 3 \text{ for all } x \in X.$$

Define the function $F : R^+ \to R$ by $F(x) = \ln(x)$ for all $x \in R^+ > 0$ and $\tau > 0$. Then the contractive condition (2.2) is satisfied. Indeed, for all $x, y \in X$, the following inequality

$$\tau + \ln(d(S(x), T(y))) \leq \ln(M(x, y))$$

holds. Particularly, for $x = 2$ and $y = 3$, we have

$$\begin{aligned} M(2,3) &= \max\left\{d(2,3), \frac{d(2,S(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,S(2))d(3,T(3))}{1+d(S2,T3)}\right\} \\ &= \max\{1,3,2\} = 3 \end{aligned}$$

and

$$d(S2, T3) = d(4, 6) = 2.$$

Thus

$$\tau + \ln(d(S(2), T(3))) = \tau + \ln 2 \leq \ln(M(2,3)) = \ln 3,$$

which implies

$$\tau + F(d(S(x), T(y))) \leq F(M(x, y)).$$

Hence all the hypotheses of Theorem 2 are satisfied and so (S, T) have a common fixed point.

By setting $S = T$, we obtain the following result.

Corollary 1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

(1) *T is a continuous mapping,*
(2) *T is a rational type F-contraction.*

Then T has a unique fixed point υ in X.

Remark 2. If we set $N(x, y) = \max\{d(x, y), d(x, T(x)), d(y, T(y))\}$ in (2.1), then Corollary 1 remains true. Similarly, if we set $M(x, y) = \max\{d(x, y), d(x, T(x)), d(y, S(y))\}$ in (2.2), then Theorem 2 remains true.

3. Application to system of integral equations

Now we discuss an application of fixed point theorem, proved in the previous section, in solving the system of Volterra type integral equations. Such a system is given by the

following equations

$$u(t) = f(t) + \int_0^t K_1(t,s,u(s))ds. \tag{3.1}$$

$$w(t) = f(t) + \int_0^t K_2(t,s,w(s))ds. \tag{3.2}$$

for all $t \in [0,a]$, and $a > 0$. We shall show, by using Theorem 2, that the solution of integral equations (3.1) and (3.2) exists. Let $C([0,a],\mathbb{R})$ be the space of all continuous functions defined on $[0,a]$. For $u \in C([0,a],\mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau t}\}$, where $\tau > 0$. Let $C([0,a],\mathbb{R})$ be endowed with the metric

$$d_\tau(u,v) = \sup_{t\in[0,a]} \| \left|u(t) - v(t)\right| e^{-\tau t}\|_\tau \tag{3.3}$$

for all $u, v \in C([0,a],\mathbb{R})$. Obviously, $C([0,a],\mathbb{R},\|\cdot\|_\tau)$ is a Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations.

Theorem 3. *Assume the following conditions are satisfied:*
(i) $K_1, K_2 : [0,a]\times[0,a]\times\mathbb{R}\to\mathbb{R}$ *and* $f, g : [0,a]\to\mathbb{R}$ *are continuous;*
(ii) Define the operators

$$Su(t) = f(t) + \int_0^t K_1(t,s,u(s))ds,$$

$$Tu(t) = f(t) + \int_0^t K_2(t,s,u(s))ds,$$

and there exists $\tau \geq 1$ *such that*

$$|K_1(t,s,u) - K_2(t,s,v)| \leq \tau e^{-\tau}[M(u,v)]$$

for all $t, s \in [0,a]$ *and* $u, v \in C([0,a],\mathbb{R})$, *where*

$$M(u,v) = \max\{|u(t)-v(t)|, \frac{|u(t)-Su(t)|\,|v(t)-Tv(t)|}{1+|u(t)-v(t)|}, \frac{|u(t)-Su(t)|\,|v(t)-Tv(t)|}{1+|Su(t)-Tv(t)|}\}.$$

Then the system of integral equations given in (3.1) *and* (3.2) *has a unique solution.*

Proof. By assumption (ii), we have

$$\begin{aligned}
|Tu(t) - Sv(t)| &= \int_0^t |K_1(t,s,u(s) - K_2(t,s,v(s)))|\, ds \\
&\leq \int_0^t \tau e^{-\tau}([M(u,v)]e^{-\tau s})e^{\tau s} ds \\
&\leq \int_0^t \tau e^{-\tau} \|M(u,v)\|_\tau e^{\tau s} ds \\
&\leq \tau e^{-\tau} \|M(u,v)\|_\tau \int_0^t e^{\tau s} ds \\
&\leq \tau e^{-2\tau} \|M(u,v)\|_\tau \frac{1}{\tau} e^{\tau t} \\
&\leq e^{-\tau} \|M(u,v)\|_\tau e^{\tau t}.
\end{aligned}$$

This implies

$$|Tu(t) - Sv(t)|\, e^{-\tau t} \leq e^{-\tau} \|M(u,v)\|_\tau,$$

that is,

$$\|Tu(t) - Sv(t)\|_\tau \leq e^{-\tau} \|M(u,v)\|_\tau.$$

So we have

$$\tau + \ln \|Tu(t) - Sv(t)\|_\tau \leq \ln \|M(u,v)\|_\tau.$$

Thus all the conditions of Theorem 2 are satisfied. Hence the system of integral equations given in (3.1) and (3.2) has a unique common solution. □

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Muhammad Nazam
Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
E-mail address: nazim.phdma47@iiu.edu.pk

Muhammad Arshad
Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
E-mail address: marshadzia@iiu.edu.pk

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Aftab Hussain
International Islamic University, Department of Mathematics and Statistics, H-10 Islamabad, Pakistan; Lahore Leads University, Department of Mathematical Sciences, Lahore - 33000, Lahore, Pakistan
E-mail address: aftabshh@gmail.com

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 02504, Korea
E-mail address: dyshin@uos.ac.kr

On continuous Fibonacci functions

Hee Sik Kim[1], J. Neggers[2] and Keum Sook So[3,*]

[1]**Department of Mathematics, Hanyang University, Seoul, 133-791, Korea**
[2]**Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A**
[3,*]**Department of Mathematics, Hallym University, Chuncheon 200-702, Korea**

Abstract. In this paper, we define and study a function $F : [0, \infty) \to \mathbf{R}$ and extensions $F : \mathbf{R} \to \mathbf{C}$, $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ which are continuous and such that if $n \in \mathbf{Z}$, the set of all integers, then $F(n) = F_n$, the n^{th} Fibonacci number based on $F_0 = F_1 = 1$. If x is not an integer and $x < 0$, then $F(x)$ may be a complex number, e.g., $F(-1.5) = \frac{1}{2} + i$. If $z = a + bi$, then $\widetilde{F}(z) = F(a) + iF(b-1)$ defines complex Fibonacci numbers. In connection with this function (and in general) we define a Fibonacci derivative of $f : \mathbf{R} \to \mathbf{R}$ as $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$ so that if $(\triangle f)(x) \equiv 0$ for all $x \in \mathbf{R}$, then f is a (real) Fibonacci function. A complex Fibonacci derivative $\widetilde{\triangle}$ is given as $\widetilde{\triangle} f(a+bi) = \triangle f(a) + i \triangle f(b-1)$ and its properties are discussed in same detail.

1. Introduction

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio (section) rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2] for a very minimal set of examples of such texts, while in [7] Kim and Neggers showed that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n\to\infty} D^n M$ is the golden section mean. Hyers-Ulam stability of Fibonacci functional equation was studied in [6]. Surprisingly novel perspectives are still available and will presumably continue to be so for the future as long as mathematical investigations continue to be made. In the following the authors of the present paper are making another small offering at the same spot many previous contributors have visited in both recent and more distant pasts.

Han et al. [4] considered several properties of Fibonacci sequences in arbitrary groupoids. They discussed Fibonacci sequences in both several groupoids and groups. The present authors [8] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed these in particular for the case where the groupoid contains idempotents and pre-idempotents. Using the notion of Smarandache-type P-algebras they obtained several relations on groupoids which are derived from generalized Fibonacci sequences.

In [5] Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1) + f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. KNS[4445] discussed Fibonacci functions using the (ultimately) periodicity and we also discuss the exponential

[0*] Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

Fibonacci functions. Especially, given a non-negative real-valued function, we obtain several exponential Fibonacci functions.

In this paper we are interested in describing properties of a function $F : \mathbf{R} \to \mathbf{C}$, the complex numbers, where $F(x) := (F_{\lfloor x-1 \rfloor})^{x-\lfloor x \rfloor} + F_{\lfloor x \rfloor} + (x - \lfloor x \rfloor - 1)$ and $\lfloor x \rfloor$ is the greatest integer function. It follows that if $x = n \in \mathbf{Z}$, then $\lfloor n \rfloor = n$ and $F(n) = (F_{n-1})^0 + F_n + (n - n - 1)$, where $(F_{n-1})^0 = 1$ implies $F(n) = F_n$ with $F_0 = F_1 = 1$ and $F_{-n} = (-1)^n F_{n-2}$ so that $F_{-1} = -F_{-1} = 0$ for example. If one computes F_1 directly, then $F_1 = F_0 + F_{-1}$ and $1 = 1 + 0$ yields $F_0 = 1$ as well.

It also follows that $F(x)$ is not itself a Fibonacci function in the sense that $F(x + 2) \neq F(x + 1) + F(x)$ if $x - \lfloor x \rfloor \neq 0$, i.e., if $x \notin \mathbf{Z}$. Nevertheless it is a very interesting function which allows one to define continuous Fibonacci numbers in an interesting manner. If one computes $F(-1.5)$ for example, then one finds that $F(-1.5) = \frac{1}{2} + i$, which suggests that the function $F(x)$ may deserve looking at in the context of the study of the zeta-function.

Given the fact that $F : \mathbf{R} \to \mathbf{C}$ and that $F(\mathbf{R}) \cap \mathbf{C} \neq \emptyset$, it also becomes a question of interest to study possible complex extensions of F to $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ where $z = a + bi$ means $\widetilde{F}(z) = F(a) + iF(b - 1)$, so that if $z = a$, then $\widetilde{F}(z) = F(a) + iF(-1)$, where $F(-1) = 0$ implies $\widetilde{F}(a) = F(a)$, i.e., $\widetilde{F}$ is an extension of F. According to this construction we find that $\widetilde{F}(1 + i) = F(1) + iF(0) = 1 + i$ as one would hope.

A second component of the paper is a study of properties of the Fibonacci derivative $\triangle f$ of a function $f : \mathbf{R} \to \mathbf{R}$, given by the formula $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$, so that $(\triangle f)(x) \equiv 0$ means that f is then a Fibonacci function. If one notes that $(\triangle f)$ exists for any function $f : \mathbf{R} \to \mathbf{R}$, then a variety of questions may be asked about properties of this operator. For example $(\triangle f)(x) \equiv f(x)$ is a simple type of Fibonacci derivative equation with many types of solutions. Other analogs of standard differential equations may also be addressed.

Given that $F : \mathbf{R} \to \mathbf{C}$ is itself a function of interest in this context, $\triangle F : \mathbf{R} \to \mathbf{C}$ is looked at below. Finally, a complex version $\widetilde{\triangle} F : \mathbf{C} \to \mathbf{C}$ defined by $\widetilde{\triangle} F(z) = \widetilde{\triangle} F(a + bi) = \triangle F(a) + i \triangle F(b - 1)$, reduces for $b = 0$ to $\widetilde{\triangle} F(a) = \triangle F(a)$, i.e., $\widetilde{\triangle}$ extends the operator $\triangle$, and thus again it is a matter of interest to study the behavior of the function $\widetilde{\triangle} \widetilde{F}(z)$ for complex numbers.

Note that because of the very rich structure of relations among the coefficients F_n, we may expect there to eventually be development of an equally rich structure of relations among the various values of $F(x)$ (and $\widetilde{F}(z)$) extending the ones already known.

2. **Preliminaries**

A function f defined on the real numbers is said to be a *Fibonacci function* ([5]) if it satisfies the formula

$$f(x + 2) = f(x + 1) + f(x)$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 2.1. ([5]) Let $f(x) := a^x$ be a Fibonacci function on $\mathbf{R}$ where $a > 0$. Then $a^x a^2 = f(x + 2) = f(x+1) + f(x) = a^x(a+1)$. Since $a > 0$, we have $a^2 = a+1$ and $a = \frac{1+\sqrt{5}}{2}$. Hence $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, and the unique Fibonacci function of this type on $\mathbf{R}$.

If we let $u_0 = 0, u_1 = 1$, then we consider the *full Fibonacci sequence*: $\cdots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \cdots$, i.e., $u_{-n} = (-1)^n u_n$ for $n > 0$, and $u_n = F_n$, the nth Fibonacci number.

Example 2.2. ([5]) Let $\{u_n\}_{n=-\infty}^{\infty}$ and $\{v_n\}_{n=-\infty}^{\infty}$ be full Fibonacci sequences. We define a function $f(x)$ by $f(x) := u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor} t$, where $t = x - \lfloor x \rfloor \in (0,1)$. Then $f(x+2) = u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor} t = u_{(\lfloor x \rfloor + 2)} + v_{(\lfloor x \rfloor + 2)} t = (u_{(\lfloor x \rfloor + 1)} + u_{\lfloor x \rfloor}) + (v_{(\lfloor x \rfloor + 1)} + v_{\lfloor x \rfloor})t = f(x+1) + f(x)$ for any $x \in \mathbf{R}$. This proves that f is a Fibonacci function.

Note that if a Fibonacci function is differentiable on $\mathbf{R}$, then its derivative is also a Fibonacci function.

Proposition 2.3. ([5]) *Let f be a Fibonacci function. If we define $g(x) := f(x+t)$ where $t \in \mathbf{R}$ for any $x \in \mathbf{R}$, then g is also a Fibonacci function.*

For example, since $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, $g(x) = (\frac{1+\sqrt{5}}{2})^{x+t} = (\frac{1+\sqrt{5}}{2})^t f(x)$ is also a Fibonacci function where $t \in \mathbf{R}$.

Theorem 2.4. ([5]) *If $f(x)$ is a Fibonacci function, then the limit of the quotient $\frac{f(x+1)}{f(x)}$ exists.*

Corollary 2.5. ([5]) *If $f(x)$ is a Fibonacci function, then*

$$\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$$

3. Continuous Fibonacci functions

Given a real number $x \notin \mathbf{Z}$, we define a map $F(x)$ by

$$F(x) := (F_{\lfloor x-1 \rfloor})^{x - \lfloor x \rfloor} + F_{\lfloor x \rfloor} + (x - \lfloor x \rfloor - 1) \tag{1}$$

where $\{F_n\}$ is the sequence of Fibonacci numbers with $F_0 = F_1 = 1$.

Example 3.1. We compute some $F(x)$ as follows: $F(1.5) = (F_{\lfloor 1.5-1 \rfloor})^{1.5-\lfloor 1.5 \rfloor} + F_{\lfloor 1.5 \rfloor} + (1.5 - \lfloor 1.5 \rfloor - 1) = (F_0)^{0.5} + F_1 + (1.5 - 1 - 1) = 1.5$ and $F(1.75) = (F_0)^{0.75} + F_1 + (1.75 - \lfloor 1.75 \rfloor - 1) = 1.75$. Moreover, $F(3.25) = (F_{\lfloor 3.25-1 \rfloor})^{0.25} + F_{\lfloor 3.25 \rfloor} + (3.25 - \lfloor 3.25 \rfloor - 1) = (F_2)^{0.25} + F_3 + (3.25 - 3 - 1) = 4\sqrt{2} + 2.25$.

Theorem 3.2. *If we define $F(n) := F_n$, the n^{th} Fibonacci function, then $F(x)$ is continuous for all $x \in \mathbf{R}$.*

Proof. Let $x := n + \epsilon$ where $n \in \mathbf{Z}$ and $0 < \epsilon < 1$. Then $F(x) = F_{n-1}^{\epsilon} + F_n + (\epsilon - 1)$. It follows that $\lim_{\epsilon \to 0^+} F(x) = \lim_{\epsilon \to 0^+} (F_{n-1}^{\epsilon} + F_n + (\epsilon - 1)) = F_n$. Let $x := n - \epsilon$ where $n \in \mathbf{Z}$ and $0 < \epsilon < 1$. Then

$$\begin{aligned} F(x) &= (F_{\lfloor n-\epsilon-1 \rfloor})^{n-\epsilon-\lfloor n-\epsilon \rfloor} + F_{\lfloor n-\epsilon \rfloor} + (n - \epsilon - \lfloor n - \epsilon \rfloor - 1) \\ &= (F_{n-2})^{n-\epsilon-(n-1)} + F_{n-1} + (n - \epsilon - (n-1) - 1) \\ &= (F_{n-2})^{1-\epsilon} + F_{n-1} - \epsilon \end{aligned}$$

It follows that $\lim_{\epsilon \to 0^+} F(x) = \lim_{\epsilon \to 0^+} [(F_{n-2})^{1-\epsilon} + F_{n-1} - \epsilon] = F_{n-2} + F_{n-1} = F_n$. □

In Theorem 3.2, we call the real number $F(x)$ the (*continuous*) *Fibonacci function* at x.

Let $f : \mathbf{R} \to \mathbf{R}$ be a real-valued function. We shall consider the expression

$$(\triangle f)(x) := f(x+2) - f(x+1) - f(x)$$

to be the *Fibonacci derivative* of $f(x)$. For example, if $\Phi := \frac{1+\sqrt{5}}{2}$, then $f(x) = \Phi^x$ yields $(\triangle f)(x) = \Phi^{x+2} - \Phi^{x+1} - \Phi^x = \Phi^x(\Phi^2 - \Phi - 1) = 0$ and similarly, if f is any Fibonacci function, then $(\triangle f)(x) = 0$ for all $x \in \mathbf{R}$.

We are next concerned with determining the Fibonacci derivative of $F(x)$ as we have defined above.

Theorem 3.3. *If $F(x)$ is a continuous Fibonacci function, then its Fibonacci derivative is*

$$(\triangle F)(x) = (F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1) \tag{2}$$

where $x = n + \epsilon$, $n \in \mathbf{Z}, 0 < \epsilon < 1$

Proof. Given $x = n + \epsilon$, $n \in \mathbf{Z}, 0 < \epsilon < 1$, by using the formula (1), we obtain

$$\begin{aligned}(\triangle F)(x) &= F(n+2+\epsilon) - F(n+1+\epsilon) - F(n+\epsilon)\\ &= (F_{\lfloor n+1+\epsilon \rfloor})^\epsilon + F_{n+2} + (\epsilon - 1)\\ &\quad -(F_{\lfloor n+\epsilon \rfloor})^\epsilon - F_{n+1} - (\epsilon - 1)\\ &\quad -(F_{\lfloor n-1+\epsilon \rfloor})^\epsilon - F_n - (\epsilon - 1)\\ &= (F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)\end{aligned}$$

□

Note that the map $F(x)$ in Theorem 3.3 is not necessarily a Fibonacci function.

The formula (2) is a function depending on ϵ, and so we need to know the value of $\frac{d}{d\epsilon}[\triangle F(x)]$.

$$\begin{aligned}\frac{d}{d\epsilon}[(\triangle F)(x)] &= \frac{d}{d\epsilon}[(F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)]\\ &= \ln(F_{n+1})(F_{n+1})^\epsilon - \ln(F_n)(F_n)^\epsilon - \ln(F_{n-1})(F_{n-1})^\epsilon - 1\end{aligned}$$

We denote $\frac{d}{d\epsilon}[\triangle F(x)]$ by $(\triangle F)'(x)$.

Proposition 3.4. *If $F(x)$ is a continuous Fibonacci function, then*

$$(\triangle(\triangle F))(x) = (F_{n+3})^\epsilon - 2(F_{n+2})^\epsilon - (F_{n+1})^\epsilon + 2(F_n)^\epsilon + (F_{n-1})^\epsilon + (\epsilon - 1) \tag{3}$$

Proof. It follows from the formula (2) that

$$\begin{aligned}(\triangle(\triangle F))(x) &= \triangle F(x+2) - \triangle F(x+1) - \triangle F(x)\\ &= (F_{n+3})^\epsilon - (F_{n+2})^\epsilon - (F_{n+1})^\epsilon - (\epsilon - 1)\\ &\quad -(F_{n+2})^\epsilon + (F_{n+1})^\epsilon - (F_n)^\epsilon + (\epsilon - 1)\\ &\quad -(F_{n+1})^\epsilon + (F_n)^\epsilon - (F_{n-1})^\epsilon + (\epsilon - 1)\\ &= (F_{n+3})^\epsilon - 2(F_{n+2})^\epsilon - (F_{n+1})^\epsilon + 2(F_n)^\epsilon + (F_{n-1})^\epsilon + (\epsilon - 1),\end{aligned}$$

proving the proposition. □

Proposition 3.5. *$(\triangle F)(x)$ is a continuous function and $(\triangle F)(n) = 0$ for all $n \in \mathbf{Z}$.*

Proof. It follows from Theorem 3.3 that $(\triangle F)(x)$ is a continuous function. Since $\lim_{\epsilon\to 0}(\triangle F)(x) = \lim_{\epsilon\to 0}[(F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)] = 0$ and $\lim_{\epsilon\to 1}(\triangle F)(x) = \lim_{\epsilon\to 1}[(F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)] = F_{n+1} - F_n - F_{n-1} = 0$ for any $n \in \mathbf{Z}$. □

Theorem 3.6. *If $F(x)$ is a continuous Fibonacci function, then there exists a $\gamma_n \in (n, n+1)$ such that $(\triangle F)'(\gamma_n) = 0$ for all $n \in \mathbf{Z}$.*

Proof. Since $(\triangle F)(x)$ is a continuous function and $(\triangle F)(n) = (\triangle F)(n+1) = 0$, by Rolle's Theorem, there exists a $\gamma_n \in (n, n+1)$ such that $(\triangle F)'(\gamma_n) = \frac{d}{d\epsilon}(\triangle F)(\gamma_n) = 0$. □

Theorem 3.7. *If $F(x)$ is a continuous Fibonacci function, then $(\triangle F)(x)$ is concave down.*

Proof. If we let $T(x) := \frac{d^2}{d\epsilon^2}[(\triangle F)(x)]$, then

$$\begin{aligned} T(x) &= \frac{d}{d\epsilon}[\frac{d}{d\epsilon}((\triangle F)(x))] \\ &= \frac{d}{d\epsilon}[\ln(F_{n+1})(F_{n+1})^\epsilon - \ln(F_n)(F_n)^\epsilon - \ln(F_{n-1})(F_{n-1})^\epsilon - 1] \\ &= \{\ln(F_{n+1})\}^2(F_{n+1})^\epsilon - \{\ln(F_n)\}^2(F_n)^\epsilon - \{\ln(F_{n-1})\}^2(F_{n-1})^\epsilon. \end{aligned}$$

Let n be very large so that $\frac{F_{k+1}}{F_k} = \Phi = \frac{1+\sqrt{5}}{2}$. It follows that

$$\begin{aligned} \frac{T(x)}{\{\ln(F_{n-1})\}^2(F_{n-1})^\epsilon} &= \{\frac{\ln(F_{n+1})}{\ln(F_{n-1})}\}^2[\frac{F_{n+1}}{F_{n-1}}]^\epsilon - \{\frac{\ln(F_n)}{\ln(F_{n-1})}\}^2[\frac{F_n}{F_{n-1}}]^\epsilon - 1 \\ &= \{\frac{2\ln\Phi + \ln(F_{n-1})}{\ln(F_{n-1})}\}^2(\Phi)^{2\epsilon} \\ &\quad -\{\frac{\ln\Phi + \ln(F_{n-1})}{\ln(F_{n-1})}\}^2(\Phi)^\epsilon - 1. \end{aligned}$$

If we let $n \to \infty$, then

$$\lim_{n\to\infty} \frac{T(x)}{\{\ln(F_{n-1})\}^2(F_{n-1})^\epsilon} = \Phi^{2\epsilon} - \Phi^\epsilon - 1 \tag{4}$$

If we let $\epsilon := \frac{1}{2}$, then $\Phi^{2\epsilon} - \Phi^\epsilon - 1 = \Phi^-\sqrt{\Phi} - 1$ and $\Phi - 1 = \frac{\sqrt{5}-1}{2}, \sqrt{\Phi} = \sqrt{\frac{1+\sqrt{5}}{2}}$, so that $(\Phi-1)^2 - (\sqrt{\Phi})^2 = \frac{4-4\sqrt{5}}{4} < 0$, proving that $T(x) < 0$. This shows that $(\triangle F)(x)$ is concave down. □

We discuss a Fibonacci derivative of a function which is not a Fibonacci function as below.

Proposition 3.8. *Let $f(x) := ax + b$ for some $a, b \in \mathbf{R}$. Then*

$$\triangle^{k+1}(f)(x) = (-1)^{k+1}ax + (-1)^k[(k+1)a - b]$$

Proof. The Fibonacci derivative $(\triangle f)(x)$ of $f(x) = ax+b$ is $f(x+2) - f(x+1) - f(x) = [a(x+2)+b] - [a(x+1)+b] - [ax+b] = -ax + a - b$. Similarly, we obtain $[\triangle^2(f)](x) = ax - 2a + b, [\triangle^3(f)](x) = -ax + 3a - b$ and $[\triangle^4(f)](x) = ax - 4a + b$. Assume $\triangle^k(f)(x) = (-1)^k ax + (-1)^{k-1}(ka - b)$. Then

$$\begin{aligned} \triangle^{k+1}(f)(x) &= \triangle[\triangle^k(f)(x)] \\ &= \triangle^k(f)(x+2) - \triangle^k(f)(x+1) - \triangle^k(f)(x) \\ &= (-1)^k a(x+2) + (-1)^{k-1}(ka - b) \\ &\quad -[(-1)^k a(x+1) + (-1)^{k-1}(ka-b)] \\ &\quad -[(-1)^k ax + (-1)^{k-1}(ka-b)] \\ &= (-1)^{k+1}ax + (-1)^k[(k+1)a - b]. \end{aligned}$$

□

Note that $[(\triangle^{k+3} + \triangle^{k+2}) - (\triangle^{k+1} + \triangle^k)](f)(x) = (-1)^{k+2}2a - (-1)^k 2a = 0$.

We need to find some conditions for a map $f : \mathbf{R} \to \mathbf{R}$ satisfying $(\triangle f)(x) = f(x)$.

Proposition 3.9. *Let $f : \mathbf{R} \to \mathbf{R}$ be a map. If it satisfies the condition either (i) $2f(x) = f(x+2) - f(x+1)$ or (ii) $f(x+1) = -f(x)$ for all $x \in \mathbf{R}$, then $(\triangle f)(x) = f(x)$.*

Proof. Straightforward. □

Example 3.10. If $f(x) := 2^x$, then $\frac{1}{2}(2^{x+2} - 2^{x+1}) = 2^x$ and hence $\triangle(2^x)(x) = 2^x$. If we let $f(x) := \sin(\pi x)$, then $f(x+1) = \sin\pi(x+1) = -\sin\pi x = -f(x)$ and hence $\triangle(\sin\pi x)(x) = \sin\pi x$.

Now, we define a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the condition: $2f(x) = f(x+2) - f(x+1)$. If we make such a function, then it satisfies the condition $(\triangle f)(x) = f(x)$.

Suppose that one defines $f(x)$ for $0 \leq x < 2$ at will. Then for $x \in [2, 3)$ one defines $f(2+\theta) := 2f(\theta) + f(1+\theta)$ where $0 \leq \theta < 1$. If $f(x)$ has been defined for $x \in [m-1, m)$, then we define $f(m+\theta) := 2f((m-2)+\theta)+f(m-1+\theta)$, where $0 \leq \theta < 1$. Then $f(x)$ is uniquely determined on $[0, \infty)$. To define $f(x)$ for $[-1, 0)$, we have $f(1+\theta) = 2f(-1+\theta) + f(\theta)$ or $f(-1+\theta) = (f(1+\theta) - f(\theta))/2$, and thus $f(-m+\theta) = [f(-m+2+\theta) - f(-m+1+\theta)]/2$ inductively as well to obtain $f(x)$ defined on the entire real line.

Example 3.11. if $f(x) := 1$ on $[0, 2)$, then $f(2+\theta) = 2f(\theta)+f(1+\theta) = 2+1 = 3$, $f(3+\theta) = 2f(1+\theta)+f(2+\theta) = 5$ and $f(4+\theta) = 2f(2+\theta) + f(3+\theta) = 2 \cdot 3 + 5 = 11$. If we take $F_1^* = 1, F_2^* = 3, F_3^* = 5, F_4^* = 11, \cdots$, then $\{F_n^*\}$ is a Fibonacci sequence type satisfying $F_{n+1}^* = 2F_{n-1}^* + F_n^*$. We have $f(4+\theta) = 2f(2+\theta) + f(3+\theta) = 5f(1+\theta) + 6f(\theta) = F_3^* f(1+\theta) + 2F_2^* f(\theta)$. Assume that $f(n+\theta) = F_{n-1}^* f(1+\theta) + 2F_{n-2}^* f(\theta)$. Then

$$\begin{aligned} f(n+1+\theta) &= 2f(n-1+\theta) + 2f(n+\theta) \\ &= 2[F_{n-2}^* f(1+\theta) + 2F_{n-3}^* f(\theta)] + [F_{n-1}^* f(1+\theta) + 2F_{n-2}^* f(\theta)] \\ &= (2F_{n-2}^* + F_{n-1}^*) f(1+\theta) + 2(2F_{n-3}^* + F_{n-2}^*) f(\theta) \\ &= F_n^* f(1+\theta) + 2F_{n-1}^* f(\theta) \end{aligned}$$

4. **Complex Fibonacci functions**

Given the Fibonacci sequence $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, \cdots$, we may compute F_{-n}, $n = 1, 2, \cdots$ via the equation

$$F_{-n+2} = F_{-n+1} + F_{-n} \tag{5}$$

so that $F_{-1} = F_1 - F_0 = 0, F_{-2} = F_0 - F_{-1} = 1 - 0 = 1, F_{-3} = F_{-1} - F_{-2} = 0 - 1 = -1, F_4 = F_{-2} - F_{-3} = 1 - (-1) = 2, F_{-5} = F_{-3} - F_{-4} = -1 - 2 = (-1)^5 F_3$ and $F_{-6} = F_{-4} - F_{-5} = (-1)^6 F_4$. Assume $F_{-n} = (-1)^n F_{n-2}$ $(n \geq 5)$. Then $F_{-(n+1)} = F_{-(n+1)+2} - F_{-(n+1)+1} = (-1)^{n-1} F_{n-1} - (-1)^n F_n = (-1)^{n+1}(F_{n-3} + F_{n-2}) = (-1)^n F_{n-1}$ so that $F_{-2} = (-1)^2 F_0 = 1, F_{-3} = (-1)^1 F_1 = -1$. For F_{-1}, the formula would yield $F_{-1} = (-1)^1 F_{-1}$, i.e., $F_{-1} = -F_{-1}$ which would imply $F_{-1} = 0$ as well. Hence we have the result: for $n \geq 1$,

$$F_{-n} = (-1)^n F_{n-2} \tag{6}$$

Thus, we may apply the formula (1) for $x < 0$ as well. For example, $F(-1.5) = (F_{\lfloor -1.5-1 \rfloor})^{(-1.5-\lfloor -1.5 \rfloor)} + F_{\lfloor -1.5 \rfloor} + (-1.5 - \lfloor -1.5 \rfloor - 1) = (F_{-3})^{0.5} + F_{-2} + (-1.5 - (-2) - 1) = \sqrt{-1} + 1 - \frac{1}{2} = \frac{1}{2} + i$, i.e., $F(-1.5) = \frac{1}{2} + i$, the complex number.

Example 4.1. We compute $F(-4+\theta)$, $(0 \leq \theta < 1)$ as follows:

$$\begin{aligned} F(-4+\theta) &= (F_{\lfloor -4+\theta-1 \rfloor})^{-4+\theta-\lfloor -4+\theta \rfloor} + F_{\lfloor -4+\theta \rfloor} + (-4+\theta-\lfloor -4+\theta \rfloor - 1) \\ &= (F_{-5})^{\theta} + F_{-4} + (\theta - 1) \\ &= (-3)^{\theta} + 2 + (\theta - 1) \\ &= 3^{\theta}(-1)^{\theta} + \theta + 1 \end{aligned}$$

where $(-1)^{\theta} = \exp^{\theta \ln(-1)}$, so that $\ln(-1) = Log(-1)$, where $Log(-1)$ is a "suitable branch of the Log-function". Note that $i^2 = -1$, so $Log(-1) = Log(i^2) = 2Log(i) = 2\ln(i)$. If we set $\ln i := a+bi$, then $\exp^{\ln i} = i = \exp^a \exp^b i$ and $a = 0$, $b = \frac{\pi}{2}$ yields $\ln i = \frac{\pi}{2}i$. Thus $2\ln i = \pi i = Log(-1)$ and $(-1)^{\theta} = \exp^{\pi\theta i}$, e.g., $\theta = 1$ yields $(-1)^1 = \exp^{\pi i} = -1$ as required. Thus we set $(-3)^{\theta} = 3^{\theta}\exp^{\pi\theta i} = 3^{\theta}(\cos\pi\theta + i\sin\pi\theta)$. Hence

$$F(-4+\theta) = (3^{\theta}\cos\pi\theta + 1 + \theta) + (3^{\theta}\sin\pi\theta)i$$

Hence the evaluation of $F(x)$ for $x < 0$ may involve complex numbers.

Definition 4.2. Given a complex number $z := a + bi \in \mathbf{C}$, we define a map $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ by

$$\widetilde{F}(z) := F(a) + iF(b-1)$$

where $F(x)$ is the continuous Fibonacci function on $\mathbf{R}$. We call such a map $\widetilde{F}$ a *complex Fibonacci function.*

Given a real number $a \in \mathbf{R}$, we have $\widetilde{F}(a) = \widetilde{F}(a+i0) = F(a)+iF(-1) = F(a)$, so that $\widetilde{F}$ extends the function F already defined on $\mathbf{R}$ to the complex numbers $\mathbf{C}$.

Proposition 4.3. *Given a Gaussian integer $z = m + in$ $(m, n \in \mathbf{Z})$, we have*

$$\widetilde{F}((m+2)+(n+3)i) = \widetilde{F}((m+1)+(n+2)i) + \widetilde{F}(m+(n+1)i))$$

Proof. Since $\widetilde{F}(m+ni) = F(m)+F(n-1)i = F_m + F_{n-1}i$, we obtain $\widetilde{F}((m+2)+(n+3)i) = F_{m+2}+F_{n+2}i = (F_{m+1}+F_{n+1}i)+(F_m+F_ni) = \widetilde{F}((m+1)+(n+2)i)+\widetilde{F}(m+(n+1)i))$. □

Using the fact that the Fibonacci derivative is a linear mapping, we define the Fibonacci derivative for complex Fibonacci numbers as follows: Given $z = a + bi$ $(a, b \in \mathbf{R})$,

$$\widetilde{\triangle}F(z) := \triangle F(a) + i \triangle F(b-1)$$

Proposition 4.4. *Given a complex Fibonacci function $\widetilde{F}(z)$, we have*

$$\widetilde{\triangle}\widetilde{F}(z) = \widetilde{F}(z+2(1+i)) - \widetilde{F}(z+1+i) - \widetilde{F}(z)$$

Proof. Given $z := a + bi \in \mathbf{R}$, we have

$$\begin{aligned} \widetilde{\triangle}\widetilde{F}(z) &= \triangle F(a) + i \triangle F(b-1) \\ &= [F(a+2) - F(a+1) - F(a)] + i[F(b+1) - F(b) - F(b-1)] \\ &= \widetilde{F}((a+2)+i(b+2)) - \widetilde{F}((a+1)+i(b+1)) - \widetilde{F}(a+ib) \\ &= \widetilde{F}(z+2(1+i)) - \widetilde{F}(z+1+i) - \widetilde{F}(z), \end{aligned}$$

i.e., $\widetilde{\triangle}$ is the complex Fibonacci derivative. □

5. Concluding remark

As was already been mentioned in the introduction and has been demonstrated in the paper, the extensions of $\{F_n\}_{n\in\mathbf{Z}}$ to $F(x)$ and $\widetilde{F}(z)$ show themselves to be rather remarkable functions. We should note that $F(0) = F(1) = 1$, only selects one among a family of functions of this type. Considering the usual property $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$ for Fibonacci function $f : \mathbf{R} \to \mathbf{R}$, it is naturally of interest to check on a variety of limit problems of this type and discover properties and solutions to these problems among others.

6. Future works

One area which needs further investigation is the adapting of the theory developed above to general groupoids. In order to do this we will need to reduce formulas such as given above, which involve two (closely related) binary operations to those using only one such binary operation. Thus, consider an arbitrary groupoid $(X, *)$ and an element $a \in X$. We consider functions $f : X \to X$ such that $(\bigtriangledown_a^c f)(x) = (f(x) * f(x * a)) * f((x * a) * a) \equiv c$ and let these be (a, c)-$(X, *)$-Fibonacci-functions. That this is a true generalization can be seen as follows. Let $(X, *) = (\mathbf{R}, -)$ and let $a = 1, c = 0$. Then $(\bigtriangledown_1^0 f)(x) = f(x) - f(x-1) - f(x-2) = 0$ means $f(x) = f(x-1) + f(x-2)$, i.e., an $(1, 0)$-$(\mathbf{R}, -)$-Fibonacci-function as defined above. At the same time, if $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$, then $(\triangle f)(x-2) = f(x) - f(x-1) - f(x-2) = (\bigtriangledown_1^0 f)(x)$, so that $\triangle f$ is a translation of $\bigtriangledown_1^0 f$ on $(\mathbf{R}, -)$. Using these approach one hopes to develop very general Fibonacci properties which may be directly applied to a great variety of situations and thus also with an improved chance for possible applications, due to a much larger range of possible models which may be available. Therefore it is among our plans to follow through with this approach as well as what has been mentioned in the concluding remark section also.

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Decomposition and improved hyperbolic cross approximation of bivariate functions on $[0,1]^2$ *

Zhihua Zhang
College of Global Change and Earth System Science, Beijing Normal University, Beijing, China
Joint Centre for Global Change Studies, Beijing, China
E-mail: zhangzh@bnu.edu.cn

Abstract. For a bivariate function on the unit square, if we extend it robustly into a periodic function on the plane, then its Fourier coefficients decay very slowly due to the discontinuity on the boundary of the unit square, therefore, we need a lot of Fourier coefficients to reconstruct this bivariate function. In order to solve this problem, for any bivariate smooth function on the unit square, we introduce a Fourier expansion with a polynomial term and several polynomial factors such that the corresponding Fourier coefficients decay fast. Using this expansion, we can construct a good approximation tool for any bivariate function on the unit square.

1. Introduction

It is well-known that smooth period functions can be approximated well by Fourier series. But, for a bivariate function f on the unit square $[0,1]^2$, if we extend it into a periodic function on the plane, then its Fourier coefficients decay very slowly due to the discontinuity on the boundary of $[0,1]^2$. So we need a lot of Fourier coefficients to reconstruct this bivariate function. In order to reconstruct f by fewest Fourier coefficients, we will develop a new approximation tool in this paper. We first construct four simple univariate polynomial $\varphi_i\,(i=1,...,4)$ of degree 3 which is independent of f. With the help of these polynomials, we express f into a sum:

$$f = f_1 + f_2 + f_3.$$

In this decomposition formula, f_1 is a linear combination of $\varphi_i(x)\varphi_j(y)\,(i,j=1,...,4)$, f_2 is a sum of products of a polynomial φ_i and a univariate function, and f_3 is a bivariate function whoes partial derivatives vanish on the boundary of $[0,1]^2$. Then we expand these univariate functions and this bivariate function into Fourier series, where the corresponding Fourier coefficients will decay fast. We call this process a Fourier expansion of f with a polynomial term and several polynomial factors. Based on this expansion, we can develop a good approximation tool of f by using the partial sums of these univariate Fourier series and the hyperbolic cross truncation of bivariate Fourier series. Precisely say, for a function f satisfying $\frac{\partial^4 f}{\partial x^2 \partial y^2} \in C([0,1]^2)$, if we use our

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approximation tool, then the approximation error is equivalent to $\frac{\log^4 N_d}{N_d^3}$, however, if we directly expand f into Fourier series, the approximation errors of its partial sums and hyperbolic cross truncation are equivalent to $\frac{1}{\sqrt{N_d}}$, $\frac{\log^2 N_d}{N_d}$, respectively, where N_d is the number of Fourier coefficients used. It is clear that our approximation tool is much better than traditional Fourier approximation. At the end of this paper, we will extend these results to the case of random processes.

Throughout this paper we always assume bivariate functions on $[0,1]^2$ are real-valued. Denote by $\{0,1\}^2$ vertexes of the unit square $[0,1]^2$ and by $\partial([0,1]^2)$ the boundary of $[0,1]^2$. We say $f \in C^{(2,2)}([0,1]^2)$ if $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ is a continuous function on $[0,1]^2$. Denote by $s_N(f;x,y)$ the Fourier series partial sum of f on $[0,1]^2$, i.e.,

$$s_N(f;x,y) = \sum_{|m|\le N} \sum_{|n|\le N} c_{mn}(f)\, e^{2\pi i m x}\, e^{2\pi i n y}.$$

Denote by $s_n^{(h)}(f;x,y)$ the Fourier series hyperbolic cross truncation of f on $[0,1]^2$, i.e.,

$$\begin{aligned} s_N^{(h)}(f;x,y) &= \sum_{|m|=0}^{N} c_{m0}(f)\, e^{2\pi i m x} \\ &+ \sum_{|n|=1}^{N} c_{0n}\, e^{2\pi i n y} + \sum_{1\le|mn|\le N} c_{mn}(f)\, e^{2\pi i (mx+ny)}. \end{aligned} \tag{1.1}$$

For a random variable ξ, denote by $E[\xi]$ and $\mathrm{Var}(\xi)$ its expectation and variance, respectively. For two random variables ξ, η, denote by $\mathrm{Cov}(\xi, \eta)$ their covariance. We also always assume that ξ is real-valued. The concept of the random calculus may refer the reference [2,6].

This paper is organized as follows: In section 2 we give a decomposition formula of a bivariate function on $[0,1]^2$. In section 3 we discuss Fourier expansions with polynomial term and polynomial factors and estimate Fourier coefficients. In Section 4 we present a new approximation tool and estimate the corresponding approximation error. In Section 5 we generalize these results to random processes on $[0,1]^2$.

2. Decomposition of bivariate functions on the unit square

Suppose that $f(x,y)$ is a real-valued function on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$. First we introduce four fundamental polynomials:

$$\begin{aligned} \varphi_1(x) &= (1+2x)(x-1)^2 = -(3-2x)x^2 + 1, \\ \varphi_2(x) &= (3-2x)x^2 = -\varphi_1(x) + 1, \\ \varphi_3(x) &= x(x-1)^2, \\ \varphi_4(x) &= x^2(x-1) \end{aligned} \tag{2.1}$$

satisfying the following conditions

$$\begin{aligned}
&\varphi_1(0)=1, \qquad \varphi_1(1)=\varphi_1'(0)=\varphi_1'(1)=0,\\
&\varphi_2(1)=1, \qquad \varphi_2(0)=\varphi_2'(0)=\varphi_2'(1)=0,\\
&\varphi_3'(0)=1, \qquad \varphi_3(0)=\varphi_3(1)=\varphi_3'(1)=0,\\
&\varphi_4'(1)=1, \qquad \varphi_4(0)=\varphi_4(1)=\varphi_4'(0)=0.
\end{aligned} \tag{2.2}$$

Define a bivariate polynomial:

$$\begin{aligned}
f_1(x,y) &= \sum_{\nu=0}^{1}\left(f(0,\nu)\varphi_1(x)+f(1,\nu)\varphi_2(x)\right)\varphi_{1+\nu}(y)\\
&+ \sum_{\nu=0}^{1}\left(\tfrac{\partial f}{\partial x}(0,\nu)\varphi_3(x)+\tfrac{\partial f}{\partial x}(1,\nu)\varphi_4(x)\right)\varphi_{1+\nu}(y)\\
&+ \sum_{\nu=0}^{1}\left(\tfrac{\partial f}{\partial y}(0,\nu)\varphi_1(x)+\tfrac{\partial f}{\partial y}(1,\nu)\varphi_2(x)\right)\varphi_{3+\nu}(y).
\end{aligned} \tag{2.3}$$

This is a linear combination of $\{\varphi_i(x)\varphi_j(y)\}_{i,j=1,\dots,4}$ whose coefficients depend only on values of f and partial derivatives of f at vertexes $\{0,1\}^2=\{(0,0),\,(0,1),\,(1,0),\,(1,1)\}$. Denote

$$g(x,y)=f(x,y)-f_1(x,y).$$

We construct a bivariate function f_2 which depends only on values of g and partial derivatives of g at the boundary of $[0,1]^2$. Define

$$\begin{aligned}
f_2(x,y) &= g(x,0)\varphi_1(y)+g(x,1)\varphi_2(y)+g(0,y)\varphi_1(x)+g(1,y)\varphi_2(x)\\
&+\tfrac{\partial g}{\partial x}(0,y)\varphi_3(x)+\tfrac{\partial g}{\partial x}(1,y)\varphi_4(x)+\tfrac{\partial g}{\partial y}(x,0)\varphi_3(y)+\tfrac{\partial g}{\partial y}(x,1)\varphi_4(y).
\end{aligned} \tag{2.4}$$

This is a sum of products of univariate functions and fundamental polynomials φ_i. Finally, we let

$$f_3(x,y)=f(x,y)-f_1(x,y)-f_2(x,y).$$

Then the following decomposition formula holds.

Theorem 2.1. Let $f\in C^{(2,2)}([0,1]^2)$. Then

$$f(x,y)=f_1(x,y)+f_2(x,y)+f_3(x,y), \tag{2.5}$$

where f_1, f_2, and f_3 are stated as above and satisfy

(i) $f_1(x,y)$ is a bivariate polynomial and for $(x,y)\in\{0,1\}^2$,

$$\begin{aligned}
&f_1(x,y)=f(x,y), \qquad &&\tfrac{\partial f_1}{\partial x}(x,y)=\tfrac{\partial f}{\partial x}(x,y),\\
&\tfrac{\partial f_1}{\partial y}(x,y)=\tfrac{\partial f}{\partial y}(x,y), \qquad &&\tfrac{\partial^2 f_1}{\partial x\partial y}(x,y)=\tfrac{\partial^2 f}{\partial x\partial y}(x,y),
\end{aligned}$$

i.e., $g = f - f_1$ satisfies

$$g(x,y) = \frac{\partial g}{\partial x}(x,y) = \frac{\partial g}{\partial y}(x,y) = \frac{\partial^2 g}{\partial x \partial y}(x,y) = 0, \qquad (x,y) \in \{0,1\}^2;$$

(ii) the remainder $f_3 \in C^{(2,2)}([0,1]^2)$ and for $(x,y) \in \partial([0,1]^2)$,

$$f_3(x,y) = \frac{\partial f_3}{\partial x}(x,y) = \frac{\partial f_3}{\partial y}(x,y) = \frac{\partial^2 f_3}{\partial x \partial y}(x,y) = 0.$$

From the definitions of f_1, f_2, f_3, and (2.2), we can directly check (i) and (ii).

With the help of this decomposition formula, we give a Fourier expansion with polynomial factors and a new approximation tool such that we can reconstruct functions on $[0,1]^2$ by fewest Fourier coefficients.

3. A kind of new Fourier expansions

In this section we give a Fourier expansion of the function on $[0,1]^2$ with a polynomial term and several polynomial factors.

Suppose that $f \in C^{(2,2)}([0,1]^2)$. By Theorem 2.1,

$$f(x,y) = f_1(x,y) + f_2(x,y) + f_3(x,y),$$

where f_1 is a polynomial which is stated in (2.3) and f_2 is stated in (2.4). We expand the first factor of each term in (2.4) into univariate Fourier series, such as we expand $g(x,\nu)\,(\nu = 0,1)$ into the Fourier series:

$$g(x,\nu) = \sum_m a_m^{(\nu)}\, e^{2\pi i m x},$$

where $a_m^{(\nu)} = \int_0^1 g(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x$ and $\sum_m = \sum_{m=-\infty}^{\infty}$. Using the integration by parts, by Theorem 2.1 (i) and Riemann-Lebesgue lemma, the Fourier coefficients satisfy

$$\begin{aligned} a_m^{(\nu)} &= \int_0^1 g(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x = \tfrac{1}{2\pi i m}\int_0^1 \tfrac{\partial g}{\partial x}(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x \\ &= -\tfrac{1}{4\pi^2 m^2}\int_0^1 \tfrac{\partial^2 g}{\partial x^2}(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x = o\left(\tfrac{1}{m^2}\right). \end{aligned}$$

Similarly, we expand $g(\nu,y)$, $\frac{\partial g}{\partial x}(\nu,y)$, $\frac{\partial g}{\partial y}(x,\nu)\,(\nu = 0,1)$ into Fourier series:

$$g(\nu,y) = \sum_n b_n^{(\nu)}\, e^{2\pi i n y},$$

$$\tfrac{\partial g}{\partial x}(\nu,y) = \sum_n \alpha_n^{(\nu)}\, e^{2\pi i n y},$$

$$\tfrac{\partial g}{\partial y}(x,\nu) = \sum_m \beta_m^{(\nu)}\, e^{2\pi i m x}.$$

From this, we see that $f_2(x,y)$ can be expanded into the following Fourier series with polynomial factors φ_i:

$$\begin{aligned} f_2(x,y) &= \varphi_1(y)\sum_m a_m^{(0)}\, e^{2\pi imx} + \varphi_2(y)\sum_m a_m^{(1)}\, e^{2\pi imx} \\ &+ \varphi_1(x)\sum_n b_n^{(0)}\, e^{2\pi iny} + \varphi_2(x)\sum_n b_n^{(1)}\, e^{2\pi iny} \\ &+ \varphi_3(x)\sum_n \alpha_n^{(0)}\, e^{2\pi iny} + \varphi_4(x)\sum_n \alpha_n^{(1)}\, e^{2\pi iny} \\ &+ \varphi_3(y)\sum_m \beta_m^{(0)}\, e^{2\pi imx} + \varphi_4(y)\sum_m \beta_m^{(1)}\, e^{2\pi imx} \end{aligned} \tag{3.1}$$

and

$$a_m^{(\nu)} = o\left(\tfrac{1}{m^2}\right), \qquad b_n^{(\nu)} = o\left(\tfrac{1}{n^2}\right),$$

$$\alpha_n^{(\nu)} = o\left(\tfrac{1}{n^2}\right), \qquad \beta_m^{(\nu)} = o\left(\tfrac{1}{m^2}\right).$$

Finally, expand f_3 into a bivariate Fourier series:

$$f_3(x,y) = \sum_{m,n} c_{mn}(f_3)\, e^{2\pi i(mx+ny)},$$

where $c_{mn}(f_3) = \int_0^1\int_0^1 f_3(x,y)\, e^{-2\pi i(mx+ny)}\,\mathrm{d}x\mathrm{d}y$ and $\sum_{m,n} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}$. By Theorem 2.1 (ii), the interior integral is equal to

$$\int_0^1 f_3(x,y)\, e^{-2\pi i(mx+ny)}\,\mathrm{d}x = \frac{1}{(2\pi im)^2}\int_0^1 \frac{\partial^2 f_3}{\partial x^2}(x,y)\, e^{-2\pi imx}\,\mathrm{d}x.$$

So the Fourier coefficients:

$$c_{mn}(f_3) = \frac{1}{(2\pi im)^2}\int_0^1 e^{-2\pi imx}\left(\int_0^1 \frac{\partial^2 f_3}{\partial x^2}(x,y)\, e^{-2\pi iny}\mathrm{d}y\right)\mathrm{d}x.$$

Again, by Theorem 2.1 (ii),

$$\begin{aligned} c_{mn}(f_3) &= \tfrac{1}{(2\pi im)^2}\int_0^1 e^{-2\pi imx}\left(\tfrac{1}{2\pi in}\int_0^1 \tfrac{\partial^3 f_3}{\partial x^2\partial y}(x,y)\, e^{-2\pi iny}\mathrm{d}y\right)\mathrm{d}x \\ &= \tfrac{1}{16\pi^4m^2n^2}\int_0^1\int_0^1 \tfrac{\partial^4 f_3}{\partial x^2\partial y^2}(x,y)\, e^{-2\pi i(mx+ny)}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.2}$$

Summarizing up the above results, we get a Fourier expansion with polynomial term and polynomial factors, where Fourier coefficients decay fast.

Theorem 3.1. Let $f \in C^{(2,2)}([0,1]^2)$ and f_1, f_2, f_3 be stated as in (2.3)-(2.5), and $\varphi_i (i = 1, ..., 4)$ be stated as in (2.1). Then f can be expanded into Fourier series with a polynomial term and several polynomial factors as follows:

$$\begin{aligned} f(x,y) = f_1(x,y) &+ \sum_{\nu=0}^{1}\left(\varphi_{1+\nu}(y)\sum_m a_m^{(\nu)}\, e^{2\pi imx} + \varphi_{1+\nu}(x)\sum_n b_n^{(\nu)}\, e^{2\pi iny}\right) \\ &+ \sum_{\nu=0}^{1}\left(\varphi_{3+\nu}(x)\sum_n \alpha_n^{(\nu)}\, e^{2\pi iny} + \varphi_{3+\nu}(y)\sum_m \beta_m^{(\nu)}\, e^{2\pi imx}\right) \\ &+ \sum_{m,n} c_{mn}(f_3)\, e^{2\pi i(mx+ny)}, \end{aligned} \tag{3.3}$$

where $f_1(x,y)$ is a polynomial which is stated in (2.3), both the second term and the third term are a combination of four univariate Fourier expansions and four fundamental polynomials, and for $\nu = 0, 1$,

$$\begin{aligned} a_m^{(\nu)} &= \int_0^1 g(x,\nu)\, e^{-2\pi imx}\mathrm{d}x = o\left(\tfrac{1}{m^2}\right), \\ b_n^{(\nu)} &= \int_0^1 g(\nu,y)\, e^{-2\pi iny}\mathrm{d}x = o\left(\tfrac{1}{n^2}\right), \\ \alpha_n^{(\nu)} &= \int_0^1 \tfrac{\partial g}{\partial x}(\nu,y)\, e^{-2\pi iny}\mathrm{d}y = o\left(\tfrac{1}{n^2}\right), \\ \beta_m^{(\nu)} &= \int_0^1 \tfrac{\partial g}{\partial y}(x,\nu)\, e^{-2\pi imx}\mathrm{d}x = o\left(\tfrac{1}{m^2}\right) \end{aligned} \tag{3.4}$$

and $g(x,y) = f(x,y) - f_1(x,y)$, and the last term is a bivariate Fourier series of f_3 whose coefficients satisfy

$$\begin{aligned} &c_{mn}(f_3) = o\left(\tfrac{1}{m^2n^2}\right) \qquad (m \to \infty \text{ or } n \to \infty), \\ &c_{0n}(f_3) = o\left(\tfrac{1}{n^2}\right) \qquad (n \to \infty), \\ &c_{m0}(f_3) = o\left(\tfrac{1}{m^2}\right) \qquad (m \to \infty). \end{aligned} \tag{3.5}$$

4. A new approximation tool

We want to reconstruct the bivariate function $f(x,y)$ by the fewest Fourier coefficients. For this purpose, we take partial sums of univariate Fourier series and hyperbolic cross truncation of the bivariate Fourier series of $\sum\limits_{m,n} c_{mn}(f_3)\, e^{2\pi i(mx+ny)}$ in (3.3), we get a hyperbolic cross truncation of Fourier expansion of f with a polynomial term and several polynomial factors. For an appropriate $N \in \mathbb{Z}_+$, we define such a combination of polynomials and trigonometric polynomials:

$$\begin{aligned} T_N^{(h)}(x,y) = f_1(x,y) &+ \sum_{\nu=0}^{1}\left(\varphi_{1+\nu}(y)\sum_{|n|\le N} a_n^{(\nu)}\, e^{2\pi inx} + \varphi_{1+\nu}(x)\sum_{|n|\le N} b_n^{(\nu)}\, e^{2\pi iny}\right) \\ &+ \sum_{\nu=0}^{1}\left(\varphi_{3+\nu}(x)\sum_{|n|\le N} \alpha_n^{(\nu)}\, e^{2\pi iny} + \varphi_{3+\nu}(y)\sum_{|n|\le N} \beta_n^{(\nu)}\, e^{2\pi inx}\right) + s_N^{(h)}(f_3;x,y), \end{aligned} \tag{4.1}$$

where the last term is the hyperbolic cross truncation of f_3 which is stated in (1.1).

From this and (3.3), it follows that

$$f(x,y) - T_N^{(h)}(x,y) = s_N^{(1)}(x,y) + s_N^{(2)}(x,y), \tag{4.2}$$

where

$$\begin{aligned} s_N^{(1)}(x,y) &= \sum_{\nu=0}^{1}\left(\varphi_{1+\nu}(y)\sum_{|n|>N} a_n^{(\nu)}\, e^{2\pi inx} + \varphi_{1+\nu}(x)\sum_{|n|>N} b_n^{(\nu)}\, e^{2\pi iny}\right) \\ &+ \sum_{\nu=0}^{1}\left(\varphi_{3+\nu}(x)\sum_{|n|>N} \alpha_n^{(\nu)}\, e^{2\pi iny} + \varphi_{3+\nu}(y)\sum_{|n|>N} \beta_n^{(\nu)}\, e^{2\pi inx}\right) \\ s_N^{(2)}(x,y) &= f_3(x,y) - s_N^{(h)}(f_3,x,y). \end{aligned} \tag{4.3}$$

Consider the square error:

$$\| f - T_N^{(h)} \|_2^2 = \int_0^1 \int_0^1 |f(x,y) - T_N^{(h)}(f;x,y)|^2 \mathrm{d}x\mathrm{d}y.$$

By (4.2), we have

$$\| f - T_N^{(h)} \|_2^2 \leq 2 \| s_N^{(1)} \|_2^2 + 2 \| s_N^{(2)} \|_2^2 . \tag{4.4}$$

By (4.3) and $\| h_1(x)h_2(y) \|_2^2 = \| h_1(x) \|_2^2 \| h_2(y) \|_2^2$, where $\| h(t) \|_2^2 = \int_0^1 |h(t)|^2 \mathrm{d}t$, we have

$$\begin{aligned} \| s_N^{(1)} \|_2^2 &\leq 64 \sum_{\nu=0}^{1} \left(\| \varphi_{1+\nu} \|_2^2 \| \sum_{|n|>N} a_n^{(\nu)} e^{2\pi i n x} \|_2^2 + \| \varphi_{1+\nu} \|_2^2 \| \sum_{|n|>N} b_n^{(\nu)} e^{2\pi i n y} \|_2^2 \right) \\ &+ 64 \sum_{\nu=0}^{1} \left(\| \varphi_{3+\nu} \|_2^2 \| \sum_{|n|>N} \alpha_n^{(\nu)} e^{2\pi i n y} \|_2^2 + \| \varphi_{3+\nu} \|_2^2 \| \sum_{|n|>N} \beta_n^{(\nu)} e^{2\pi i n x} \|_2^2 \right). \end{aligned}$$

By the Parseval identity of the univariate Fourier series, we get

$$\| s_N^{(1)} \|_2^2 \leq 64 \sum_{\nu=0}^{1} \left(\| \varphi_{1+\nu} \|_2^2 \sum_{|n|>N} (|a_n^{(\nu)}|^2 + |b_n^{(\nu)}|^2) + \| \varphi_{3+\nu} \|_2^2 \sum_{|n>N|} (|\alpha_n^{(\nu)}| + |\beta_n^{(\nu)}|^2) \right).$$

Again, by (3.4),

$$\| s_N^{(1)} \|_2^2 = o\left(\sum_{|n|>N} \frac{1}{n^4} \right) = o\left(\frac{1}{N^3} \right). \tag{4.5}$$

For $s_N^{(2)}$, by (1.1), we have

$$\begin{aligned} s_N^{(2)}(f_3;x,y) &= \sum_{|m|>N} c_{m0}(f_3)\, e^{2\pi i m x} + \sum_{|n|>N} \sum_{|m|=0}^{\infty} c_{mn}(f_3)\, e^{2\pi i (mx+ny)} \\ &+ \sum_{|n|=1}^{N} \sum_{|m|>\frac{N}{|n|}} c_{mn}(f_3)\, e^{2\pi i (mx+ny)}. \end{aligned}$$

By the Parseval identity and (3.5), we have $\| s_N^{(2)} \|_2^2 = J_N^{(1)} + J_N^{(2)} + J_N^{(3)}$, where

$$J_N^{(1)} = \sum_{|m|>N} |c_{m0}(f_3)|^2 = o\left(\tfrac{1}{m^4}\right) = o\left(\tfrac{1}{N^3}\right),$$

$$J_N^{(2)} = \sum_{|n|>N} \sum_{|m|=1}^{\infty} |c_{mn}(f_3)|^2 = o\left(\sum_{|n|>N} \tfrac{1}{n^4} \right) \left(\sum_{|m|=1}^{\infty} \tfrac{1}{m^4} \right) = o\left(\tfrac{1}{N^3}\right),$$

$$\begin{aligned} J_N^{(3)} &= \sum_{|n|=1}^{N} \sum_{|m|>\left[\frac{N}{|n|}\right]} |c_{mn}(f_3)|^2 = O\left(\sum_{|n|=1}^{N} \sum_{|m|>\frac{N}{|n|}} \tfrac{1}{m^4 n^4} \right) \\ &= O\left(\sum_{|n|=1}^{N} \tfrac{1}{n^4} \sum_{|m|>\frac{N}{|n|}} \tfrac{1}{m^4} \right) = O\left(\sum_{|n|>N} \tfrac{1}{n^4} \tfrac{|n|^3}{N^3} \right) = O\left(\tfrac{1}{N^3} \log N \right). \end{aligned}$$

Therefore,

$$\| s_N^{(2)} \|_2^2 = O\left(\frac{1}{N^3}\log N\right).$$

From this and (4.4), and (4.5), it follows that

$$\| f - T_N^{(h)} \|_2^2 = O\left(\frac{\log N}{N^3}\right).$$

Since $\sum_{1\le|mn|\le N} 1 \sim N\log N$, by (4.1), we see that the number N_d of Fourier coefficients in the approximation tool $T_N^{(h)}(x,y)$ is equivalent to $N\log N$, i.e.,

$$N_d \sim N\log N.$$

This implies the following:

Theorem 4.1. Let $f \in C^{(2,2)}([0,1]^2)$ and $T_N^{(h)}$ be the hyperbolic cross truncation of its Fourier expansion with a polynomial term and several polynomial factors which are stated in (4.1). Then

$$\| f - T_N^{(h)} \|_2^2 = O\left(\frac{\log^4 N_d}{N_d^3}\right), \tag{4.6}$$

where N_d is the number of Fourier coefficients in $T_N^{(h)}$.

For $f \in C^{(2,2)}([0,1]^2)$, consider the partial sums of the Fourier series of f:

$$s_N(x,y) = \sum_{|m|\le N}\sum_{|n|\le N} c_{mn}(f)\, e^{2\pi i(mx+ny)}.$$

By the Parseval identity,

$$\| f - s_N \|_2^2 = \sum_{|n|>N}\sum_{|m|=0}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=0}^{N}\sum_{|m|>N} |c_{mn}(f)|^2.$$

From this and $c_{m0}(f) = O\left(\frac{1}{m}\right)$, $c_{0n}(f) = O\left(\frac{1}{n}\right)$, $c_{mn}(f) = O\left(\frac{1}{mn}\right)$, it follows that

$$\| f - s_N \|_2^2 = O\left(\frac{1}{N}\right).$$

Note that the number N_d of Fourier coefficients in the partial sum is equivalent to N^2, i.e., $N_d \sim N^2$. So

$$\| f - s_N \|_2^2 = O\left(\frac{1}{\sqrt{N_d}}\right). \tag{4.7}$$

Consider the hyperbolic cross truncation of the Fourier series of f. By (1.1) and the Parseval identity,

$$\| f - s_N^{(h)} \|_2^2 = \sum_{|m|>N} |c_{m0}(f)|^2 + \sum_{|n|>N}\sum_{|m|=1}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=1}^{N}\sum_{|m|>\frac{N}{|n|}} |c_{mn}(f)|^2 = o\left(\frac{\log N}{N}\right). \tag{4.8}$$

From this and $N_d \sim N\log N$,

$$\| f - s_N^{(h)} \|_2^2 = o\left(\frac{\log^2 N_d}{N_d}\right).$$

Comparing (4.7), (4.8) with (4.6), we see that the approximation tool $T_N^{(h)}$ is such that we reconstruct f by the fewest Fourier coefficients.

5. Uncertainty analysis

Now we extend the above results to the case of random processes. Suppose that f is a real-valued random process on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$ (Refer to [6] for random calculus). Then the decomposition formula (2.5) is still valid:

$$f = f_1 + f_2 + f_3,$$

where f_1 and f_2 are stated as in (2.3) and (2.4), respectively, and f_3 is the residual. However, now f_1 is a random polynomial, f_2 is a sum of products of univariate random processes and fundamental polynomials φ_i. Theorem 2.1 and the expansion (3.3) are still valid. However, Fourier coefficients in (3.3): $a_m^{(\nu)}, b_n^{(\nu)}, \alpha_n^{(\nu)}, \beta_m^{(\nu)}$ $(\nu = 0, 1)$, and $c_{mn}(f_3)$ are now all random variables. Consider their expectations and variances. Note that

$$a_m^{(\nu)} = \int_0^1 g(x,\nu)\, e^{-2\pi i m x} \mathrm{d}x \qquad (\nu = 0, 1). \tag{5.1}$$

Since the expectation and the integral can be exchanged,

$$E[a_m^{(\nu)}] = \int_0^1 E[g(x,\nu)]\, e^{-2\pi i m x} \mathrm{d}x \qquad (\nu = 0, 1). \tag{5.2}$$

Since the random process $g = f - f_1$ belongs to $C^{(2,2)}([0,1]^2)$ and the expectation and the partial derivatives can be exchanged, the deterministic function $E[g(x,y)] \in C^{(2,2)}([0,1]^2)$. Noticing that Theorem 2.1 is still valid, for $(x,y) \in \{0,1\}^2$,

$$E[g(x,y)] = E\left[\frac{\partial g}{\partial x}(x,y)\right] = E\left[\frac{\partial g}{\partial y}(x,y)\right] = E\left[\frac{\partial^2 g}{\partial x \partial y}(x,y)\right] = 0.$$

Exchanging the expectation and the partial derivatives, for $(x,y) \in \{0,1\}^2$, we get

$$E[g(x,y)] = \frac{\partial}{\partial x} E[g(x,y)] = \frac{\partial}{\partial y} E[g(x,y)] = \frac{\partial^2}{\partial x \partial y} E[g(x,y)] = 0.$$

Therefore,

$$\begin{aligned} E[a_m^{(\nu)}] &= -\tfrac{1}{2\pi i m}\, E[g(x,\nu)]|_{x=0}^1 + \tfrac{1}{2\pi i m} \int_0^1 \tfrac{\partial}{\partial x}(E[g(x,\nu)])\, e^{-2\pi i m x} \mathrm{d}x \\ &= -\tfrac{1}{(2\pi i m)^2}\, \tfrac{\partial}{\partial x} E[g(x,\nu)]\big|_{x=0}^1 + \tfrac{1}{(2\pi i m)^2} \int_0^1 \tfrac{\partial^2}{\partial x^2}(E[g(x,\nu)])\, e^{-2\pi i m x} \mathrm{d}x \\ &= \tfrac{1}{(2\pi i m)^2} \int_0^1 \tfrac{\partial^2}{\partial x^2}(E[g(x,\nu)])\, e^{-2\pi i m x} \mathrm{d}x = o\left(\tfrac{1}{m^2}\right) \qquad (\nu = 0, 1). \end{aligned}$$

This implies that

$$E[a_m^{(\nu)}] \le \frac{1}{4\pi^2 m^2} \max_{0 \le x \le 1} |\frac{\partial^2}{\partial x^2} E[g(x,\nu)]| \qquad (\nu = 0, 1).$$

Similarly, we compute $E[b_n^{(\nu)}]$, $E[\alpha_n^{(\nu)}]$, $E[\beta_m^{(\nu)}]$, and $E[c_{mn}(f_3)]$. So we have the following:

Theorem 5.1. Let f be a random process on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$. Then, in the Fourier expansion (3.3) with a random polynomial and several random polynomial factors, Fourier coefficients satisfy

$$E[a_m^{(\nu)}] = o\left(\tfrac{1}{m^2}\right), \qquad E[a_m^{(\nu)}] \le \tfrac{1}{4\pi^2 m^2} \max_{0\le x\le 1} |\tfrac{\partial^2}{\partial x^2} E[g(x,\nu)]|,$$

$$E[b_n^{(\nu)}] = o\left(\tfrac{1}{n^2}\right), \qquad E[b_n^{(\nu)}] \le \tfrac{1}{4\pi^2 n^2} \max_{0\le y\le 1} |\tfrac{\partial^2}{\partial y^2} E[g(\nu,y)]|,$$

$$E[\alpha_n^{(\nu)}] = o\left(\tfrac{1}{n^2}\right), \qquad E[\alpha_n^{(\nu)}] \le \tfrac{1}{4\pi^2 n^2} \max_{0\le y\le 1} |\tfrac{\partial^3}{\partial x \partial y^2} E[g(\nu,y)]|,$$

$$E[\beta_m^{(\nu)}] = o\left(\tfrac{1}{m^2}\right), \qquad E[\beta_m^{(\nu)}] \le \tfrac{1}{4\pi^2 m^2} \max_{0\le x\le 1} |\tfrac{\partial^3}{\partial x^2 \partial y} E[g(x,\nu)]| \qquad (\nu = 0,1),$$

$$E[c_{mn}(f_3)] = o\left(\tfrac{1}{m^2 n^2}\right), \qquad E[c_{mn}(f_3)] \le \tfrac{1}{16\pi^4 m^2 n^2} \max_{0\le x,y\le 1} |\tfrac{\partial^4}{\partial x^2 \partial y^2} E[f_3(x,y)]|,$$

where $g = f - f_1$ and f_1, f_2 are stated as above.

Now we consider the variances of Fourier coefficients in (3.3).

Since g is a real-valued, by (5.1), we deduce that for $\nu = 0, 1$,

$$|a_m^{(\nu)}|^2 = \int_0^1 \int_0^1 g(x,\nu) g(t,\nu)\, e^{-2\pi i m(x-t)} \mathrm{d}x\mathrm{d}t,$$

$$E[\,|a_m^{(\nu)}|^2\,] = \int_0^1 \int_0^1 E[\,g(x,\nu)g(t,\nu)\,]\, e^{-2\pi i m(x-t)} \mathrm{d}x\mathrm{d}t,$$

Since $E[g(x,\nu)g(t,\nu)] \in C^{(2,2)}([0,1]^2)$, by Theorem 2.1 (i) and using integration by parts, it follows that

$$E[\,|a_m^{(\nu)}|^2\,] = \frac{1}{16\pi^4 m^4} \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^2 \partial t^2} E[g(x,\nu)g(t,\nu)]\, e^{-2\pi i m(x-t)} \mathrm{d}x\mathrm{d}t \qquad (\nu = 0,1), \tag{5.3}$$

and so

$$E[\,|a_m^{(\nu)}|^2\,] = o\left(\frac{1}{m^4}\right) \qquad (\nu = 0,1).$$

Noticing that $\mathrm{Var}(a_m^{(\nu)}) = E[\,|a_m^{(\nu)}|^2\,] - (\,E[a_m^{(\nu)}]\,)^2$, we get

$$\mathrm{Var}(a_m^{(\nu)}) \le E[(a_m^{(\nu)})^2] = o\left(\frac{1}{m^4}\right) \qquad (\nu = 0,1). \tag{5.4}$$

From (5.2), it follows that

$$(E[\,a_m^{(\nu)}\,])^2 = \frac{1}{16\pi^4 m^4} \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^2 \partial t^2} (E[g(x,\nu)]E[g(t,\nu)])\, e^{-2\pi i m(x-t)} \mathrm{d}x\mathrm{d}t \qquad (\nu = 0,1). \tag{5.5}$$

Again, by the covariance formula: $\mathrm{Cov}(g(x,\nu),\ g(t,\nu)) = E[g(x,\nu)g(t,\nu)] - E[g(x,\nu)]E[g(t,\nu)]$, we have

$$\mathrm{Var}(a_m^{(\nu)}) = \frac{1}{16\pi^4 m^4} \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^2 \partial t^2} \mathrm{Cov}(g(x,\nu),\ g(t,\nu))\, e^{-2\pi i m(x-t)} \mathrm{d}x\mathrm{d}t \qquad (\nu = 0,1).$$

Similarly, we compute $\mathrm{Var}(b_n^{(\nu)}), \mathrm{Var}(\alpha_n^{(\nu)}), \mathrm{Var}(\beta_m^{(\nu)})$, and $\mathrm{Var}(c_{mn}(f_3))$. So we have

Theorem 5.2. Under conditions of Theorem 5.1, for $\nu = 0, 1$, we have

$$\mathrm{Var}(a_m^{(\nu)}) \le \frac{1}{16\pi^4 m^4} \max_{0\le x,t\le 1} |\frac{\partial^4}{\partial x^2 \partial t^2}\mathrm{Cov}(g(x,\nu),\ g(t,\nu))|,$$

$$\mathrm{Var}(b_n^{(\nu)}) \le \frac{1}{16\pi^4 n^4} \max_{0\le y,t\le 1} |\frac{\partial^4}{\partial y^2 \partial t^2}\mathrm{Cov}(g(\nu,y),\ g(\nu,t))|,$$

$$\mathrm{Var}(\alpha_n^{(\nu)}) \le \frac{1}{16\pi^4 n^4} \max_{0\le y,t\le 1} |\frac{\partial^4}{\partial y^2 \partial t^2}\mathrm{Cov}\left(\frac{\partial g}{\partial x}(\nu,y),\ \frac{\partial g}{\partial x}(\nu,t)\right)|,$$

$$\mathrm{Var}(\beta_m^{(\nu)}) \le \frac{1}{16\pi^4 m^4} \max_{0\le x,t\le 1} |\frac{\partial^4}{\partial x^2 \partial t^2}\mathrm{Cov}\left(\frac{\partial g}{\partial y}(x,\nu),\ \frac{\partial g}{\partial y}(t,\nu)\right)|,$$

$$\mathrm{Var}(c_{mn}(f_3)) \le \frac{1}{256\pi^8 m^4 n^4} \max_{0\le x,y,t,s\le 1} |\mathrm{Cov}\left(\frac{\partial^4 f_3}{\partial x^2 \partial y^2}(x,y),\ \frac{\partial^4 f_3}{\partial t^2 \partial s^2}(t,s)\right)|,$$

Similar to (5.5), we can obtain that the second-order moments are as follows. For $\nu = 0, 1$,

$$\begin{aligned} &E[|a_m^{(\nu)}|^2] = o\left(\tfrac{1}{m^4}\right), && E[|b_n^{(\nu)}|^2] = o\left(\tfrac{1}{n^4}\right), \\ &E[|\alpha_n^{(\nu)}|^2] = o\left(\tfrac{1}{n^4}\right), && E[|\beta_m^{(\nu)}|^2] = o\left(\tfrac{1}{m^4}\right), \\ &E[|c_{mn}(f_3)|^2] = o\left(\tfrac{1}{m^4 n^4}\right). \end{aligned} \tag{5.6}$$

Finally, for a random process, we still define an approximation tool $T_N^{(h)}(x,y)$ as in (4.1). Now $T_N^{(h)}(x,y)$ is a combination of random polynomials of degree 3 and random trigonometric polynomials of degree N. Using the Parseval identity, by (5.6), we have the following:

Theorem 5.3. Let f be a random process on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$, and $T_N^{(h)}$ be stated as above. Then the mean square error of approximation by $T_N^{(h)}$ satisfies

$$E[\,\|\, f - T_N^{(h)} \,\|_2^2\,] = o\left(\frac{\log^4 N_d}{N_d^3}\right),$$

where N_d is the number of Fourier coefficients in $T_N^{(h)}$.

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Gronwall-Bellman type inequalities for the distributional Henstock-Kurzweil integral and applications *

Wei Liu, Guoju Ye, Dafang Zhao
College of Science, Hohai University, Nanjing 210098, P.R. China
E-mails: liuw626@hhu.edu.cn, yegj@hhu.edu.cn, zhaodafang@hhu.edu.cn

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Abstract: This paper is devoted to studying the Gronwall-Bellman type inequalities involving the distributional Henstock-Kurzweil integral. Moreover, an linear differential equation with distributional coefficients is considered as an application.

Keywords: distributional derivative, distributional Henstock-Kurzweil integral, Gronwall-Bellman inequality.

MSC 2010: 26D15, 26A39, 46G12.

1 Introduction

It is well-known that the Gronwall-Bellman inequality (also called the Gronwall's lemma or the Gronwall's inequality) has played a fundamental role in the study of the qualitative behaviour of solutions of differential and integral equations.

In 1919, T. H. Gronwall [1] firstly established the following integral inequality.

Lemma 1.1. *Let $f(t)$ be a continuous function defined on $[a, a+\delta] \subset \mathbb{R}$, for which the inequality*

$$0 \leq f(t) \leq \int_a^t (\alpha f(s) + \beta) ds, \quad t \in [a, a+\delta] \tag{1.1}$$

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holds, where α, β, δ *are nonnegative constants. Then*

$$0 \le f(t) \le \beta\delta \exp(\alpha\delta), \quad t \in [a, a+\delta]. \tag{1.2}$$

In 1943, R. Bellman [2] generalized Lemma 1.1 to the following result.

Lemma 1.2. *Let* $f(t)$ *and* $g(t)$ *be nonnegative, continuous functions on* $[a, b] \subset \mathbb{R}$*, for which the inequality*

$$0 \le f(t) \le \eta + \int_a^t g(s)f(s)ds, \quad t \in [a, b] \tag{1.3}$$

holds, where η *is a nonnegative constant. Then*

$$0 \le f(t) \le \eta \exp\left(\int_a^t g(s)ds\right), \quad t \in [a, b]. \tag{1.4}$$

Because of the importance of this inequality, over the years investigators have discovered many useful generalizations in order to achieve a diversity of desired goals in various branches of differential and integral equations [3–18].

However, almost all the generalizations are based on continuous functions under Riemann and Lebesgue integrals, they are not applicable to generalized ordinary differential equations [19, 20]. This did not really change until Š. Schwabik [11] presented the Gronwall-Bellman inequality for the Henstock-Kurzweil integral in 1985, while K. Ostaszewski and J. Sochacki [12] gave a simpler and significant proof in 1987. As for the Gronwall-Bellman type inequalities for the Stieltjes integrals, we refer the reader to [14–18].

In this paper, we study the Gronwall-Bellman type inequalities for the distributional Henstock-Kurzweil integral, which defined by using Schwartz distributional derivative. It is a very wide integral form including the Riemann integral, the Lebesgue integral, and the Henstock-Kurzweil integral (see [19–22, 27–30] for details). The space of such integrable distributions, denoted by D_{HK}, is a completion of the space of Henstock-Kurzweil integrable functions (shortly, HK).

This paper is organized as follows. Section 2 is devoted to the basic notations of the distributional Henstock-Kurzweil integral. Section 3 contains our main results on the Gronwall-Bellman type inequalities involving the distributional Henstock-Kurzweil integral, while Section 4 sets forth an application to an linear differential equation with distributional coefficients.

2 The Distributional Henstock-Kurzweil Integral

Let (a,b) be an open interval in $\mathbb{R}$, we define

$$\mathcal{D}((a,b)) = \{\phi : (a,b) \to \mathbb{R} \mid \phi \in C_c^\infty \text{ and } \phi \text{ has a compact support in } (a,b)\}.$$

The distributions on (a,b) are defined to be the continuous linear functionals on $\mathcal{D}((a,b))$. The dual space of $\mathcal{D}((a,b))$ is denoted by $\mathcal{D}'((a,b))$.

For all $f \in \mathcal{D}'((a,b))$, we define the distributional derivative f' of f to be a distribution satisfying $\langle f', \phi\rangle = -\langle f, \phi'\rangle$, where $\phi \in \mathcal{D}((a,b))$ is a test function. Further, we write distributional derivative as f' and its pointwise derivative as $f'(t)$ where $t \in \mathbb{R}$. From now on, all derivative in this paper will be distributional derivatives unless stated otherwise.

Denote the space of continuous functions on $[a,b]$ by $C[a,b]$. Let

$$C_0 = \{F \in C[a,b] : F(a) = 0\}. \tag{2.1}$$

Then C_0 is an Banach space under the norm

$$\|F\|_\infty = \sup_{t\in[a,b]} |F(t)| = \max_{t\in[a,b]} |F(t)|.$$

Definition 2.1 ([29, Definition 1])**.** A distribution $f \in \mathcal{D}'((a,b))$ is said to be Henstock–Kurzweil integrable (shortly D_{HK}) on an interval $[a,b]$ if there exists a continuous function $F \in C_0$ such that $F' = f$, i.e., the distributional derivative of F is f. The distributional Henstock–Kurzweil integral of f on $[a,b]$ is denoted by $\int_a^b f(t)dt = F(b) - F(a)$. The function F is called the primitive of f.

For every $f \in D_{HK}$, $\phi \in \mathcal{D}((a,b))$, we write

$$\langle f, \phi\rangle = \int_a^b f(t)\phi(t)dt = -\int_a^b F(t)\phi'(t)dt.$$

The distributional Henstock–Kurzweil integral is very wide and it includes Riemann integral, Lebesgue integral, Henstock–Kurzweil integral, restricted and wide Denjoy integral (see [21, 22, 27–29]). From now on, we write " $\int_a^b f(t)dt$ " as " $\int_a^b f$ " for short.

For $f \in D_{HK}$, define the Alexiewicz norm in D_{HK} as

$$\|f\| = \|F\|_\infty = \sup_{t\in[a,b]} |F(t)| = \max_{t\in[a,b]} |F(t)|.$$

Under the Alexiewicz norm, D_{HK} is a Banach space, see [28, Theorem 2].

For $F \in C_0$, the positive part $F^+ = \max_{t\in[a,b]}\{F(t), 0\}$, the negative part $F^- = \max_{t\in[a,b]}\{-F(t), 0\}$, and hence $F = F^+ - F^-$ and the absolute value $|F| = F^+ + F^-$. Moreover, F^+, F^-, $|F|$ all belong to C_0. Let $f \in D_{HK}$ with the primitive $F \in C_0$, as in [28], define

$$f^+ = (F^+)', \quad f^- = (F^-)', \quad |f| = |F|'. \tag{2.2}$$

Then,

$$f = f^+ - f^-, \quad |f| = f^+ + f^-. \tag{2.3}$$

In C_0 there exists a pointwise order: for $F, G \in C_0$, $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in [a, b]$. For $f, g \in D_{HK}$ with primitives $F, G \in C_0$, respectively, we say

$$f \overset{(p)}{\preceq} g \text{ (or } g \overset{(p)}{\succeq} f) \quad \text{if and only if} \quad F(t) \le G(t),\ \forall t \in [a, b], \tag{2.4}$$

and

$$f \overset{(m)}{\preceq} g \text{ (or } g \overset{(m)}{\succeq} f) \quad \text{if and only if} \quad \int_I f \le \int_I g, \tag{2.5}$$

where I is arbitrary subinterval of $[a, b]$. Obviously,

$$f \overset{(m)}{\preceq} g \quad \Rightarrow \quad f \overset{(p)}{\preceq} g, \tag{2.6}$$

but the converse is not true. Particularly, if f, g are functions, then

$$f(t) \le g(t)\ (\forall t \in [a, b]) \quad \Leftrightarrow \quad f \overset{(m)}{\preceq} g \quad \Rightarrow \quad f \overset{(p)}{\preceq} g. \tag{2.7}$$

Lemma 2.2. *Let $f, g \in D_{HK}$. Then*

(I) $|f| \in D_{HK}$ and $|\int_a^t f| \le \int_a^t |f|$ for all $t \in [a, b]$;

(II) $\| |f| \| = \| |F'| \| = \| |F| \|_\infty = \|f\|$;

(III) $|f + g| \overset{(p)}{\preceq} |f| + |g|$.

Proof. (I) and (II) see [28, Theorem 24].

Since $|F + G| \le |F| + |G|$ in C_0, (III) follows immediately from (2.3) and (2.4). □

If $g : [a, b] \to \mathbb{R}$, its variation is $Vg = \sup \sum_n |g(t_n) - g(s_n)|$, where the supremum is taken over every sequence $\{(t_n, s_n)\}$ of disjoint intervals in $[a, b]$. If $Vg < \infty$ then g is called a function with bounded variation. Denote the set of functions with bounded variation by $\mathcal{BV}$ (see [21–23]).

Lemma 2.3 ([23, Theorem 2.2]). *Let $g,\ h \in \mathcal{BV}$. Then*

(i) $g \pm h \in \mathcal{BV}$ and $V(g \pm h) \leq Vg + Vh$;

(ii) $gh \in \mathcal{BV}$;

(iii) $gh^{-1} = \frac{g}{h} \in \mathcal{BV}$ if there exists constant $\delta > 0$ such that $|h| \geq \delta$.

Moreover, we have the following result.

Lemma 2.4 ([25, Lemma 1.5]). *Let $F \in C[a,b]$ and $g \in \mathcal{BV}$. Then*

$$F'g = \left(\int_a^t g dF \right)', \tag{2.8}$$

and

$$Fg' = \left(\int_a^t F dg \right)'. \tag{2.9}$$

Lemma 2.5 ([29, Lemma 2, Integration by parts]). *Let $f \in D_{HK}$, and $g \in \mathcal{BV}$. Then $fg \in D_{HK}$ and*

$$\int_a^b fg = F(b)g(b) - \int_a^b F dg.$$

By Lemmas 2.4 and 2.5, it is easy to see that

Lemma 2.6. *Let $f \in D_{HK}$ be the distributional derivative of $F \in C[a,b]$, and $g \in \mathcal{BV}$. Then*

$$(Fg)' = fg + Fg'.$$

From (2.5) and Lemma 2.5, the following lemma holds.

Lemma 2.7. *Let $f \in D_{HK}$ and let g be a nonnegative function on $[a,b]$.*

(I) If $f \overset{(m)}{\succeq} 0$ and g is monotone on $[a,b]$, then

$$fg \overset{(m)}{\succeq} 0.$$

(II) If $f \overset{(p)}{\succeq} 0$ and g is nonincreasing on $[a,b]$, then

$$fg \overset{(p)}{\succeq} 0.$$

Proof. (I) Let $F(t) = \int_a^t f$, $t \in [a,b]$. Then, $F \in C[a,b]$, and $F \geq 0$ on $[a,b]$, because $f \overset{(m)}{\geq} 0$. Since $g \geq 0$ is monotone, then $g \in \mathcal{BV}$. By the first mean value theorem for Riemann integrals, there exists $\xi \in [c,d] \subset [a,b]$ such that

$$\int_c^d F dg = F(\xi)(g(d) - g(c)), \quad \xi \in [c,d].$$

In view of Lemma 2.5 and (2.5), one has $fg \in D_{HK}$, and

$$\begin{aligned}\int_c^d fg &= Fg\,|_c^d - \int_c^d F dg \\ &= F(d)g(d) - F(c)g(c) - F(\xi)(g(d) - g(c)) \\ &= g(d)\int_\xi^d f + g(c)\int_c^\xi f \geq 0, \quad \forall\ [c,d] \subset [a,b].\end{aligned}$$

Hence, by (2.5), (I) follows.

(II) Let $F(t) = \int_a^t f$, $t \in [a,b]$. Then, $F \in C[a,b]$, and $F \geq 0$ on $[a,b]$, because $f \overset{(p)}{\geq} 0$. Since $g \geq 0$ is nonincreasing on $[a,b]$, then $g \in \mathcal{BV}$ and $fg \in D_{HK}$. Moreover,

$$\begin{aligned}\int_a^t fg &= Fg|_a^t + \int_a^t F d(-g) \\ &\geq F(t)g(t) - F(a)g(a) + F(\eta)(g(a) - g(t)) \geq 0, \quad t \in [a,b],\end{aligned}$$

where $F(\eta) = \min_{s\in[a,t]} F(s)$. Thus, by(2.4), (II) holds. The proof is therefore complete. □

Remark 2.8. In Lemma 2.7, (II) is not true if g is nondecreasing on $[a,b]$. For example, let

$$f = \sin t,\ t \in \left[0, \frac{5\pi}{4}\right],\ \text{and } g(t) = \begin{cases} 0, & t \in [0,\pi), \\ 1, & t \in \left[\pi, \frac{5\pi}{4}\right]. \end{cases}$$

It is easy to see that $f \overset{(p)}{\geq} 0$ and g is nonnegative and nondecreasing on $\left[0, \frac{5\pi}{4}\right]$. However, $\int_0^{\frac{5\pi}{4}} fg = \int_\pi^{\frac{5\pi}{4}} \sin t = \frac{\sqrt{2}}{2} - 1 < 0$. This implies by (2.4) that (II) is not true.

3 Main Results

In this section, we shall prove that the Gronwall-Bellman type inequalities involving the distributional Henstock-Kurzweil integral remain valid.

Theorem 3.1. *Let $f \in D_{HK}$, $g : [a,b] \to \mathbb{R}$ be a nonnegative nonincreasing function. If there is a constant η such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta + \int_a^t fg, \quad t \in [a,b]. \tag{3.1}$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta \exp\left(\int_a^t g\right), \quad t \in [a,b]. \tag{3.2}$$

Proof. Since g is a nonnegative nonincreasing function on $[a,b]$, then $g\exp\left(-\int_a^t g\right)$ is also nonnegative and nonincreasing on $[a,b]$. This together with Lemma 2.5 implies that

$$fg \in D_{HK},\ fg\exp\left(-\int_a^t g\right) \in D_{HK}.$$

Let $x(t) = \int_a^t fg$, then $x(t) \in C_0,\ x' = fg$, and (3.1) can be transformed into

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta + x, \quad \text{on } [a,b]. \tag{3.3}$$

Furthermore, by Lemma 2.7, $0 \le x(t),\ t \in [a,b]$. Multiplying by $g\exp\left(-\int_a^t g\right)$ on both sides of (3.3), one has

$$(fg - xg)\exp\left(-\int_a^t g\right) \overset{(p)}{\preceq} \eta g\exp\left(-\int_a^t g\right), \quad t \in [a,b]. \tag{3.4}$$

By Lemma 2.6,

$$\left(x\exp\left(-\int_a^t g\right)\right)' \overset{(p)}{\preceq} \left(\eta - \eta\exp\left(-\int_a^t g\right)\right)', \quad t \in [a,b]. \tag{3.5}$$

Taking in account (2.4), we get

$$0 \le x(t) \le \eta\left(\exp\left(\int_a^t g\right) - 1\right), \quad \forall t \in [a,b]. \tag{3.6}$$

It follows from (2.7), (3.3) and (3.6) that

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta + \eta\left(\exp\left(\int_a^t g\right) - 1\right) = \eta\exp\left(\int_a^t g\right), \quad t \in [a,b].$$

This completes the proof. □

As Lemma 1.1, we have the following consequence.

Corollary 3.2. *Let $f \in D_{HK}$. If there exist positive constants K and η such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \int_a^t (Kf(s) + \eta)ds, \quad t \in [a, b].$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta(b-a)\exp\left(K(b-a)\right).$$

Remark 3.3. If $f(t)$ is nonnegative and Henstock-Kurzweil integrable on $[a, b]$, then Theorem 3.1 and Corollary 3.2 are still valid. Therefore, Lemma 1.1 and the corresponding result in [12] are only special cases of our results.

For the ordering (2.5), we have the following result.

Theorem 3.4. *Let $f \in D_{HK}$, $g : [a, b] \to \mathbb{R}$ be a positive monotone function. If there is a constant η such that*

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} \eta + \int_a^t fg, \quad t \in [a, b]. \tag{3.7}$$

Then

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} \eta \exp\left(\int_a^t g\right), \quad t \in [a, b]. \tag{3.8}$$

Proof. The proof is similar to Theorem 3.1, so we omit it. □

Remark 3.5. Assume that f, g are nonnegative continuous functions. Since the ordering (2.5) equals to the pointwise ordering (see (2.7)), it is easy to see that Theorem 3.4 is a generalization of Lemma 1.2.

Next we give a more general version of the Gronwall-Bellman type inequality due to H. E. Gollwitzer [13].

Theorem 3.6. *Let $f \in D_{HK}$, $g : [a, b] \to \mathbb{R}$ be a nonnegative nonincreasing function. If there exist $l \in D_{HK}$, $l \overset{(p)}{\succeq} 0$ on $[a, b]$, and $h \in HK$, $h \geq 0$ on $[a, b]$, such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\int_a^t fg, \quad t \in [a, b]. \tag{3.9}$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\int_a^t lg\exp\left(\int_a^t gh\right), \quad t \in [a, b]. \tag{3.10}$$

Proof. Since $g(t) \in \mathcal{BV}$ and $f, l \in D_{HK}$. Then $\exp\left(-\int_a^t g\right) \in \mathcal{BV}$, $fg \in D_{HK}$, and $lg \in D_{HK}$. Suppose that $x(t) = \int_a^t fg$, one has $x(a) = 0$ and

$$x' = fg, \quad \text{on } [a, b]. \tag{3.11}$$

According to (3.9),

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + hx, \quad \text{on } [a,b]. \tag{3.12}$$

It turns out from (3.11), (3.12) and Lemma 2.7 that

$$x' - ghx \overset{(p)}{\preceq} lg, \quad \text{on } [a,b]. \tag{3.13}$$

Multiplying $\exp\left(-\int_a^t gh\right)$ on both sides of (3.13), we have

$$\left(\exp\left(-\int_a^t gh\right)x\right)' \overset{(p)}{\preceq} lg\exp\left(-\int_a^t gh\right) \leq lg, \quad t \in [a,b]. \tag{3.14}$$

Applying (2.5) yields that

$$0 \leq x(t) \leq \exp\left(\int_a^t gh\right)\int_a^t gl, \quad t \in [a,b], \tag{3.15}$$

and hence, by (2.7) and (3.12),

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\exp\left(\int_a^t gh\right)\int_a^t gl, \quad t \in [a,b],$$

which completes the proof. □

Corollary 3.7. *Let $f \in D_{HK}$, K be a positive constant. If there exist $l \in D_{HK}$, $l \overset{(p)}{\succeq} 0$ on $[a,b]$, and $h \in HK$, $h \geq 0$ on $[a,b]$, such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\int_a^t fK, \quad t \in [a,b]. \tag{3.16}$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \|l\|Kh\exp\left(\int_a^t Kh\right), \quad t \in [a,b]. \tag{3.17}$$

Proof. Let $F(t) = \int_a^t f$ and $L(t) = \int_a^t l$. By Theorem 3.6 and Lemma 2.7, we get

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + Kh\exp\left(\int_a^t Kh\right)\int_a^t l, \quad t \in [a,b]. \tag{3.18}$$

Moreover, in view of (2.4),

$$\begin{aligned} 0 \leq F(t) &\leq L(t) + \int_a^t \left(Kh\exp\left(\int_a^s Kh\right)\int_a^s l\right) \\ &\leq \|l\|\left(1 + \int_a^t \left(Kh\exp\left(\int_a^s Kh\right)\right)\right) \\ &= \|l\|\exp\left(\int_a^t Kh\right), \quad t \in [a,b]. \end{aligned} \tag{3.19}$$

Therefore, by (2.4) and Lemma 2.4, the assertion follows. □

Remark 3.8. In Corollary 3.7, without loss of generality, let $h \in C[a,b]$. Obviously, the inequality (3.16) implies by (2.5) that

$$0 \leq F(t) \leq L(t) + \int_a^t KhF \leq \|l\| + \int_a^t KhF, \quad t \in [a,b]. \tag{3.20}$$

Since $F,\ h \in C[a,b]$ are nonnegative, $K,\ \|l\|$ are positive constants, then by Lemma 1.2,

$$0 \leq F(t) \leq \|l\| \exp\left(\int_a^t Kh\right), \quad t \in [a,b],$$

which is the same result as in (3.19). Hence, our results are extensions of Lemma 1.2.

Furthermore, we have another result for the ordering (2.5).

Theorem 3.9. *Let $f \in D_{HK}$, $g : [a,b] \to \mathbb{R}$ be a nonnegative nonincreasing function. If there exist $l \in D_{HK}$, $l \overset{(m)}{\geq} 0$ on $[a,b]$, and $h \in HK$, $h \geq 0$ on $[a,b]$, such that*

$$0 \overset{(m)}{\leq} f \overset{(m)}{\leq} l + h\int_a^t fg, \quad t \in [a,b]. \tag{3.21}$$

Then

$$0 \overset{(m)}{\leq} f \overset{(m)}{\leq} l + h\int_a^t lg \exp\left(\int_a^t gh\right), \quad t \in [a,b]. \tag{3.22}$$

Remark 3.10. If $f,\ g,\ h,\ l$ are nonnegative continuous functions, the inequalities in Theorem 3.6 also hold for the pointwise order in $C[a,b]$, because of (2.7). Therefore, Theorem 3.6 extends the corresponding result in [13].

4 Application

In this section, we will give an application concerned about the Gronwall-Bellman type inequalities.

We consider the system

$$A_1(A_0x)' - A_2'x = F', \tag{4.1}$$

where the derivatives, products and equality are understood in the sense of distributions, see [26].

Assumptions 4.1. The function $A_0 \in C[0,T]$, $A_0 \neq 0$ on $[0,T]$, $A_1^{-1} \in \mathcal{BV}$ with $|A_1^{-1}| \geq \delta_1 > 0$, and $A_2 \in C[0,T] \cap \mathcal{BV}$. Furthermore, $F \in C[0,T]$.

Let us notice that under Assumptions 4.1 the products $A_1(A_0x)'$ and $A_2'x$, by Lemma 2.4, are well defined for any $x \in C[0,T]$.

Definition 4.2. A function $x(t)$ is called a solution to the equation (4.1) on the interval $[0,T]$ if $x \in C[0,T]$ and $A_1(A_0x)' - A_2'x - F'$ is the zero distribution.

Firstly, we show the estimate of solutions to the equation (4.1).

Theorem 4.3. *Let the assumptions 4.1 be satisfied. If $x(t) \in C[0,T]$ is a solution to the equation (4.1) on $[0,T]$, then*

$$|x(t)| \le |A_0^{-1}(t)|\|l\| \exp\left(\int_0^t h\right), \quad t \in [0,T], \tag{4.2}$$

where

$$l = |A_1^{-1}F'|, \qquad h = |A_1^{-1}A_2'A_0^{-1}|. \tag{4.3}$$

Proof. By Assumptions 4.1 and (4.1),

$$(A_0x)' = A_1^{-1}F' + A_1^{-1}A_2'A_0^{-1}(A_0x). \tag{4.4}$$

Hence, by Lemma 2.2,

$$|(A_0x)'| \overset{(p)}{\preceq} \left|A_1^{-1}F'\right| + |A_1^{-1}A_2'A_0^{-1}| \int_0^t |(A_0x)'|. \tag{4.5}$$

Let

$$l = \left|A_1^{-1}F'\right|, \quad g = 1, \quad h = |A_1^{-1}A_2'A_0^{-1}|. \tag{4.6}$$

It is easy to see that $l \in D_{HK}$, $h \in HK$. Therefore, by Corollary 3.7,

$$|(A_0x)'| \overset{(p)}{\preceq} \|l\|h \exp\left(\int_0^t h\right), \quad t \in [0,T], \tag{4.7}$$

which yields by (2.4) and Lemma 2.2 that

$$|x(t)| \le |A_0^{-1}(t)|\|l\| \exp\left(\int_0^t h\right), \quad t \in [0,T]. \tag{4.8}$$

The proof is therefore complete. □

Finally, we give an existence and uniqueness result.

Theorem 4.4. *Let the Assumptions 4.1 be satisfied. Moreover, either* $A_0^{-1} \in \mathcal{BV}$ *with* $|A_0^{-1}| \geq \delta_0 > 0$ *or* $A_2 \in AC$ *holds. Then (4.1) has a unique solution* $x(t) \in C[0,T]$ *satisfying*

$$x(t) = A_0^{-1}(t)k^{-1}(t)\int_0^t kA_1^{-1}dF, \quad t \in [0,T], \tag{4.9}$$

where

$$k(t) = \exp\left(-\int_0^t A_0^{-1}A_1^{-1}dA_2\right), \quad t \in [0,T]. \tag{4.10}$$

Proof. We only prove the necessity, the sufficiency is easy to prove.

By Assumptions 4.1 and (4.1),

$$(A_0x)' - A_1^{-1}A_2'A_0^{-1}(A_0x) = A_1^{-1}F'. \tag{4.11}$$

Let

$$k(t) = \exp\left(-\int_0^t A_0^{-1}A_1^{-1}dA_2\right), \quad t \in [0,T]. \tag{4.12}$$

It is easy to see that $k(t) \in C[0,T] \cap \mathcal{BV}$. Multiplying both side of (4.11) by $k(t)$, we get

$$k(A_0x)' - kA_1^{-1}A_2'A_0^{-1}(A_0x) = kA_1^{-1}F'. \tag{4.13}$$

By Lemma 2.6,

$$(kA_0x)' = kA_1^{-1}F'. \tag{4.14}$$

Therefore,

$$x(t) = A_0^{-1}(t)k^{-1}(t)\int_0^t kA_1^{-1}dF, \quad t \in [0,T].$$

Let $y(t)$ be another solution to (4.1). Then,

$$x(t) - y(t) = 0, \quad t \in [0,T].$$

Therefore, $x(t)$ satisfying (4.9) is a unique solution of (4.1). □

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Formulas and properties of some class of nonlinear difference equations

E. M. Elsayed[1,2], Faris Alzahrani[1], and H. S. Alayachi[1]
[1]Mathematics Department, Faculty of Science,
King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mails: emmelsayed@yahoo.com, faris.kau@hotmail.com,
hazas2010@hotmail.com.

ABSTRACT

We obtain the formulas of the solutions of the recursive sequences

$$x_{n+1}=\frac{x_n x_{n-5}}{x_{n-4}(\pm 1\pm x_n x_{n-5})},\quad n=0,1,...,$$

where the initial conditions are arbitrary non zero real numbers. Also, we discuss and illustrate the stability of the solutions in the neighborhood of the critical points and the periodicity of the considered equations.

Keywords: equilibrium point, recursive sequences, periodicity.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

In recent years, the qualitative study of difference equations has become an active research area among a considerable number of mathematicians. Some economical and biological examples can be seen in [9,36,40,47,48,54]. It is commonly known that nonlinear difference equations are able to produce and present sophisticated behaviors regardless their orders.

Some articles show that a great effort has been done to demonstrate and explore the dynamics of nonlinear difference equations (see [40]-[61]). In fact, investigating these equations is a challenge and still new in the mathematical world. It is strongly believed that the rational difference equations are significant in their own right.

Abo-Zeid and Cinar [1] illustrated the global stability, cyclical behavior, oscillation of all acceptable solutions of the equation

$$x_{n+1}=\tfrac{Ax_{n-1}}{B-Cx_n x_{n-2}}.$$

In [7], [8] Cinar considered the solutions of the equations

$$y_{n+1}=\tfrac{y_{n-1}}{1+ay_n y_{n-1}},\qquad y_{n+1}=\tfrac{y_{n-1}}{-1+ay_n y_{n-1}}.$$

A. El-Moneam, and Alamoudy [16] examined the positive solutions of the equation in terms of its periodicity, boundedness and the global stability. The considered difference equation is given by

$$x_{n+1}=ax_n+\tfrac{bx_{n-1}+cx_{n-2}+fx_{n-3}+rx_{n-4}}{dx_{n-1}+ex_{n-2}+gx_{n-3}+sx_{n-4}}.$$

Khatibzadeh and Ibrahim [42] studied the boundedness, asymptotic stability, oscillatory behavior and discovered the closed form of solutions of the equation

$$x_{n+1}=ax_n+\frac{bx_nx_{n-1}}{cx_n+dx_{n-1}}.$$

Simsek et al. [49] has found and explored solutions for the recursive formula

$$y_{n+1}=\frac{y_{n-3}}{1+y_{n-1}}.$$

For other related papers, see [25–46].

We analyze and explore the solutions of the following nonlinear recursive equation

$$x_{n+1}=\frac{x_nx_{n-5}}{x_{n-4}(\pm1\pm x_nx_{n-5})},\qquad n=0,1,..., \tag{1}$$

with conditions posed on the initial values are arbitrary non zero real numbers. Also, we will survey some dynamic behaviors of its solutions.

The linearized equation of equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),\quad n=0,1,..., \tag{2}$$

about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1}=\sum_{i=0}^{k}\frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}}y_{n-i}.$$

Theorem A [43]: Assume that $p_i\in R,\ i=1,2,...,k$ and $k\in\{0,1,2,...\}$. Then $\sum_{i=1}^{k}|p_i|<1,$ is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k}+p_1x_{n+k-1}+...+p_kx_n=0,\ \ n=0,1,... .$$

2. THE FIRST EQUATION $X_{N+1}=\frac{X_NX_{N-5}}{X_{N-4}(1+X_NX_{N-5})}$

This section is devoted to give a specific solution of the first difference equation which is

$$x_{n+1}=\frac{x_nx_{n-5}}{x_{n-4}(1+x_nx_{n-5})}. \tag{3}$$

THEOREM 2.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(3). Then*

$$\begin{aligned}
x_{10n-5} &= f\prod_{i=0}^{n-1}\left(\frac{1+(10i)af}{1+(10i+5)af}\right), \qquad x_{10n-4}=e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right),\\
x_{10n-3} &= d\prod_{i=0}^{n-1}\left(\frac{1+(10i+2)af}{1+(10i+7)af}\right), \qquad x_{10n-2}=c\prod_{i=0}^{n-1}\left(\frac{1+(10i+3)af}{1+(10i+8)af}\right),\\
x_{10n-1} &= b\prod_{i=0}^{n-1}\left(\frac{1+(10i+4)af}{1+(10i+9)af}\right), \qquad x_{10n}=a\prod_{i=0}^{n-1}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right),\\
x_{10n+1} &= \frac{af}{e(1+af)}\prod_{i=0}^{n-1}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right), \quad x_{10n+2}=\frac{af}{d(1+2af)}\prod_{i=0}^{n-1}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right),\\
x_{10n+3} &= \frac{af}{c(1+3af)}\prod_{i=0}^{n-1}\left(\frac{1+(10i+8)af}{1+(10i+13)af}\right), \quad x_{10n+4}=\frac{af}{b(1+4af)}\prod_{i=0}^{n-1}\left(\frac{1+(10i+9)af}{1+(10i+14)af}\right),
\end{aligned}$$

where we put $x_{-5}=f,\ x_{-4}=e,\ \ x_{-3}=d,\ \ x_{-2}=c,\ \ x_{-1}=b,\ \ x_0=a.$

Proof: The result holds for $n=0$. Assume that $n>0$ and our assumption true for $n-1$. Then;

$$x_{10n-15} = f\prod_{i=0}^{n-2}\left(\frac{1+(10i)af}{1+(10i+5)af}\right), \qquad x_{10n-14} = e\prod_{i=0}^{n-2}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right),$$

$$x_{10n-13} = d\prod_{i=0}^{n-2}\left(\frac{1+(10i+2)af}{1+(10i+7)af}\right), \qquad x_{10n-12} = c\prod_{i=0}^{n-2}\left(\frac{1+(10i+3)af}{1+(10i+8)af}\right),$$

$$x_{10n-11} = b\prod_{i=0}^{n-2}\left(\frac{1+(10i+4)af}{1+(10i+9)af}\right), \qquad x_{10n-10} = a\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right),$$

$$x_{10n-9} = \frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right), \quad x_{10n-8} = \frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right),$$

$$x_{10n-7} = \frac{af}{c(1+3af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+8)af}{1+(10i+13)af}\right), \quad x_{10n-6} = \frac{af}{b(1+4af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+9)af}{1+(10i+14)af}\right).$$

Now, it follows from Eq.(3) that

$$x_{10n-5} = \frac{x_{10n-6}x_{10n-11}}{x_{10n-10}(1+x_{10n-6}x_{10n-11})}$$

$$= \frac{\frac{af}{b(1+4af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+9)af}{1+(10i+14)af}\right)b\prod_{i=0}^{n-2}\left(\frac{1+(10i+4)af}{1+(10i+9)af}\right)}{\left[a\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)\right]\left[1+\frac{af}{b(1+4af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+9)af}{1+(10i+14)af}\right)b\prod_{i=0}^{n-2}\left(\frac{1+(10i+4)af}{1+(10i+9)af}\right)\right]}$$

$$= \frac{\frac{f}{(1+4af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+4)af}{1+(10i+14)af}\right)}{\left[\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)\right]\left[1+\frac{af}{(1+4af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+4)af}{1+(10i+14)af}\right)\right]}$$

$$= \frac{\left(\frac{f}{1+(10n-6)af}\right)}{\left[\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)\right]\left[1+\frac{af}{1+(10n-6)af}\right]}$$

$$= \frac{f}{\left[\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)\right][1+(10n-6)af+af]}$$

$$= \frac{f}{[1+(10n-5)af]}\prod_{i=0}^{n-2}\left(\frac{1+(10i+10)af}{1+(10i+5)af}\right) = f\prod_{i=0}^{n-1}\left(\frac{1+(10i)af}{1+(10i+5)f}\right).$$

Also, we have

$$x_{10n-4} = \frac{x_{10n-5}x_{10n-10}}{x_{10n-9}(1+x_{10n-5}x_{10n-10})}$$

$$= \frac{f\prod_{i=0}^{n-1}\left(\frac{1+(10i)af}{1+(10i+5)af}\right)a\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)}{\left[\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right]\left[1+f\prod_{i=0}^{n-1}\left(\frac{1+(10i)af}{1+(10i+5)af}\right)a\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)\right]}$$

$$= \frac{af\left(\frac{1}{1+(10n-5)af}\right)}{\left[\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right]\left[1+\frac{af}{1+(10n-5)af}\right]}$$

$$= \frac{af}{\left[\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right][1+(10n-5)af+af]}$$

$$= \frac{e(1+af)}{[1+(10n-4)af]}\prod_{i=0}^{n-2}\left(\frac{1+(10i+11)af}{1+(10i+6)af}\right) = e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right).$$

Similarly

$$x_{10n-3} = \frac{x_{10n-4}x_{10n-9}}{x_{10n-8}(1+x_{10n-4}x_{10n-9})}$$

$$= \frac{e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right)\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)}{\left[\frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right]\left[1+e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right)\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right]}$$

$$= \frac{\prod_{i=0}^{n-1}\left(\frac{af}{1+(10i+6)af}\right)\prod_{i=0}^{n-2}(1+(10i+6)af)}{\left[\frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right]\left[1+\prod_{i=0}^{n-1}\left(\frac{af}{1+(10i+6)af}\right)\prod_{i=0}^{n-2}(1+(10i+6)af)\right]}$$

$$= \frac{\left(\frac{af}{1+(10n-4)af}\right)}{\left[\frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right]\left[1+\left(\frac{af}{1+(10n-4)af}\right)\right]}$$

$$= \frac{d}{\left[\frac{1}{(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right][1+(10n-4)af+af]}$$

$$= \frac{d}{\left[\frac{1}{(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right][1+(10n-3)af]} = d\prod_{i=0}^{n-1}\left(\frac{1+(10i+2)af}{1+(10i+7)af}\right).$$

Similarly, one can simply find the other relations. Thus, the proof is done.

THEOREM 2.2. *The unique equilibrium point of Eq.(3) is the number zero which is not locally asymptotically stable.*

Proof: The equilibrium points of Eq.(3) obtained by

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(1+\overline{x}^3\right)}.$$

Arranging the previous equation gives $\overline{x}^4 = 0$. Thus $\overline{x} = 0$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function takes the form

$$f(u,v,w) = \frac{uw}{v\left(1+uw\right)}.$$

Therefore

$$f_u(u,v,w) = \frac{w}{v\left(1+uw\right)^2},\ f_v(u,v,w) = -\frac{uw}{v^2\left(1+uw\right)},\ f_w(u,v,w) = \frac{u}{v\left(1+uw\right)^2}.$$

So

$$f_u(\overline{x},\overline{x},\overline{x}) = 1, \quad f_v(\overline{x},\overline{x},x) = 1, \quad f_w(\overline{x},\overline{x},\overline{x}) = 1.$$

Then by using Theorem A the proof follows.

Example 1. We assume $x_{-5} = 6,\ x_{-4} = 11,\ x_{-3} = 3,\ x_{-2} = 2,\ x_{-1} = 1.8,\ x_0 = -7$**.** See Fig. 1.

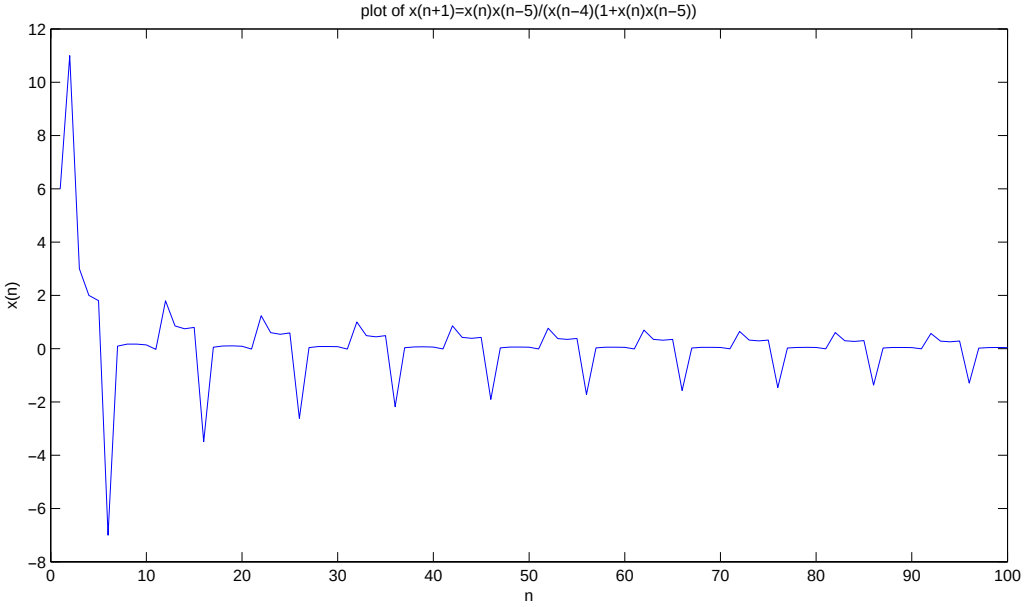


Figure 1.

Example 2. See Fig. 2, since $x_{-5} = 1.6,\ x_{-4} = 1.2,\ x_{-3} = -3,\ x_{-2} = .7,\ x_{-1} = 1.8,\ x_0 = 3$.

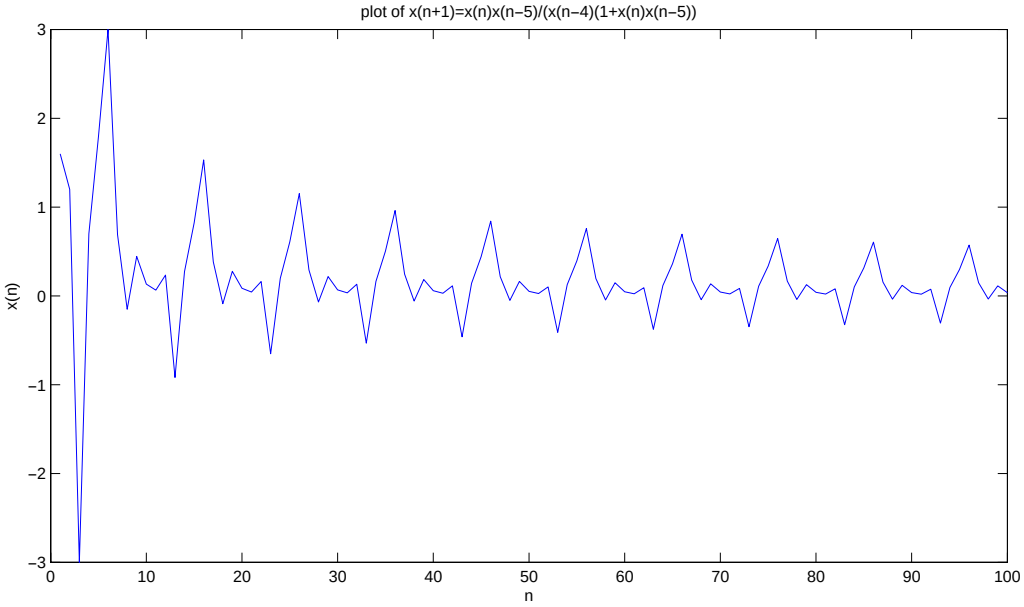


Figure 2.

3. THE SECOND EQUATION $X_{N+1} = \frac{X_N X_{N-5}}{X_{N-4}(-1+X_N X_{N-5})}$

This section is devoted to obtain the solution of the difference equation which is

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(-1 + x_n x_{n-5})}, \tag{4}$$

where $x_0 x_{-5} \neq 1$.

THEOREM 3.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(4). Then for*

$$\begin{aligned}
x_{10n-5} &= \frac{f}{(-1+af)^n}, & x_{10n-4} &= e(-1+af)^n, \\
x_{10n-3} &= \frac{d}{(-1+af)^n}, & x_{10n-2} &= c(-1+af)^n, \\
x_{10-1} &= \frac{b}{(-1+af)^n}, & x_{10n} &= a(-1+af)^n, \\
x_{10n+1} &= \frac{af}{e(-1+af)^{n+1}}, & x_{10n+2} &= \frac{af(-1+af)^n}{d}, \\
x_{10n+3} &= \frac{af}{c(-1+af)^{n+1}}, & x_{10n+4} &= \frac{af(-1+af)^n}{b}.
\end{aligned}$$

Proof: The result holds for $n = 0$. Assume that $n > 0$ and that our assumption true for $n - 1$. Then;

$$\begin{aligned}
x_{10n-15} &= \frac{f}{(-1+af)^{n-1}}, & x_{10n-14} &= e(-1+af)^{n-1},\\
x_{10n-13} &= \frac{d}{(-1+af)^{n-1}}, & x_{10n-12} &= c(-1+af)^{n-1},\\
x_{10-11} &= \frac{b}{(-1+af)^{n-1}}, & x_{10n-10} &= a(-1+af)^{n-1},\\
x_{10n-9} &= \frac{af}{e(-1+af)^{n}}, & x_{10n-8} &= \frac{af(-1+af)^{n-1}}{d},\\
x_{10n-7} &= \frac{af}{c(-1+af)^{n}}, & x_{10n-6} &= \frac{af(-1+af)^{n-1}}{b}.
\end{aligned}$$

It follows from (4) that

$$x_{10n-5} = \frac{x_{10n-6}x_{10n-11}}{x_{10n-10}(-1+x_{10n-6}x_{10n-11})} = \frac{af}{a(-1+af)^{n-1}\left[-1+af\right]} = \frac{af}{a(-1+af)^{n}},$$

$$\begin{aligned}
x_{10n-4} &= \frac{x_{10n-5}x_{10n-10}}{x_{10n-9}(-1+x_{10n-5}x_{10n-10})}\\
&= \frac{\frac{af}{a(-1+af)^{n}}a(-1+af)^{n-1}}{\left[\frac{af}{e(-1+af)^{n}}\right]\left[-1+\frac{af}{a(-1+af)^{n}}a(-1+af)^{n-1}\right]}\\
&= \frac{e(-1+af)^{n}}{(-1+af)\left[-1+\frac{af}{(-1+af)}\right]} = \frac{e(-1+af)^{n}}{[1-af+af]} = e(-1+af)^{n}.
\end{aligned}$$

Similarly one can simply prove the other relations.

THEOREM 3.2. *Eq.(4) has a period ten solution iff* $af = 2$ *and will be in the following form*

$$\left\{f, e, d, c, b, a, \frac{af}{e}, \frac{af}{d}, \frac{af}{c}, \frac{af}{b}, f, e, d, ...\right\}.$$

Proof: Firstly, assume that there exists a period ten solution

$$\left\{f, e, d, c, b, a, \frac{af}{e}, \frac{af}{d}, \frac{af}{c}, \frac{af}{b}, f, e, d, ...\right\},$$

of Eq.(4). Then, we can notice from the solution of Eq.(4) that

$$\begin{aligned}
f &= \frac{f}{(-1+af)^{n}}, \quad e = e(-1+af)^{n}, \quad d = \frac{d}{(-1+af)^{n}}, \quad c = c(-1+af)^{n},\\
b &= \frac{b}{(-1+af)^{n}}, \quad a = a(-1+af)^{n}, \quad \frac{af}{e} = \frac{af}{e(-1+af)^{n+1}},\\
\frac{af}{d} &= \frac{af(-1+af)^{n}}{d}, \quad \frac{af}{c} = \frac{af}{c(-1+af)^{n+1}}, \quad \frac{af}{b} = \frac{af(-1+af)^{n}}{b}.
\end{aligned}$$

or,

$$(-1+af)^{n} = 1.$$

Then

$$af = 2.$$

Secondly, suppose that $af = 2$. Then, it is easily seen from the solution of Eq.(4) that

$$\begin{aligned} x_{10n-5} &= f, \quad x_{10n-4} = e, \quad x_{10n-3} = d, \quad x_{10n-2} = c, \quad x_{10-1} = b, \\ x_{10n} &= a, \quad x_{10n+1} = \frac{af}{e}, \qquad x_{10n+2} = \frac{af}{d}, \quad x_{10n+3} = \frac{af}{c}, \qquad x_{10n+4} = \frac{af}{b}. \end{aligned}$$

Thus, the periodic solution of period ten is obtained and this proves the theorem.

THEOREM 3.3. *Eq.(4) has two equilibrium points which are* $0, \sqrt[3]{2}$ *and these equilibrium points are not locally asymptotically stable.*

Proof: The equilibrium points of Eq.(4) can be written in the following form

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(-1+\overline{x}^2\right)}.$$

Arranging this gives

$$\begin{aligned} \overline{x}^2\left(-1+\overline{x}^2\right) &= \overline{x}^2 \\ &\Rightarrow \qquad \overline{x}^2\left(\overline{x}^2-2\right) = 0. \end{aligned}$$

Therefore, the fixed points are $0, \pm\sqrt{2}$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{uw}{v\left(-1+uw\right)}.$$

Then it follows that

$$\begin{aligned} f_u(u,v,w) &= -\frac{w}{v\left(-1+uw\right)^2}, \\ f_v(u,v,w) &= -\frac{uw}{v^2\left(-1+uw\right)}, \\ f_w(u,v,w) &= -\frac{u}{v\left(-1+uw\right)^2}, \end{aligned}$$

It can be seen that

$$f_u(\overline{x},\overline{x},\overline{x}) = -1, \quad f_v(\overline{x},\overline{x},x) = \pm 1, \quad f_w(\overline{x},\overline{x},\overline{x}) = -1.$$

Then by using Theorem A the proof follows.

Example 3. We consider $x_{-5} = .8,\ x_{-4} = 1.7,\ x_{-3} = .3,\ x_{-2} = 2,\ x_{-1} = 1.8,\ x_0 = .7$. See Fig. 3.

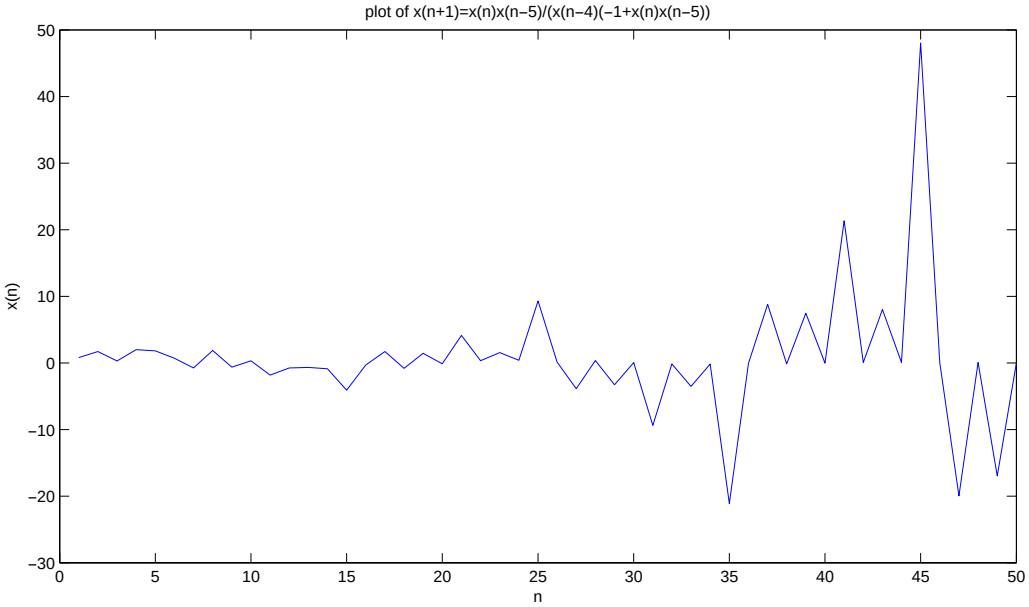


Figure 3.

Example 4. See Fig. 4, since $x_{-5} = 8,\ x_{-4} = 1.7,\ x_{-3} = .3,\ x_{-2} = 2,\ x_{-1} = 1.8,\ x_0 = 1/4.$

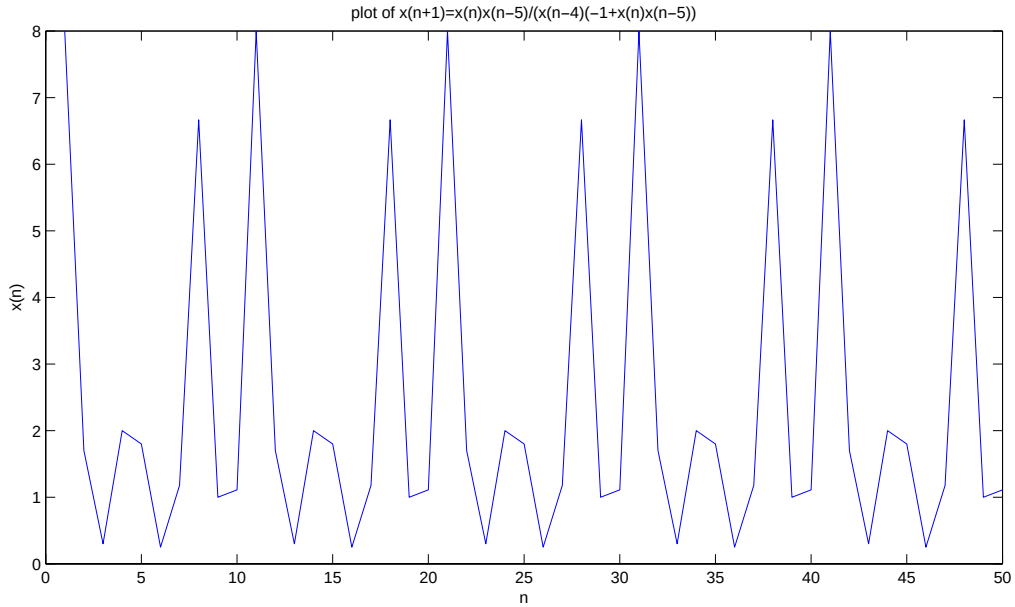


Figure 4.

4. THE THIRD EQUATION $X_{N+1} = \frac{X_N X_{N-5}}{X_{N-4}(1-X_N X_{N-5})}$

In this section we will obtain and present the solution of the third difference equation which is

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(1 - x_n x_{n-5})}. \tag{5}$$

THEOREM 4.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, ...$*

$$
\begin{aligned}
x_{10n-5} &= f\prod_{i=0}^{n-1}\left(\frac{1-(10i)af}{1-(10i+5)af}\right), \qquad x_{10n-4} = e\prod_{i=0}^{n-1}\left(\frac{1-(10i+1)af}{1-(10i+6)af}\right),\\
x_{10n-3} &= d\prod_{i=0}^{n-1}\left(\frac{1-(10i+2)af}{1-(10i+7)af}\right), \qquad x_{10n-2} = c\prod_{i=0}^{n-1}\left(\frac{1-(10i+3)af}{1-(10i+8)af}\right),\\
x_{10n-1} &= b\prod_{i=0}^{n-1}\left(\frac{1-(10i+4)af}{1-(10i+9)af}\right), \qquad x_{10n} = a\prod_{i=0}^{n-1}\left(\frac{1-(10i+5)af}{1-(10i+10)af}\right),\\
x_{10n+1} &= \frac{af}{e(1-af)}\prod_{i=0}^{n-1}\left(\frac{1-(10i+6)af}{1-(10i+11)af}\right),\\
x_{10n+2} &= \frac{af}{d(1-2af)}\prod_{i=0}^{n-1}\left(\frac{1-(10i+7)af}{1-(10i+12)af}\right),\\
x_{10n+3} &= \frac{af}{c(1-3af)}\prod_{i=0}^{n-1}\left(\frac{1-(10i+8)af}{1-(10i+13)af}\right),\\
x_{10n+4} &= \frac{af}{b(1-4af)}\prod_{i=0}^{n-1}\left(\frac{1-(10i+9)af}{1-(10i+14)af}\right).
\end{aligned}
$$

THEOREM 4.2. *The unique critical point of Eq.(5) is the number zero which is not locally asymptotically stable.*

Example 5. Suppose that $x_{-5} = 8,\ x_{-4} = 1.7,\ x_{-3} = .3,\ x_{-2} = 2,\ x_{-1} = 1.8,\ x_0 = 1/4$ see Fig. 5.

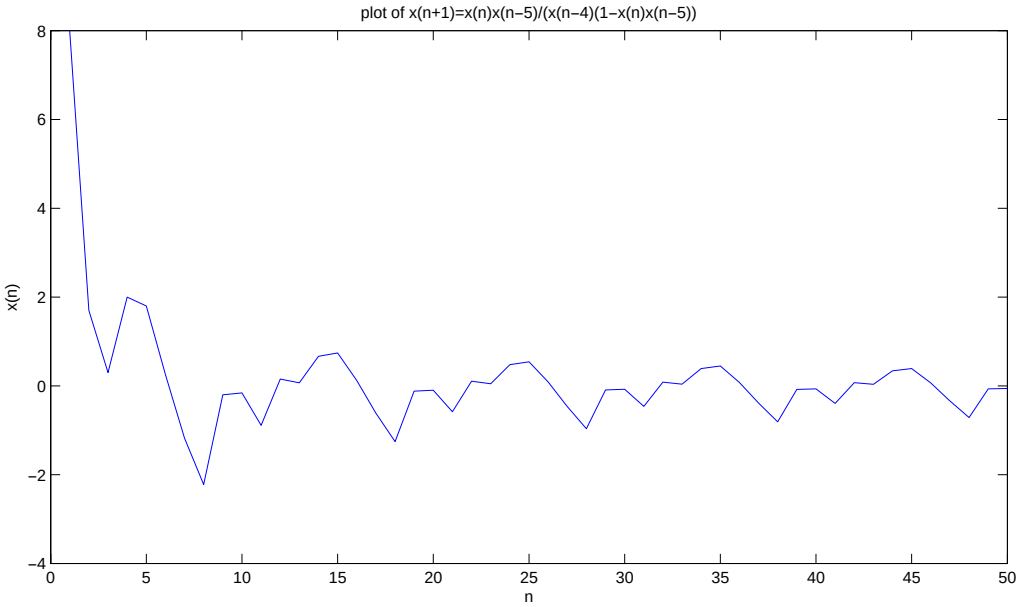


Figure 5.

Example 6. See Fig. 6 since $x_{-5} = -7,\ x_{-4} = 1.5,\ x_{-3} = -3,\ x_{-2} = 2,\ x_{-1} = 12,\ x_0 = 4.$

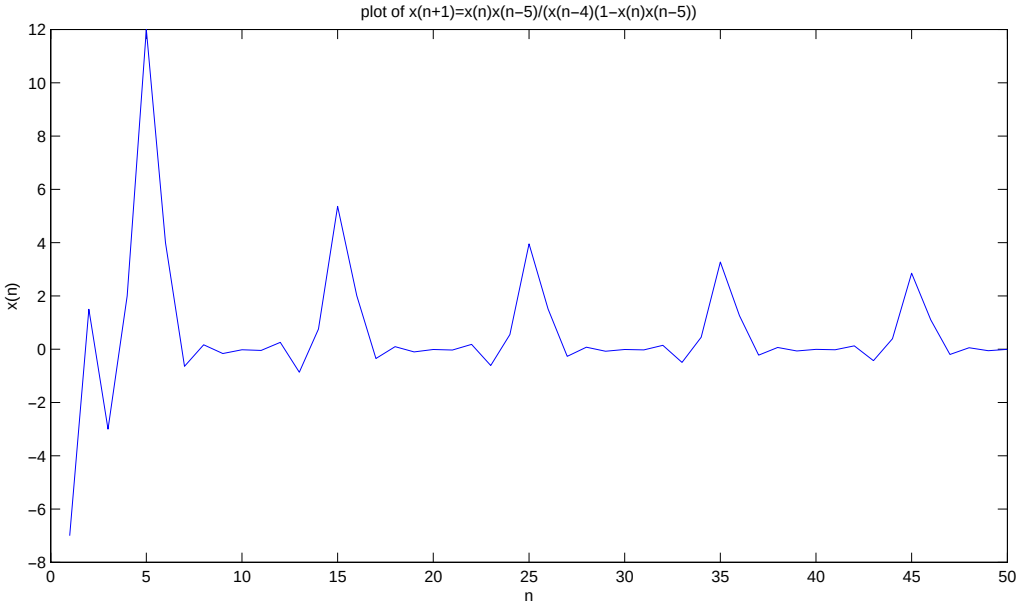


Figure 6.

5. THE FOURTH EQUATION $X_{N+1} = \frac{X_N X_{N-5}}{X_{N-4}(-1-X_N X_{N-5})}$

Now, we will explore and discover the solution of the following difference equation

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(-1 - x_n x_{n-5})}, \quad n = 0, 1, ..., \tag{6}$$

where $x_{-5}x_0 \neq -1$.

THEOREM 5.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(6). Then Eq.(6) has unboundedness solution (except in the case if $af = -2$) and for $n = 0, 1, ...$*

$$\begin{aligned}
x_{10n-5} &= \frac{f}{(-1-af)^n}, & x_{10n-4} &= e(-1-af)^n, \\
x_{10n-3} &= \frac{d}{(-1-af)^n}, & x_{10n-2} &= c(-1-af)^n, \\
x_{10-1} &= \frac{b}{(-1-af)^n}, & x_{10n} &= a(-1-af)^n, \\
x_{10n+1} &= \frac{af}{e(-1-af)^{n+1}}, & x_{10n+2} &= \frac{af(-1-af)^n}{d}, \\
x_{10n+3} &= \frac{af}{c(-1-af)^{n+1}}, & x_{10n+4} &= \frac{af(-1-af)^n}{b}.
\end{aligned}$$

THEOREM 5.2. *Eq.(6) has a periodic solution of period ten iff $af = -2$ and written in the following form* $\left\{f, e, d, c, b, a, \frac{af}{e}, \frac{af}{d}, \frac{af}{c}, f, e, d, c, ...\right\}$.

THEOREM 5.3. *The unique equilibrium of Eq.(6) is the number zero which is not locally asymptotically stable.*

Example 7. Consider $x_{-5} = -7,\ x_{-4} = 1.5,\ x_{-3} = -3,\ x_{-2} = 2,\ x_{-1} = 12,\ x_0 = 4$ see Fig. 7.

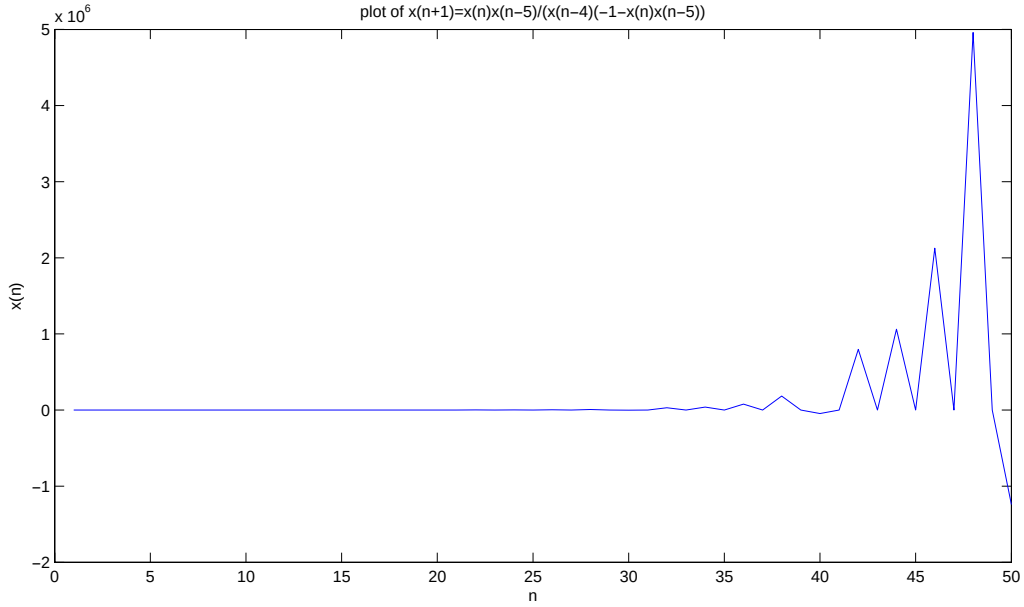


Figure 7.

Example 8. Fig. 8 illustrates the solutions when $x_{-5} = -7,\ x_{-4} = 1.5,\ x_{-3} = -3,\ x_{-2} = 2,\ x_{-1} = 12,\ x_0 = 2/7$.

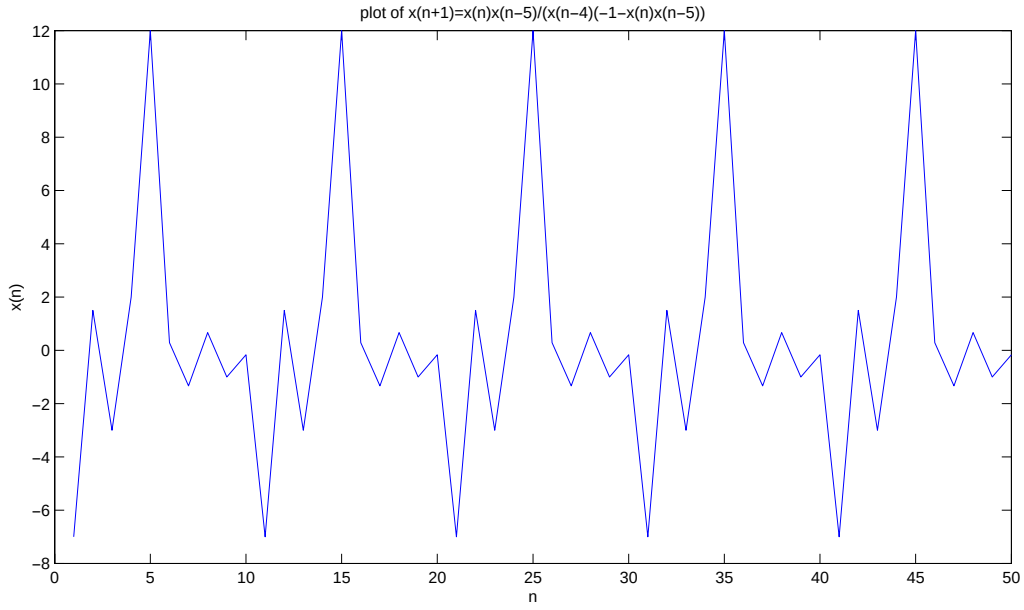


Figure 8.

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A rational bicubic spline for visualization of shaped data

Xilian Fu
College of Science, Shandong University of Technology, Zibo, 255049, China
E-mail: xilianfu001@163.com

Abstract

The shaped data usually needs to be represented in such a way that its visual display looks smooth and pleasant, its shape is preserved everywhere and the computation cost is economical. This work contributes to the graphical display of positive or monotone data. For this purpose, a new bicubic rational interpolating spline with biquadratic denominator is developed based on function values and partial derivatives, and simple sufficient conditions are derived on the shape parameters in the description of the rational function to visualize the positive or monotone data in the view of positive or monotone surfaces.

Keywords: Rational bicubic spline, shape parameter, positivity, monotonicity.

1 Introduction

The construction method of curve and surface and the mathematical description of them is a important issue in Computer-Aided Geometry Design (CAGD). Generally speaking, the interpolating data are often given as a set of values, in order to display these data, it is first necessary to construct an interpolant through those data; and then this interpolant is used in the subsequent contouring or curve and surface drawing. Thus, for the data obtained from some complex function or from some scientific phenomena, smooth curve or surface expression becomes crucial to incorporate the inherited features of the data. In many problems of industrial design and manufacturing, the given data often have some special shape properties, such as positivity, monotonicity and convexity, it is usually needed to generate a smooth function, which passes through the given set of data and preserves those certain shape properties of the data.

In recent years, a good amount of work has been published that focuses on shape preserving curves and surfaces. Goodman and Ong [7] presented a local convexity preserving interpolation scheme using parametric C^2 cubic splines with uniform knots produced by a vector subdivision scheme. In [2, 4, 8, 9, 13, 14, 16, 18], several shape-preserving rational curves were shown for shaped data, such as positive data, monotonic data and convex data. Beatson and Ziegler [1] interpolated monotone data, given on a rectangular grid, with a C^1 monotone quadratic spline, and derived necessary and sufficient conditions to visualize monotone data. Floater and Pena [6] defined three kinds of monotonicity preservation of systems of bivariate functions on a triangle, and investigated some geometric applications. In [10], authors proposed a kind of monotonicity-preserving interpolating schemes for 2D/3D monotone data by constraints on shape parameters in the description of rational spline interpolants. In [12], Hussain et al. presented the C^1 rational bi-cubic local interpolation schemes for the shape preservation of convex, monotone and positive surface data. In [11], A bi-quadratic trigonometric interpolation scheme with four free parameters is developed for the positive and monotone 3D data. Piah et al. [15] discussed the problem of positivity preserving for scattered data interpolation.

In [5], Duan et al. developed a C^1 bivariate rational spline interpolant based on function values and partial derivatives under some suitable hypotheses. In [17], Sun et al. proposed a surface modeling method by using C^2 piecewise rational spline interpolation. This paper is concerned with the preservation of 3D positive data and monotone data. To solve the problem, motivated by [5, 17], we will first construct a new bicubic rational interpolating spline with biquadratic denominator based function values and derivative values of an original function. Further more, a positivity-preserving scheme and a monotonicity-preserving scheme are developed to visualize 3D positive data and monotone data in the view of positive surfaces and monotone surfaces, respectively.

This paper is arranged as follows. Section 2 describes about the C^1 bicubic rational spline interpolant to be used in the surface schemes. In Section 3, the positivity problem is discussed for the generation of a smooth surface which can preserve the shape of positive data. In Section 4, a method is developed to preserve the shape of monotone data in the view of monotone surfaces by making constraints on shape parameters in the description of bicubic rational interpolant. Finally, Numerical examples are presented to discuss and demonstrate the performance of the method in Section 5.

2 Bicubic rational interpolant

Let $\Omega = [a, b; c, d]$ be the plane region, and $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a given set of data points, where $a = x_1 < x_2 < ... < x_n = b$, $c = y_1 < y_2 < \cdots < y_m = d$ are the knot spacings, $f_{i,j}$ represents $f_{i,j}(x, y)$ at the point (x_i, y_j). Let $d^*_{i,j}$ and $d_{i,j}$ be chosen partial derivative values $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ at the knots (x_i, y_j), respectively. Denote $h_i = x_{i+1} - x_i$, $l_j = y_{j+1} - y_j$, $I = \{1, 2, \cdots, n\}$, $J = \{1, 2, \cdots, m\}$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$, $\theta := \frac{x-x_i}{h_i}$, $\eta := \frac{y-y_j}{l_j}$. Let $\alpha^*_{i,j}, \beta^*_{i,j}$ and $\gamma^*_{i,j}$ be the positive parameters. First, we construct the x-direction interpolating curve $P^*_{i,j}(x)$ in $[x_i, x_{i+1}]$, this is given by

$$P^*_{i,j}(x) = \frac{(1-\theta)^3\alpha^*_{i,j}f_{i,j} + \theta(1-\theta)^2V^*_{i,j} + \theta^2(1-\theta)W^*_{i,j} + \theta^3\beta^*_{i,j}f_{i+1,j}}{(1-\theta)^2\alpha^*_{i,j} + \theta(1-\theta)\gamma^*_{i,j} + \theta^2\beta^*_{i,j}}, \tag{1}$$

where

$$V^*_{i,j} = (\alpha^*_{i,j} + \gamma^*_{i,j})f_{i,j} + h_i\alpha^*_{i,j}d^*_{i,j},$$
$$W^*_{i,j} = (\beta^*_{i,j} + \gamma^*_{i,j})f_{i+1,j} - h_i\beta^*_{i,j}d^*_{i+1,j},$$

The interpolant $P^*_{i,j}(x)$ is called a rational cubic spline interpolation in $[x_1, x_n]$, and which satisfies:

$$P^*_{i,j}(x_i) = f_{i,j},\ P^*_{i,j}(x_{i+1}) = f_{i+1,j},\ P^{*\prime}_{i,j}(x_i) = d^*_{i,j},\ P^{*\prime}_{i,j}(x_{i+1}) = d^*_{i+1,j}.$$

Using the x-direction $P^*_{i,j}(x)$ defines the bivariate function on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follow:

$$P(x, y) \equiv P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \tag{2}$$

where

$$p_{i,j}(x, y) = (1-\eta)^3\alpha_{i,j}P^*_{i,j}(x) + \eta(1-\eta)^2V_{i,j} + \eta^2(1-\eta)W_{i,j} + \eta^3\beta_{i,j}P^*_{i,j+1}(x),$$
$$q_{i,j}(y) = (1-\eta)^2\alpha_{i,j} + \eta(1-\eta)\gamma_{i,j} + \eta^3\beta_{i,j},$$

with

$$V_{i,j} = (\alpha_{i,j} + \gamma_{i,j})P^*_{i,j}(x) + l_j\alpha_{i,j}D_{i,j}(x),$$
$$W_{i,j} = (\beta_{i,j} + \gamma_{i,j})P^*_{i,j+1}(x) - l_j\beta_{i,j}D_{i,j+1}(x),$$

and

$$D_{i,j}(x)=\frac{(1-\theta)^2(\alpha^*_{i,j}+\theta\gamma^*_{i,j})d_{i,j}}{(1-\theta)^2\alpha^*_{i,j}+\theta(1-\theta)\gamma^*_{i,j}+\theta^2\beta^*_{i,j}}+\frac{\theta^2((1-\theta)\gamma^*_{i,j}+\beta^*_{i,j})d_{i+1,j}}{(1-\theta)^2\alpha^*_{i,j}+\theta(1-\theta)\gamma^*_{i,j}+\theta^2\beta^*_{i,j}},\tag{3}$$

and $\alpha_{i,j}>0,\beta_{i,j}>0,\gamma_{i,j}>0$. The interpolant $P(x,y)$ defined by (2) is called a bicubic rational interpolation in $[x_1,x_n;y_1,y_m]$, and which satisfies

$$P(x_r,y_s)=f(x_r,y_s),\ \frac{\partial P(x_r,y_s)}{\partial x}=d^*_{r,s},$$
$$\frac{\partial P(x_r,y_s)}{\partial y}=d_{r,s},\ r=i,i+1;s=j,j+1.$$

It is easy to test that the interpolant $P(x,y)$ is C^1 in the interpolating region $[a,b;c,d]$ if the shape parameters satisfy $\alpha_{i,j}$ =constant, $\beta_{i,j}$ =constant and $\gamma_{i,j}$ =constant, for each $j\in J$ and all $i\in I$, no matter what the shape parameters $a^*_{i,j},\beta^*_{i,j}$ and $\gamma^*_{i,j}$ might be.

3 Positivity-preserving surface interpolating scheme

In engineering, industrial, and scientific problems, the construction of shape preserving interpolants is an everlasting demand and one of the major research areas of computer aided design. In this section, we identify suitable values for the shape parameters involved in $P(x,y)$ defined by (2), which make the interpolating surface to preserve positive property of given data.

Let $\{(x_i,y_j,f_{i,j}):i=1,2,\cdots,n;j=1,2,\cdots,m\}$ be a monotone data set defined over the rectangular grid $[x_i,x_{i+1};y_j,y_{j+1}]$ such that $f_{i,j}>0$ for all i,j. Here, the aim is to construct a piecewise rational bivariate function $P(x,y)$ on $\Omega=[x_1,x_n;y_1,y_m]$ such that

$$P(x_i,y_j)=f_{i,j},\quad i=1,2,\cdots,n;j=1,2,\cdots,m,$$

and $P(x,y)>0$ for $(x,y)\in\Omega$. From (2), $q_{i,j}(y)>0$ for the positive shape parameters, therefore, $P(x,y)$ is positive if the following constraints hold:

$$P^*_{i,j}(x)>0,\quad V_{i,j}>0,\quad W_{i,j}>0,\quad P^*_{i,j+1}(x)>0.$$

From (1), it is easy to see that $P^*_{i,j}(x)>0$ holds if the following equalities are satisfied:

$$V^*_{i,j}=(\alpha^*_{i,j}+\gamma^*_{i,j})f_{i,j}+h_i\alpha^*_{i,j}d^*_{i,j}>0,$$
$$W^*_{i,j}=(\beta^*_{i,j}+\gamma^*_{i,j})f_{i+1,j}-h_i\beta^*_{i,j}d^*_{i+1,j}>0.$$

Thus, $P^*_{i,j}(x)>0$ if

$$\frac{\gamma^*_{i,j}}{\alpha^*_{i,j}}>-\frac{h_id^*_{i,j}}{f_{i,j}}-1,\quad \frac{\gamma^*_{i,j}}{\beta^*_{i,j}}>\frac{h_id^*_{i+1,j}}{f_{i+1,j}}-1.\tag{4}$$

Similarly, $P^*_{i,j+1}(x)>0$ if

$$\frac{\gamma^*_{i,j+1}}{\alpha^*_{i,j+1}}>-\frac{h_id^*_{i,j+1}}{f_{i,j+1}}-1,\quad \frac{\gamma^*_{i,j+1}}{\beta^*_{i,j+1}}>\frac{h_id^*_{i+1,j+1}}{f_{i+1,j+1}}-1.\tag{5}$$

Consider $V_{i,j}$ which, after simplification, leads to

$$V_{i,j}=\frac{(1-\theta)^3\kappa_1+\theta(1-\theta)^2\kappa_2+\theta^2(1-\theta)\kappa_3+\theta^3\kappa_4}{(1-\theta)^2\alpha^*_{i,j}+\theta(1-\theta)\gamma^*_{i,j}+\theta\beta^*_{i,j}},$$

where

$$\begin{aligned}
&\kappa_1 = \alpha^*_{i,j}((\alpha_{i,j}+\gamma_{i,j})f_{i,j} + l_j d_{i,j}\alpha_{i,j}),\\
&\kappa_2 = (\alpha_{i,j}+\gamma_{i,j})((\alpha^*_{i,j}+\gamma^*_{i,j})f_{i,j} + h_i d^*_{i,j}\alpha^*_{i,j}) + l_j d_{i,j}\alpha_{i,j}(\alpha^*_{i,j}+\gamma^*_{i,j}),\\
&\kappa_3 = (\alpha_{i,j}+\gamma_{i,j})((\beta^*_{i,j}+\gamma^*_{i,j})f_{i+1,j} - h_i d^*_{i+1,j}\beta^*_{i,j}) + l_j d_{i+1,j}\alpha_{i,j}(\beta^*_{i,j}+\gamma^*_{i,j}),\\
&\kappa_4 = \beta^*_{i,j}((\alpha_{i,j}+\gamma_{i,j})f_{i+1,j} + l_j d_{i+1,j}\alpha_{i,j}).
\end{aligned}$$

Note that $V_{i,j} > 0$ if

$$\begin{cases}
\dfrac{\gamma^*_{i,j}}{\alpha^*_{i,j}} > -\dfrac{2h_i d^*_{i,j}}{f_{i,j}} - 1, \quad \dfrac{\gamma^*_{i,j}}{\beta^*_{i,j}} > \dfrac{2h_i d^*_{i+1,j}}{f_{i+1,j}} - 1,\\
\dfrac{\gamma_{i,j}}{\alpha_{i,j}} > \max\{0, -\dfrac{2h_i d_{i,j}}{f_{i,j}} - 1, -\dfrac{2h_i d_{i+1,j}}{f_{i+1,j}} - 1\}.
\end{cases} \tag{6}$$

Moreover, $W_{i,j}$ can be rewritten as

$$W_{i,j} = \frac{(1-\theta)^3\tau_1 + \theta(1-\theta)^2\tau_2 + \theta^2(1-\theta)\tau_3 + \theta^3\tau_4}{(1-\theta)^2\alpha^*_{i,j} + \theta(1-\theta)\gamma^*_{i,j} + \theta\beta^*_{i,j}},$$

where

$$\begin{aligned}
&\tau_1 = \alpha^*_{i,j+1}((\beta_{i,j}+\gamma_{i,j})f_{i,j+1} - l_j d_{i,j+1}\beta_{i,j}),\\
&\tau_2 = (\beta_{i,j}+\gamma_{i,j})((\alpha^*_{i,j+1}+\gamma^*_{i,j+1})f_{i,j+1} + h_i d^*_{i,j+1}\alpha^*_{i,j+1}) - l_j d_{i,j+1}\beta_{i,j}(\alpha^*_{i,j+1}+\gamma^*_{i,j+1}),\\
&\tau_3 = (\beta_{i,j}+\gamma_{i,j})((\beta^*_{i,j+1}+\gamma^*_{i,j+1})f_{i+1,j+1} - h_i d^*_{i+1,j+1}\beta^*_{i,j+1}) - l_j d_{i+1,j+1}\beta_{i,j}(\beta^*_{i,j+1}+\gamma^*_{i,j+1}),\\
&\tau_4 = \beta^*_{i,j+1}((\beta_{i,j}+\gamma_{i,j})f_{i+1,j+1} - l_j d_{i+1,j+1}\beta_{i,j}).
\end{aligned}$$

Thus, it is easy to derive that $V_{i,j} > 0$ if

$$\begin{cases}
\dfrac{\gamma^*_{i,j+1}}{\alpha^*_{i,j+1}} > -\dfrac{2h_i d^*_{i,j+1}}{f_{i,j+1}} - 1, \quad \dfrac{\gamma^*_{i,j+1}}{\beta^*_{i,j+1}} > \dfrac{2h_i d^*_{i+1,j+1}}{f_{i+1,j+1}} - 1,\\
\dfrac{\gamma_{i,j}}{\beta_{i,j}} > \max\{0, \dfrac{2l_j d_{i,j+1}}{f_{i,j+1}} - 1, \dfrac{2l_j d_{i+1,j+1}}{f_{i+1,j+1}} - 1\}.
\end{cases} \tag{7}$$

Based on the analysis above, from (4)-(7), the following theorem can be obtained.

Theorem 1. *Let $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a positive data set defined over the plane region $[x_1, x_n; y_1, y_m]$ such that $f_{i,j} > 0$ for all i and j, where $f_{i,j}$ represents $f_{i,j}(x, y)$ at the point (x_i, y_j). Let $d^*_{i,j}$ and $d_{i,j}$ be the chosen partial derivatives. Then the bivariate rational spline interpolant $P(x, y)$ defined in (2) visualize positive data in the view of positive surface if the positive shape parameters satisfy the following constraints:*

$$\begin{aligned}
&\frac{\gamma^*_{i,j}}{\alpha^*_{i,j}} > \max\{0, -\frac{2h_i d^*_{i,j}}{f_{i,j}} - 1\},\\
&\frac{\gamma^*_{i,j}}{\beta^*_{i,j}} > \max\{0, \frac{2h_i d^*_{i+1,j}}{f_{i+1,j}} - 1\},\\
&\frac{\gamma_{i,j}}{\alpha_{i,j}} > \max\{0, -\frac{2h_i d_{i,j}}{f_{i,j}} - 1, -\frac{2h_i d_{i+1,j}}{f_{i+1,j}} - 1\},\\
&\frac{\gamma_{i,j}}{\beta_{i,j}} > \max\{0, \frac{2l_j d_{i,j+1}}{f_{i,j+1}} - 1, \frac{2l_j d_{i+1,j+1}}{f_{i+1,j+1}} - 1\}.
\end{aligned}$$

4 Monotonicity-preserving surface interpolating scheme

In this section, we will develop a monotonicity preserving surface interpolating scheme for the given monotone interpolating data.

Denote $\Delta_{i,j}^{(x)} = (f_{i+1,j} - f_{i,j})/h_i$, $\Delta_{i,j}^{(y)} = (f_{i,j+1} - f_{i,j})/l_j$. Let $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a monotone data set defined over the rectangular grid $[x_i, x_{i+1}; y_j, y_{j+1}]$ such that $f_{i+1,j} > f_{i,j}$, $f_{i,j+1} > f_{i,j}$ for all i, j, or equivalently $\Delta_{i,j}^{(x)} > 0$, $\Delta_{i,j}^{(y)} > 0$. For a monotone surface $P(x, y)$, it is necessary that the corresponding first partial derivatives $d_{i,j}^*$ and $d_{i,j}$ should meet:

$$d_{i,j}^* > 0, \quad d_{i,j} > 0, \quad \text{for all} \quad i = 1, 2, \cdots, n; j = 1, 2, \cdots, m.$$

Using the result developed in [3]: *Bicubic partially blended surface patch inherits all the properties of network of boundary curves*, we just need to consider the monotonicity of the boundary curves in the interpolating surface.

The function $P(x, y_j)$ is monotonic increasing if and only if $P'(x, y_j) > 0$. From (2), we can derive that

$$P'(x, y_j) = \frac{(1-\theta)^4 C_1 + \theta(1-\theta)^3 C_2 + \theta^2(1-\theta)^2 C_3 + \theta^3(1-\theta) C_4 + \theta^4 C_5}{((1-\theta)^2 \alpha_{i,j}^* + \theta(1-\theta)\gamma_{i,j}^* + \theta^2 \beta_{i,j}^*)^2}, \tag{8}$$

with

$$\begin{aligned}
C_1 &= \alpha_{i,j}^{*2} d_{i,j}, \\
C_2 &= 2\alpha_{i,j}^* [(\beta_{i,j}^* + \gamma_{i,j}^*)\Delta_{i,j}^{(x)} - \beta_{i,j}^* d_{i+1,j}], \\
C_3 &= (\gamma_{i,j}^{*2} + \gamma_{i,j}^*(\alpha_{i,j}^* + \beta_{i,j}^*) + 4\alpha_{i,j}^* \beta_{i,j}^*)\Delta_{i,j}^{(x)} - \alpha_{i,j}^*(\beta_{i,j}^* + \gamma_{i,j}^*) d_{i,j} - \beta_{i,j}^*(\alpha_{i,j}^* + \gamma_{i,j}^*) d_{i+1,j}, \\
C_4 &= 2\beta_{i,j}^* [(\alpha_{i,j}^* + \gamma_{i,j}^*)\Delta_{i,j}^{(x)} - \alpha_{i,j}^* d_{i,j}], \\
C_5 &= \beta_{i,j}^{*2} d_{i+1,j}.
\end{aligned}$$

It is evident that C_1 and C_5 are positive, and $C_2 > 0$ if $\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \frac{d_{i,j}^*}{\Delta_{i,j}^{(x)}} - 1$, $C_4 > 0$ if $\frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{d_{i+1,j}^*}{\Delta_{i,j}^{(x)}} - 1$. Further, since

$$\begin{aligned}
C_3 &= \frac{1}{2}(C_2 + C_4) + (\gamma_{i,j}^{*2} + 2\alpha_{i,j}^* \beta_{i,j}^*)\Delta_{i,j}^{(x)} - \alpha_{i,j}^* \gamma_{i,j}^* d_{i,j} - \beta_{i,j}^* \gamma_{i,j}^* d_{i+1,j} \\
&\geq \frac{1}{2}(C_2 + C_4) + \gamma_{i,j}^*(\frac{1}{2}\gamma_{i,j}^* \Delta_{i,j}^{(x)} - \alpha_{i,j}^* d_{i,j}) + \gamma_{i,j}^*(\frac{1}{2}\gamma_{i,j}^* \Delta_{i,j}^{(x)} - \beta_{i,j}^* d_{i+1,j}),
\end{aligned}$$

it is easy to see that $C_3 > 0$ if $\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \frac{2d_{i,j}^*}{\Delta_{i,j}^{(x)}}$ and $\frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{2d_{i+1,j}^*}{\Delta_{i,j}^{(x)}}$.

Similarly, we have

$$P'(x_i, y) = \frac{(1-\eta)^4 K_1 + 2\eta(1-\eta)^3 K_2 + \eta^2(1-\eta)^2 K_3 + \eta^3(1-\eta) K_4 + \eta^4 K_5}{((1-\eta)^2 \alpha_{i,j} + \eta(1-\eta)\gamma_{i,j} + \eta^2 \beta_{i,j})^2}, \tag{9}$$

where

$$\begin{aligned}
K_1 &= \alpha_{i,j}^2 d_{i,j}, \\
K_2 &= 2\alpha_{i,j} [(\beta_{i,j} + \gamma_{i,j})\Delta_{i,j}^{(y)} - \beta_{i,j} d_{i,j+1}], \\
K_3 &= (\gamma_{i,j}^2 + \gamma_{i,j}(\alpha_{i,j} + \beta_{i,j}) + 4\alpha_{i,j}\beta_{i,j})\Delta_{i,j}^{(y)} - \beta_{i,j}(\alpha_{i,j} + \gamma_{i,j}) d_{i,j+1} - \alpha_{i,j}(\beta_{i,j} + \gamma_{i,j}) d_{i,j}, \\
K_4 &= 2\beta_{i,j} [(\alpha_{i,j} + \gamma_{i,j})\Delta_{i,j}^{(y)} - \alpha_{i,j} d_{i,j}], \\
K_5 &= \beta_{i,j}^2 d_{i,j+1}.
\end{aligned}$$

Hence, $K_i > 0$ $(k = 1, 2, \cdots, 5)$ hold if $\frac{\gamma_{i,j}}{\alpha_{i,j}} > \frac{2d_{i,j}}{\Delta_{i,j}^{(y)}}$ and $\frac{\gamma_{i,j}}{\beta_{i,j}} > \frac{2d_{i,j+1}}{\Delta_{i,j}^{(y)}}$.

Thus, the above discussion is epitomized in the form of following theorem.

Theorem 2. *Let $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a monotone data set defined over the plane region $[x_1, x_n; y_1, y_m]$ such that $f_{i+1,j} > f_{i,j}$, $f_{i,j+1} > f_{i,j}$ for all i and j, where $a = x_1 < x_2 < ... < x_n = b$, $c = y_1 < y_2 < \cdots < y_m = d$ are the knot spacings, $f_{i,j}$ represents $f_{i,j}(x, y)$ at the point (x_i, y_j). Let $d_{i,j}^*$ and $d_{i,j}$ be the chosen partial derivatives, so that $d_{i,j}^* > 0$ and $d_{i,j} > 0$. Then the bivariate rational spline interpolant $P(x, y)$ defined in (2) visualize monotonic data in the view of monotone surface if the positive shape parameters satisfy the following constraints:*

$$\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \frac{2d_{i,j}^*}{\Delta_{i,j}^{(x)}}, \quad \frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{2d_{i+1,j}^*}{\Delta_{i,j}^{(x)}}, \quad \frac{\gamma_{i,j}}{\alpha_{i,j}} > \frac{2d_{i,j}}{\Delta_{i,j}^{(y)}}, \quad \frac{\gamma_{i,j}}{\beta_{i,j}} > \frac{2d_{i,j+1}}{\Delta_{i,j}^{(y)}}.$$

5 Demonstration

In this section, we shall illustrate the positivity preserving scheme and the monotonicity preserving scheme developed in Sections 3 and 4 with some examples, respectively.

Example 1. First of all, let us take the example of a positive data in Table 1. This data is generated approximately from the following smooth positive function by taking the values truncated to four decimal places:

$$f(x, y) = \exp(-x^2 - 2y) + 0.01, \quad 0 \le x, y \le 8. \tag{10}$$

The interpolant $P(x, y)$ defined by (2) is identified uniquely by the given interpolating data and

Table 1: A positive data set taken from function (10).

y/x	0	2	4	6	8
0	1.0100	0.0283	0.0103	0.0100	0.0100
2	0.0283	0.0103	0.0100	0.0100	0.0100
4	0.0100	0.0100	0.0100	0.0100	0.0100
6	0.0100	0.0100	0.0100	0.0100	0.0100
8	0.0100	0.0100	0.0100	0.0100	0.0100

the values of shape parameters. We take $\alpha_{i,j}^* = \beta_{i,j}^* = \alpha_{i,j} = \beta_{i,j} = 1$ and $\gamma_{i,j}^* = \gamma_{i,j} = 2$, the interpolant coincides with the bicubic Hermite interpolant. Figure 1 shows the graph of corresponding interpolating surface $P_1(x, y)$, which loses the positivity.

For the same data set in table 1, we employ Theorem 1 to compute the values of shape parameters, which are taken as: $\alpha_{i,j}^* = \beta_{i,j}^* = \alpha_{i,j} = \beta_{i,j} = 0.4$ and $\gamma_{i,j}^* = \gamma_{i,j} = 3$. Figure 2 provides the graph of the corresponding bicubic rational interpolant $P_2(x, y)$. It is obvious to see from Figure 2 that the shape of the data has been preserved by the surface representation.

Example 2. Here, we consider the example of a monotonic data in Table 2. This data has been generated approximately from the following smooth function:

$$f(x, y) = \sin(x^2 + y^2 + xy), \quad 0 \le x, y \le 0.6. \tag{11}$$

This example will illustrate how visualization of 3D monotone data can be achieved in the view of monotone surfaces only by selecting suitable shape parameters for unchanged interpolating data in Table 2. For any values of the shape parameters, it cannot be guaranteed that the bicubic rational surface generated by (2) is monotone. For example, we take $\alpha_{i,j}^* = \alpha_{i,j} = 1$, $\beta_{i,j}^* = \beta_{i,j} = 8$ and

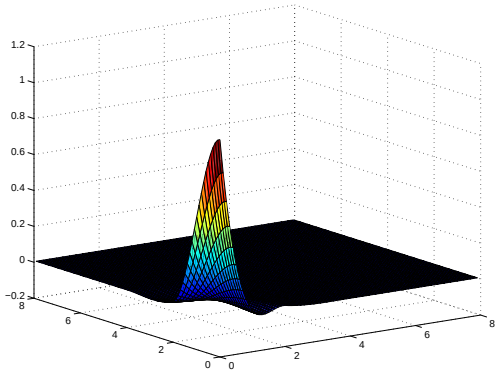


Figure 1: Non-positive Hermite surface P_1.

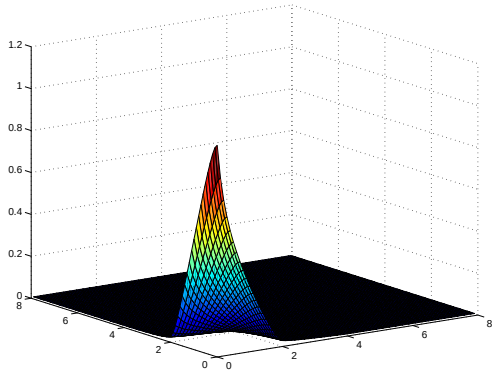


Figure 2: Positive rational surface P_2.

Table 2: A monotone data set taken from function (11).

y/x	0	0.2	0.4	0.6
0	0	0.0400	0.1593	0.3523
0.2	0.0400	0.1197	0.2764	0.4969
0.4	0.1593	0.2764	0.4618	0.6889
0.6	0.3523	0.4969	0.6889	0.8820

$\gamma^*_{i,j} = \gamma_{i,j} = 0.1$. Figure 3 shows the graph of the corresponding rational surface $P_3(x, y)$, which loses the monotonicity in its display. Figure 4 is a different view of Figure 3 obtained after making a rotation, it confirms quite clearly that the surface is not preserving monotonicity feature.

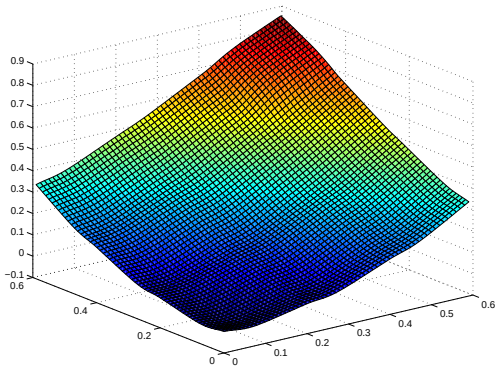


Figure 3: Non-monotonic rational surface P_3.

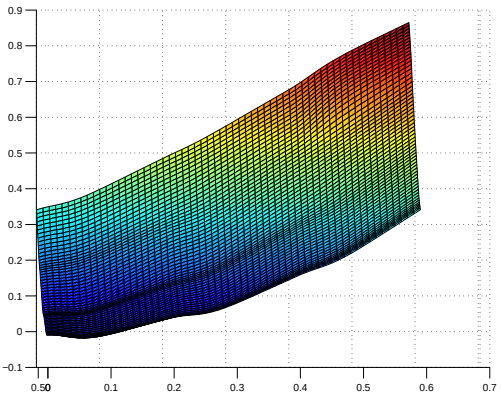


Figure 4: A different view of surface P_3.

Now, we employ Theorem 2 to compute the values of shape parameters, which are taken as: $\alpha^*_{i,j} = \beta^*_{i,j} = \alpha_{i,j} = \beta_{i,j} = 0.6$ and $\gamma^*_{i,j} = \gamma_{i,j} = 1.2$. Figure 5 provides the graph of the corresponding bicubic rational interpolant $P_4(x, y)$. It is obvious to see from Figure 5 that the shape of the monotone data in Table 2 has been preserved by the surface representation. Figure 6 is produced from this data set using a bicubic Hermite interpolant.

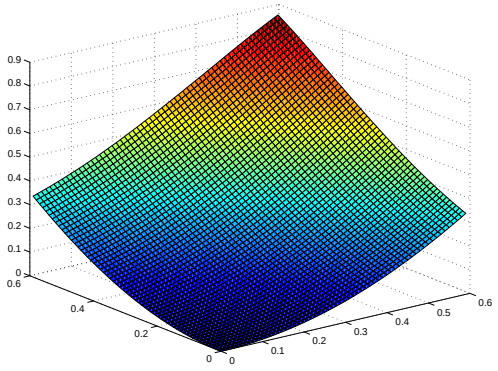


Figure 5: Monotonic rational surface P_4.

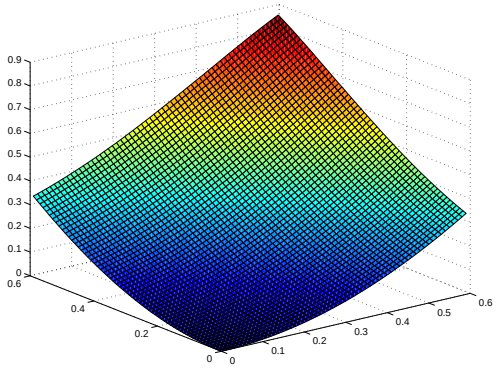


Figure 6: Monotonic Hermite surface P_5.

6 Concluding remarks

In engineering, industrial and scientific problems, the construction of shape preserving interpolants is an everlasting demand and one of the major research areas of computer aided design. This work is a contribution towards the graphical display of data when it is positive or monotone. To overcome the problem, we present a new bicubic rational interpolating spline with the shape parameters based on function values and partial derivatives. Further, a positivity-preserving scheme and a monotonicity-preserving scheme are developed to visualize positive data and monotone data in the view of positive surface and monotone surface, respectively, and the simple sufficient conditions are derived on the shape parameters in the description of the rational function.

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Explicit Viscosity Rules and Applications of Nonexpansive Mappings

Young Chel Kwun[1], Waqas Nazeer[2,*], Mobeen Munir[3] and Shin Min Kang[4,5,*]

[1]Department of Mathematics, Dong-A University, Busan 49315, Korea
yckwun@dau.ac.kr

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

Abstract

This article presents a new explicit viscosity rule for nonexpansive mappings in Hilbert spaces. The strong convergence theorems of the rule is proved under certain assumptions imposed on the sequence of parameters. Moreover, we give applications to a more general system of variational inequalities, the constrained convex minimization problem and K-mapping.

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Key words and phrases: Explicit viscosity rule, nonexpansive mappings, variational inequality, minimization problem, K-mapping

1 Introduction

In this paper, we will take H as a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and C as a nonempty closed subset of the Hilbert space H. A mapping $T : H \to H$ is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

A mapping $f : H \to H$ is called a *contraction* if there exists $\theta \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \theta \|x - y\|, \quad \forall x, y \in H.$$

Note that $F(T)$ is the set of fixed points of T. The following strong convergence theorem for nonexpansive mappings in real Hilbert spaces is given by Moudafi [8] in 2000.

*Corresponding author

Theorem 1.1. *Let C be a noneempty closed convex subset of the a Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} T(x_n), \quad n \geq 0,$$

where the sequence $\{\epsilon_n\}$ in $(0,1)$ satisfies

(i) $\lim_{n\to\infty} \epsilon_n = 0$,

(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$,

(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.

Then $\{x_n\}$converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle (I-f)x, y-x \rangle \leq 0, \quad \forall \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*)$ $= x^*$.

This type of method for approximation of fixed points is called the *viscosity approximation method.*

In 2015, Xu et al. [11] applied the viscosity method on the midpoint rule for nonexpansive mappings and give the following generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

This use contraction to regularize the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of T, which can also solved variational inequality.

Ke and Ma [6] motivated and inspired by the idea of Xu et al. [11] and they proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(s_n x_n + (1-s_n) x_{n+1}\right)$$

and

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n) x_{n+1})$$

for $n \geq 0$.

In [3, 7], new viscosity rules and applications are developed. But they correspond to one step viscosity rule.

In this paper, we give the following new two step explicit viscosity rule:

$$\begin{cases} x_{n+1} = (1-\alpha_n) f(x_n) + \alpha_n T(y_n), \\ y_n = (1-\beta_n) x_n + \beta_n T(x_n). \end{cases}$$

We also give many applications of above rule.

2 Preliminaries

Now, we recall the properties of the metric projection.

Definition 2.1. $P_C : H \to C$ is called a *metric projection* if for every point $x \in H$, there exist a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The following lemma gives the condition for a projection mapping to be nonexpansive.

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to H$ be a metric projection. Then*
(1) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$ *for all* $x, y \in H$.
(2) P_C *is a nonexpansive mapping, that is,* $\|x - P_C x\| \leq \|x - y\|$ *for all* $y \in C$.
(3) $\langle x - P_C x, y - P_C x\rangle \leq 0$ *for all* $x \in H$ *and* $y \in C$.

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

Lemma 2.3. (The demiclosedness principle) ([2]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \textit{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$, where $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

In addition, we also need the following convergence lemma.

Lemma 2.4. ([11]) *Assume that $\{x_n\}$ is a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and$\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0 or \sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$ as $n \to \infty$.

3 Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(1) $\lim_{n\to\infty}\alpha_n = 1$ *and* $\lim_{n\to\infty}\beta_n = 1$,
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$,
(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\forall n \geq 0$,
(iv) $\lim_{n\to\infty}\|x_n - T(x_n)\| = 0$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the mapping* T *which is also the unique solution of the variational inequality*

$$\langle (I-f)x, y-x\rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* *is the unique fixed point of the contraction* $P_{F(T)}f$*, that is,* $P_{F(T)}f(x^*) = x^*$.

Proof. We divide the proof into the following five steps.

Step 1. Firstly, we show that x_n is bounded.

Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n)f(x_n) + \alpha_n T(y_n) - p\| \\
&= \|(1-\alpha_n)f(x_n) - (1-\alpha_n)p + \alpha_n T(y_n) - \alpha_n p\| \\
&\leq (1-\alpha_n)\|f(x_n)-p\| + \alpha_n\|T(y_n)-p\| \\
&\leq (1-\alpha_n)\|f(x_n)-f(p)\| + (1-\alpha_n)\|f(p)-p\| \\
&\quad + \alpha_n\|y_n-p\| \\
&\leq (1-\alpha_n)\theta\|x_n-p\| + (1-\alpha_n)\|f(p)-p\| \\
&\quad + \alpha_n\|y_n-p\|.
\end{aligned} \tag{3.2}$$

Now, consider

$$\begin{aligned}
\|y_n-p\| &= \|(1-\beta_n)x_n + \beta_n T(x_n) - p\| \\
&= \|(1-\beta_n)x_n - (1-\beta_n)p + \beta_n T(x_n) - \beta_n p\| \\
&\leq (1-\beta_n)\|x_n-p\| + \beta_n\|T(x_n)-p\| \\
&\leq (1-\beta_n)\|x_n-p\| + \beta_n\|x_n-p\| \\
&\leq \|x_n-p\|.
\end{aligned}$$

Using this in (3.2) we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\alpha_n)\theta\|x_n-p\| + (1-\alpha_n)\|f(p)-p\| + \alpha_n\|x_n-p\| \\
&= [(1-\alpha_n)\theta + \alpha_n]\|x_n-p\| + (1-\alpha_n)\|f(p)-p\| \\
&= [1-1+\alpha+(1-\alpha_n)\theta]\|x_n-p\| + (1-\alpha_n)\|f(p)-p\| \\
&= [1-(1-\alpha)+(1-\alpha_n)\theta]\|x_n-p\| + (1-\alpha_n)\|f(p)-p\| \\
&= [1-(1-\alpha_n)(1-\theta)]\|x_n-p\| \\
&\quad + (1-\alpha_n)(1-\theta)\left(\frac{1}{1-\theta}\|f(p)-p\|\right).
\end{aligned}$$

Thus, we have

$$\|x_{n+1}-p\| \leq \max\left\{\|x_n-p\|, \left(\frac{1}{1-\theta}\|f(p)-p\|\right)\right\}.$$

Similarly

$$\|x_n - p\| \leq \max\left\{\|x_{n-1} - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\}.$$

From this

$$\begin{aligned}\|x_{n+1} - p\| &\leq \max\left\{\|x_n - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\} \\ &\leq \max\left\{\|x_{n-1} - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\} \\ &\vdots \\ &\leq \max\left\{\|x_0 - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\},\end{aligned}$$

which shows that $\{x_n\}$ is bounded. From this we deduce immediately that $\{f(x_n)\}$, $\{T(x_n)\}$ are bounded.

Step 2. Next, we want to prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

For this consider

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &= \|(1-\alpha_n)f(x_n) + \alpha_n T(y_n) - (1-\alpha_{n-1})f(x_{n-1}) - \alpha_{n-1}T(y_{n-1})\| \\ &= \|(1-\alpha_n)(f(x_n) - f(x_{n-1})) - (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\ &\quad + \alpha_n(T(y_n) - T(y_{n-1})) + (\alpha_n - \alpha_{n-1})T(y_{n-1})\| \\ &\leq (1-\alpha_n)\theta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T(y_{n-1}) - f(x_{n-1})\| \\ &\quad + \alpha_n\|y_n - y_{n-1}\|.\end{aligned} \tag{3.3}$$

Now, consider

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ &= \|(1-\beta_n)x_n + \beta_n T(x_n) - (1-\beta_{n-1})x_{n-1} - \beta_{n-1}T(x_{n-1})\| \\ &= \|(1-\beta_n)(x_n - x_{n-1}) - (\beta_n - \beta_{n-1})x_{n-1} + \beta_n(T(x_n) - T(x_{n-1})) \\ &\quad + (\beta_n - \beta_{n-1})T(x_{n-1})\| \\ &\leq (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\|.\end{aligned}$$

Using this in (3.3) we get

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\leq (1-\alpha_n)\theta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T(y_{n-1}) - f(x_{n-1})\| \\ &\quad + \alpha_n\|x_n - x_{n-1}\| + \alpha_n|\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\| \\ &= [(1-\alpha_n)\theta + \alpha_n]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T(y_{n-1}) - f(x_{n-1})\| \\ &\quad + \alpha_n|\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\|.\end{aligned}$$

Let $\lambda_n = (1-\alpha_n)$ so $\lambda_n \in (0,1)$ since $\alpha_n \in (0,1)$ $\sum_{n=0}^{\infty}\lambda_n = \infty$, $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=0}^{\infty}|\beta_n - \beta_{n-1}| < \infty$. By using Lemma 2.4, we get $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. Now we want to prove that $\lim_{n\to\infty}\|x_n - T(y_n)\| = 0$.

Now, consider

$$\begin{aligned}\|x_n - T(y_n)\| &\leq \|x_n - T(x_n)\| + \|T(x_n) - T(y_n)\| \\ &\leq \|x_n - T(x_n)\| + \|x_n - y_n\| \\ &= \|x_n - T(x_n)\| + \|x_n - (1-\beta_n)x_n - \beta_n T(x_n)\| \\ &\leq \|x_n - T(x_n)\| + \beta_n\|x_n - T(x_n)\| \\ &\leq (1+\beta_n)\|x_n - T(x_n)\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Step 4. In this step, we claim that $\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle \leq 0$, where $x^* = P_{F(T)}f(x^*)$.

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and Lemma 2.3 we have $p = Tp$. This together with the property of the metric projection implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_{n_i}\rangle \\ &= \langle x^* - f(x^*), x^* - p\rangle \\ &\leq 0.\end{aligned}$$

Step 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$.

Here again $x^* \in F(T)$ is the unique fixed point of the contraction $P_{F(T)}f$. Consider

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)f(x_n) + \alpha_n T(y_n) - x^*\|^2 \\ &= \|(1-\alpha_n)[f(x_n) - x^*] + \alpha_n[T(y_n) - x^*]\|^2 \\ &= (1-\alpha_n)_n^2\|f(x_n) - x^*\|^2 + (1-\alpha_n)^2\|T(x_n) - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - x^*, T(y_n) - x^*\rangle \\ &\leq \alpha_n^2\|y_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - f(x^*), T(y_n) - x^*\rangle \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \qquad (3.4) \\ &\leq \alpha_n^2\|y_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\|f(x_n) - f(x^*)\|\|T(y_n) - x^*\| \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\ &\leq \alpha_n^2\|y_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\theta\|x_n - x^*\|\|y_n - x^*\| \\ &\quad + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle.\end{aligned}$$

Now, consider

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n T(x_n) - x^*\| \\ &= \|(1-\beta_n)x_n - (1-\beta_n)x^* + \beta_n T(x_n) - \beta_n x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|T(x_n) - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\ &\leq \|x_n - x^*\|.\end{aligned}$$

Using (3.5) in (3.4) we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq \alpha_n^2\|x_n-x^*\|^2+2\alpha_n(1-\alpha_n)\theta\|x_n-x^*\|\|x_n-x^*\|\\
&\quad+(1-\alpha_n)^2\|f(x_n)-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle\\
&\leq \alpha_n^2\|x_n-x^*\|^2+2\alpha_n(1-\alpha_n)\theta\|x_n-x^*\|\|x_n-x^*\|\\
&\quad+(1-\alpha_n)^2\|f(x_n)-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle\\
&\leq [\alpha_n^2+2\alpha_n(1-\alpha_n)\theta]\|x_n-x^*\|^2+(1-\alpha_n)^2\|f(x_n)-x^*\|^2\\
&\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle.
\end{aligned}$$

Note that $\alpha_n\theta<\alpha_n$ since $\alpha_n\in(0,1)$ and $\theta\in[0,1)$.

$$2\alpha_n\theta<2\alpha_n$$

implies

$$2\alpha_n\theta(1-\alpha_n)<2\alpha_n(1-\alpha_n)$$

implies

$$\alpha_n^2+2\alpha_n\theta(1-\alpha_n)<\alpha_n^2+2\alpha_n(1-\alpha_n).$$

So we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&\leq [\alpha_n^2+2\alpha_n(1-\alpha_n)]\|x_n-x^*\|^2+(1-\alpha_n)^2\|f(x_n)-x^*\|^2\\
&\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle\\
&\leq [2\alpha_n-\alpha_n^2)]\|x_n-x^*\|^2+(1-\alpha_n)^2\|f(x_n)-x^*\|^2\\
&\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle\\
&\leq 2\alpha_n\|x_n-x^*\|^2+(1-\alpha_n)^2\|f(x_n)-x^*\|^2\\
&\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle\\
&\leq 2[1-(1-\alpha_n)]\|x_n-x^*\|^2+(1-\alpha_n)^2\|f(x_n)-x^*\|^2\\
&\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle.
\end{aligned}\tag{3.6}$$

By $\lim_{n\to\infty}\alpha_n=1$ we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{(1-\alpha_n)^2\|f(x_n)-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,T(y_n)-x^*\rangle}{(1-\alpha_n)}\\
&=\limsup_{n\to\infty}[(1-\alpha_n)\|f(x_n)-x^*\|^2+2\alpha_n\langle f(x^*)-x^*,T(y_n)-x^*\rangle]\leq 0.
\end{aligned}\tag{3.7}$$

From (3.6), (3.7) and Lemma 2.4 we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|^2=0,$$

which implies that $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

4 Applications

4.1 A more general system of variational inequalities

Let Cbe a nonempty closed convex subset of a real Hilbert space H and $\{A_i\}_{i=1}^N : C \to H$ be a family of mappings. In [1], Cai and Bu considered the problem of finding $x_1^*, x_2^*, ..., x_N^* \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{4.1}$$

The equation (4.1) can be written as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N) x_N^*, x - x_1^* \rangle \geq 0, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1}) x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad \vdots \\ \langle x_3^* - (I - \lambda_2 A_2) x_2^*, x - x_3^* \rangle \geq 0, \\ \langle x_2^* - (I - \lambda_1 A_1) x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases}$$

which is a more general system of variational inequalities in a Hilbert space, where $\lambda_i > 0$ for all $i \in \{1, 2, 3, ..., N\}$. We also have following lemmas.

Lemma 4.1. ([6]) *Let C be a nonempty closed convex subset of a real Hilbert spaces H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i, namely,*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C$$

Let $G : C \to C$ be a mapping defined by

$$\begin{aligned} G(x) = P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x, \quad \forall x \in C. \end{aligned} \tag{4.2}$$

If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, ..., N\}$, then G is nonexpansive.

Lemma 4.2. ([5]) *Let C be a nonempty closed convex subject of a real Hilbert space H. Let $A_i : C \to H$ be a nonlinear mapping, where $i \in \{1, 2, 3, ..., N\}$. For given $x_i^* \in C$, $i \in \{1, 2, 3, ..., N\}$, $(x_1^*, x_2^*, x_3^*, ..., x_N^*)$ is a solution of the problem (4.1) if and onli if*

$$\begin{aligned} x_1^* &= P_C(I - \lambda_N A_N) x_N^*, \\ x_i^* &= P_C(I - \lambda_{i-1} A_{i-1}) x_{i-1}^*, \quad i = 2, 3, 4, ..., N, \end{aligned} \tag{4.3}$$

that is,

$$\begin{aligned} x_1^* = P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x_1^*, \quad \forall x \in C. \end{aligned}$$

From Lemma 4.2, we know that $x_1^* = G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by (4.2). Moreover, if we find the fixed point x_1^*, it is easy to get the other points by (4.3). Applying Theorem 3.1 we get the result

Theorem 4.3. *Let C be a nonempty closed convex subject of a real Hilbert space H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n G(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n G(x_n), \end{cases}$$

where $\lambda_i \in (0, 2\delta_i)$, $i = 1, 2, 3, ..., N$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping G, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.

4.2 The constrained convex minimization problem

Now, we consider the following constrained convex minimization problem:

$$\min_{x \in C} \phi(x), \tag{4.4}$$

where $\phi : C \to R$ is a real-valued convex function and assumes that the problem (4.4) is consistent. Let Ω denote its solution set.

For the minimization problem (4.4), if ϕ is (Fréchet)differentiable, then we have the following lemma.

Lemma 4.4. (Optimality Condition) ([5]) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (4.4) *is that x^* solves the variational inequality*

$$\langle \nabla\phi(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{4.4}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\left(x^* - \lambda\nabla\phi(x^*)\right)$$

for every constant $\lambda > 0$. If, in a addition ϕ is convex, then the optimality condition (4.5) *is also sufficient.*

It is well known that the mapping $P_C(I-\lambda A)$ is nonexpansive when the mapping A is δ-inverse-strongly monotone and $0<\lambda<2\delta$. We therefore have the following result.

Theorem 4.5. *Let C be a nonempty closed convex subset of the real Hilbert Space H. For the minimization problem* (4.4), *assume that ϕ is (Fréchet) differentiable and the gradient $\nabla\phi$ is a δ-inverse-strongly monotone mapping for some positive real number δ. Let $f: C\to C$ be a contraction with coefficient $\theta\in[0,1)$. Pick any $x_0\in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1}=(1-\alpha_n)f(x_n)+\alpha_n P_C(1-\lambda\nabla\phi)(y_n), \\ y_n=(1-\beta_n)x_n+\beta_n P_C(1-\lambda\nabla\phi)(x_n), \end{cases}$$

where $\lambda_i\in(0,2\delta)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a solution x^* of the minimization problem (4.4), *which is also the unique solution of the variational inequality*

$$\langle (I-f)x, y-x\rangle\geq 0, \quad \forall y\in\Omega.$$

In other words, x^* is the unique fixed point of the contraction $P_\Omega f$, that is, $P_\Omega f(x^*)=x^*$.

4.3 The K-mapping

In 2009, Kangtunyakarn and Suantai [4] gave a K-mapping generated by $T_1,T_2,T_3,...,T_N$ and $\lambda_1,\lambda_2,\lambda_3,...,\lambda_N$ as follows.

Definition 4.6. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a family of mappings of C into itself and let $\lambda_1,\lambda_2,\lambda_3,...,\lambda_N$ be real numbers such that $0\leq\lambda_i\leq 1$ for every $i=1,2,3,...,N$. We define a mapping $K: C\to C$ as follows:

$$\begin{cases} U_1=\lambda_1T_1+(1-\lambda_1)I, \\ U_2=\lambda_2T_2U_1+(1-\lambda_2)U_1, \\ \quad\vdots \\ U_{N-1}=\lambda_{N-1}T_{N-1}U_{N-2}+(1-\lambda_{N-1})U_{N-2}, \\ K=U_N=\lambda_NT_NU_{N-1}+(1-\lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a *K-mapping* generated by $T_1,T_2,T_3,...,T_N$ and $\lambda_1,\lambda_2,\lambda_3,...,\lambda_N$.

In 2014, Suwannaut and Kangtunyakarn [10] established the following result for K-mapping generated by $T_1,T_2,T_3,...,T_N$ and $\lambda_1,\lambda_2,\lambda_3,...,\lambda_N$.

Lemma 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,3,...,N$, let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself with $k_i\leq\omega_1$ and $\bigcap_{i=1}^N F(T_i)\neq\emptyset$, namely, there exist constants $k_i\in[0,1)$ such that*

$$\|T_ix-T_iy\|^2\leq\|x-y\|^2+k_i\|(I-T_i)x-(I-T_i)y\|^2, \quad \forall x,y\in C.$$

Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2$ for all $i = 1, 2, 3, ..., N$ and $\omega_1 + \omega_2 < 1$. Let K be the K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. Then the following properties hold:

(a) $F(K) = \bigcap_{i=1}^{N} F(T_i)$,

(b) K is a nonexpansive mapping.

On the bases of above lemma, we have the following results.

Theorem 4.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, 3, ..., N$, let $\{T_i\}_{i=1}^{N}$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself with $k_i \leq \omega_1$ and $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2$ for all $i = 1, 2, 3, ..., N$ and $\omega_1 + \omega_2 < 1$. Let K be the K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be sequence generated by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) f(x_n) + \alpha_n K(y_n), \\ y_n = (1 - \beta_n) x_n + \beta_n K(x_n). \end{cases},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mappings $\{T_i\}_{i=1}^{N}$, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle, \quad \forall y \in F(K) = \bigcap_{i=1}^{N} F(T_i).$$

In other words, x^* is the unique fixed point of the contraction $P_{\bigcap_{i=1}^{N} F(T_i)} f$, that is, $P_{\bigcap_{i=1}^{N} F(T_i)} f(x^*) = x^*$.

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Applications and Strong Convergence Theorems of Asymptotically Nonexpansive Non-self Mappings

Young Chel Kwun[1], Waqas Nazeer[2,*], Mobeen Munir[3] and Shin Min Kang[4,5,*]

[1]Department of Mathematics, Dong-A University, Busan 49315, Korea
yckwun@dau.ac.kr

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan
e-mail: sm.kang@mail.cmuh.org.tw

[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

Abstract

We give some strong convergence theorems about asymptotically nonexpansive non-self mappings. In the end we give some applications of our results in the form of examples. Our results extend the results already proved.

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1 Introduction

Fixed points of special mappings like asymptotically nonexpansive, nonexpansive, contractive and other mappings has become a field of interest on its on and has a variety of applications in related fields like image recovery, signal processing and geometry of objects. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now.

*Corresponding author

Let C be a nonempty subset of a real normed space E of A mapping $T : C \to C$ is *asymptotically nonexpansive* if there exist a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and $n \geq 1$. This class was introduced by Goebel and Kirk in 1972, see [4]. Ever since it has occupied a central place in fixed point theory and other related fields concerning about special mappings.

Theorem 1.1. *If C is a nonempty closed convex subset of a real uniformly convex Banach space E and $T : C \to C$ is an asymptotically nonexpansive mapping, then T has a unique fixed point in C.*

However, in 1991, Schu [9] developed the modified Mann process for the approximation of fixed points of an asymptotically nonexpansive self mapping which is defined on a nonempty closed convex bounded subset of a Hilbert space given as follows:

Theorem 1.2. *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ be a completely continuous and asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty]$ having $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) \leq \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ with condition $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n > 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ defined recursively by*

$$\begin{cases} x_1 \in C \quad (\textit{arbitrarily}), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall\, n \geq 1 \end{cases} \tag{1.1}$$

converges strongly to some fixed point of T.

In the above theorem, T is a self mapping of C, where C is a nonempty closed convex subset of H. If, however, the domain $D(T)$ of T is a proper subset of H and $T : D(T) \to H$ is a mapping the modified iteration $\{x_n\}$ may fail to be well defined.

To overcome this, in 2003, Chidume et al. [3] introduce the concept of asymptotically nonexpansive non-self mappings.

Let E be a real Banach space. A subset C of E is called a *retract* of E if there exist a continuous mapping $P : E \to E$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniform convex Banach space is a retract. A mapping $P : E \to E$ is called a *retraction* if $P^2 = P$. It is clear that if P is a retraction, then $Py = y$ for all y in the the range of P (see [3]).

Definition 1.3. Let C be a nonempty subset of a normed linear space E. Let $P : E \to C$ be a nonexpansive retraction of E onto C. A non-self mapping $T : C \to E$ is said to be *asymptotically nonexpansive* if there exist a sequence $k_n \subset [0, 1)$ with $\lim_{n\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq k_n \|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$.

Chidume et al. [3] introduced the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1} x_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$ and proved some results of strong convergence and weak convergence for asymptotic nonexpansive non-self mapping.

Throughout this paper, we denote $Fix(T) = \{x \in C : Tx = x\}$.

Remark 1.4. If $T : C \to C$ is a self mapping, then P becomes identity and we have

1. The non-self mapping with (1.2) coincides with an asymptotically nonexpansive self mapping in (1.1).

2. Both iterations (1.3) and (1.1) coincide.

After Chidume et al. many authors proved weak and strong convergence theorems for asymptotically nonexpansive non-self mapping in Banach spaces [6–8, 10]. Guo and Guo [7] introduced following new iterative scheme which is given as:

Let E be a real Banach space, C be a nonempty closed convex subset of E and $P : C \to E$ be a nonexpansive retraction of E onto C. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping defined by

$$\begin{cases} x_1 \in C, \\ y_n = P((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$. They proved some theorems of strong convergence and weak convergence of the above iteration for an asymptotically nonexpansive non-self mapping $T : C \to E$.

In this paper, we first introduce a new iterative scheme $\{x_n\}$ defined as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ z_n = P((1-\gamma_n)y_n + \gamma_n T(PT)^{n-1}y_n), \\ x_{n+1} = P\big((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}\big(\frac{y_n+z_n}{2}\big)\big) \end{cases} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$. We first use the condition which is weaker than the completely continuous mappings, given in [5] named as the condition (BP). Secondly, we prove some strong convergence theorems for our iteration scheme for an asymptotically nonexpansive non-self mapping. It is important to remark that our results extend the results in [3, 5]. Finally, we give examples to explain the main results of this paper.

2 Some lemmas

In this section we give some key lemmas which will be used to prove the main results of this paper.

Lemma 2.1. ([11]) *Let $p > 1$ and $R > 0$ and E be a Banach space. Then E is uniformly convex if and only if there exist a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B(0,R) = \{x \in E : |x| \leq R\}$ *with* $\lambda \in [0,1]$, *where* $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 2.2. *Let* E *be a real uniformly convex Banach space and* C *be a nonempty convex subset of* E. *Let* $T : C \to E$ *be an asymptotically nonexpansive non-self mapping with* $\{h_n\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ *and* $Fix(T) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence defined by* (1.5), *where* $\{\alpha_n\} \subset [0,1)$ *and* $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$. *Then*

(*a*) $\|x_{n+1} - p\| \leq h_n^2 \|x_n - p\|$ *for all* $p \in Fix(T)$.

(*b*) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in Fix(T)$.

Proof. Take $T_2 = T_1 = T$ and $S_2 = S_1 = I$ in [6], we obtain the required result (see [5]). □

Lemma 2.3. *Let* E *be a real uniform convex Banach space and* C *be a nonempty closed convex subset of* E. *Let* $T : C \to E$ *be an asmyptotically nonexpansive non-self mapping with* $\{h_n\}, \{h'_n\} \subset [1,\infty)$ *and* $h'_n \leq h_n$ *such that* $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, $\sum_{n=1}^{\infty}(h'_n - 1) < \infty$ *and* $Fix(T) \neq \emptyset$. *Let* $\{x_n\}$ *a sequence defined in* (1.5), *where*

$$0 < \liminf_{n\to\infty} \alpha_n, \quad \limsup_{n\to\infty} \alpha_n < 1, \limsup_{n\to\infty} \beta_n < 1, \quad \limsup_{n\to\infty} \gamma_n < 1.$$

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. By Lemma 2.2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in Fix(T)$. So $\{x_n - p\}, \{y_n - p\}, \{z_n - p\}, \{T(PT)^{n-1}x_n - p\}, \{T(PT)^{n-1}y_n - p\}$ and $\{T(PT)^{n-1}z_n - p\}$ are all bounded so we have a real number $R > 0$ such that

$$\{x_n - p, y_n - p, z_n - p, T(PT)^{n-1}x_n - p, T(PT)^{n-1}y_n - p\, T(PT)^{n-1}z_n - p\} \subset B(0,R),$$

for all $n \geq 1$. It follows from (1.5) and Lemma 2.1 that

$$\begin{aligned}
\|y_n - p\|^2 &\leq \|(1-\beta_n)(x_n - p) + \beta_n(T(PT)^{n-1}x_n - p)\|^2 \\
&\leq (1-\beta_n)\|(x_n - p)\|^2 + \beta_n\|(T(PT)^{n-1}x_n - p)\|^2 \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - T(PT)^{n-1}x_n\|) \\
&\leq (1-\beta_n)h_n^2\|(x_n - p)\|^2 + \beta_n h_n^2\|x_n - p\|^2 \\
&= h_n^2\|(x_n - p)\|^2,
\end{aligned}$$

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|(1-\gamma_n)(y_n - p) + \gamma_n(T(PT)^{n-1}y_n - p)\|^2 \\
&\leq (1-\gamma_n)\|(y_n - p)\|^2 + \gamma_n\|(T(PT)^{n-1}y_n - p)\|^2 \\
&\quad - \gamma_n(1-\gamma_n)g(\|y_n - T(PT)^{n-1}y_n\|) \\
&\leq (1-\gamma_n)h'^2_n\|(y_n - p)\|^2 + \gamma_n h'^2_n\|y_n - p\|^2 \\
&= h'^2_n|(y_n - p)\|^2 \\
&\leq h'^2_n h_n^2|(x_n - p)\|^2 \\
&\leq h_n^4\|(x_n - p)\|^2.
\end{aligned}$$

Also

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \left\|(1-\alpha_n)(x_n-p)+\alpha_n\left(T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)-p\right)\right\|^2\\
&\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\left\|\left(T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)-p\right)\right\|^2\\
&\quad-\alpha_n(1-\alpha_n)g\left(\left\|x_n-T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)\right\|\right)\\
&\le (1-\alpha_n)h_n^8\|x_n-p\|^2+\alpha_n h'^2_n h_n^2\left\|\frac{y_n+z_n}{2}-p\right\|^2\\
&\quad-\alpha_n(1-\alpha_n)g\left(\left\|x_n-T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)\right\|\right)\\
&\le (1-\alpha_n)h_n^8\|x_n-p\|^2+\frac{1}{2}\alpha_n h'^2_n h_n^2(\|y_n-p\|^2+\|z_n-p\|^2)\\
&\quad-\alpha_n(1-\alpha_n)g\left(\left\|x_n-T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)\right\|\right)\\
&\le (1-\alpha_n)h_n^8\|x_n-p\|^2+\alpha_n h_n^8\|x_n-p\|^2\\
&\quad-\alpha_n(1-\alpha_n)g\left(\left\|x_n-T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)\right\|\right)\\
&= h_n^8\|x_n-p\|^2-\alpha_n(1-\alpha_n)g\left(\left\|x_n-T(PT)^{n-1}\left(\frac{y_n+z_n}{2}\right)\right\|\right),
\end{aligned}$$

where $g:[0,\infty)\to[0,\infty)$ is a continuous, strictly increasing and convex function with the condition $g(0)=0$. Since we have condition $0<\liminf_{n\to\infty}\alpha_n$ and $\limsup_{n\to\infty}\alpha_n<1$. So there exist two real numbers $a,b\in(0,1)$ and a positive integer n_0 such that $a\le\alpha_n\le b$ for all $n\ge n_0$, so

$$a(1-b)g(\|\,x_n-T(PT)^{n-1}y_n\,\|)\le h^4\,\|\,x_n-p\,\|^2-\|\,x_{n+1}-p\,\|^2$$

and

$$a(1-b)g(\|\,x_n-T(PT)^{n-1}z_n\,\|)\le h'^4\,\|\,x_n-p\,\|^2-\|\,x_{n+1}-p\,\|^2\,.$$

Since $\lim_{n\to\infty}h_n=1$ and $\lim_{n\to\infty}h'_n=1$, there exist positive integer m_0, to and real number s, $s'\in(0,1)$ such that $\beta_n h_n\le s$ and $\gamma_n h'_n\le s'$ for all $n>m_0$ and

$$\begin{aligned}
\|y_n-x_n\| &\le \beta_n\|T(PT)^{n-1}x_n-x_n\|\\
&\le \beta_n\|T(PT)^{n-1}x_n-T(PT)^{n-1}y_n\|+\beta_n\|x_n-T(PT)^{n-1}y_n\|\\
&\le \beta_n h_n\|x_n-y_n\|+\|x_n-T(PT)^{n-1}y_n\|.
\end{aligned}$$

Similarly,

$$\|z_n-x_n\|\le\gamma_n h'_n\|x_n-z_n\|+\|x_n-T(PT)^{n-1}z_n\|.$$

Hence

$$(1-s)\|y_n-x_n\|\le(1-\beta_n h_n)\|y_n-x_n\|\le\|x_n-T(PT)^{n-1}y_n\|$$

and

$$(1-s')\|z_n-x_n\|\le(1-\gamma_n h'_n)\|z_n-x_n\|\le\|x_n-T(PT)^{n-1}z_n\|.$$

So we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Furthermore from

$$\begin{aligned}\|x_n - T(PT)^{n-1}x_n\| \leq \|x_n - T(PT)^{n-1}y_n\| + \|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\| \\ + \|x_n - T(PT)^{n-1}y_n\| + h_n\|y_n - x_n\|,\end{aligned}$$

it follows

$$\lim_{n\to\infty}\|x_n - T(PT)^{n-1}x_n\| = 0.$$

Using equations, we get

$$\lim_{n\to\infty}\|x_{n+1} - T(PT)^{n-1}y_n\| = 0.$$

$$\lim_{n\to\infty}\|x_{n+1} - T(PT)^{n-1}z_n\| = 0.$$

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0.$$

$$\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0.$$

Now

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - Tx_n\| \\ &\leq \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_{n-1}\| \\ &\quad + \|T(PT)^{n-1}y_{n-1} - Tx_n\| + \|T(PT)^{n-1}x_n - T(PT)^{n-1}z_{n-1}\| \\ &\quad + \|T(PT)^{n-1}z_{n-1} - Tx_n\|,\end{aligned}$$

and using above condition we get required result $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The completes the proof. □

3 The conditions (BP)

Here we recall the condition introduced in [5]. Let E be a Banach space and $T : E \to E$ be a bounded linear operator. In 1966, Browdin and Petryshyn [2] considered the existence of a solution of the equation $f = u - Tu$ by iteration of Picard-Poincoré-Newmann,

$$\begin{cases} x_0 \in E \\ x_{n+1} = Tx_n + f, \end{cases} \tag{3.1}$$

or

$$\begin{cases} x_0 \in E \\ x_{n+1} = T^n x_0 + (f + Tf + \cdots + T^{n-1}f), \quad \forall n \geq 0,\ f \in E. \end{cases} \tag{3.2}$$

In fact, in 1958, Browder [1] proved the following.

Theorem 3.1. *Let E be a reflexive Banach space. Then a solution u of the equation $u - Tu = f$ exists for a given point $f \in E$ and an operator T which is asmyptotic bounded if and only if the sequence $\{x_n\}$ defined by (3.1) is bounded for any fixed $x_0 \in E$.*

Relaxing the assumption of reflexivity on E, under a slight sharp condition on T, Browden and Petryshyn proved the following:

Theorem 3.2. *Let E be a Banach space, $T : E \to E$ be a bounded linear operator which is asymptotically convergent, that is, $\{T^n x\}$ converges in E for all $x \in E$. Then we have the following:*

(1) *If $f \in R(1-T)$, the sequence $\{x_n\}$ converges to a solution u of the equation $u + Tu = f$.*

(2) *If any subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ converges to an element $y \in E$, then y is a solution of the equation $y - Ty = f$.*

(3) *If E is a reflexive Banach space and the sequence $\{x_n\}$ is bounded, then the sequence $\{x_n\}$ converges to a solution of the equation $u + Tu = f$.*

Motivated by the above theorem, we have the concept of the condition (BP) as in [5] given by

Let E be a real normed linear space, C be a nonempty subset of E and $T : C \to E$ be a mapping

Definition 3.3. (Condition) The pair (T, C) is said to satisfy the *condition* (BP) if for any bounded closed subset G of C, $\{z : z = x - Tx, x \in G\}$ is a closed subset of E.

Let E and F be Banach spaces. Recall that a mapping $T : E \to F$ is completely continuous if it is continuous and compact (that is, C is bounded implies that $\overline{T(C)}$ is compact) or a weakly convergent sequence ($x_n \to x$ weakly) implies a strong convergent sequence ($Tx_n \to Tx$).

Next we establish a relation between the condition (BP) and completely continuous mapping given as:

Proposition 3.4. *Let E be a real normed linear space, C be a nonempty subset of E and $T : C \to E$ be a completely continuous mapping. Then the pair (T, C) satisfy the condition (BP).*

Proof. It is similar as in [5]. □

Remark 3.5. ([5]) The converse of the above proposition does not holds in general.

4 Strong convergence theorems

Now we turn to strong convergence theorems for the asymptotically nonexpansive non-self mappings with condition in the real uniformly convex Banach spaces. First two results correspond to our new scheme where as remaining results are the same results in [3] and [5], which are the simple derivations of our result.

Theorem 4.1. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping*

with $\{h_n\}, \{h'_n\} \subset [1, \infty)$ *and* $h'_n \leq h_n$ *such that* $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, $\sum_{n=1}^{\infty}(h'_n - 1) < \infty$ *and* $Fix(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_1 \in C, \\ y_n = P((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ z_n = P((1-\gamma_n)y_n + \gamma_n T(PT)^{n-1}y_n), \\ x_{n+1} = P\big((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}\big(\frac{y_n+z_n}{2}\big)\big) \end{cases} \tag{4.1}$$

for all $n \geq 1$, *where*

$$0 \leq \liminf_{n\to\infty} \alpha_n, \quad \limsup_{n\to\infty} \alpha_n < 1, \quad \limsup_{n\to\infty} \beta_n < 1, \quad \limsup_{n\to\infty} \gamma_n < 1.$$

If the pair (T, C) *satisfy the condition* (BP), *then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. Let $G = \{\overline{x}_n\}$, where $\overline{A}$ denotes the closure of A. Since the sequence $\{x_n\}$ is bounded in C by Lemma 2.2 and so G is a bounded closed subset of C. As pair (T, C) satisfy (BP) conditions, it follows that

$$N = \{z = x - Tx : x \in G\}$$

is closed. Lemma 2.3 guarantees $\{x_n - Tx_n\} \subset N$ and $x_n - Tx_n \to 0$, $\{y_n - Ty_n\} \subset N$ and $y_n - Ty_n \to 0$ as $n \to \infty$. Clearly the zero vector $0 \in N$ so there exist a $q \in G$ such that $q = Tq$ so $q \in Fix(T)$. Since $q \in G$ so there exists a positive integer n_0 such that $x_{n_0} = q$ or there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$.

If $x_{n_k} = q$, then it follows from Lemma 2.2 that $x_n = q$ for all $n \geq n_0$ and so $x_n \to q$ as $n \to \infty$.

If $x_{n_k} \to q$, then, since $\lim_{n\to\infty} \|x_n - q\|$ exists, we have $x_n \to q$ as $n \to \infty$. This completes the proof. □

Using Theorem 4.1 and Proposition 3.4, we have

Corollary 4.2. *Let* E *be a real uniformly convex Banach space and* C *be a nonempty closed convex subset of* E. *Let* $T : C \to E$ *be an asymptotically nonexpansive non-self mapping with* $\{h_n\}, \{h'_n\} \subset [1, \infty)$ *and* $h'_n \leq h_n$ *such that* $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, $\sum_{n=1}^{\infty}(h'_n - 1) < \infty$ *and* $Fix(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined by* (4.1), *where*

$$0 \leq \liminf_{n\to\infty} \alpha_n, \quad \limsup_{n\to\infty} \alpha_n < 1, \quad \limsup_{n\to\infty} \beta_n < 1, \quad \limsup_{n\to\infty} \gamma_n < 1.$$

If T *is completely continuous, then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T.

Theorem 4.3. *Let* E *be a real uniformly convex Banach space and* C *be a nonempty closed convex subset of* E. *Let* $T : C \to E$ *be an asymptotically nonexpansive non-self mapping with* $\{h_n\} \subset [1, \infty)$ *such that* $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ *and* $Fix(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined by* (1.3), *where*

$$0 \leq \liminf_{n\to\infty} \alpha_n \quad \textit{and} \quad \limsup_{n\to\infty} \alpha_n < 1.$$

If the pair (T, C) satisfy the condition (BP), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. In Theorem 4.1, take $\beta_n = 0$ and $\gamma_n = 0$ for all $n \geq 1$, we arrive at the result. □

Using Theorem 4.3 and Proposition 3.4, we have

Corollary 4.4. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping with $\{h_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.3), where*

$$0 \leq \liminf_{n\to\infty} \alpha_n \quad \text{and} \quad \limsup_{n\to\infty} \alpha_n < 1.$$

If T is completely continuous, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 4.5. The results proved in [3] can also be obtained from our Theorem 4.1 under special assumptions of sequences on α_n, β_n and γ_n.

5 Examples

Here we focus on the families of examples to apply on our results. First example also extends the example presented in [5].

Example 5.1. Let X be a Hilbert space and

$$C = \{x \in X : \|x\| \leq r, \forall r > 0\}.$$

Let $P : C \to C$ by

$$Px = \begin{cases} x, & \text{if } x \in C, \\ \frac{rx}{\|x\|}, & \text{if } x \in X - C. \end{cases}$$

Then P is a nonexpansive retraction of X onto C (see [5]).

Take $X = R^n$ with

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

and

$$\|x\| = \left(\sum_{i=1}^{n} (x_i)^2\right)^{\frac{1}{2}}.$$

Then X is a Hilbert space.

Let $C = \{x \in X : \|x\| \leq 1, \}$.

Take $P : C \to C$ by

$$Px = \begin{cases} x, & \text{if } x \in C, \\ \frac{x}{\|x\|}, & \text{if } x \in X - C. \end{cases}$$

Then P is a nonexpansive retraction of X onto C by [8], taking $r = 1$.

Define $T_i : C \to X$ by $T_i = (0, 0, ..., 1 - x_i, 0, 0, ..., 0)$ for all $X = (x_1, x_2, x_3, ..., x_n) \in C$. Then

$$\begin{aligned}\|Tx_i - Ty_i\| &= \|(0, 0, ...1 - x_i, 0, 0, ..., 0) - (0, 0, ..., 1 - y_i, 0, 0, ..., 0)\| \\ &= \|(0, 0, ..., x_i - y_i, 0, 0, ..., 0)\| \\ &\leq \|x - y\|.\end{aligned}$$

So $T_i x$ are a family of nonexpansive non-self mappings, so

$$\begin{aligned}\|T_i(PT_i)^{2-1}x - T_i(PT_i)^{2-1}y\| &\leq \|P(T_i x) - P(T_i y)\| \\ &\leq \|T_i x - T_i y\| \\ &\leq \|x - y\|.\end{aligned}$$

Suppose that

$$\|T_i(PT_i)^{k-1}x - T_i(PT_i)^{k-1}y\| \leq \|x - y\|, \quad \forall n = k.$$

Taking $n = k + 1$, we have

$$\begin{aligned}\|T_i(PT_i)^{(k+1)-1}x - T_i(PT_i)^{(k+1)-1}y\| &\leq \|P(T_i x)^{(k+1)-1} - P(T_i y)^{(k+1)-1}\| \\ &= \|P(T_i)P(T_i x)^{k-1} - P(T_i)P(T_i y)^{k-1}\| \\ &\leq \|T_i P(T_i x)^{k-1} - T_i P(T_i y)^{k-1}\| \\ &\leq \|x - y\|.\end{aligned}$$

It follws that from Mathematical Induction, T_i is a family of an asymptotically non expensive non-self mapping with sequence $\{h_{i,n}\}$ defined by $h_{i,n} = 1$ for all $n \geq 1$. Put

$$Fix(T_i) = \Big\{(0, 0, ..., \frac{1}{2}, 0, 0, ..., 0)\Big\}.$$

Now, we prove that the pairs (T_i, C) for each i satisfy the condition. For any closed subset G of C, we denoted $N_i = \{z = x - T_i x : x \in G\}$ Then N_i are closed. Reality is that, for any $z_n \in N_i$ with $z_n \to z$, there exist $x_n \in G$ such that $z_n = x_n - T_i x_n$. As G is bounded and closed in C so is compact. Therefore, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Letting $x_{n_k} \to x_0$ as $k \to \infty$, we have $x_0 \in G$. Also as T_i are continuous so

$$z = \lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} (x_{n_k} - T_i x_{n_k}) = x_0 - T_i x_0 \in N_i.$$

For any given $x_1 \in C$, take a sequence $\{x_n\}$

$$\begin{cases} y_n = P((1 - \beta_1)x_n + \beta_n T_i(PT_i)^{n-1}x_n), \\ z_n = P((1 - \gamma_1)y_n + \gamma_n T_i(PT_i)^{n-1}y_n), \\ x_{n+1} = P\big((1 - \alpha_1)x_n + \alpha_n T_i(PT_i)^{n-1}\big(\frac{y_n + z_n}{2}\big)\big) \end{cases}$$

for any $n \geq 1$, where

$$\begin{aligned}\alpha_n &= \frac{4}{5} + \frac{1}{6n}, \quad n = 0 \pmod 2, \\ \alpha_n &= \frac{1}{10} + \frac{1}{2n}, \quad n = 1 \pmod 2, \\ \beta_n &= \gamma_n = \frac{2n}{3n + 2}, \quad n \geq 1.\end{aligned}$$

Clearly

$$\liminf_{n\to\infty}\alpha_n=\frac{1}{10},\quad \limsup_{n\to\infty}\alpha_n=\frac{4}{5},$$
$$\limsup_{n\to\infty}\beta_n=\limsup_{n\to\infty}\gamma_n=\frac{2}{3}.$$

So all condition of Theorem 4.1 are satisfied and so $\{x_n\}$ converge strongly to a fixed point $(0,0,0,...,\frac{1}{2},0,0,...,0)$ of T_i.

Remark 5.2. If we take $T_1(x_1,x_2)=(1-x_1,0)$ with $x_1=(0,0)$, then $\{x_n\}$ converges strongly to $Fix(T_i)=(\frac{1}{2},0)$ after single iteration.

Example 5.3. Let $X=l^2$ with $\langle x,y\rangle=\sum_{i=1}^{\infty}x_iy_i$ and $\|x\|=\left(\sum_{i=1}^{\infty}x_i^2\right)^{\frac{1}{2}}$. Then X is a real infinite dimensional Hilbert space. Let $C=\{x\in X:\|x\|\leq 1\}$. Define a mapping $P:X\to C$ by

$$Px=\begin{cases}x, & \text{if } x\in X,\\ \frac{x}{\|x\|}, & \text{if } x\in X-C.\end{cases}$$

Then P is nonexpansive retraction of X onto C. Define $T:C\to X$ by

$$Tx=(-x_1,-x_2,...,-x_i,...),\quad \forall x\in C.$$

Then we have

$$\begin{aligned}\|Tx-Ty\|&=\|(y_1-x_1,y_2-x_2,...,y_i-x_i,...)\|\\&=\left(\sum_{i=1}^{\infty}(y_i-x_i)^2\right)^{\frac{1}{2}}=\|x-y\|\end{aligned}$$

for all $x=(x_1,x_2,...,x_i,...)$, $y=(y_1,y_2,...,y_i,...)\in C$ so T is an asymptotically nonexpansive non-self mapping with sequence $\{h_n\}$ defined by $h_n=1$ for all $n\geq 1$ and $Fix(T)=\{(0,0,...,0)\}$.

Now we prove that the pair (T,C) satisfy our condition and T is not completely continuous. In fact for any closed subset G of C, we denote $N=\{z=x-Tx:c\in G\}$. For any $z_n\in N$ with $z_n\to z$ as $n\to\infty$, there exists $z_n=x_n-Tx_n=2x_n$. It follows from $z_n\to z$ that $x_n\to\frac{1}{2}z$ as $n\to\infty$. Since G is closed in C, it follows that $\frac{1}{2}z\in G$. Since T is continuous, it follows that

$$z=\lim_{n\to\infty}z_n=\lim_{n\to\infty}(x_n-Tx_n)=\frac{1}{2}z-T\left(\frac{1}{2}z\right)\in N.$$

This shows that the pair (T,C) satisfy the condition (BP). Since T is surjective, and unit ball $C=\{x\in X:\|x\|\leq 1\}$, is not sequentially compact, so T is not completely continuous (see [5]).

For $x_1\in C$, define sequence $\{x_n\}$ by

$$\begin{cases}y_n=P((1-\beta_n)x_n+\beta_nT(PT)^{n-1}x_n),\\ z_n=P((1-\gamma_n)y_n+\gamma_nT(PT)^{n-1}y_n),\\ x_{n+1}=P\big((1-\alpha_n)x_n+\alpha_nT(PT)^{n-1}\big(\frac{y_n+z_n}{2}\big)\big),\end{cases}$$

where $\alpha_n = \frac{3}{5} + \frac{1}{4n}$, $n = 1 \pmod 2$, and $\alpha_n = \frac{1}{7} + \frac{1}{2n}$, $n = 0 \pmod 2$ for all $c \geq 1$, $\beta_n = \frac{3n}{4n+2}$, $\gamma_n = \frac{3n}{4n+2}$ for all $n \geq 1$. Clearly $\liminf_{n\to\infty} \alpha_n = \frac{1}{7}$ and $\limsup_{n\to\infty} \alpha_n = \frac{3}{5}$ and $\limsup_{n\to\infty} \beta_n = \limsup_{n\to\infty} \gamma_n = \frac{3}{4}$. So all conditions of Theorem 4.1 are satisfied. Hence the sequence $\{x_n\}$ converges strongly to fixed point $(0, 0, ..., 0)$ of T.

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On existence of nondecreasing solutions of q-quadratic integral equations

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Mohamed Abdalla Darwish[a,b], Maryam Al-Yami[b] and Kishin Sadarangani[c]

[a]Department of Mathematics, Faculty of Science, Damanhour University
Damanhour, Egypt
e-mail: dr.madarwish@gmail.com

[b]Department of Mathematics, Sciences Faculty for Girls, King Abdulaziz University
Jeddah, Saudi Arabia
e-mail: mohad20020@hotmail.com

[c]Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria
Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
e-mail: ksadaran@dma.ulpgc.es

Abstract

We investigate a q-fractional integral equation with supremum and prove an existence theorem for it. We will prove that our q-integral equation has a solution in $C[0,1]$ which is monotonic on $[0,1]$. The monotonicity measure of noncompactness due to Banaś and Olszowy and Darbo's theorem are the main tools used in the proof our main result.

MSC: 45G10, 47H09, 45M99.
Keywords: q-fractional; integral equation; monotonic solutions; Darbo theorem; monotonicity measure of noncompactness.

1 Introduction

Jackson in [20, 21] introduced the concept of quantum calculus (q-calculus). This area of research has rich history and several applications, see [1, 3, 22, 23] and references therein. There are several developments and applications of the q-calculus in mathematical physics,

especially concerning quantum mechanics, the theory of relativity and special functions [1, 3, 17, 23, 24]. Recently, several researchers attracted their attention by the concept of q-calculus, and we can find several new results in [2, 3, 18, 25] and the references therein.

In several papers among them [4, 19], differential equations with supremum as well as integral equations with supremum have been studied. In [10, **?**, 12, 13, 14, 15] Darwish *et al.* studied differential and integral equations of arbitrary orders with supremum. Also, Caballero *et al.* [8, 9] introduced and studied the quadratic Volterra equations with supremum. They showed that these equations have monotonic solutions in the space $C[0,1]$. In [10], Darwish generalized and extend Caballero *et al.* [8] results to the quadratic integral equations of arbitrary orders with supremum.

In this paper we will study the q-quadratic integral equation with supremum

$$y(t) = f(t) + \frac{t^{\beta-1}(Ty)(t)}{\Gamma_q(\beta)} \int_0^t (qs/t;q)_{\beta-1}\, \kappa(t,s) \max_{[0,\sigma(s)]} |y(\tau)|\, d_q s,\ t \in J = [0,1], \tag{1.1}$$

where $0 < \beta$, $q \in (0,1)$, $f : J \to \mathbb{R}$, $T : C(J) \to C(J)$, $\kappa : J \times J \to \mathbb{R}_+$ and $\sigma : J \to J$.

By using Darbo fixed point theorem and the monotonicity measure of noncompactness due to Banaś and Olszowy [6] we prove the existence of monotonic solution to Eq.(1.1) in $C[0,1]$.

2 Fractional q-calculus

We collect basic definitions and results of the q-fractional integrals and q-derivatives, for more details, see [2, 3, 7, 17, 18, 24, 25] and references therein.

First, for a real parameter $q \in (0,1)$, we define a q-real number $[a]_q$ by

$$[a]_q = \frac{1-q^a}{1-q},\ a \in \mathbb{R},$$

and a q-analog of the Pochhammer symbol (q-shifted factorial) is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1} (1-aq^k), & n \in \mathbb{N}. \end{cases}$$

Also, the q-analog of the power $(a-b)^n$ is given by

$$(a-b)^{(n)} = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1} (a-bq^k), & n \in \mathbb{N}; a, b \in \mathbb{R}. \end{cases}$$

Moreover,

$$(a-b)^{(n)} = a^n (b/a;q)_n,\ a \neq 0.$$

Notice that, $\lim_{n\to\infty}(a;q)_n$ exists and we will denote it by $(a;q)_\infty$.

More generally, for $\beta \in \mathbb{R}$, $aq^\beta \neq q^{-n}$ $(n \in \mathbb{N})$, we define

$$(a;q)_\beta = \frac{(a;q)_\infty}{(aq^\beta;q)_\infty}$$

and

$$(a-b)^{(\beta)} = a^\beta \frac{(b/a;q)_\infty}{(q^\beta b/a;q)_\infty}.$$

Notice that $(a-b)^{(\beta)} = a^\beta (b/a;q)_\beta$. Therefore, if $b=0$, then $a^{(\beta)} = a^\beta$.

Now, the q-gamma function is given by

$$\Gamma_q(x) = \frac{G(q^x)}{(1-q)^{x-1}G(q)},\ x \in \mathbb{R}\backslash\{0,-1,-2,\cdots\},$$

where $G(q^x) = \frac{1}{(q^x;q)_\infty}$. Or, equivalently, $\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$ and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

Next, the q-derivative of a function f is given by

$$(D_q f)(t) = \frac{f(t)-f(qt)}{t-qt},\ \ (D_q f)(0) = \lim_{t\to 0}(D_q f)(t),$$

and the q-derivative of higher order of a function f is defined by

$$(D_q^n f)(t) = \begin{cases} f(t), & n=0, \\ \\ D_q(D_q^{n-1}f)(t), & n \in \mathbb{N}. \end{cases}$$

Let f be a function defined on $[0,b]$. The q-integral of f is defined as follows

$$(I_q f)(t) = \int_0^t f(s)\ d_q s = t(1-q)\sum_{n=0}^{\infty} q^n f(tq^n),\ t \in [0,b]. \tag{2.2}$$

If f is given on the interval $[0,b]$ and $a \in [0,b]$, then

$$\int_a^b f(s)\ d_q s = \int_0^b f(s)\ d_q s - \int_0^a f(s)\ d_q s.$$

The operator I_q^n is defined by

$$(I_q^n f)(t) = \begin{cases} f(t), & n=0, \\ \\ I_q(I_q^{n-1}f)(t), & n \in \mathbb{N}. \end{cases}$$

The fundamental theorem of calculus satisfies for D_q and I_q, i.e., $(D_q I_q f)(t) = f(t)$, and if f is continuous at $t=0$, then $(I_q D_q f)(t) = f(t) - f(0)$.

The following four formulas will be used later in this paper

$$[a(t-s)]^{(\beta)} = a^{\beta}(t-s)^{(\beta)},$$
$$_tD_q(t-s)^{(\beta)} = [\beta]_q(t-s)^{(\beta-1)},$$
$$_sD_q(t-s)^{(\beta)} = -[\beta]_q(t-qs)^{(\beta-1)}$$

and

$$_tD_q \int_0^t f(t,s)\ d_qs = \int_0^t {}_tD_q f(t,s)\ d_qs + f(qt,t),$$

where $_tD_q$ denotes the derivative with respect to variable t.

Notice that, if $\beta > 0$ and $a \leq b \leq t$, then $(t-b)^{(\beta)} \leq (t-a)^{(\beta)}$.

Definition 1. [1] Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \geq 0$ is given by

$$(I_q^{\beta} f)(t) = \begin{cases} f(t), & \beta = 0, \\ \frac{1}{\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} f(s)\ d_qs = t^{\beta}(1-q)^{\beta} \sum_{n=0}^{\infty} q^n \frac{(q^{\beta};q)_n}{(q;q)_n} f(tq^n), & \beta > 0,\ t \in [0,1]. \end{cases}$$

Notice that, for $\beta = 1$, the above q-integral reduces to (2.2).

Definition 2. [1] The fractional q-derivative of the Riemann-Liouville type of order $\beta \geq 0$ is given by

$$(D_q^{\beta} f)(t) = \begin{cases} f(t), & \beta = 0, \\ (D_q^{[\beta]} I_q^{[\beta]-\beta} f)(t), & \beta > 0, \end{cases}$$

where $[\beta]$ stands for the smallest integer equal or greater than β.

In $q-$calculus, the derivative rule for the product of two functions and integration by parts formulas are

$$(D_q fg)(t) = (D_q f)(t)g(t) + f(qt)(D_q g)(t),$$
$$\int_0^t f(s)(D_q g)(s)\ d_qs = [f(s)g(s)]_0^t - \int_0^t (D_q f)(s)g(qs)\ d_qs.$$

Lemma 1. *Let $\beta, \gamma \geq 0$. Then the following are verified for a function f defined on $[0,1]$:*
(1) $(I_q^{\gamma} I_q^{\beta} f)(t) = (I_q^{\beta+\gamma} f)(t)$,
(2) $(D_q^{\beta} I_q^{\beta} f)(t) = f(t)$.

Lemma 2. *[24] For $\beta > 0$. Then q-integration by parts allows us to have*

$$(I_q^{\beta} 1)(t) = \frac{t^{(\beta)}}{\Gamma_q(\beta+1)}$$

or

$$\int_0^t (t-qs)^{(\beta-1)}\ d_qs = \frac{t^{(\beta)}}{[\beta]_q}.$$

3 Measure of noncompactness

We assume that $(E, \|.\|)$ is a real Banach space with zero element θ and we denote by $B(x, r)$ the closed ball with radius r and centre x, where $B_r \equiv B(\theta, r)$.

Now, let $X \subset E$ and denote by $\overline{X}$ and ConvX the closure and convex closure of X, respectively. Also, the symbols $X+Y$ and λY stands for the usual algebraic operators on sets. Moreover, the families $\mathfrak{M}_E$ and $\mathfrak{N}_E$ are defined by $\mathfrak{M}_E = \{A \subset E : A \neq \emptyset,\ A \text{ is bounded}\}$ and $\mathfrak{N}_E = \{B \subset \mathfrak{M}_E : B \text{ is relatively compact}\}$, respectively.

Definition 3. [5] Let $\mu : \mathfrak{M}_E \to [0, +\infty)$. If the following conditions

1° $\emptyset \neq \{X \in \mathfrak{M}_E : \mu(X) = 0\} = \ker\mu \subset \mathfrak{N}_E$,

2° if $X \subset Y$, then $\mu(X) \leq \mu(Y)$,

3° $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv}X)$,

4° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, $0 \leq \lambda \leq 1$ and

5° if (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ with $X_{n+1} \subset X_n$, $n = 1,\ 2,\ 3,\ ...$, and $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$

hold. Then, the mapping μ is said to be a measure of noncompactness in E.

Here, $\ker\mu$ is the kernel of the measure of noncompactness μ.

Our result will establish in $C(J)$ the Banach space of all defined, continuous and real functions on $J \equiv [0, 1]$ with $\|y\| = \max\{|y(\tau)| : \tau \in J\}$.

Next, we defined the measure of noncompactness related to monotonicity in $C(J)$, see [5, 6].

We fix a bounded subset $Y \neq \emptyset$ of $C(J)$. For $\varepsilon \geq 0$ and $y \in Y$, $\omega(y, \varepsilon)$ denotes the modulus of continuity of the function y given by

$$\omega(y, \varepsilon) = \sup\{|y(t) - y(s)| : t, s \in J,\ |t - s| \leq \varepsilon\}.$$

Moreover, we let

$$\omega(Y, \varepsilon) = \sup\{\omega(y, \varepsilon) : y \in Y\}$$

and

$$\omega_0(Y) = \lim_{\varepsilon\to 0} \omega(Y, \varepsilon).$$

Define

$$d(y) = \sup_{t,s\in J,\ s\leq t} (|y(t) - y(s)| - [y(t) - y(s)])$$

and

$$d(Y) = \sup_{y\in Y} d(y).$$

Notice that all functions in Y are nondecreasing on J if and only if $d(Y) = 0$.

Now, we define the map μ on $\mathfrak{M}_{C(J)}$ as

$$\mu(Y) = d(Y) + \omega_0(Y).$$

Clearly, μ verifies all conditions in Definition 3 and, therefore it is a measure of noncompactness in $C(J)$ [6].

Definition 4. Let $\emptyset \neq M \subset E$. Let $\mathcal{P} : M \to E$ be a continuous operator. Suppose that $\mathcal{P}$ maps bounded sets onto bounded ones. If there exists a bounded $Y \subset M$ with $\mu(\mathcal{P}Y) \leq \alpha\mu(Y)$, $\alpha \geq 0$, then $\mathcal{P}$ is said to be satisfies the Darbo condition with respect to a measure of noncompactness μ.

If $\alpha < 1$, then $\mathcal{P}$ is called a contraction operator with respect to μ.

Theorem 1. *[16] Let $\Omega \neq \emptyset$ be a bounded, convex and closed subset of E. If $\mathcal{P} : \Omega \to \Omega$ is a contraction operator with respect to μ. Then $\mathcal{P}$ has at least one fixed point belongs to Ω.*

We will need the following two lemmas throughout our proof [8].

Lemma 3. *Let $r : J \to J$ be a continuous function and $y \in C(J)$. If, for $t \in J$,*

$$(Fy)(t) = \max_{[0,\sigma(t)]} |y(\tau)|,$$

then $Fy \in C(J)$.

Lemma 4. *Let (y_n) be a sequence in $C(J)$ and $y \in C(J)$. If (y_n) converges to $y \in C(J)$, then (Fy_n) converges uniformly to Fy uniformly J.*

4 Main Theorem

Let us consider the following hypotheses:

- (h_1) $f \in C(J)$. Moreover, f is nondecreasing and nonnegative on J.
- (h_2) The operator $T : C(J) \to C(J)$ is continuous and satisfies the Darbo condition with a constant c for the measure of noncompactness μ . Moreover, $Tx \geq 0$ if $x \geq 0$.
- (h_3) $\exists\ a, b \geq 0\ s.t.\ |(Tx)(t)| \leq a + b\|x\|\ \forall x \in C(J),\ t \in J$.
- (h_4) The function $\kappa : J \times J \to \mathbb{R}_+$ is continuous on $J \times J$ and nondecreasing $\forall t$ and s separately. Moreover, $\kappa^* = \sup_{(t,s)\in J\times J} \kappa(t, s)$.
- (h_5) The function $\sigma : J \to J$ is nondecreasing and continuous on J.
- (h_6) $\exists\ r_0 > 0$ such that

$$\|f\| + \frac{\kappa^* r_0(a + br_0)}{\Gamma_q(\beta + 1)} \leq r_0 \tag{4.3}$$

and $\frac{ck^* r_0}{\Gamma_q(\beta+1)} < 1$.

Now, we rewrite Eq.(1.1) as

$$x(t) = f(t) + \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)\,(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s,\ 0 < \beta \leq 1,\ t \in J, \tag{4.4}$$

and define the two operators $\mathcal{K}$ and $\mathcal{F}$ on $C(J)$ as follows

$$(\mathcal{K}y)(t) = \frac{1}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |y(\tau)|\, d_q s \tag{4.5}$$

and

$$(\mathcal{F}y)(t) = f(t) + (Ty)(t) \cdot (\mathcal{K}y)(t), \tag{4.6}$$

respectively. Finding a fixed point of the operator $\mathcal{F}$ is equivalent to solving Eq.(4.4).

Under the above hypotheses, we state and prove our main theorem.

Theorem 2. *Assume the hypotheses $(h_1) - (h_6)$ be verified. Then Eq.(4.4) has at least one solution $x \in C(J)$ which is nondecreasing on J.*

Proof. First, we will show that the operator $\mathcal{F}$ maps $C(J)$ into itself. For this, it is sufficient to show that $\mathcal{K}x \in C(J)$ if $x \in C(J)$. Fix $\varepsilon > 0$ and let $x \in C(J)$ and $t_1,\ t_2 \in J$ $(t_1 \leq t_2)$ with $|t_2 - t_1| \leq \varepsilon$. We have

$$\begin{aligned}
|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| &= \Big| \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad - \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big| \\
&\leq \Big| \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad - \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big| \\
&\quad + \Big| \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad - \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big| \\
&\quad + \Big| \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad - \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big| \\
&\leq \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} |\kappa(t_2,s) - \kappa(t_1,s)|(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + \frac{1}{\Gamma_q(\beta)} \int_{t_1}^{t_2} |\kappa(t_1,s)|(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma_q(\beta)}\int_0^{t_1}|\kappa(t_1,s)|\,|(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}|\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\\
\leq\;& \frac{\|x\|}{\Gamma_q(\beta)}\,\omega_\kappa(\varepsilon,.)\int_0^{t_2}(t_2-qs)^{(\beta-1)}\,d_qs\\
&+\frac{\kappa^*\|x\|}{\Gamma_q(\beta)}\left\{\int_0^{t_1}[(t_1-qs)^{(\beta-1)}-(t_2-qs)^{(\beta-1)}]d_qs+\int_{t_1}^{t_2}(t_2-qs)^{(\beta-1)}d_qs\right\}\\
=\;& \frac{\|x\|}{\Gamma_q(\beta+1)}\,\omega_\kappa(\varepsilon,.)\,t_2^{(\beta)}+\frac{\kappa^*\,\|x\|}{\Gamma_q(\beta+1)}[t_1^{(\beta)}-t_2^{(\beta)}+2(t_2-t_1)^{(\beta)}]\\
\leq\;& \frac{\|x\|}{\Gamma_q(\beta+1)}\,\omega_\kappa(\varepsilon,.)\,t_2^{(\beta)}+\frac{2\kappa^*\,\|x\|}{\Gamma_q(\beta+1)}(t_2-t_1)^{(\beta)}\\
\leq\;& \frac{\|x\|}{\Gamma_q(\beta+1)}\,\omega_\kappa(\varepsilon,.)\,t_2^{\beta}+\frac{\kappa^*\,\|x\|}{\Gamma(\beta+1)}\,\varepsilon^{\beta},
\end{aligned}
\tag{4.7}
$$

where we used

$$\omega_\kappa(\varepsilon,.)=\sup_{t,\,\tau\in J,\,|t-\tau|\leq\varepsilon}|\kappa(t,s)-\kappa(\tau,s)|.$$

Notice that, since the function κ is uniformly continuous on $J\times J$, then when $\varepsilon\to 0$ we have that $\omega_\kappa(\varepsilon,.)\to 0$.

Therefore, $\mathcal{K}x\in C(J)$ and consequently, $\mathcal{F}x\in C(J)$.

Now, $\forall t\in J$, we have

$$
\begin{aligned}
|(\mathcal{F}x)(t)| &\leq \left|f(t)+\frac{(Tx)(t)}{\Gamma_q(\beta)}\int_0^t\kappa(t,s)(t-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\right|\\
&\leq \|f\|+\frac{a+b\|x\|}{\Gamma_q(\beta)}\int_0^t\kappa(t,s)(t-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\\
&\leq \|f\|+\frac{a+b\|x\|}{\Gamma_q(\beta+1)}\kappa^*\|x\|.
\end{aligned}
$$

Hence

$$\|\mathcal{F}x\|\leq\|f\|+\frac{a+b\|x\|}{\Gamma_q(\beta+1)}\kappa^*\|x\|.$$

From hypothesis (h_6), if $\|x\|\leq r_0$, we get

$$
\begin{aligned}
\|\mathcal{F}x\| &\leq \|f\|+\frac{a+br_0}{\Gamma_q(\beta+1)}\kappa^*r_0\\
&\leq r_0.
\end{aligned}
$$

Therefore, $\mathcal{F}$ maps B_{r_0} into itself.

Next, we consider the operator $\mathcal{F}$ on the set $B_{r_0}^+=\{x\in B_{r_0}: x(t)\geq 0,\ \forall t\in J\}$. It is clear that $B_{r_0}^+\neq\emptyset$ is closed, convex and bounded. By this facts and hypotheses (h_1), (h_3) and (h_5), we obtain $\mathcal{F}$ transforms $B_{r_0}^+$ into itself.

In what follows, we will show that $\mathcal{F}$ is continuous on $B_{r_0}^+$. For, let (x_n) be us a sequence in $B_{r_0}^+$ such that $x_n\to x$ and we will show that $\mathcal{F}x_n\to\mathcal{F}x$. We have, $\forall t\in J$,

$$|(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)| = \left|\frac{(Tx_n)(t)}{\Gamma_q(\beta)}\int_0^t\kappa(t,s)(t-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x_n(\tau)|d_qs\right.$$

$$
\begin{aligned}
&\quad - \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| d_q s \Big| \\
\leq &\quad \Big| \frac{(Tx_n)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s \\
&\quad - \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s \Big| \\
&\quad + \Big| \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s \\
&\quad - \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_q s \Big| \\
\leq &\quad \frac{|(Tx_n)(t) - (Tx)(t)|}{\Gamma_q(\beta)} \int_0^t |\kappa(t,s)|(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s \\
&\quad + \frac{|(Tx)(t)|}{\Gamma_q(\beta)} \int_0^t |\kappa(t,s)|(t-qs)^{(\beta-1)} \Big| \max_{[0,\sigma(s)]} |x_n(\tau)| - \max_{[0,\sigma(s)]} |x(\tau)| \Big| \, d_q s.
\end{aligned}
$$

Applying Lemma 4, we obtain

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leq \frac{\kappa^* r_0}{\Gamma_q(\beta+1)} \|Tx_n - Tx\| + \frac{\kappa^*(a + b\, r_0)}{\Gamma_q(\beta+1)} \|x_n - x\|. \tag{4.8}$$

By the continuity of T, $\exists n_1 \in \mathbb{N}$ such that

$$\|Tx_n - Tx\| \leq \frac{\varepsilon \Gamma_q(\beta+1)}{2\kappa^* r_0} \; \forall n \geq n_1.$$

Also, $\exists n_2 \in \mathbb{N}$ such that

$$\|x_n - x\| \leq \frac{\varepsilon \Gamma_q(\beta+1)}{2\kappa^*(a + b r_0)} \; \forall n \geq n_2.$$

Now, take $\max\{n_1, n_2\} \leq n$, then (4.8) gives us that

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leq \varepsilon.$$

This shows that $\mathcal{F}$ is continuous in $B_{r_0}^+$.

Now, we take $\emptyset \neq X \subset B_{r_0}^+$. Let us fix an arbitrarily number $\varepsilon > 0$ and choose $x \in X$ and $t_1,\ t_2 \in J$ with $|t_2 - t_1| \leq \varepsilon$. We will assume that $t_1 \leq t_2$ because no generality will be loss. Then, by using our hypotheses and inequality (4.7), we get

$$
\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| &\leq |f(t_2) - f(t_1)| + |(Tx)(t_2)\ (\mathcal{K}x)(t_2) - (Tx)(t_2)\ (\mathcal{K}x)(t_1)| \\
&\quad + |(Tx)(t_2)\ (\mathcal{K}x)(t_1) - (Tx)(t_1)\ (\mathcal{K}x)(t_1)| \\
&\leq \omega(f,\varepsilon) + |(Tx)(t_2)|\ |(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\
&\quad + |(Tx)(t_2) - (Tx)(t_1)|\ |(\mathcal{K}x)(t_1)| \\
&\leq \omega(f,\varepsilon) + \frac{(a + b\|x\|)}{\Gamma_q(\beta+1)} \left[\|x\| \omega_\kappa(\varepsilon, .) + 2\kappa^* \|x\| \varepsilon^\beta \right] + \frac{\omega(Tx,\varepsilon)}{\Gamma_q(\beta+1)} \|x\| \kappa^*
\end{aligned}
$$

$$\leq \quad \omega(f,\varepsilon)+\frac{r_0(a+br_0)}{\Gamma_q(\beta+1)}[\omega_\kappa(\varepsilon,.)+2\ \kappa^*\varepsilon^\beta]+\frac{\kappa^* r_0}{\Gamma_q(\beta+1)}\omega(Tx,\varepsilon).$$

Hence,

$$\omega(\mathcal{F}x,\varepsilon)\leq \omega(f,\varepsilon)+\frac{r_0(a+br_0)}{\Gamma_q(\beta+1)}[\omega_\kappa(\varepsilon,.)+2\kappa^*\varepsilon^\beta]+\frac{\kappa^*\ r_0}{\Gamma_q(\beta+1)}\omega(Tx,\varepsilon).$$

Consequently,

$$\omega(\mathcal{F}X,\varepsilon)\leq \omega(f,\varepsilon)+\frac{r_0(a+br_0)}{\Gamma_q(\beta+1)}[\omega_\kappa(\varepsilon,.)+2\kappa^*\varepsilon^\beta]+\frac{\kappa^* r_0}{\Gamma_q(\beta+1)}\omega(TX,\varepsilon).$$

Since the function κ is uniformly continuous on $J\times J$ and the function f is continuous on J, then the last inequality gives us that

$$\omega_0(\mathcal{F}X)\leq \frac{\kappa^* r_0}{\Gamma_q(\beta+1)}\omega_0(TX). \tag{4.9}$$

Further, fix arbitrary $x\in X$ and $t_1,t_2\in J$ with $t_2>t_1$. Then, by our hypotheses, we have

$$\begin{aligned}
&|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)|-[(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)]\\
&=\Bigg|f(t_2)+\frac{(Tx)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\\
&\quad -f(t_1)-\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}\kappa(t_1,s)(t_1-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\Bigg|\\
&\quad -\Bigg[f(t_2)+\frac{(Tx)(t_2)}{\Gamma(_q\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\\
&\quad -f(t_1)-\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}\kappa(t_1,s)(t_1-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\Bigg]\\
&\leq \{|f(t_2)-f(t_1)|-[f(t_2)-f(t_1)]\}+\Bigg|\frac{(Tx)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\\
&\quad -\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\Bigg|\\
&\quad +\Bigg|\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\\
&\quad -\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}\kappa(t_1,s)(t_1-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\Bigg|\\
&\quad -\Bigg\{\Bigg[\frac{(Tx)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\\
&\quad -\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs\Bigg]\\
&\quad +\Bigg[\frac{(Tx)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\ d_qs
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big] \Big\} \\
&\leq \{|(Tx)(t_2)-(Tx)(t_1)| - [(Tx)(t_2)-(Tx)(t_1)]\} \\
&\quad \times \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \Big\{ \Big| \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad - \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big| \\
&\quad - \Big[\int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad - \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \Big] \Big\}. \qquad (4.10)
\end{aligned}
$$

Now, we will prove that

$$
\int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s - \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \geq 0.
$$

In fact, we have

$$
\begin{aligned}
&\int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s - \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&= \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s - \int_0^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + \int_0^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s - \int_0^{t_1} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + \int_0^{t_1} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s - \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&= \int_0^{t_2} (\kappa(t_2,s)-\kappa(t_1,s))(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + + \int_{t_1}^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + \int_0^{t_1} \kappa(t_1,s)[(t_2-qs)^{(\beta-1)} - (t_1-qs)^{(\beta-1)}] \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s.
\end{aligned}
$$

But, $\kappa(t_1,s) \leq \kappa(t_2,s)$ because $\kappa(t,s)$ is increasing with respect to t, then

$$
\int_0^{t_2} (\kappa(t_2,s)-\kappa(t_1,s))(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \geq 0 \qquad (4.11)
$$

and, since $(t_2-qs)^{(\beta-1)} - (t_1-qs)^{(\beta-1)} \geq 0$ for $s \in [0,t_1)$ then

$$
\int_0^{t_1} \kappa(t_1,s)[(t_2-qs)^{(\beta-1)} - (t_1-qs)^{(\beta-1)}] \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s
$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\kappa(t_1,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\\
&\geq\int_0^{t_1}\kappa(t_1,t_1)[(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}]\max_{[0,\sigma(t_1)]}|x(\tau)|\,d_qs\\
&+\int_{t_1}^{t_2}\kappa(t_1,t_1)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(t_1)]}|x(\tau)|\,d_qs\\
&=\kappa(t_1,t_1)\max_{[0,\sigma(t_1)]}|x(\tau)|\left[\int_0^{t_2}(t_2-qs)^{(\beta-1)}\,d_qs-\int_0^{t_1}(t_1-qs)^{(\beta-1)}\,d_qs\right]\\
&=\kappa(t_1,t_1)\frac{t_2^{\beta}-t_1^{\beta}}{[\beta]_q}\max_{[0,\sigma(t_1)]}|x(\tau)|\\
&\geq 0. \qquad (4.12)
\end{aligned}
$$

Finally, (4.11) and (4.12) imply that

$$\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs-\int_0^{t_1}\kappa(t_1,s)(t_1-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\geq 0.$$

The above inequality and (4.10) leads us to

$$
\begin{aligned}
&|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)|-[(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)]\\
&=\{|(Tx)(t_2)-(Tx)(t_1)|-[(Tx)(t_2)-(Tx)(t_1)]\}\\
&\times\frac{1}{\Gamma_q(\beta)}\int_0^{t_2}\kappa(t_2,s)(t_2-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\\
&\leq\frac{\kappa^*r_0}{\Gamma_q(\beta+1)}d(Tx).
\end{aligned}
$$

Thus,

$$d(\mathcal{F}x)\Gamma_q(\beta+1)\leq\kappa^*r_0d(Tx)$$

and therefore,

$$d(\mathcal{F}X)\Gamma_q(\beta+1)\leq\kappa^*r_0d(TX). \qquad (4.13)$$

Finally, (4.9) and (4.13) gives us that

$$\omega_0(\mathcal{F}X)+d(\mathcal{F}X)\leq\frac{\kappa^*r_0}{\Gamma_q(\beta+1)}(\omega_0(\mathcal{F}X)+d(TX))$$

or

$$
\begin{aligned}
\mu(\mathcal{F}X) &\leq \frac{r_0\kappa^*}{\Gamma_q(\beta+1)}\mu(TX)\\
&\leq \frac{\kappa^*cr_0}{\Gamma_q(\beta+1)}\mu(X).
\end{aligned}
$$

But $\frac{\kappa^*r_0c}{\Gamma_q(\beta+1)}<1$, then

$$\mu(\mathcal{F}X)\leq\mu(X). \qquad (4.14)$$

Inequality (4.14) enables us to use Theorem 1, then there are solutions to Eq.(1.1) in $C(J)$. This finishes our proof. □

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